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Direct and inverse problem in multiple-scattering of small obstacles in homogeneous media.

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Journ ee "Jeunes Chercheur-e-s" sur les probl emes d'ondes harmoniques de grande taille .

UPMC, Nov 2017.

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Motivation

Theme

Simulating propagation of acoustic waves through *(highly) heterogeneous media* .

Goal

Create effective numerical tools for

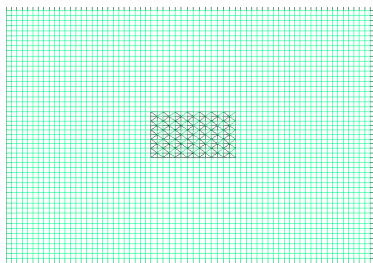
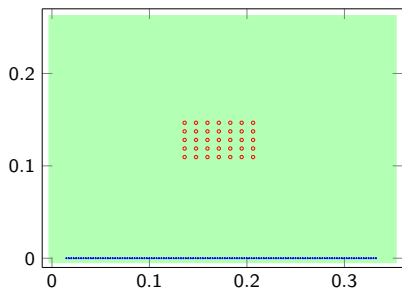
- direct simulation
- and inverse problems

Collaboration with the acoustic lab I2M (i2m.u-bordeaux.fr)

Motivation (cont)

Heterogeneities produced by obstacles

- Domains of size ≥ 100 incidence wavelength λ
- Obstacles of radius $\leq 0.3\lambda$.



Volume-discretization based methods lose their robustness in these settings :

large linear systems, numerical pollution caused by dispersion, etc.

- I2M uses COMSOL (commercial software, finite-element based).
- Highly-optimized Software in Magique3D : MONTJOIE, HOU10NI.

Overview

- 1 Introduction of method
- 2 Comparison with Montjoie
- 3 Solver's robustness comparison
 - Closely spaced obstacles
 - Far away obstacles
- 4 Discussion of the inverse problems
 - An example of an localization problem and data
 - Discussion of reconstruction method
- 5 Numerical inversion experiments
 - Periodic configuration of 6 obstacles with 30dB
 - Periodic configuration of 12 obstacles with 25dB
 - Random configuration of 12 obstacles with 30 dB

Plan

- 1 Introduction of method

Multiple obstacle scattering as Exterior Boundary Value problems

Propagation of acoustic waves of freq. \mathbf{f} in a hom. medium with sound speed \mathbf{c} .

$$u_{\text{total}} = u_{\text{inc}} + u_{\text{scatt}}.$$

1. PDE satisfied by u_{scatt} outside of the obstacles:

$$(-\Delta - \kappa^2) u_{\text{scatt}} = 0, \quad \kappa = \frac{2\pi\mathbf{f}}{\mathbf{c}}.$$

2. Conditions on the boundary of the obstacles:

$$\text{Dirichlet} \quad \gamma_0^+ u_{\text{total}} = 0$$

$$\text{Neumann} \quad \gamma_1^+ u_{\text{total}} = 0$$

$$\text{Impedance} \quad \gamma_1^+ u_{\text{total}} + i\lambda\gamma_0^+ u_{\text{total}} = 0$$

3. (Outgoing) Sommerfeld radiation condition at ∞ :

$$\lim_{r \rightarrow \infty} \sqrt{r} (\partial_r u_{\text{scatt}} - i\kappa u_{\text{scatt}}) = 0 \quad ; \quad r = |x|$$

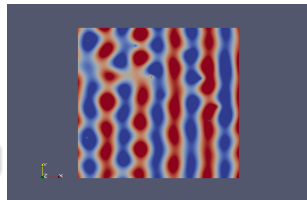
∃! solution for the exterior BVPs (all parameters > 0).

Time-harmonic Planewave :

$$u_{\text{pw}}(x) \exp(-i 2\pi \mathbf{f} t)$$

$$u_{\text{pw}}(x) = \exp(\kappa x \cdot \begin{pmatrix} \cos \alpha_{\text{inc}} \\ \sin \alpha_{\text{inc}} \end{pmatrix})$$

$$\alpha_{\text{inc}} = 0^\circ, \quad 2\pi\mathbf{f} = 1.0, \quad \kappa = 1.0.$$



Motivation

$$\mathbb{R}^2 = \underbrace{\Omega^-}_{\text{bounded}} \cup \Gamma \cup \Omega^+$$

$$v \text{ satisfies } \begin{cases} (-\Delta - \kappa^2)v = 0 & \text{in } \Omega^- \\ (-\Delta - \kappa^2)v = 0 & \text{in } \Omega^+, \quad v \text{ outgoing} \end{cases}$$

For $x \notin \Gamma$

$$v(x) = \underbrace{- \int_{\Gamma} G_{\kappa}(x, y) [\gamma_1 v](y) ds(y)}_{\mathcal{S}[\gamma_1 v]} + \underbrace{\int_{\Gamma} \frac{\partial}{\partial n(y)} G_{\kappa}(x, y) [\gamma_0 v](y) ds(y)}_{\mathcal{D}[\gamma_0 v]},$$

$$G_{\kappa}(x, y) = \frac{i}{4} H_0^{(1)}(\kappa|x - y|)$$

$$[\gamma_0 u] = \gamma_0^+ u - \gamma_0^- u ; \quad [\gamma_1 u] = \gamma_1^+ u - \gamma_1^- u$$

Choices of solution representation and trace operators

$u_{\text{total}} = u + u_{\text{inc}}$			
Choice of ext. for u	Choice of trace op.	Dirichlet $\gamma_0^+ u_{\text{total}} = 0$	Neumann $\gamma_1^+ u_{\text{total}} = 0$
$u _{\Omega^-} = 0$		$u _{\Omega^-} = -\mathcal{S} \gamma_1^+ u + \mathcal{D} \gamma_0^+ u$	
$[\gamma_0 u] = 0$	$u = -\mathcal{S}[\gamma_1 u]$		
	Outer	Apply $\gamma_0^+ \rightarrow$ an equation = EFIE	Apply γ_1^+
	Inner	Null field method : Extend $u_{\text{total}} = 0$ on Ω^- $\gamma_0^- \rightarrow$ Electric Field IE (EFIE) $\gamma_1^- \rightarrow$ Magnetic Field IE (MFIE) $\gamma_1^- + \eta \gamma_0^- \rightarrow$ Combined Field IE (CFIE)	
$[\gamma_1 u] = 0$	$u = \mathcal{D}[\gamma_0 u]$		
	Outer	Apply γ_0^+	Apply γ_0^+
	Inner		Null field method $u_{\text{total}} = 0$ on Ω^- $\gamma_1^- \rightarrow$ EFIE 2 $\gamma_0^- \rightarrow$ MFIE 2 $\eta \gamma_1^- + \gamma_0^- \rightarrow$ CFIE 2
Brackhage - Werner	Outer	$u _{\Omega^+} = (\eta \mathcal{S} + \mathcal{D}) \phi$ Apply γ_0^+	$u _{\Omega^+} = (\mathcal{S} + \eta \mathcal{D}) \phi$ Apply γ_1^+

Properties of the outgoing Green kernel

$$G_\kappa(x, y) = \frac{i}{4} H_0^{(1)}(\kappa|x - y|) \text{ is } \begin{cases} \text{smooth off the diagonal } \{x = y\} \\ \text{weakly singular around the diagonal} \end{cases} .$$

$$|G_\kappa(x, y)| \leq C|x - y|^{-1+\epsilon} \quad 0 < \epsilon < 1.$$

$$H_0^{(1)}(z) = \frac{2i}{\pi} \left(\ln \frac{|z|}{2} + \frac{\text{Euler constant}}{2} - \frac{\pi i}{2} \right) + \mathbf{O} \left(|z|^2 \ln \frac{1}{|z|} \right), \quad |z| \rightarrow 0.$$

$$\mathcal{S} : H^s(\Gamma) \rightarrow H_{\text{loc}}^{s+\frac{3}{2}}(\mathbb{R}^2) \quad \text{is bounded for } \begin{cases} -1 < s < 0 & , \Gamma \text{ Lipschitz} \\ -1 < s & , \Gamma \mathcal{C}^\infty \end{cases}$$

The definition $\mathcal{S}\phi := \int_\Gamma G_\kappa(x, y) \phi(y) ds(y)$, $x \notin \Gamma$, $\phi \in L^1(\Gamma)$

extended to $\mathcal{S} := \mathcal{N} \gamma'_0$

\mathcal{N} is the Newton potential $\mathcal{N}f := \int_{\mathbb{R}^2} G_\kappa(x, y) f(y) dy$, $f \in L^2_{\text{comp}}(\mathbb{R}^2)$.

Jump of single-layer potential

 Γ Lipschitz, $\phi \in H^{-1/2}(\Gamma)$

$$[\gamma_0 \mathcal{S}\phi] = 0 \quad , \quad \text{in } H^{1/2}(\Gamma) \quad ; \quad [\gamma_1 \mathcal{S}\phi] = -\phi \quad , \quad \text{in } H^{-1/2}(\Gamma)$$

Zero-th trace of single-layer potential

$S := \gamma_0 \mathcal{S} : H^s(\Gamma) \rightarrow H^{s+1}(\Gamma)$ is bounded for $\begin{cases} -1 < s < 0 & , \Gamma \text{ Lipschitz} \\ -1 < s < r + \frac{1}{2} & , \Gamma \in \mathcal{C}^{r+1,1} \end{cases}$

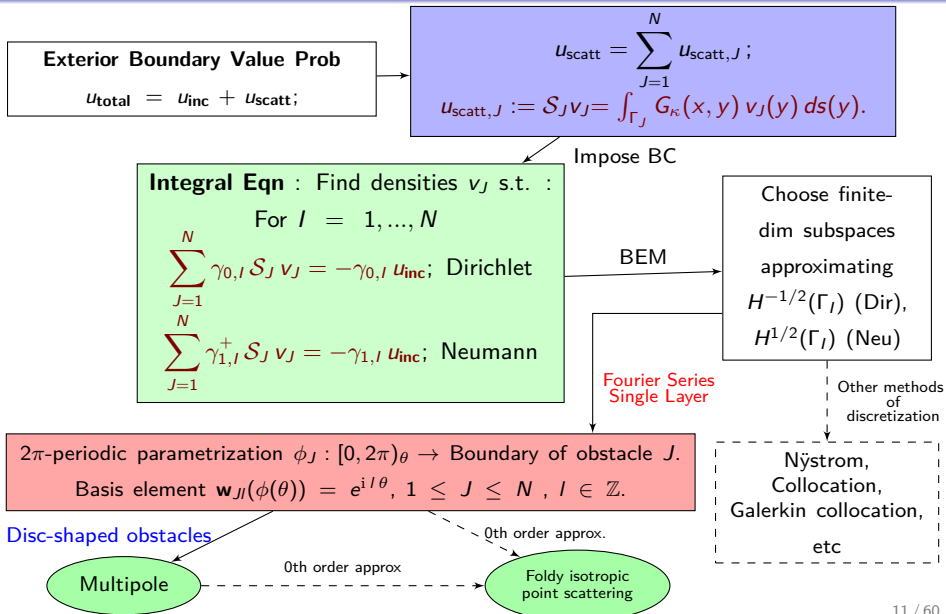
Integral presentation (for $\Gamma \in \mathcal{C}^2$) $(S\phi)(x) := \int_{\Gamma} G_{\kappa}(x, y) \phi(y) ds(y)$, $x \in \Gamma$, $\phi \in L^{\infty}(\Gamma)$.

Conormal derivative of single-layer potential

$\gamma_1^{\pm} \mathcal{S} = \mp \frac{1}{2} \text{Id} + D'$, $D' : H^s(\Gamma) \rightarrow H^s(\Gamma)$ bounded for $\begin{cases} -1 < s < 0 & , \Gamma \text{ Lipschitz} \\ -1 < s < r + \frac{1}{2} & , \Gamma \in \mathcal{C}^{r+1,1} \end{cases}$

Integral presentation (for $\Gamma \in \mathcal{C}^2$) $(D'\phi)(x) := \int_{\Gamma} \phi(y) \frac{\partial}{\partial n(x)} G_{\kappa}(x, y) ds(y)$, $x \in \Gamma$, $\phi \in L^{\infty}(\Gamma)$.

Single layer potential formulation for multi-scattering .



Fourier Series Single Layer method.

The scattered and approx. wave

$$u_{\text{scatt}} = \sum_{J=1}^N u_{\text{scatt};J},$$

$$u_{\text{scatt},h} = \sum_{J=1}^N u_{h,\text{scatt};J}.$$

The exact and app. wave scattered by Obs J

$$u_{\text{scatt};J} = \mathcal{S}_J v_J; \quad u_{h,\text{scatt};J} = \mathcal{S}_J v_{h,J}.$$

In basis elements

$$\mathbf{w}_{J,k}(x) = e^{ik\theta_J(x)},$$

$$u_{\text{scatt};J} = \sum_{k \in \mathbb{Z}} V_{J,k} \mathcal{S}_J \mathbf{w}_{J,k}$$

$$u_{h,\text{scatt};J} = \sum_{k=-m}^m V_{J,k} \mathcal{S}_J \mathbf{w}_{J,k}.$$

The unknowns are the Fourier coefficients of density v_J

$$V = (V_{J,k}), \quad k \in \mathbb{Z}, 1 \leq J \leq N,$$

and the truncated ones for the approx. $v_{h,J}$.

$$V_h = (V_{J,k}), \quad -m \leq k \leq m, 1 \leq J \leq N.$$

For $\alpha = D, N, \text{Im}$, they solve

$$\mathbf{A}_\alpha V = F_\alpha, \quad \mathbf{A}_{h,\alpha} V_h = F_{\alpha,h}.$$

$$\mathbf{A}_\alpha = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \cdots & \mathbf{A}_{1(N-1)} & \mathbf{A}_{1N} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \cdots & \mathbf{A}_{2(N-1)} & \mathbf{A}_{2N} \\ \vdots & & \ddots & & \vdots \\ \mathbf{A}_{(N-1)1} & \mathbf{A}_{(N-1)2} & \cdots & \mathbf{A}_{(N-1)(N-1)} & \mathbf{A}_{(N-1)N} \\ \mathbf{A}_{N1} & \mathbf{A}_{N2} & \cdots & \mathbf{A}_{N(N-1)} & \mathbf{A}_{NN} \end{pmatrix}$$

$\mathbf{A}_{h,\alpha}$ square matrix of size $(2m+1) \times N$.

$\mathbf{A}_{\alpha,l}$ self-interaction of obstacle l

$\mathbf{A}_{\alpha,lJ}$ diffraction by obs. l of wave emitted by obs. J

Multi-scattering with circular obstacles.

Single-layer potential with density $\mathbf{w}_{J,k}$ can be written in **multipole expansions**,

$$(\mathcal{S}_J \mathbf{w}_{J,k})(x) = \frac{i\pi r_J}{2} J_k(\kappa r_J) \underbrace{H_k^{(1)}(\kappa r_J(x)) e^{ik\theta_J(x)}}_{\substack{\text{multiple pole of order } k \\ \text{placed at the center of } \mathcal{O}_J}}.$$

Same obstacle interaction

$$(\mathbf{A}_I)_{kl} = i\pi r_I J_k(\kappa r_I) \delta_{kl} \begin{cases} H_k^{(1)}(\kappa r_I) & \text{Dirichlet} \\ \kappa H_k^{(1)'}(\kappa r_I) & \text{Neumann} \end{cases}, \quad k, l \in \mathbb{Z}.$$

Interaction between two different obstacles $I \neq J$

$$(\mathbf{A}_{IJ})_{kl} = i\pi r_J e^{i(l-k)\theta_{\mathbf{x}_J}(\mathbf{x}_I)} H_{l-k}^{(1)}(\kappa d_{IJ}) J_k(\kappa r_I) \begin{cases} J_l(\kappa r_J) & \text{Dirichlet} \\ \kappa J_l'(\kappa r_J) & \text{Neumann} \end{cases},$$

$$d_{IJ} = |\mathbf{x}_I - \mathbf{x}_J| \quad ; \quad k, l \in \mathbb{Z}.$$

Obstacle I of radius r_I .

Relative polar coordinates $(r_J(\cdot), \theta_J(\cdot))$ with respect to obstacle \mathbf{x}_J

$$x = \mathbf{x}_J + r_J(x)(\cos \theta_J(x), \sin \theta_J(x))$$

Well-posedness

$$0 \leq \kappa < \infty \quad ; \quad \lambda \in \mathbb{R} .$$

If κ^2 is not a Dirichlet eigenvalues (EV) of $-\Delta$ for \mathcal{O}_l for $1 \leq l \leq N$, then the following maps are injective

$$\mathbf{A}_\alpha : \mathbb{H}^{1/2}(\Gamma_{\text{Obs}}) \longrightarrow \mathbb{H}^{1/2}(\Gamma_{\text{Obs}}) , \text{ Impedance, Neumann}$$

$$\mathbf{A}_\alpha : \mathbb{H}^{-1/2}(\Gamma_{\text{Obs}}) \longrightarrow \mathbb{H}^{1/2}(\Gamma_{\text{Obs}}) , \text{ Dirichlet}$$

$$\mathbb{H}^s(\Gamma_{\text{Obs}}) = H^s(\Gamma_1) \times \dots \times H^s(\Gamma_N)$$

Well-posedness and small obstacles

Circular obstacles

$$\text{Dirichlet EV : } \lambda_{n,m} = \left(\frac{j_{n,m}}{r} \right)^2,$$

$j_{n,m}$ m -th positive root of $J_n(r) = 0$,

r = radius of obstacle.

$$\text{Injectivity : } \kappa_e^2 r^2 \neq j_{n,m}.$$

The first 4 roots :

$$j_{0,1} \sim 2.40, \quad j_{1,1} \sim 3.83, \quad j_{2,1} \sim 5.13, \quad j_{1,2} \sim 5.52.$$

General shape obstacles

Isoperimetric inequality gives

$$\lambda_1(\mathcal{O}) \geq \frac{\pi}{\text{Area}(\mathcal{O})} j_{0,1}^2.$$

Injectivity for small obstacles

$$\kappa_e r_{\text{circumvent}}(\mathcal{O}) < 2$$

Multiple Scattering Literature

- **Single-layer method**

Thierry, Bertrand. *Analyse et simulations numériques du retournement temporel et de la diffraction multiple*. Diss. Université Henri Poincaré-Nancy I, 2011.

Thierry, Bertrand, et al. *μ -diff: an open-source Matlab toolbox for computing multiple scattering problems by disks*. (2015): 348-362.

- **Modified single-layer method** (single + double layer)

Ganesh, Mahadevan, and Stuart Collin Hawkins. *An efficient algorithm for simulating scattering by a large number of two dimensional particles*. (2011).

- **T-matrix method**

Amirkulova, Feruza A., and Norris, Andrew. *Acoustic multiple scattering using recursive algorithms*. (2015).

- **Approximation methods for small obstacles**

Challa, D.P., and Sini, Mourad. *On the justification of the Foldy–Lax approximation for the acoustic scattering by small rigid bodies of arbitrary shapes*. (2014).

Bendali, A., Cocquet, P-H and Tordeux, S. *Approximation by Multipoles of the Multiple Acoustic Scattering by Small Obstacles in Three Dimensions and Application to the Foldy Theory of Isotropic Scattering*. (2016).

Plan

2 Comparison with Montjoie

Feature of Direct Simulation Codes

- Written in Fortran90.
- **Parallelized** using MPI,
- Runs on **the platform Plafrim** of Inria.
- Multi-frequency option.
- Choices of both **direct and iterative linear system solvers**.

Mumps , Lapack , Scalapack

GMRES with restart³ with various preconditioners

- Validated and compared with highly optimized Montjoie.

³GMRES with restart without preconditioner was developed by Luc Giraud's team (Cerfacs). L. Giraud, et al. , *A set of GMRES routines for real and complex arithmetics on high performace computers*, Technical report, CERFACS, tR/PA/03/3 (1997).

Calculation time costs (CPU time)

$$u_{h,\text{scatt}}(x) = \frac{i\pi}{2} \sum_{J=1}^{N_{\text{Obs}}} \mathbf{r}_J \sum_{l=-m}^m \mathbf{V}_{J,l} H_k^{(1)}(k r_J(x)) e^{il\theta_J(x)} \quad (*)$$

Pre-processing time = Time to resolve the linear system for V_h .
 Linear system is dense but small : $N_{\text{Obs}} \times (2m + 1)$.

Post-processing time = Eval. time of LHS of (*)
 at each point of visualization grid

Evaluation of Hankel is costly (~ 540 times more expensive than '+' operation).

Cost $\sim N_{\text{Obs}} \times (\# \text{ points of visualization grid})$.

* Reduce the cost (associated with second factor) by

parallelization and **interpolation**
 e.g. **Hermite interpolation** \subset cubic spline

Experiment 1: Small obstacles on medium domain

Soft-scattering of PW with angle 90°

of wavelength $\kappa = 10$, $\lambda \sim 0.63$

by **200 obstacles**

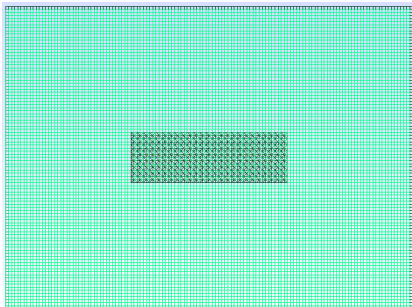
of radius = 0.03, with distanced by 0.3.

Domain size : $31\lambda \times 23\lambda$

$$\kappa \times (\text{Obs Rad}) = 0.3,$$

$$\frac{\lambda}{\text{Obs Rad}} \sim 21, \quad \frac{\lambda}{\text{Obs. Dist.}} \sim 2,$$

$$\frac{\text{Obs. Dist.}}{\text{Obs. Rad.}} \sim 10.$$



Montjoie initial mesh has mesh size of 0.13.

Montjoie

(montjoie.gforge.inria.fr)

Bases: Curved finite element (FE) with Lagrange polynomials based on Gauss-Lobatto points.

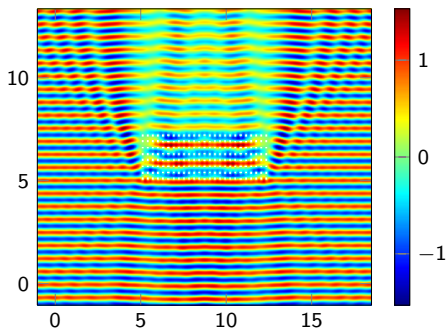
Q-n denotes the n^{th} order FE on **quadrangular meshes**.

Domain truncation: Perfectly Matched Layers.

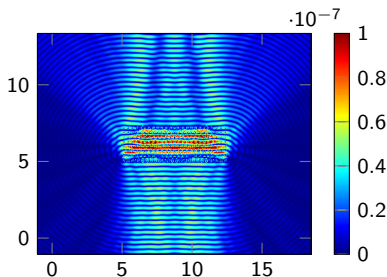
Experiment 1: Reference solutions

Soft-scattering of 200 obstacles on domain of size : $31\lambda \times 23\lambda$

$$\kappa \times (\text{Obs Rad}) = 0.3, \quad \frac{\lambda}{\text{Obs Rad}} \sim 21, \quad \frac{\lambda}{\text{Obs. Dist.}} \sim 2, \quad \frac{\text{Obs. Dist.}}{\text{Obs. Rad.}} \sim 10.$$

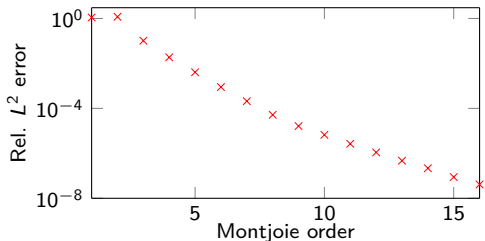


(a) Real part of FSSL 14 total wave



(b) Abs. difference compared with Montejoie Q17. Relative L^2 err. = 3.38×10^{-8} .

Experiment 1: Convergence curve

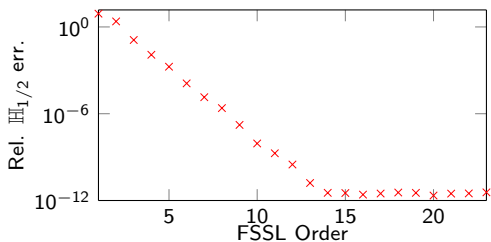


(c) Rel. consecutive err. : Montjoie

Candidates for comparison at precision 10^{-3}

Compare between		Rel. L^2 error
FSSL 14	FSSL 2	4.65×10^{-5}
MJ Q17	MJ Q6	6.52×10^{-4}
MJ Q6	FSSL2	6.84×10^{-4}

Hermite interp. precision is 10^{-6} .



(d) Rel. consecutive err : FSSL densities

Compare between		Rel. L^2 error
FSSL 2 Inter	FSSL 2	1.76×10^{-5}
FSSL 2 Inter	MJ Q6	6.85×10^{-4}

Solvers for both Montjoie and FSSL are Mumps.

Experiment 1: Comparison at precision 10^{-3}

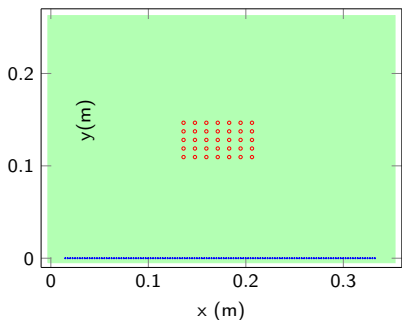
Pre-processing by Mumps	FSSL Order 2	MJ Q6
Size of lin. sys.	1000	842677
Task	Time (s)	
Construction	0.055	1.97
Factorization	0.44	29.8
Resolution	0.003	0.35
Total time	0.498	32.12

Evaluation on 400×400 grid			
	Exact eval	Inter. eval	MJ Q6
Post-proc.	26.2	4.30	0.72
Pre-proc. + Post-proc.	26.70	4.80	33.82

At precision 10^{-3} , FSSL using Hermite interpolation takes 7 times less than MJ.

Experiment 2: sizable obstacles on a large domain

Acoustic vibration, produced by a block transducer, is diffracted by 35 thin aluminum wires (of radius 0.5 mm) immersed in water.



Horizontal-view cut.

The phenomenon is approximated by the **hard scattering** of acoustic sound in fluid.

- The incident wave (from the transducer) is simulated by a PW of angle 90° .
- Input pulse's central freq. = 500 kHz.
- The sound speed in water = 1478 m s^{-1} .
- The wavenumber $\kappa = 2125.57 \text{ m}^{-1}$.
- The spatial wavelength $\lambda = 2.96 \times 10^{-3} \text{ m}$.

Domain size = $117\lambda \times 87\lambda$.

$$\kappa \times (\text{Obs Rad}) \sim 1.1, \quad \frac{\text{Obs Dist}}{\text{Obs Rad}} \sim (23, 19),$$

$$\frac{\lambda}{\text{Obs Rad}} \sim 5.91, \quad \frac{\lambda}{\text{Obs. Dist.}} \sim 0.3.$$

Exp 2: Computational time comparison at precision 10^{-4}

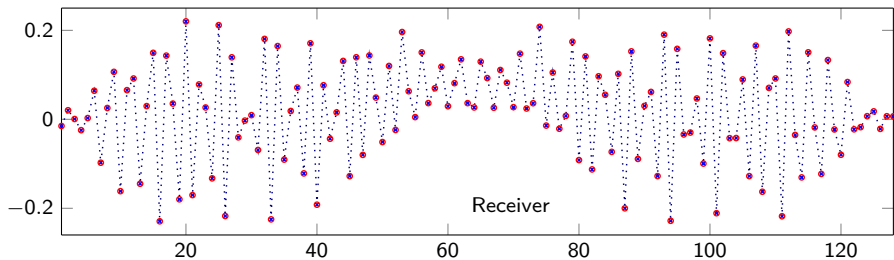
Regarding the value of the diffracted wave at 128 receptors,

Rel. L^2 error : FSSL 12 and FSSL 4 = 2.82×10^{-6} ,

Rel. L^2 error : MJ Q12 and MJ Q8 Ref 2 = 1.42×10^{-4} .

Rel. L^2 error : FSSL 4 and MJ Q8 Ref 2 = 1.48×10^{-4} .

Q8 Ref 2 = Q8
with one time
mesh refinement.



Real of part of diffracted wave at 128 receptors : FSSL 4 \cdots and MJ Q8 Ref2 \circ .

Exp 2: Candidates for comparison at precision 10^{-4}

	Size of LS	Pre-proc. Time (s)	Post-proc. Time at 128 receivers (s)	Total time (s)
FSSL 4	315	0.024	6.58×10^{-3}	0.031
MJ Q8 Ref 2	993870	61.27	0.13	61.4

FSSL (with exact evaluation) is 2046 times faster than MJ.

Plan

- 3 Solver's robustness comparison
 - Closely spaced obstacles
 - Far away obstacles

Restart GMRES (generalized minimal residual method)

Restart with Krylov space size m .

Initial guess x_0 .

Initial residue $r_0 = b - A_0$.

No preconditioning : $A p_\star = r_0$.

- Use **Arnoldi process**, to find, approximate sol. p_j in Krylov space $K_j(A, r_0)$, $j \leq m$, minimizes

$$p_j = \underset{p \in K_j(A, r_0)}{\operatorname{argmin}} \|A p - r_0\|_2 \quad (*).$$

- **Stop** if p_j satisfies the **residue error criteria**.

If not, and if $j = m$, *restart the process* with initial guess $r_0 = p_m$.

- **Final stop criteria** : **NiterMax** .

Right preconditioning : $(AP^{-1})(Pp_\star) = r_0$.

Left preconditioning : $(P^{-1}A)p_\star = P^{-1}r_0$.

GMRES Preconditioners

L = strictly lower part of matrix A

$$M_u = U + D, N_u = -L$$

Splittings of A :

D = diagonal of matrix A

$$M_l = L + D, N_l = -U,$$

$$A = L + D + U = M_u - N_u$$

U = strictly upper part of A

$$R = -L - U.$$

$$= M_l - N_l = D - R.$$

The **backward Gauss-Seidel (BGS)** preconditioner is $\mathcal{P} = M_u$.

The **forward Gauss-Seidel (FGS)** preconditioner is $\mathcal{P} = M_l$.

The **Jacobi** preconditioner is $\mathcal{P} = D$.

The **2nd-order Jacobi (2Jacobi)** preconditioner is

$$\mathcal{P} = D(R + D)^{-1}D.$$

Formally, \mathcal{P}^{-1} is the 2nd approx. of the Neumann series of $A^{-1} = (D - R)^{-1}$.

The **Symmetric Gauss-Seidel (SGS)** preconditioner is

$$\mathcal{P} = M_u D^{-1} M_l.$$

Interpretation: $u = \mathcal{P}^{-1}f$ solves

$$M_u \tilde{u} = f, M_l u = N_l \tilde{u} + f.$$

The **2nd-order Forward Gauss-Seidel (2FGS)** preconditioner is

$$\mathcal{P} = M_l (N_l + M_l)^{-1} M_l.$$

Formally, \mathcal{P}^{-1} is the 2nd approx. of the Neumann series of $A^{-1} = (M_l - N_l)^{-1}$.

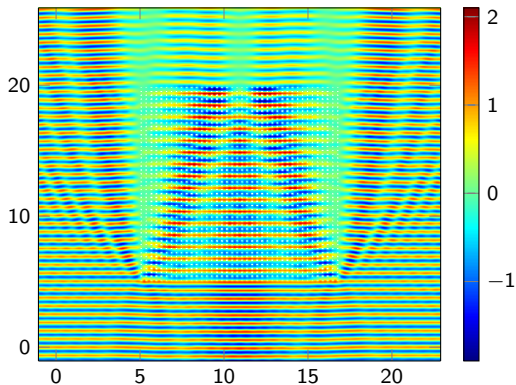
The **Lower-Upper Symmetric Gauss-Seidel (LUSGS)** preconditioner is

$$\mathcal{P} = M_l D^{-1} M_u.$$

Interpretation: $u = \mathcal{P}^{-1}f$ solves

$$M_l \tilde{u} = f, M_u u = N_u \tilde{u} + f.$$

Closely-spaced obstacles comparison



FSSL order 2 with Mumps for 2000 obstacles.

Planewave (PW) with 90° .

Wavenumber $\kappa = 10$.

Radius of obstacle 0.03.

Distance btwn obs 0.3.

$$\kappa \times (\text{Obs Rad}) = 0.3;$$

$$\frac{\lambda}{\text{Obs. Rad}} \sim 21; \quad \frac{\lambda}{\text{Obs Dis}} \sim 2$$

$$\frac{\text{Obs Dist}}{\text{Obs Rad}} = 10.$$

GMRES stop criteria : Residue error tolerance, Niter Max, Size of Krylov.

Exp 3a: Closely-spaced obstacles comparison (Dirichlet)

Name Method	Case 200 obstacles				Case 1616 obstacles			
	Cv	δ_{err} in $\mathbb{H}_{1/2}$	# Iter	Time (s)	Cv	δ_{err} in $\mathbb{H}_{1/2}$	# Iter	Time (s)
Mumps	n/a	0	n/a	0.5	n/a	0	n/a	130
Lapack	n/a	10^{-12}	n/a	0.1	n/a	10^{-10}	n/a	42.7
	GMRES stop criteria (10^{-6} , 2000,100)				GMRES stop criteria (10^{-6} , 2000,150)			
NoPreCond	Y	5×10^{-3}	820	0.9	N	n/a	n/a	n/a
L_Jacobi	Y	5×10^{-3}	656	0.8	N	n/a	n/a	n/a
L_FGS	Y	2×10^{-3}	239	0.5	N	n/a	n/a	n/a
L_BGS	Y	4×10^{-3}	197	0.4	N	n/a	n/a	n/a
L_2Jacobi	Y	5×10^{-3}	594	2.2	N	n/a	n/a	n/a
L_2FGS	Y	1×10^{-3}	169	1.0	N	n/a	n/a	n/a
L_SGS	Y	2×10^{-3}	76	0.3	Y	4×10^{-1}	757	274
L_LUSGS	Y	1×10^{-3}	77	0.3	Y	1×10^{-1}	897	325
R_Jacobi	Y	4×10^{-3}	660	1.1	N	n/a	n/a	n/a
R_FGS	Y	3×10^{-3}	199	0.5	N	n/a	n/a	n/a
R_BGS	Y	3×10^{-3}	198	0.4	N	n/a	n/a	n/a
R_2Jacobi	Y	4×10^{-3}	600	1.7	N	n/a	n/a	n/a
R_2FGS	Y	3×10^{-3}	155	0.9	N	n/a	n/a	n/a
R_SGS	Y	3×10^{-3}	75	0.3	Y	2×10^{-1}	886	321
R_LUSGS	Y	3×10^{-3}	74	0.3	Y	2×10^{-1}	897	325

Exp 3b: Closely-spaced obstacles comparison (Dirichlet)

FSSL order =2 ; Size matrix = $10^4 \times 10^4$; GMRES stop criteria (10^{-6} , 5000, 400)

Solver	Post-proc (n16)	Rel $H_{1/2}$ diff	Rel L^2 diff	# iter	Preproc. time (s)	Postproc. time (s)	Total (s)
Mumps (n16)	Exact	3×10^{-10}	8×10^{-14}	n/a	242	96.0	338
Mumps (n16)	Inter	3×10^{-10}	9×10^{-6}	n/a	242	36.0	278
Lapack (n1)	Exact	0	0	n/a	80.4	96.0	176
Lapack (n1)	Inter	0	9×10^{-6}	n/a	80.4	37.5	118
R_LUSGS (n1)	Exact	1×10^{-1}	4×10^{-5}	1146	573	95.8	669
R_LUSGS (n1)	Inter	1×10^{-1}	4×10^{-5}	1146	573	36.2	609
R_SGS (n1)	Exact	1×10^{-1}	4×10^{-5}	1151	598	95.8	694
R_SGS (n1)	Inter	1×10^{-1}	4×10^{-5}	1151	598	36.2	635
Scala (n16)	Exact	3×10^{-10}	8×10^{-14}	n/a	34.6	95.6	130
Scala (n16)	Inter	3×10^{-10}	9×10^{-6}	n/a	34.6	36.1	70.9

PW of 90° ; $\kappa = 10.0$; $N_{\text{Obs}} = 2000$; Obs. Rad. = 0.03 ; Obs. Dist. = 0.30 ;

$$\kappa \times (\text{Obs Rad}) = 0.3, \quad \frac{\lambda}{\text{Obs. Rad}} \sim 21, \quad \frac{\lambda}{\text{Obs Dis}} \sim 2, \quad \frac{\text{Obs Dist}}{\text{Obs Rad}} = 10.$$

Exp 4: Far apart obstacles (Dirichlet)

FSSL order 2 ; Size matrix = 10000 × 10000;

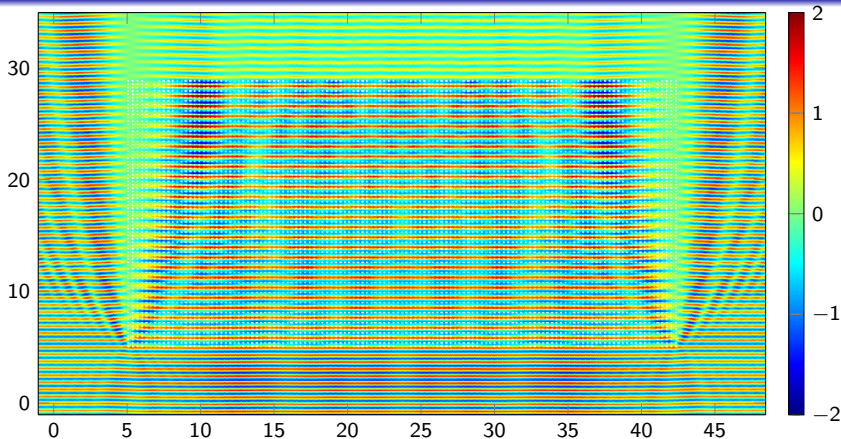
Post-processing on 800 × 800 regular grid; GMRES stop criteria (10^{-7} , 5000,500).

Solver	Post-proc (n16)	Rel $\mathbb{H}_{1/2}$ diff	Rel L^2 diff	# iter	Pre-pro. time (s)	Post-pro. time (s)	Total (s)
Mumps (n1)	Exact	0.0	0.0	n/a	251	96.0	347
Mumps (n1)	Inter	0.0	1×10^{-5}	n/a	251	37.5	289
Lapack (n1)	Exact	4×10^{-12}	2×10^{-15}	n/a	79.9	96.0	176
Lapack (n1)	Inter	4×10^{-12}	1×10^{-5}	n/a	79.9	37.5	118
R_LUSGS (n1)	Exact	3×10^{-4}	1×10^{-7}	57	37.5	96.0	134
R_LUSGS (n1)	Inter	3×10^{-4}	1×10^{-5}	57	37.5	37.5	75.3
R_SGS (n1)	Exact	4×10^{-4}	1×10^{-7}	56	37.0	96.0	133
R_SGS (n1)	Inter	4×10^{-4}	1×10^{-5}	56	37.0	37.5	74.6
Scala (n16)	Exact	1×10^{-11}	4×10^{-15}	n/a	34.9	96.0	131
Scala (n16)	Inter	1×10^{-11}	1×10^{-5}	n/a	34.9	37.5	72.5

PW of 90.0° ; $\kappa = 10.0$; # obs = 2000; Obs.Rad. = 0.01; Obs. Dist. = 2.00;

$$\kappa \times (\text{Obs Rad}) = 0.1, \quad \frac{\lambda}{\text{Obs. Rad.}} \sim 63, \quad \frac{\lambda}{\text{Obs Dist.}} \sim 0.3, \quad \frac{\text{Obs Dist.}}{\text{Obs Rad.}} = 200.$$

Exp 5



Soft scattering of a planewave coming from the south by 10^4 obstacles.

$$\kappa \times r = 0.03 \quad , \quad \frac{\lambda}{r} \sim 21 \quad , \quad \frac{d}{r} = 10 \quad , \quad \frac{\lambda}{d} \sim 2.$$

Preprocessing uses FSSL order 2 + Scalapack. **Dense matrix of size 50000×50000 .**

Post-processing on 800×800 grid of size $79\lambda \times 57\lambda$ uses Hermit inter

Total simulation time = 24 mins 40 secs on 48 processors (of Plafrim).

Plan

- 4 Discussion of the inverse problems
 - An example of an localization problem and data
 - Discussion of reconstruction method

An example of localization problem

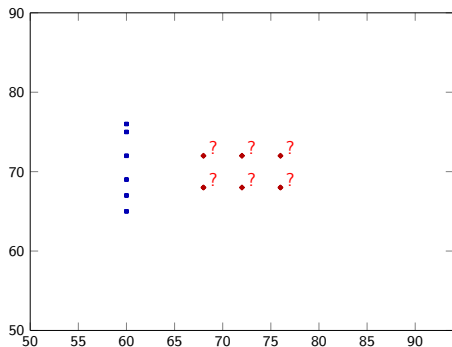
On $[51, 93]_x \times [51, 89]_y$, locate 6

hard-scattering obstacles

of radius 0.5 positioned at ♦

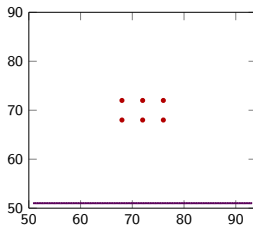
$(68, 68)$, $(68, 72)$, $(72, 68)$
 $(72, 72)$, $(76, 68)$, $(76, 72)$.

Initial guesses are placed at ■.

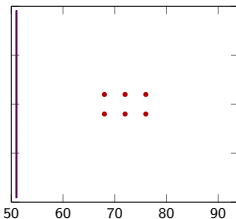


Synthetic data

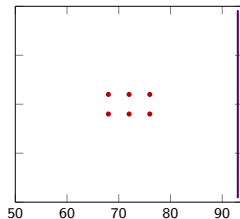
The positions of 128 equally-spaced receivers vary with the angle of incidence.



(a) $\alpha_{\text{inc}} = 90^\circ$



(b) $\alpha_{\text{inc}} = 0^\circ$



(c) $\alpha_{\text{inc}} = 180^\circ$

Complex Gaussian white noise is added by Matlab routine

Synthetic Data is produced

`awgn(data, SNRdB, 'measured')`.

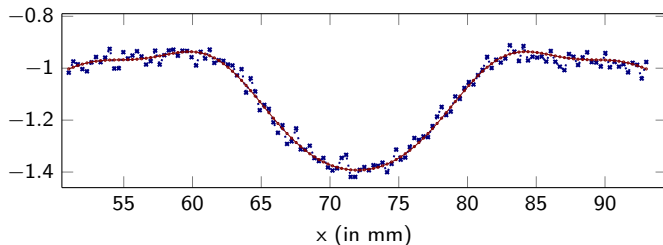
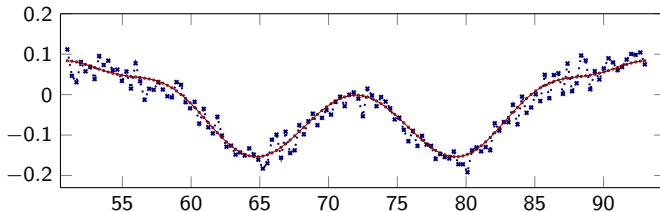
by FSSL order 12
with solver Lapack.

SNR_{dB} = signal-to-noise ratio per sample in decibel.

$$\text{SNR}_{\text{dB}} = 10 \log_{10} \frac{\|\text{Data Vector}\|}{\|\text{Noise Vector}\|}.$$

Noise Vector is generated using Gaussian probability distribution.

Synthetic data (cont) : Noise - 30dB

(d) Real part of total wave at 128 receivers at $\kappa = 0.8$ with PW 90° 

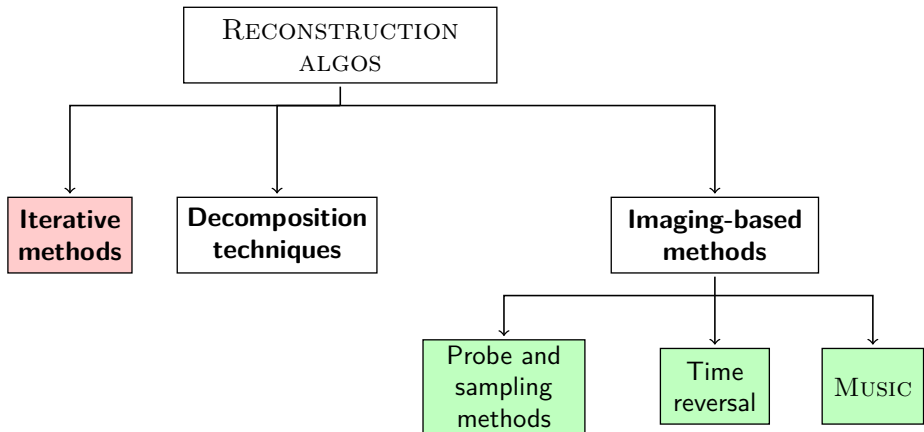
(e) Imaginary part .

Rel. error in
norm

$$l^2 = 5\%$$

$$l^\infty = 8\%$$

Inversion Literature



Quantitative gradient-based inversion

Find the minimizer of the (reduced) cost function $\hat{\mathcal{J}}$,

$$\hat{\mathcal{J}}(\mathbf{m}) = \frac{1}{2} \|\Phi(\mathbf{m}) - \mathbf{d}_{\text{obs}}\|^2.$$

Trace operator at the receptors $\mathcal{R}_{\text{rec}} : u|_{\text{receptor}}, \partial_n u|_{\text{receptor}}, \text{etc.}$

Observed data at receptors : \mathbf{d}_{obs} .

Forward map $\Phi : \text{model } \mathbf{m} \mapsto \text{simulated data at receptors.}$

Main features

Use **line-search optimization** strategy.

Calculate **gradient** $\nabla_p \hat{\mathcal{J}}$ by **adjoint method** (with FSSL formulation).

Use **frequency-hopping** to escape from stagnation in local minima.

Motivation

- Approximate f by second-order Taylor poly \mathfrak{M} .

$$f(\mathbf{m} + \mathbf{s}) = \mathfrak{M}(\mathbf{s}) + o(\|\mathbf{s}\|^2).$$

$$\mathfrak{M}(\mathbf{s}) := f(\mathbf{m}) + \mathbf{s}^t \nabla f(\mathbf{m}) + \frac{1}{2} \mathbf{s}^t \nabla^2 f(\mathbf{m}) \mathbf{s}.$$

- Rate of change of f along direction \mathbf{s} at \mathbf{m} is : $\mathbf{s}^t \nabla f(\mathbf{m})$.
- Direction \mathbf{s} is called a **descent direction** at \mathbf{m} if $\mathbf{s}^t \nabla f(\mathbf{m}) < 0$.

Steepest descent

$$\mathbf{s} = -\nabla f(\mathbf{m})$$

- ★ **Pros** : does not require second derivatives ; ★ **Cons**: slow convergence.

Newton

Newton direction is defined by the minimum of \mathfrak{M} (Assuming $\nabla^2 f_k$ pos def.)

$$\nabla \mathfrak{M} = 0 \Leftrightarrow \nabla f + \nabla^2 f \mathbf{s} = 0 \Leftrightarrow \mathbf{s}_{k, \text{Newton}} := -(\nabla^2 f_k)^{-1} \nabla f_k.$$

Search Dir. $\mathbf{s}_{k, \text{Newton}}$ is a **descent direction**, if $\nabla f_k \neq 0$ and $\nabla^2 f_k$ pos. def.

- ★ **Pros**: fast rate of local convergence ; ★ **Cons**: needs Hessian.

Search directions

- ★ Do not need the Hessian ;
- ★ faster than Steepest Descent.

Quasi-Newton

Use an approximation B_k (positive and definite) of the Hessian $\nabla^2 f_k$

$$\mathbf{s}_k = -B_k^{-1} \nabla f_k.$$

A popular formula is by BFGS (Broyden, Fletcher, Goldfarb and Shannon)

- ★ require storage of matrix.

Nonlinear Conjugate gradient

$$\mathbf{s}_{k+1} = -\nabla f_k + \beta_k \mathbf{s}_k$$

A popular formula for β_k is by Polak-Ribière

$$\beta_k = \frac{\nabla f_k^t (\nabla f_k - \nabla f_{k-1})}{\nabla f_{k-1}^t \nabla f_{k-1}}.$$

- ★ storage of matrix not required.

Adjoint method for calculating the gradient

$$\begin{aligned}
 u_{h,\text{scatt}}(x) &= \frac{i\pi}{2} \sum_{J=1}^{N_{\text{Obs}}} r_J \sum_{l=-n}^n V_{J,l} H_k^{(1)}(\kappa r_J(x)) e^{i l \theta_J(x)} \quad (*) \\
 &= T(\mathbf{m})^t V(\mathbf{m}) .
 \end{aligned}$$

Forward map Φ	:	Model space	\longrightarrow	Simulated data space
		\mathbf{m}	\mapsto	$u_{h,\text{scatt}} _{\text{receivers}}$

$$\Phi(\mathbf{m}) = \mathcal{R}_{\text{rec}} u_{h,\text{scatt}} = \mathcal{R}_{\text{rec}} T(\mathbf{m})^t V(\mathbf{m}) = \mathfrak{R}(\mathbf{m}) V(\mathbf{m}).$$

$$\nabla_{\mathbf{m}} \hat{\mathcal{J}} = \text{Re} [\partial_{\mathbf{m}} \Phi^* (\Phi(\mathbf{m}) - \mathbf{d}_{\text{obs}})]$$

Adjoint method for calculating the gradient (cnt)

- Avoid calculating the Jacobian $\partial_{\mathbf{m}}\Phi$
- Avoid calculating $\partial_{\mathbf{m}}\mathbf{A}^{-1}$.

Forward linear system

$$\mathbf{A}V = F$$

Adjoint linear system

$$\mathbf{A}^*\gamma_1 = -\mathfrak{R}^*(\mathbf{m}) (\Phi(\mathbf{m}) - \mathbf{d}_{\text{obs}}).$$

$$\hat{\mathcal{J}}'(\mathbf{m}) =$$

expression in terms of
 γ_1 , V , \mathbf{d}_{obs}

Inexact Line search algorithm

$$\mathbf{m}_{k+1} = \mathbf{m}_k + \alpha_k \mathbf{s}_k.$$

Exact minimization can be expensive

⇒ use a **line-search algorithm** to obtain an approximate minimum of

$$\min_{\alpha > 0} \phi(\alpha),$$

$$\phi(\alpha) = \widehat{\mathcal{J}}(\mathbf{m}_k + \alpha \mathbf{s}_k).$$

Strategy : Adequate reduce in $\widehat{\mathcal{J}}$ with minimal cost.

Make a 'trade off'

- ★ Choose α_k so that ϕ reduces substantially;
- ★ Minimize the time making that choice.

Algo 1: Simple backtracking

$$\phi(\alpha) < \phi(0)$$

$$\alpha \mapsto 0.5 \alpha$$

Algo 2: Sufficient decrease

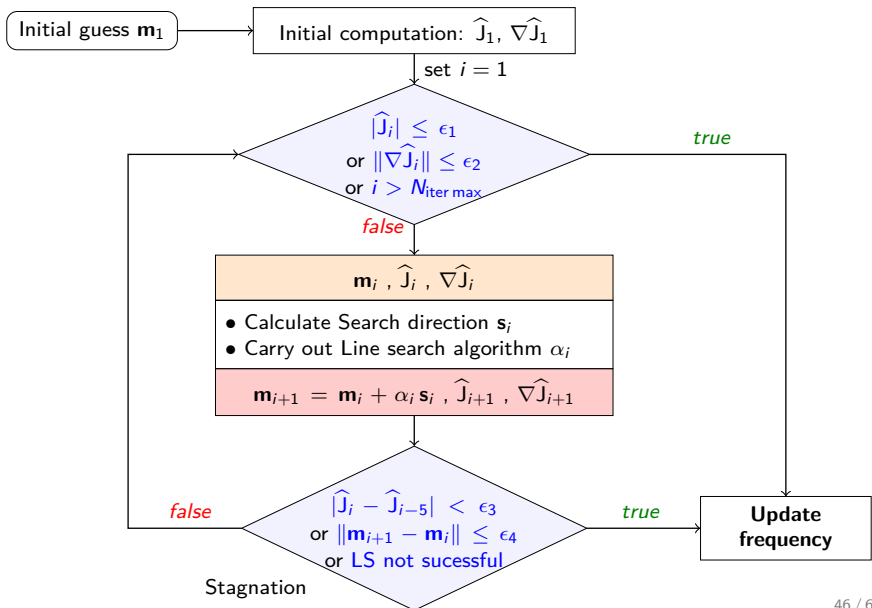
$$\phi(\alpha) < \phi(0) + c_1 \alpha \phi'(0)$$

$\alpha \mapsto$ quadratic interpolation on $[0, \alpha]$

Algo 3 :

Strong Wolfe

Optimization algorithm at a frequency



Plan

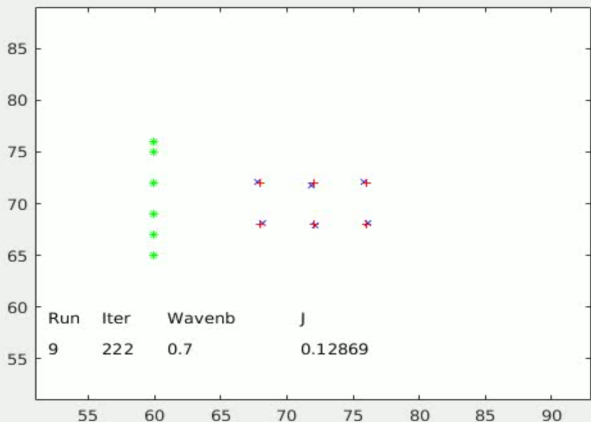
- 5 Numerical inversion experiments
 - Periodic configuration of 6 obstacles with 30dB
 - Periodic configuration of 12 obstacles with 25dB
 - Random configuration of 12 obstacles with 30 dB

Features of Inverse Problem codes

Latest version

- written in Fortran90
- offers choices of different optimization schemes.
- currently uses Mumps.
- integrates a copy of the principal part of the direct simulation codes.

Group 1 - Exp 1 - 6 obstacles with 30dB data



Run time = 2.27s .

Nb of iter = 224.

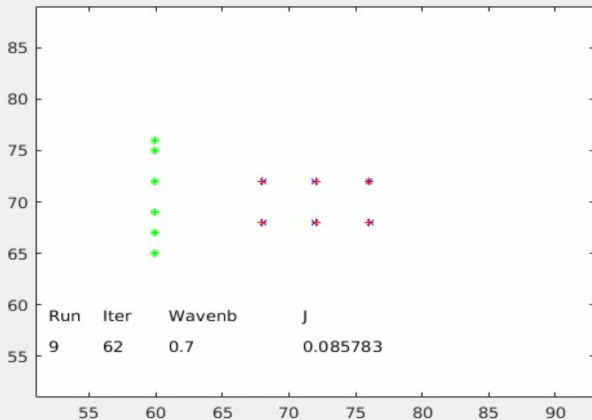
2 incidence angles: 90° , 0°

9 wavenumbers used: 0.08, 0.09 , 0.1- 0.7.

Quasi-Newton and Simple backtracking.

	Err Pos	scaled Err Pos
Initial guess	20.49	53.9%.
Final position	0.108	0.2%.

Group 1 - Exp 2 - 6 obstacles with 30dB data



Run time = 0.30s .

Nb of iter =57 .

2 incidence angle: 90° , 0°

NL Conjugate gradient and Sufficient decrease
Backtracking.

	Err Pos	scaled Err Pos
Initial guess	20.49	53.9%
Final position	0.125	0.33% . 50 / 60

Comparison among the methods

Search Dir (SD)	Line Search (LS)	Linesearch parameters	# wn	# lter	Final ErrPos	Final Scaled ErrPos	Run time (secs)
1	1	n/a	9	224	0.108	0.28 %	2.27
1	2	(0.0001, n/a)	9	147	0.130	0.34 %	0.58
1	3	(0.0001, 0.4)	9	46	0.090	0.24%	0.34
1	3	(0.0001, 0.9)	9	64	0.131	0.34 %	0.43
2	1	n/a	10	84	0.171	0.45 %	0.58
2	2	(0.0001, n/a)	9	57	0.125	0.33 %	0.30
2	3	(0.0001, 0.4)	9	61	0.104	0.27 %	0.42
2	3	(0.0001, 0.9)	9	73	0.142	0.37 %	0.41

SD 1 : Quasi-Newton ;

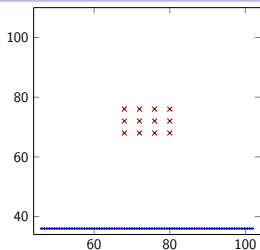
SD 2 : NL Conjugate gradient.

LS 1: simple backtracking;

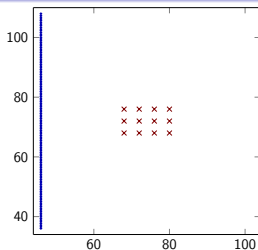
LS 2: backtracking with sufficient descent and quadratic interpolation.

LS 3 : Strong Wolfe.

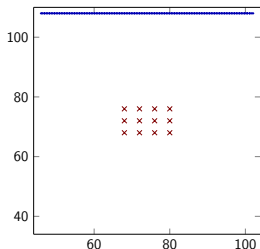
Inversion Exp 2 :



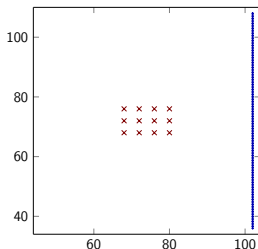
(f) 90° incidence



(g) 0° incidence



(h) 270° incidence

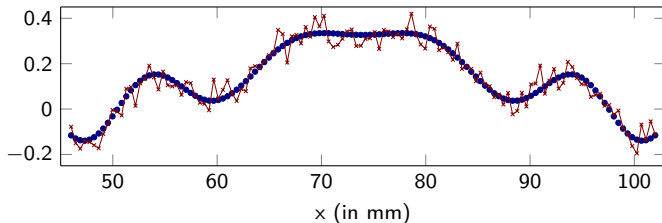


(i) 180° incidence

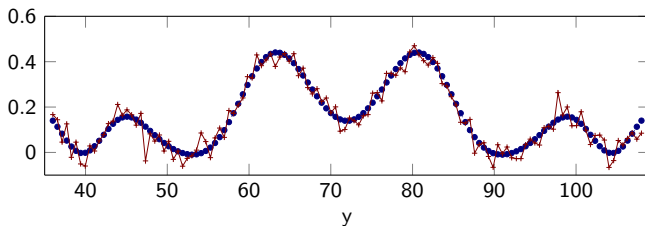
Locate **12 soft-scattering obs** of radius 0.5 placed at between 68 and 80 (in x); between 68 and 76 (in y)

For each angle of incidence:
128 receivers on one corresponding side of $[46, 102]_x \times [36, 108]_y$.

Noisy Data at 25dB



(j) Real part of data for 270° incidence. Total relative error in $l^2 = 5.43\%$ and $l^\infty = 9.92\%$.

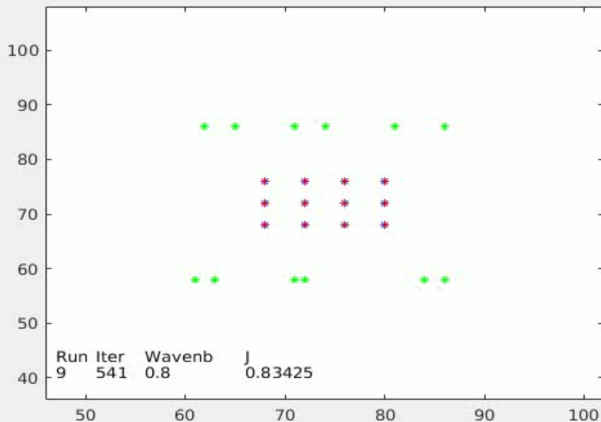


(k) Imag. part for 180° . Rel. error in $l^2 = 5.24\%$ and in $l^\infty = 10.95\%$.

White Gaussian noise is added by using `wgn` in Matlab.

Rel. error in norm
 l^2 : 4.9 - 6.2% ,
 l^∞ : 7.3 - 15.4 %

Inversion result for data with 25dB noise



Run time = 17.2s .

Nb of iter = 552.

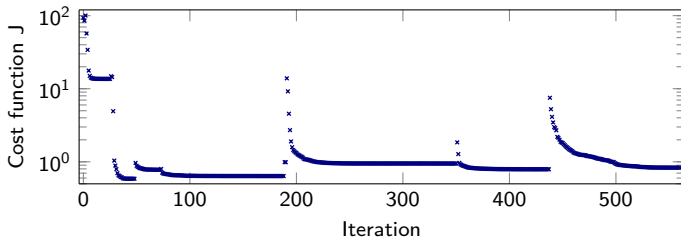
Four angles of acquisitions: 90° , 0° , 180° , 270°

9 wavenumbers used: 0.08, 0.09, 0.1-0.6, 0.8.

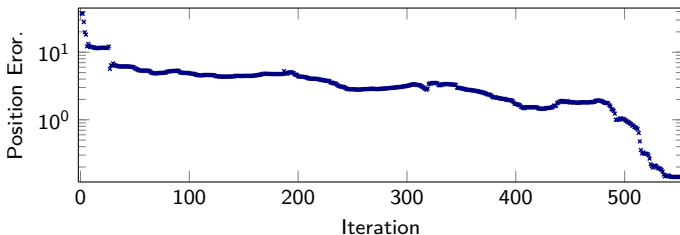
Quasi-Newton and strong Wolfe linesearch.

	Err Pos	scaled Err Pos
Initial Guess	37.6	67.15%
Final position	0.14	0.25%

Inversion result for data with 25dB noise (cnt)



Initial $J = 93.45$ at $\kappa = 0.08$; Final $J = 0.83$ at $\kappa = 0.8$



Initial guess: Err Pos = 37.6; rel. err. = 67.15%; Final construction: Err Pos = 0.14; Rel. err = 0.25%.

★ Four angles of acquisitions:

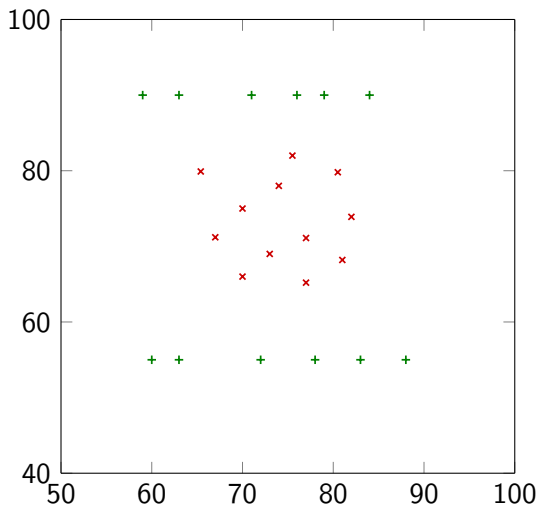
$90^\circ, 0^\circ, 180^\circ, 270^\circ$.

★ Niter total = 552;

★ Use 9 freqs :
0.08, 0.09,
0.1, ..., 0.6, 0.8

★ Run time: 17.2 s

Numerical exp 3: 12 ran Obs and data w/ 30 dB noise



★ Locate 12 hard-scattering obstacles of radius 0.5 on domain $[50, 100]_x \times [40, 100]_y$.

★ Ratios

$$0.04 \leq \kappa r \leq 0.25$$

$$0.34 \leq \kappa d_{\min} \leq 2.12$$

$$1.56 \leq \kappa d_{\max} \leq 9.75$$

★ Four angles of acquisitions:

$$90^\circ, 0^\circ, 180^\circ, 270^\circ$$

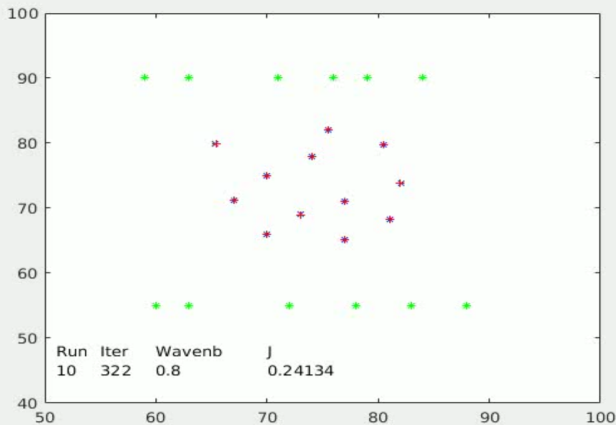
★ 128 receivers for each angle of incidence, equally on a corresponding side of the domain.

★ **Noise** :

$$2.7\% \leq l^2 \text{ rel. err} \leq 3.6\%,$$

$$3.9\% \leq l^\infty \leq 8.8\%.$$

Inversion exp 3 results.



Run time = 7.7 s .

Nb of iter = 317.

4 incidence angles:

10 wavenumbers used: 0.08-0.09, 0.1-0.8.

Quasi-Newton and strong Wolfe linesearch.

	Err Pos	scaled Err Pos
Initial guess	39.7	79.4%
Final position	0.17	0.3%

Conclusions

- FSSL is robust in simulating the multi-scattering by **small circular obstacles** in **large homogeneous media**.

- **Direct Solvers (Lapack and Scalapack)** are more efficient when the obstacles are close together.

- Iterative solvers are more preferable when the obstacles are far apart.

In particular, **GMRES with LUSGS and SGS** are faster than Lapack and as fast as Scalapack.

- **LUSGS and SGS** are the most robust among the preconditioners considered.

- Direct problem resolution using FSSL and direct solvers are robust in FWI.

- Successful reconstruction in presence of noise.

- Although NL conjugate gradient with cheaper linesearch can be faster in some cases, the more reliable method is **Quasi-Newton** with **strong Wolfe linesearch**.

Future directions

- Compare with other optimization, e.g. Newton-like methods, 2nd order.
- Compare with imaging-based methods, in particular MUSIC.
- Use in combination with such methods for a good initial guess (even without knowledge of the number of obstacles), and then use the current method for precise reconstruction.
- Other inverse problems: determining material parameters within the obstacles.
- Extension to elastic inclusions.

Thank you for your attention !

Questions?

Parameters for experiment : 6 Obs 30dB Exp 1

- Error tolerance and stagnation parameters:

$$\epsilon_{\text{Stag Pos}} = \begin{cases} 0.00005 & \text{for run 1-6} \\ 0.0005 & \text{for run 7-9} \end{cases} \quad \begin{aligned} \epsilon_J = \epsilon_{\nabla J} = \epsilon_{\text{Stag LS}} &= 0.00001 \\ \epsilon_{\text{Stag J}} &= 0.1 \end{aligned}$$

- Order FSSL = 3.
- Niter max = 50 ; Niter LS max = 10.

Run	κ	Order FSSL	Step size
1	0.08	3	15
2	0.09	3	12
3	0.1	3	12
4	0.2	3	12
5	0.3	3	8

Run	κ	Order FSSL	Step size
6	0.4	3	8
7	0.5	3	5
8	0.6	3	5
9	0.7	3	3

Parameters for experiment : 6 Obs 30dB Exp 2

- Error tolerance and stagnation parameters:

$$\epsilon_{\text{Stag Pos}} = \begin{cases} 0.5 & \text{for run 1-6} \\ 0.001 & \text{for run 7-9} \end{cases}, \quad \begin{aligned} \epsilon_J &= \epsilon_{\nabla J} = \epsilon_{\text{Stag LS}} = 0.00001 \\ \epsilon_{\text{Stag J}} &= 0.1 \end{aligned}$$

- Wolfe Line search parameters : $c_1 = 0.0001$.
- $\#$ Iter Max = 300 , $\#$ Iter Linesearch Max = 30.
- Order FSSL = 3.

Run	κ	$\epsilon_{\text{Stag Pos}}$	Init. Step size
1	0.08	0.5	10
2	0.09	0.5	10
3	0.1	0.5	10
4	0.2	0.5	8
5	0.3	0.5	8

Run	κ	$\epsilon_{\text{Stag Pos}}$	Init. Step size
6	0.4	0.5	5
7	0.5	0.01	5
8	0.6	0.01	3
9	0.7	0.01	3

Parameters of experiment : 12 structured Obs 25dB

- Error tolerance and stagnation parameters

$$\begin{aligned}\epsilon_J &= 0.5 & \epsilon_{\nabla J} &= 5.0 \times 10^{-4} \\ \epsilon_{\text{Stag LS}} &= 1.0 \times 10^{-3} & \epsilon_{\text{Stag J}} &= 0.1 & \epsilon_{\text{Stag Pos}} &= 1.0 \times 10^{-8}\end{aligned}$$

- Wolfe Line search parameters : $c_1 = 0.0001$, $c_2 = 0.4$.
- $\#$ Iter Max = 300 , $\#$ Iter Line search Max = (30 , 30).

Run	κ	Order FSSL	Step size
1	0.08	3	1
2	0.09	3	24
3	0.1	3	24
4	0.2	3	19
5	0.3	3	18

Run	κ	Order FSSL	Step size
6	0.4	3	16
7	0.5	4	16
8	0.6	4	10
9	0.8	6	8

Parameters of experiment : 12 rand obs 30dB

- Error tolerance and stagnation parameters

$$\begin{aligned} \epsilon_J &= 0.05 & \epsilon_{\nabla J} &= 5.0 \times 10^{-4} \\ \epsilon_{\text{Stag LS}} &= 0.05 & \epsilon_{\text{Stag J}} &= 0.1 & \epsilon_{\text{Stag Pos}} &= 1.0 \times 10^{-8} \end{aligned}$$

- Wolfe Line search parameters : $c_1 = 0.0001$, $c_2 = 0.4$.
- # Iter Max = 300 , # Iter Linesearch Max = (30 , 30).

Run	κ	Ord FSSL	Init. Step size
1	0.08	3	50
2	0.09	3	30
3	0.1	3	10

Run	κ	Ord FSSL	Init. Step size
4 - 6	0.2 - 0.4	3	10
7 - 10	0.5 - 0.8	4	10