# Addendum to Pontryagin's maximum principle for dynamic systems on time scales

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# Addendum to *Pontryagin's maximum principle for dynamic systems on time scales*

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May 9, 2017

**Abstract**

This note is an addendum to [1, 2], pointing out the differences between these papers and raising open questions.

**Keywords:** time scale; optimal control; Pontryagin maximum principle; Ekeland variational principle; packages of needle-like variations.

**AMS Classification:** 34K35; 34N99; 39A12; 39A13; 49K15; 93C15; 93C55.

**The main differences.** In view of establishing a time scale version of the Pontryagin Maximum Principle (PMP), the authors of [1, Theorem 1] have developed in 2013 a strategy of proof based on the *Ekeland variational principle*. This strategy was originally considered for the classical continuous case by Ivar Ekeland in his seminal paper [3].

The authors of [2, Theorem 2.11] developed in 2017 a different approach, with *packages of needle-like variations* and *necessary conditions for an extreme in a cone*. Note that the authors of [2] prove moreover in [2, Theorem 2.13] that the necessary conditions derived in the PMP are also sufficient in the linear-convex case.

In the sequel of this paragraph, we focus on the major pros and cons of each approach:

1. In [1]:
   (a) The set $\Omega$ of control constraints is assumed to be closed. This is in order to apply the Ekeland variational principle on a complete metric space.
   (b) There is no assumption on the time scale $\mathbb{T}$.
2. In [2]:
   (a) The set $\Omega$ of control constraints is assumed to be convex, but need not to be closed.

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(b) The time scale $\mathbb{T}$ is assumed to satisfy *density conditions* (see [2, Definition 2.4]) of the kind

$$\lim_{\substack{\beta\to 0^+\\ s+\beta\in\mathbb{T}}} \frac{\mu(s+\beta)}{\beta} = 0,$$

for every right-dense points $s$, in order to guarantee that

$$\lim_{\beta\to 0^+} \frac{1}{\beta}\int_{[s,s+\beta)_{\mathbb{T}}} x(\tau)\,\Delta\tau = x(s),$$

for $\Delta$-integrable function $x$ and for right-dense $\Delta$-Lebesgue points $s$, even for $\beta > 0$ such that $s+\beta \notin \mathbb{T}$. Note that a discussion about this issue was provided in [1, Section 3.1].

Hence, the method developed in [2] allows to remove the closedness assumption done on $\Omega$ in [1], but this is at the price of an additional assumption on the time scale $\mathbb{T}$.

In [1, Section 3.1], the authors explained why other approaches (other than the Ekeland variational principle), based for instance on implicit function arguments, or on Brouwer fixed point arguments, or on separation (Hahn-Banach) arguments, fail for general time scales.

As a conclusion, a time scale version of the PMP without closedness assumption on $\Omega$ and without any assumption on the time scale $\mathbb{T}$ still remains an open challenge.

**Additional comments on the terminal constraints.** In [1] the authors considered constraints on the initial/final state of the kind $g(x(t_0), x(t_1)) \in \mathrm{S}$, where $\mathrm{S}$ is a nonempty closed convex set and $g$ is a general smooth function.

In [2] the authors considered constraints on the initial/final state of the kind $\Phi_i(x(t_0), x(t_1)) = 0$ for $i = 1,\dots,k$, and $\Phi_i(x(t_0), x(t_1)) \leq 0$ for $i = k+1,\dots,n$, where $\Phi_i$ are general smooth functions.

Contrarily to what is claimed in [2], the terminal constraints considered in [2] are only a particular case of the ones considered in [1]. Indeed, it suffices to take

$$g = (\Phi_1,\dots,\Phi_k,\Phi_{k+1},\dots,\Phi_n)$$

and

$$\mathrm{S} = \{0\}\times\ldots\times\{0\}\times\mathbb{R}^-\times\ldots\times\mathbb{R}^-.$$

Moreover, note that the necessary condition $-\Psi \in \mathcal{O}_{\mathrm{S}}(g(x(t_0), x(t_1)))$ obtained in [1, Theorem 1] encompasses both the *sign condition (1)* and the *complementary slackness (2)* obtained in [2, Theorem 2.11]. For the sign condition, it is sufficient to recall that the orthogonal of $\mathbb{R}^-$ at a point $x \in \mathbb{R}^-$ is included in $\mathbb{R}^+$. For the complementary slackness, it is sufficient to recall that the orthogonal of $\mathrm{S}$ at $g(x(t_0), x(t_1))$ is reduced to $\{0\}$ when $g(x(t_0), x(t_1))$ belongs to the interior of $\mathrm{S}$.

**Additional comments on the convexity of $\Omega$.** The set $\Omega$ is assumed to be convex in [2], while it is not in [1]. As explained in [1, Section 3.1], in order to apply necessary conditions of an extreme in a cone, the authors of [2] require that the parameters of perturbations live in intervals. As a consequence, in order to remove the convexity assumption on $\Omega$, one would need (local-directional) convexity of the set $\Omega$ for perturbations at right-scattered points, which is a concept that differs from the stable $\Omega$-dense directions used in [1]. Hence, in spite of the claim done in [2], the convexity assumption on $\Omega$ does not seem to be easily removable.

**On the universal Lagrange multipliers.** This paragraph is devoted to providing more details on the existence of universal Lagrange multipliers claimed in [2, page 25]. In the sequel, we use the notations of [2], and we denote by $\mathcal{S}$ the unit sphere of $\mathbb{R}^{n+1}$.

A package P consists of:

- $N \in \mathbb{N}$ and $\nu \in \mathbb{N}$;
- $\overline{\tau} = (\tau_1, \dots, \tau_N)$ where $\tau_i$ are right-dense points of $\mathbb{T}$;
- $\overline{v} = (v_1, \dots, v_N)$ where $v_i \in U$;
- $\overline{r} = (r_1, \dots, r_\nu)$ where $r_i$ are right-scattered points of $\mathbb{T}$.
- $\overline{z} = (z_1, \dots, z_\nu)$ where $z_i \in U$.

Let $(\mathrm{P}_i)_{i \in I}$ denotes the set of all possible packages.

Following the proof of [2, Theorem 2.11], for every $i \in I$, there exists a nonzero vector $\lambda = (\lambda_0, \dots, \lambda_n)$ (that we renormalize in $\mathcal{S}$) of Lagrange multipliers such that:

(i) (1) and (2) in [2, Theorem 2.11] are satisfied;

(ii) the adjoint vector $\Psi$ solution of (2.9), with the final condition (3.65) which depends on $\lambda$, satisfies the initial condition $\Psi(t_0) = L_{x_0}$;

(iii) (4a) and (4b) in [2, Theorem 2.11] are satisfied, but only at the points contained in $\overline{\tau}$ and $\overline{r}$ respectively.

For every $i \in I$, the above vector $\lambda$ is not necessarily unique. Then, for every $i \in I$, we denote by $\mathrm{K}_i$ the set of all nonzero and renormalized Lagrange multiplier vectors associated with $\mathrm{P}_i$ satisfying the above properties.

By continuity of the adjoint vector $\Psi$ with respect to the Lagrange multipliers (dependence from its final condition), we infer that $\mathrm{K}_i$ is a nonempty closed subset contained in the compact $\mathcal{S}$. This is true for every $i \in I$.

Now, let us prove that the family $(\mathrm{K}_i)_{i \in I}$ satisfies the finite intersection property. Let $J \subset I$ be a finite subset and let us prove that $\cap_{i \in J} \mathrm{K}_i \neq \emptyset$. Note that we can construct a package P corresponding to the union of all packages $\mathrm{P}_i$ with $i \in J$. It follows that $\mathrm{P} \in (\mathrm{P}_i)_{i \in I}$, and thus there exists a nonzero and renormalized Lagrange multiplier vector $\lambda$ associated with P satisfying the above properties. Since $\lambda \in \mathrm{K}_i$ for every $i \in J$, we conclude that $\cap_{i \in J} \mathrm{K}_i \neq \emptyset$.

It follows from the lemma of a centered system in a compact set that $\cap_{i \in I} \mathrm{K}_i \neq \emptyset$, and we deduce the existence of a universal Lagrange multiplier vector.

**On the density conditions and the Cantor set.** Contrarily to what is claimed in [2, Example 2.5], the classical Cantor set does not satisfy the density conditions. However, generalized versions of the Cantor set (see, e.g., [4]) that satisfy density conditions can be constructed as follows.

Let $(\alpha_k)_{k \in \mathbb{N}}$ be a real sequence such that $0 < \alpha_k < \frac{1}{2}$ for all $k \in \mathbb{N}$, and such that $\lim_{k \to +\infty} \alpha_k = \frac{1}{2}$. Let $(A_k)_{k \in \mathbb{N}}$ be the sequence of compact subsets defined by the induction

$$A_0 = [0, 1], \qquad A_{k+1} = \mathcal{T}_k(A_k) \quad \forall k \in \mathbb{N},$$

where $\mathcal{T}_k$ denotes the operator removing the open $(\alpha_k, 1-\alpha_k)$-central part of all intervals. Note that the classical Cantor set corresponds to the case where $\alpha_k = \frac{1}{3}$ for every $k \in \mathbb{N}$.

In our situation, we obtain

$$A_1 = [0, \alpha_0] \cup [1-\alpha_0, 1],$$

$$A_2 = \Big( [0, \alpha_1 \alpha_0] \cup [(1-\alpha_1)\alpha_0, \alpha_0] \Big) \cup \Big( [1-\alpha_0, 1-(1-\alpha_1)\alpha_0)] \cup [1-\alpha_1\alpha_0, 1] \Big),$$

etc. We define the generalized Cantor set $\mathbb{T} = \cap_{k \in \mathbb{N}} A_k$. In order to prove that the time scale $\mathbb{T}$ satisfies the density conditions, from the fractal properties of $\mathbb{T}$, it suffices to prove that the density condition is satisfied at the right-dense point $0 \in \mathbb{T}$. More precisely, it is sufficient to prove that

$$\lim_{\substack{\beta \to 0^+ \\ \beta \in \mathbb{T}}} \frac{\mu(\beta)}{\beta} = 0.$$

Since $\mu(\beta) = 0$ for every right-dense point $\beta$, we only have to consider the case where $\beta$ is a right-scattered point of $\mathbb{T}$. In that case, one can easily see that $\frac{\mu(\beta)}{\beta} \leq \frac{1-2\alpha_k}{\alpha_k}$ for some $k \in \mathbb{N}$ and that $k$ tends to $+\infty$ when $\beta$ tends to 0. The conclusion follows from the fact that $\lim_{k \to +\infty} \alpha_k = \frac{1}{2}$.

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