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# Addendum to *Pontryagin's maximum principle for dynamic systems on time scales*

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## Abstract

This note is an addendum to [1, 2], pointing out the differences between these papers and raising open questions.

**Keywords:** time scale; optimal control; Pontryagin maximum principle; Ekeland variational principle; packages of needle-like variations.

**AMS Classification:** 34K35; 34N99; 39A12; 39A13; 49K15; 93C15; 93C55.

**The main differences.** In view of establishing a time scale version of the Pontryagin Maximum Principle (PMP), the authors of [1, Theorem 1] have developed in 2013 a strategy of proof based on the *Ekeland variational principle*. This strategy was originally considered for the classical continuous case by Ivar Ekeland in his seminal paper [3].

The authors of [2, Theorem 2.11] developed in 2017 a different approach, with *packages of needle-like variations* and *necessary conditions for an extreme in a cone*. Note that the authors of [2] prove moreover in [2, Theorem 2.13] that the necessary conditions derived in the PMP are also sufficient in the linear-convex case.

In the sequel of this paragraph, we focus on the major pros and cons of each approach:

1. In [1]:
  - (a) The set  $\Omega$  of control constraints is assumed to be closed. This is in order to apply the Ekeland variational principle on a complete metric space.
  - (b) There is no assumption on the time scale  $\mathbb{T}$ .
2. In [2]:
  - (a) The set  $\Omega$  of control constraints is assumed to be convex, but need not to be closed.

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- (b) The time scale  $\mathbb{T}$  is assumed to satisfy *density conditions* (see [2, Definition 2.4]) of the kind

$$\lim_{\substack{\beta \rightarrow 0^+ \\ s+\beta \in \mathbb{T}}} \frac{\mu(s+\beta)}{\beta} = 0,$$

for every right-dense points  $s$ , in order to guarantee that

$$\lim_{\beta \rightarrow 0^+} \frac{1}{\beta} \int_{[s, s+\beta)_{\mathbb{T}}} x(\tau) \Delta\tau = x(s),$$

for  $\Delta$ -integrable function  $x$  and for right-dense  $\Delta$ -Lebesgue points  $s$ , even for  $\beta > 0$  such that  $s + \beta \notin \mathbb{T}$ . Note that a discussion about this issue was provided in [1, Section 3.1].

Hence, the method developed in [2] allows to remove the closedness assumption done on  $\Omega$  in [1], but this is at the price of an additional assumption on the time scale  $\mathbb{T}$ .

In [1, Section 3.1], the authors explained why other approaches (other than the Ekeland variational principle), based for instance on implicit function arguments, or on Brouwer fixed point arguments, or on separation (Hahn-Banach) arguments, fail for general time scales.

As a conclusion, a time scale version of the PMP without closedness assumption on  $\Omega$  and without any assumption on the time scale  $\mathbb{T}$  still remains an open challenge.

**Additional comments on the terminal constraints.** In [1] the authors considered constraints on the initial/final state of the kind  $g(x(t_0), x(t_1)) \in S$ , where  $S$  is a nonempty closed convex set and  $g$  is a general smooth function.

In [2] the authors considered constraints on the initial/final state of the kind  $\Phi_i(x(t_0), x(t_1)) = 0$  for  $i = 1, \dots, k$ , and  $\Phi_i(x(t_0), x(t_1)) \leq 0$  for  $i = k+1, \dots, n$ , where  $\Phi_i$  are general smooth functions.

Contrarily to what is claimed in [2], the terminal constraints considered in [2] are only a particular case of the ones considered in [1]. Indeed, it suffices to take

$$g = (\Phi_1, \dots, \Phi_k, \Phi_{k+1}, \dots, \Phi_n)$$

and

$$S = \{0\} \times \dots \times \{0\} \times \mathbb{R}^- \times \dots \times \mathbb{R}^-.$$

Moreover, note that the necessary condition  $-\Psi \in \mathcal{O}_S(g(x(t_0), x(t_1)))$  obtained in [1, Theorem 1] encompasses both the *sign condition (1)* and the *complementary slackness (2)* obtained in [2, Theorem 2.11]. For the sign condition, it is sufficient to recall that the orthogonal of  $\mathbb{R}^-$  at a point  $x \in \mathbb{R}^-$  is included in  $\mathbb{R}^+$ . For the complementary slackness, it is sufficient to recall that the orthogonal of  $S$  at  $g(x(t_0), x(t_1))$  is reduced to  $\{0\}$  when  $g(x(t_0), x(t_1))$  belongs to the interior of  $S$ .

**Additional comments on the convexity of  $\Omega$ .** The set  $\Omega$  is assumed to be convex in [2], while it is not in [1]. As explained in [1, Section 3.1], in order to apply necessary conditions of an extreme in a cone, the authors of [2] require that the parameters of perturbations live in intervals. As a consequence, in order to remove the convexity assumption on  $\Omega$ , one would need (local-directional) convexity of the set  $\Omega$  for perturbations at right-scattered points, which is a concept that differs from the stable  $\Omega$ -dense directions used in [1]. Hence, in spite of the claim done in [2], the convexity assumption on  $\Omega$  does not seem to be easily removable.

**On the universal Lagrange multipliers.** This paragraph is devoted to providing more details on the existence of universal Lagrange multipliers claimed in [2, page 25]. In the sequel, we use the notations of [2], and we denote by  $\mathcal{S}$  the unit sphere of  $\mathbb{R}^{n+1}$ .

A package  $P$  consists of:

- $N \in \mathbb{N}$  and  $\nu \in \mathbb{N}$ ;
- $\bar{\tau} = (\tau_1, \dots, \tau_N)$  where  $\tau_i$  are right-dense points of  $\mathbb{T}$ ;
- $\bar{v} = (v_1, \dots, v_N)$  where  $v_i \in U$ ;
- $\bar{r} = (r_1, \dots, r_\nu)$  where  $r_i$  are right-scattered points of  $\mathbb{T}$ .
- $\bar{z} = (z_1, \dots, z_\nu)$  where  $z_i \in U$ .

Let  $(P_i)_{i \in I}$  denotes the set of all possible packages.

Following the proof of [2, Theorem 2.11], for every  $i \in I$ , there exists a nonzero vector  $\lambda = (\lambda_0, \dots, \lambda_n)$  (that we renormalize in  $\mathcal{S}$ ) of Lagrange multipliers such that:

- (i) (1) and (2) in [2, Theorem 2.11] are satisfied;
- (ii) the adjoint vector  $\Psi$  solution of (2.9), with the final condition (3.65) which depends on  $\lambda$ , satisfies the initial condition  $\Psi(t_0) = L_{x_0}$ ;
- (iii) (4a) and (4b) in [2, Theorem 2.11] are satisfied, but only at the points contained in  $\bar{\tau}$  and  $\bar{r}$  respectively.

For every  $i \in I$ , the above vector  $\lambda$  is not necessarily unique. Then, for every  $i \in I$ , we denote by  $K_i$  the set of all nonzero and renormalized Lagrange multiplier vectors associated with  $P_i$  satisfying the above properties.

By continuity of the adjoint vector  $\Psi$  with respect to the Lagrange multipliers (dependence from its final condition), we infer that  $K_i$  is a nonempty closed subset contained in the compact  $\mathcal{S}$ . This is true for every  $i \in I$ .

Now, let us prove that the family  $(K_i)_{i \in I}$  satisfies the finite intersection property. Let  $J \subset I$  be a finite subset and let us prove that  $\bigcap_{i \in J} K_i \neq \emptyset$ . Note that we can construct a package  $P$  corresponding to the union of all packages  $P_i$  with  $i \in J$ . It follows that  $P \in (P_i)_{i \in I}$ , and thus there exists a nonzero and renormalized Lagrange multiplier vector  $\lambda$  associated with  $P$  satisfying the above properties. Since  $\lambda \in K_i$  for every  $i \in J$ , we conclude that  $\bigcap_{i \in J} K_i \neq \emptyset$ .

It follows from the lemma of a centered system in a compact set that  $\bigcap_{i \in I} K_i \neq \emptyset$ , and we deduce the existence of a universal Lagrange multiplier vector.

**On the density conditions and the Cantor set.** Contrarily to what is claimed in [2, Example 2.5], the classical Cantor set does not satisfy the density conditions. However, generalized versions of the Cantor set (see, e.g., [4]) that satisfy density conditions can be constructed as follows.

Let  $(\alpha_k)_{k \in \mathbb{N}}$  be a real sequence such that  $0 < \alpha_k < \frac{1}{2}$  for all  $k \in \mathbb{N}$ , and such that  $\lim_{k \rightarrow +\infty} \alpha_k = \frac{1}{2}$ . Let  $(A_k)_{k \in \mathbb{N}}$  be the sequence of compact subsets defined by the induction

$$A_0 = [0, 1], \quad A_{k+1} = \mathcal{T}_k(A_k) \quad \forall k \in \mathbb{N},$$

where  $\mathcal{T}_k$  denotes the operator removing the open  $(\alpha_k, 1 - \alpha_k)$ -central part of all intervals. Note that the classical Cantor set corresponds to the case where  $\alpha_k = \frac{1}{3}$  for every  $k \in \mathbb{N}$ .

In our situation, we obtain

$$A_1 = [0, \alpha_0] \cup [1 - \alpha_0, 1],$$

$$A_2 = \left( [0, \alpha_1 \alpha_0] \cup [(1 - \alpha_1) \alpha_0, \alpha_0] \right) \cup \left( [1 - \alpha_0, 1 - (1 - \alpha_1) \alpha_0] \cup [1 - \alpha_1 \alpha_0, 1] \right),$$

etc. We define the generalized Cantor set  $\mathbb{T} = \bigcap_{k \in \mathbb{N}} A_k$ . In order to prove that the time scale  $\mathbb{T}$  satisfies the density conditions, from the fractal properties of  $\mathbb{T}$ , it suffices to prove that the density condition is satisfied at the right-dense point  $0 \in \mathbb{T}$ . More precisely, it is sufficient to prove that

$$\lim_{\substack{\beta \rightarrow 0^+ \\ \beta \in \mathbb{T}}} \frac{\mu(\beta)}{\beta} = 0.$$

Since  $\mu(\beta) = 0$  for every right-dense point  $\beta$ , we only have to consider the case where  $\beta$  is a right-scattered point of  $\mathbb{T}$ . In that case, one can easily see that  $\frac{\mu(\beta)}{\beta} \leq \frac{1 - 2\alpha_k}{\alpha_k}$  for some  $k \in \mathbb{N}$  and that  $k$  tends to  $+\infty$  when  $\beta$  tends to 0. The conclusion follows from the fact that  $\lim_{k \rightarrow +\infty} \alpha_k = \frac{1}{2}$ .

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