Numerical Robustness of Single Layer method with Fourier basis in 2D multiple obstacle scattering.

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Wave 2017 Conference, University of Minnesota,
May 2017.

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Overview

Plan

Multiple obstacle scattering as Exterior Boundary Value problems

Propagation of acoustic waves of freq. **f** in a hom. medium with sound speed **c**.

$$u_{\text{total}} = u_{\text{inc}} + u_{\text{scatt}}.$$

1. PDE satisfied by u_{scatt} outside of the obstacles:

$$(-\Delta - \kappa^2)\, u_{\text{scatt}} = 0\,,\ \kappa = \frac{2\pi \mathbf{f}}{\mathbf{c}}.$$

For transmission prob, $(-\Delta - \kappa_{\text{int}})u_{\text{int}} = 0$ inside obstacles.

2. Conditions on the boundary of the obstacles:

Dirichlet	$u_{\text{total}} = 0$
Neumann	$\partial_n u_{\text{total}} = 0$
Impedance	$\partial_n u_{\text{total}} + i\lambda u_{\text{total}} = 0$
Transmission	$u_{\text{total}} - u_{\text{int}} = 0\,;\ \partial_n u_{\text{total}} - \mu u_{\text{int}} = 0$

3. (Outgoing) Sommerfeld radiation condition at ∞:

$$\lim_{r\to\infty} \sqrt{r}\,(\partial_r\, u_{\text{scatt}} - i\,\kappa u_{\text{scatt}}) = 0 \quad ; \quad r = |x|$$

$\exists!$ solution for the exterior BVPs (all parameters > 0).

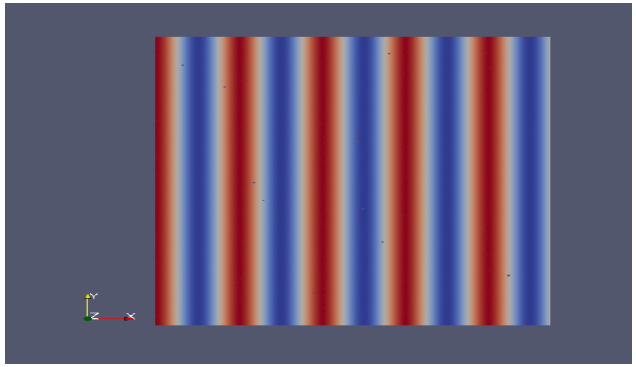
References: Hettlich, *Fréchet derivatives in inverse obstacle scattering*.
Colton, Kress, *Integral equation methods in scattering theory*.

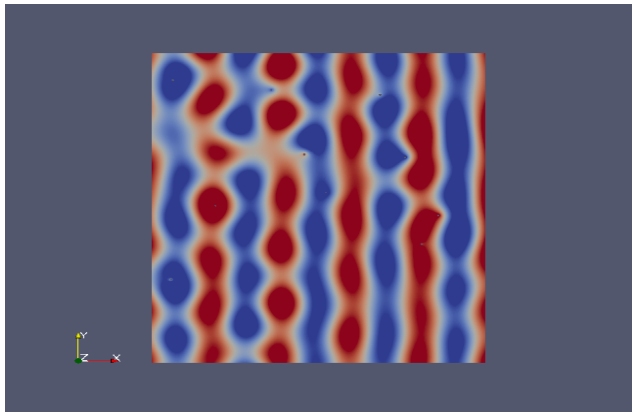
Time-harmonic Planewave :

$$u_{\text{pw}}(x)\ \exp(-i\,2\pi\,\mathbf{f}\,t)$$

$$u_{\text{pw}}(x) = \exp\left(\kappa\, x \cdot \begin{pmatrix}\cos\alpha_{\text{inc}} \\ \sin\alpha_{\text{inc}}\end{pmatrix}\right)$$

$$\alpha_{\text{inc}} = 0^\circ\,,\ 2\pi\mathbf{f} = 1.0\,,\ \kappa = 1.0.$$





Single layer potential formulation.

Ext. Dir. Prob : $u_{\text{total}} = u_{\text{inc}} + u_{\text{scatt}}$;
$(-\Delta - \kappa^2) u_{\text{scatt}} = 0$ outside of Obs;
u_{scatt} satisfies Som. rad. cond;
$u_{\text{scatt}}|_{\Gamma_{\text{Obs}}} = -u_{\text{inc}}$.

Integral Equation (IE)

$u_{\text{scatt}} = \sum_{J=1}^{N_{\text{Obs}}} u_{\text{scatt},J}$;
$u_{\text{scatt},J} := \tilde{S}_J v_J = \int_{\Gamma_J} G_\kappa(x,y)\, v_J(y)\, ds(y)$.

IE Problem: Find densities v_J so that
$\sum_{J=1}^{N_{\text{Obs}}} S_{IJ}\, v_J = -\gamma_{0,I}\, u_{\text{inc}}$, $I = 1,\ldots,N_{\text{Obs}}$;
$S_{IJ} = \gamma_{0,I}\, \tilde{S}_J$; $\gamma_{0,I}$ 0-th trace along Γ_I.

Variational IE : Find v_J so that
$\sum_{J=1}^{N_{\text{Obs}}} (S_{IJ} v_J\,,\, \phi) = \left\langle -u_{\text{i},I}\,,\, \phi \right\rangle_{H^{1/2}(\Gamma_I)\,,\, H^{-1/2}(\Gamma_I)}$
$\forall\, I = 1,\ldots,N_{\text{Obs}}$ and test func $\phi \in H^{-1/2}(\Gamma_I)$.

Variational formulation

Boundary Element (BEM)

Variational IE for test func in finite-dim subspaces $\{V_{I,\mathbf{m}}\}$ approximating $H^{-1/2}(\Gamma_I)$

Fourier Series Single Layer

Curved BEM Galerkin

$V_{\mathbf{m}}$ given by piecewise $\mathbf{P_m}$ functions

$V_{\mathbf{m}}$ is given by $\{\sum_{k=-\mathbf{m}}^{\mathbf{m}} a_k e^{ik\theta}\}$

Disc-shaped obstacles

0th order approximation

Multipole

0th order approximation

Foldy isotropic point scattering

Fourier Series Single Layer method.

The scattered and approx. wave

$$u_{\mathbf{scatt}} = \sum_{J=1}^{N_{\mathrm{Obs}}} u_{\mathbf{scatt};J} \,,$$

$$u_{\mathbf{scatt},h} = \sum_{J=1}^{N_{\mathrm{Obs}}} u_{h,\mathbf{scatt};J} \,.$$

The exact and app. wave scattered by Obs J

$$u_{\mathbf{scatt};J} = \tilde{S}_J v_J \;;\; u_{h,\mathbf{scatt};J} = \tilde{S}_J v_{h,J} \,.$$

In basis elements

$$\mathbf{w}_{J,k}(x) = e^{i\,k\,\theta_J(x)},$$

$$u_{\mathrm{scatt};J} = \sum_{k\in\mathbb{Z}} V_{J,k}\,\tilde{S}_J\,\mathbf{w}_{J,k}$$

$$u_{h,\mathrm{scatt};J} = \sum_{k=-\mathbf{m}}^{\mathbf{m}} V_{J,k}\,\tilde{S}_J\,\mathbf{w}_{J,k} \,.$$

The unknowns are the Fourier coeff. of density v_J

$$V = (V_{J,k}) \;,\; k \in \mathbb{Z}, 1 \le J \le N_{\mathrm{Obs}} \,,$$

and the truncated ones for the approx. $v_{h,J}$.

$$V_h = (V_{J,k}) \,, -\mathbf{m} \le k \le \mathbf{m} \;,\; 1 \le J \le N_{\mathrm{Obs}} \,.$$

For $\alpha = \mathrm{D}, \mathrm{N}, \mathrm{Im}, \mathrm{T}$, they solve

$$\mathbf{A}_\alpha\, V = F_\alpha \quad , \quad \mathbf{A}_{h,\alpha}\, V_h = F_{\alpha,h} \,.$$

$$\mathbf{A}_\alpha = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \cdots & \mathbf{A}_{1(\mathbf{N}-1)} & \mathbf{A}_{1\mathbf{N}} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \cdots & \mathbf{A}_{2(\mathbf{N}-1)} & \mathbf{A}_{2\mathbf{N}} \\ \vdots & & \ddots & & \vdots \\ \mathbf{A}_{(\mathbf{N}-1)1} & \mathbf{A}_{(\mathbf{N}-1)2} & \cdots & \mathbf{A}_{(\mathbf{N}-1)(\mathbf{N}-1)} & \mathbf{A}_{(\mathbf{N}-1)\mathbf{N}} \\ \mathbf{A}_{\mathbf{N}1} & \mathbf{A}_{\mathbf{N}2} & \cdots & \mathbf{A}_{\mathbf{N}(\mathbf{N}-1)} & \mathbf{A}_{\mathbf{NN}} \end{pmatrix}$$

$\mathbf{A}_{h,\alpha}$ square matrix of size $(2\mathbf{m}+1) \times N_{\mathrm{Obs}}$.
$\mathbf{A}_{\alpha,I}$ self-interaction of obstacle I
$\mathbf{A}_{\alpha,IJ}$ diffraction by obs. I of wave emitted by obs. J

Multi-scattering matrix coefficient for circular obstacles.

For circular obstacles, single-layer densities $\tilde{S}_J \mathbf{w}_{J,k}$ can be written in multipole expansions,

$$\left(\tilde{S}_J \mathbf{w}_{J,k}\right)(x) = \frac{i\pi \mathbf{r}_J}{2} J_k(\kappa \mathbf{r}_J) \underbrace{H_k^{(1)}(\kappa r_J(x))\, e^{i k \theta_J(x)}}_{\text{multiple pole of order k placed at the center of } \mathcal{O}_J}.$$

For Dirichlet : Same obstacle interaction

$$(\mathbf{A}_{D;I})_{kl} = i\pi \mathbf{r}_I H_k^{(1)}(\kappa \mathbf{r}_I) J_k(\kappa \mathbf{r}_I) \delta_{kl} \quad , \quad k, l \in \mathbb{Z}.$$

Interaction between two different obstacles $I \neq J$

$$(\mathbf{A}_{D;IJ})_{kl} = i\pi \mathbf{r}_J e^{i(l-k)\theta_{\mathbf{x}_J}(\mathbf{x}_I)} H_{l-k}^{(1)}(\kappa \mathbf{d}_{IJ}) J_k(\kappa \mathbf{r}_I) J_l(\kappa \mathbf{r}_J),$$

$$\mathbf{d}_{IJ} = |\mathbf{x}_I - \mathbf{x}_J| \quad ; \quad k, l \in \mathbb{Z}.$$

Right-hand-side corresponding to planewave

$$u_{\mathbf{pw}}(x) = \exp(i\kappa x \cdot (\cos\alpha_{\mathbf{inc}}, \sin\alpha_{\mathbf{inc}})),$$

$$(\mathbf{F}_{D;I})_k = -2\, u_{\mathbf{pw}}(\mathbf{x}_I)\, i^k\, e^{-ik\alpha_{\mathbf{inc}}} J_k(\kappa \mathbf{r}_I).$$

N_{Obs} circular obstacles.

Obstacle $\mathcal{O}_I$ centered at $\mathbf{x}_I$ with radius $\mathbf{r}_I$

Relative polar coordinates

$$(r_J(\cdot), \theta_J(\cdot))$$

with respect to $\mathbf{x}_J$

$$x = \mathbf{x}_J + r_J(x)(\cos\theta_J(x), \sin\theta_J(x))$$

$$\mathbf{x}_I = \mathbf{x}_J + \mathbf{d}_{IJ}(\cos\theta_{JI}, \sin\theta_{JI})$$

Well-posedness

$$0 \leq \kappa_e < \infty \quad ; \quad \lambda \in \mathbb{R} \quad ; \quad 0 \leq \kappa_{\text{int}} < \infty\,,\ 0 < \mu < \infty,\ \mu \neq 1.$$

If κ_e^2 is not a Dirichlet eigenvalues (EV) of $-\Delta$ for $\mathcal{O}_l$ for $1 \leq l \leq N_{\text{Obs}}$, then $\mathbf{A}_\alpha$ is injective for $\alpha = \text{D, N, Imp, T}$.

Circular obstacles

Dirichlet EV : $\lambda_{n,m} = \left(\dfrac{j_{n,m}}{\mathbf{r}}\right)^2$,

$j_{n,m}$ m-th positive root of $J_n(r) = 0$,

$\mathbf{r}$ = radius of obstacle.

Injectivity : $\kappa_e^2\,\mathbf{r}^2 \neq j_{n,m}$.

General shape obstacles

Isoperimetric inequality gives

$$\lambda_1(\mathcal{O}) \ \geq \ \frac{\pi}{\text{Area}\,(\mathcal{O})}\, j_{0,1}^2\,.$$

Injectivity small obs. : $\kappa_e\ \mathbf{r}_{\text{circumvent}}(\mathcal{O}) < 2.$

The first 4 roots :

$j_{0,1} \sim 2.40$, $j_{1,1} \sim 3.83$, $j_{2,1} \sim 5.13$, $j_{1,2} \sim 5.52$.

Plan

Calculation time costs

$$u_{h,\mathbf{scatt}}(x) = \frac{i\pi}{2} \sum_{J=1}^{N_{\mathrm{Obs}}} \mathbf{r}_J \sum_{l=-\mathbf{m}}^{\mathbf{m}} V_{J,l} \quad H_k^{(1)}(\kappa\, r_J(x)) \quad e^{i\, l\, \theta_J(x)} \quad (\star)$$

$$\text{Unknowns} \quad V_h = (V_{J,l}) \quad , \quad 1 \leq J \leq N_{\mathrm{Obs}}\,,\ -\mathbf{m} \leq l \leq \mathbf{m}\,.$$

Pre-processing time = Time to resolve the linear system for V_h.

Linear system is dense but small : $N_{\mathrm{Obs}} \times (2\mathbf{m}+1)$.

Post-processing time = Eval. time of LHS of $(\star)$ at each point of visualization grid.

Evaluation of Hankel is costly.

- Cost increases with N_{Obs} and $\sharp$ points of visualization grid.
- Can reduce the cost by parallelization and interpolation (e.g. Hermite interpolation).

Experiment 1: Small obstacles on medium domain

Soft-scattering of PW with angle 90°

of wavelength $\kappa = 10$, $\lambda \sim 0.63$

by 200 obstacles

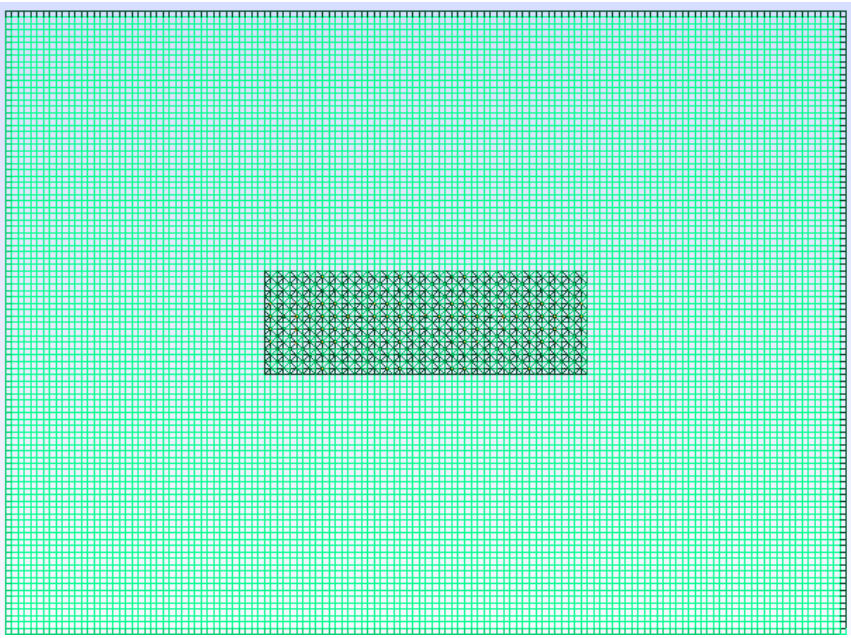
of radius $= 0.03$, with distanced by 0.3.

Domain size : $31\lambda \times 23\lambda$

$\kappa \times (\text{Obs Rad}) = 0.3$,

$\frac{\lambda}{\text{Obs Rad}} \sim 21$, $\frac{\lambda}{\text{Obs. Dist.}} \sim 2$,

$\frac{\text{Obs. Dist.}}{\text{Obs. Rad.}} \sim 10$.



Montjoie initial mesh has mesh size of 0.13.

Montjoie

(`montjoie.gforge.inria.fr`)

Bases: Curved finite element (FE) with Lagrange polynomials based on Gauss-Lobatto points.

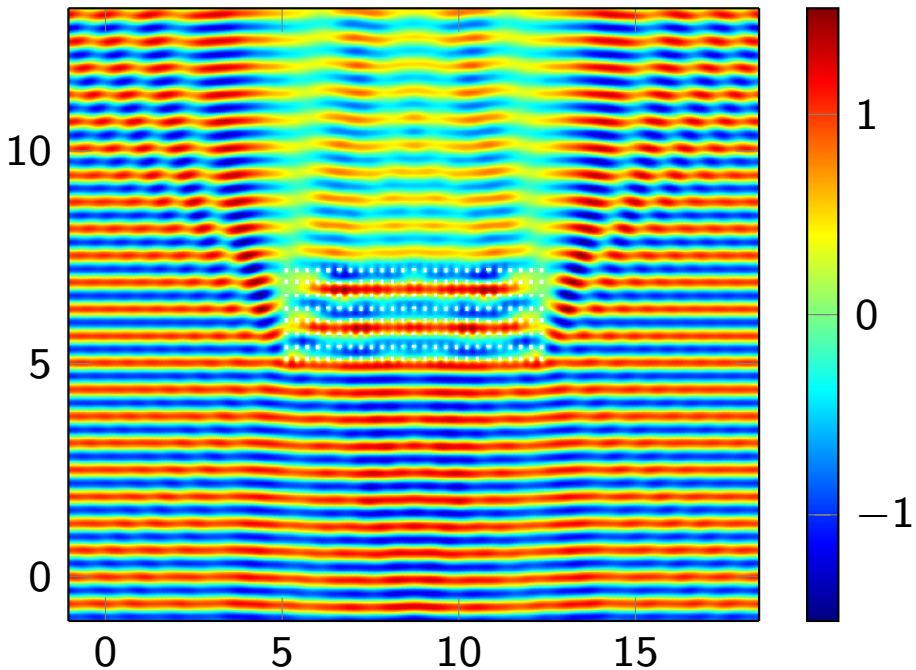
`Q-n` denotes the n^{th} order FE on quadrangular meshes.

Domain truncation: Perfectly Matched Layers.

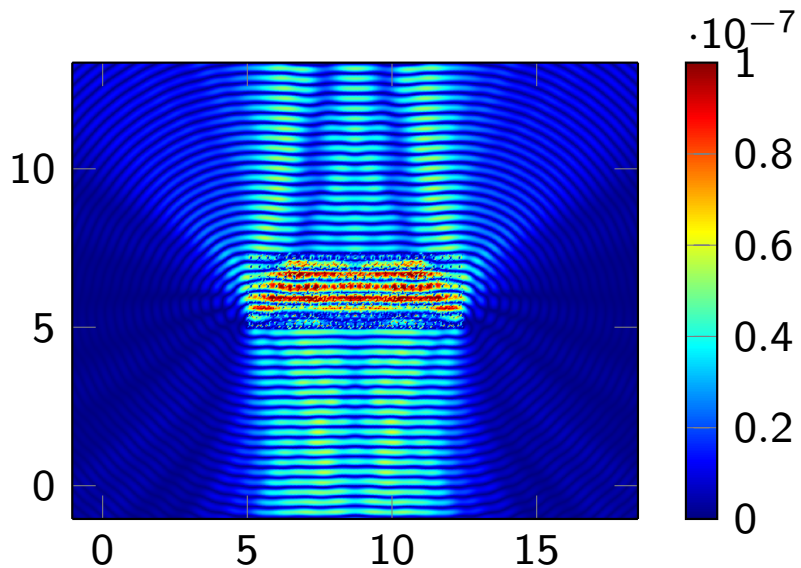
Experiment 1: Reference solutions

Soft-scattering of 200 obstacles on domain of size : $31\lambda \times 23\lambda$

$$\kappa \times (\text{Obs Rad}) = 0.3\ , \quad \frac{\lambda}{\text{Obs Rad}} \sim 21\ , \quad \frac{\lambda}{\text{Obs. Dist.}} \sim 2\ , \quad \frac{\text{Obs. Dist.}}{\text{Obs. Rad.}} \sim 10.$$

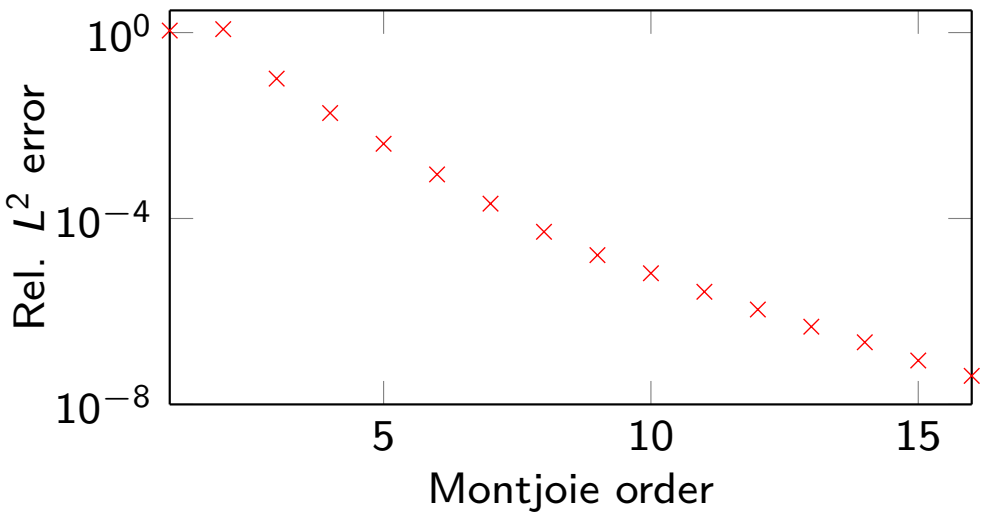


(a) Real part of FSSL 14 total wave

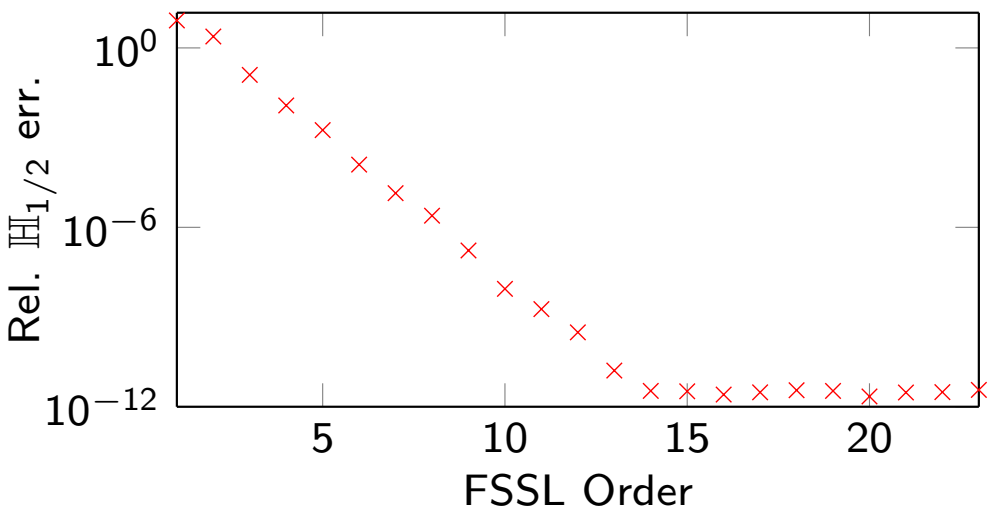


(b) Abs. difference compared with Montjoie Q17. Relative L^2 err. $= 3.38 \times 10^{-8}$.

Experiment 1: Convergence curve



(c) Rel. consecutive err. : Montjoie



(d) Rel. consecutive err : FSSL densities

Candidates for comparison at precision 10^{-3}

Compare between		Rel. L^2 error
FSSL 14	FSSL 2	4.65×10^{-5}
MJ Q17	MJ Q6	6.52×10^{-4}
MJ Q6	FSSL2	6.84×10^{-4}

Hermite interp. precision is 10^{-6}.

Compare between		Rel. L^2 error
FSSL 2 Inter	FSSL 2	1.76×10^{-5}
FSSL 2 Inter	MJ Q6	6.85×10^{-4}

Solvers for both Montjoie and FSSL are Mumps.

Experiment 1: Comparison at precision 10^{-3}

Pre-processing by Mumps	**FSSL Order 2**	**MJ Q6**
Size of lin. sys.	1000	842677
Task	**Time (s)**	
Construction	0.055	1.97
Factorization	0.44	29.8
Resolution	0.003	0.35
Total time	**0.498**	**32.12**

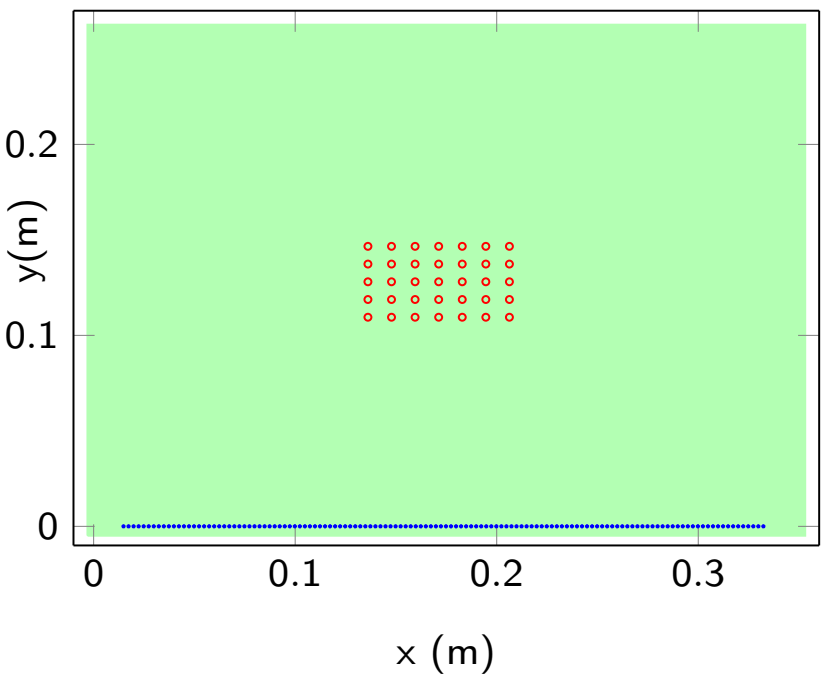
Evaluation on 400 $\times$ 400 grid

	Exact eval	Inter. eval	MJ Q6
Post-proc.	26.2	4.30	0.72
Pre-proc. + Post-proc.	**26.70**	**4.80**	**33.82**

At precision 10^{-3}, FSSL using Hermite interpolation takes 7 times less than MJ.

Experiment 2: sizable obstacles on a large domain

Acoustic vibration, produced by a block transducer ·, is diffracted by 35 thin aluminum wires ◦ (of radius 0.5 mm) immersed in water.



Horizontal-view cut.

The phenomenon is approximated by the hard scattering of acoustic sound in fluid.

The incident wave (from the transducer) is simulated by a PW of angle 90°.

Input pulse's central freq. = 500 kHz.

The speed of sound in water c = 1478 $\mathrm{m\,s^{-1}}$.

The wavenumber $\kappa = 2125.57\ \mathrm{m^{-1}}$.

The spatial wavelength $\lambda = 2.96 \times 10^{-3}$ m.

Domain size $= 117\lambda \times 87\lambda$.

$$\kappa \times (\text{Obs Rad}) \sim 1.1\ , \quad \frac{\text{Obs Dist}}{\text{Obs Rad}} \sim (23, 19)\,,$$

$$\frac{\lambda}{\text{Obs Rad}} \sim 5.91 \quad , \quad \frac{\lambda}{\text{Obs. Dist.}} \sim 0.3\,.$$

Exp 2: Computational time comparison at precision 10^{-4}

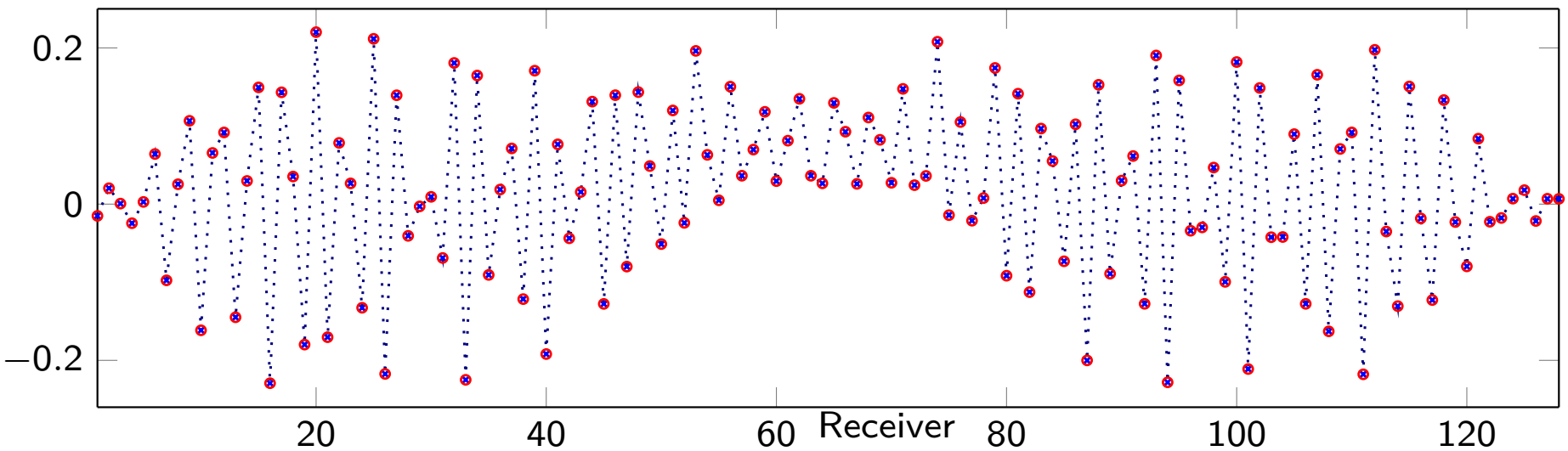
Regarding the value of the diffracted wave at 128 receptors,

Rel. L^2 error : FSSL 12 and FSSL 4 $= 2.82 \times 10^{-6}$,

Rel. L^2 error : MJ Q12 and MJ Q8 Ref 2 $= 1.42 \times 10^{-4}$.

Rel. L^2 error : FSSL 4 and MJ Q8 Ref 2 $= 1.48 \times 10^{-4}$.

Q8 Ref 2 = Q8 with one time mesh refinement.



Real of part of diffracted wave at 128 receptors : FSSL 4 ····×···· and MJ Q8 Ref2 ◦.

Exp 2: Candidates for comparison at precision 10^{-4}

	Size of LS	**Pre-proc. Time (s)**	**Post-proc. Time at 128 receivers (s)**	**Total time (s)**
FSSL 4	315	0.024	6.58×10^{-3}	0.031
MJ Q8 Ref 2	993870	61.27	0.13	61.4

FSSL using Hermite interpolation is 2046 times faster than MJ.

Plan

Restart GMRES (generalized minimal residual method)

Consider $Ax = b$, A matrix of size $N \times N$.

$$\overset{\text{Minimal poly}}{\Rightarrow} A^{-1}b \in \underbrace{K_n(A,b) := \text{span}\{b, Ab, \dots, A^{n-1}y\}}_{\text{Krylov space}}.$$

For $d \leq m$, Arnoldi process constructs

$$A\,Q_d = \underbrace{Q_{d+1}}_{\text{orthonormal , } N\times d} \overline{H}_d \; ; \; \overline{H}_n = \begin{pmatrix} H_d \\ 0_{1\times(d-1)} \; h_{(d+1)d} \end{pmatrix}.$$

H_d Hessenberg of size $d \times d$

$$\|Ax - b\|_2 = \|A\,Q_d y - b\|_2 = \left\|\overline{H}_d y - \|b\|_2 \mathbf{e}_1\right\|_2.$$

A sequence of approx. sol.

$$x_d = \underset{z \in K_d(A\,,\,b)}{\text{argmin}} \|Az - b\|_2.$$

$$\Leftrightarrow y_d \text{ with } x_d = Q_d y_d$$

$$y_d = \underset{y\in\mathbb{R}^d}{\text{argmin}} \left\|\overline{H}_d y - \|b\|_2 \mathbf{e}_1\right\|_2.$$

Fixed Krylov size m. Initial guess (IG) x_0. Initial residue $r_0 = b - A_0$.

No preconditioning : $A\,p_\star = r_0$.

For $j \leq m$, approximate sol. $p_j \in K_j(A, r_0)$ minimizes

$$p_j = \underset{p \in K_j(A\,,\,r_0)}{\text{argmin}} \|Ap - r_0\|_2 \quad (*).$$

Stop if p_j satisfies the residue error criteria.

If not, and if $j = m$, *restart the process* with IG $r_0 = p_m$.

Final stop criteria : NiterMax .

Right preconditioning

$$(A\mathcal{P}^{-1})(\mathcal{P}p_\star) = r_0.$$

Left preconditioning

$$(\mathcal{P}^{-1}A)p_\star = \mathcal{P}^{-1}r_0.$$

GMRES Preconditioners

L = strictly lower part of matrix A
D = diagonal of matrix A
U = strictly upper part of A

$M_u = U + D$, $N_u = -L$
$M_l = L + D$, $N_l = -U$,
$R = -L - U$.

Splitings of A :

$$A = L + D + U = M_u - N_u = M_l - N_l = D - R.$$

The **backward Gauss-Seidel (BGS)** preconditioner is $\mathcal{P} = M_u$.

The **Jacobi** preconditioner is $\mathcal{P} = D$.

The **2nd-order Jacobi (2Jacobi)** preconditioner is

$$\mathcal{P} = D(R + D)^{-1} D.$$

Formally, $\mathcal{P}^{-1}$ is *the 2nd approx. of the Neumann series of* $A^{-1} = (D - R)^{-1}$.

The **2nd-order Forward Gauss-Seidel (2FGS)** preconditioner is

$$\mathcal{P} = M_l \, (N_l + M_l)^{-1} \, M_l \, .$$

Formally, $\mathcal{P}^{-1}$ is *the 2nd approx. of the Neumann series of* $A^{-1} = (M_l - N_l)^{-1}$.

The **forward Gauss-Seidel (FGS)** preconditioner is $\mathcal{P} = M_l$.

The **Symmetric Gauss-Seidel (SGS)** preconditioner is

$$\mathcal{P} = M_u \, D^{-1} \, M_l \, .$$

Interpretation: $u = \mathcal{P}^{-1} f$ solves

$$M_u \, \tilde{u} = f \, , \; M_l \, u = N_l \tilde{u} + f \, .$$

The **Lower-Upper Symmetric Gauss-Seidel (LUSGS)** preconditioner is

$$\mathcal{P} = M_l \, D^{-1} \, M_u \, .$$

Interpretation: $u = \mathcal{P}^{-1} f$ solves

$$M_l \, \tilde{u} = f \, , \; M_u \, u = N_u \, \tilde{u} + f \, .$$

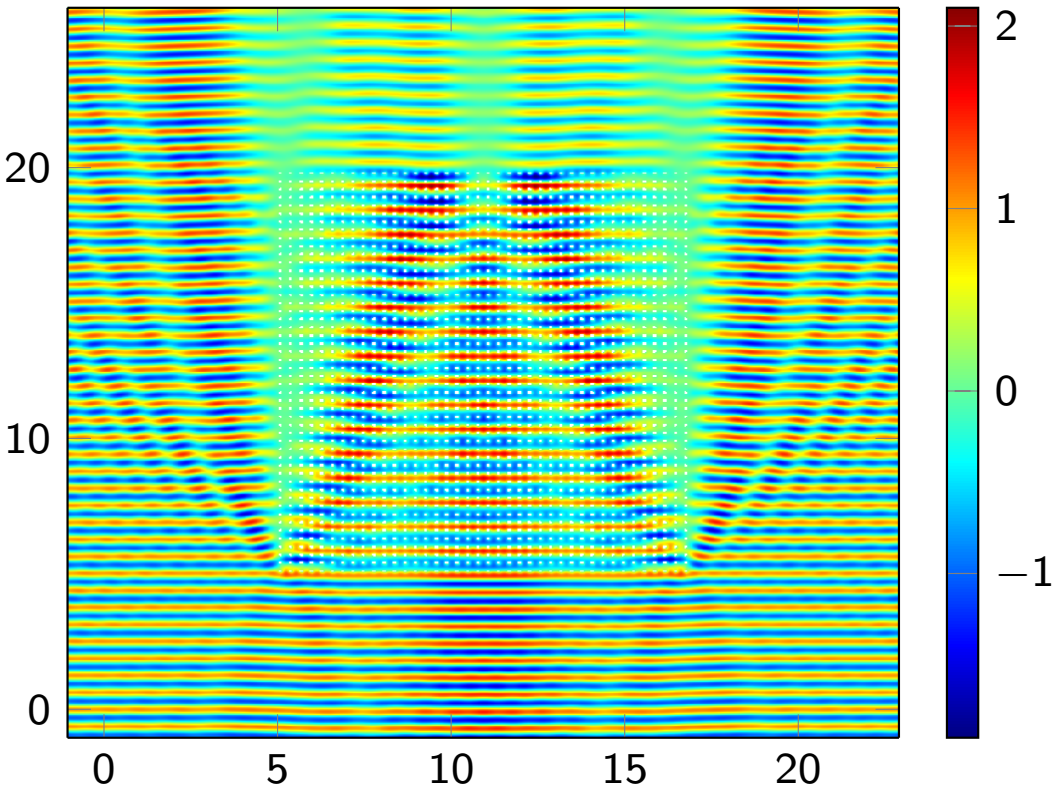
Solvers in comparison

- Direct solvers : MUMPS, LAPACK, SCALAPACK.
- Code for GMRES solver is obtained from :

  L. Giraud, et al. , *A set of GMRES routines for real and complex arithmetics on high performace computers*, Technical report, CERFACS, tR/PA/03/3 (1997).

  The code allows user to define
  - multiplication by the coefficient matrix.
  - multiplication by a preconditioner with choices of positions.
- Parallel tests are run on cluster plafrim (`www.plafrim.fr`).

Closely-spaced obstacles comparison



FSSL order 2 with Mumps for 2000 obstacles.

Planewave (PW) with 90°.

Wavenumber $\kappa = 10$.

Radius of obstacle 0.03.

Distance btwn obs 0.3.

$$\kappa \times (\text{Obs Rad}) = 0.3;$$

$$\frac{\lambda}{\text{Obs. Rad}} \sim 21 \; ; \; \frac{\lambda}{\text{Obs Dis}} \sim 2$$

$$\frac{\text{Obs Dist}}{\text{Obs Rad}} = 10.$$

GMRES stop criteria : Residue error tolerance, Niter Max, Size of Krylov.

Exp 4: Closely-spaced obstacles comparison (Dirichlet)

	Case 200 obstacles				Case 1616 obstacles			
Name Method	**Cv**	δ_{err} in $\mathbb{H}_{1/2}$	♯ **Iter**	**Time** (s)	**Cv**	δ_{err} in $\mathbb{H}_{1/2}$	♯ **Iter**	**Time** (s)
Mumps	n/a	0	n/a	0.05	n/a	0	n/a	130
Lapack	n/a	10^{-12}	n/a	0.01	n/a	10^{-10}	n/a	42.7
	GMRES stop criteria (10^{-6}, 2000,100)				GMRES stop criteria (10^{-6}, 2000,150)			
NoPreCond	Y	5×10^{-3}	820	0.09	N	n/a	n/a	n/a
L_Jacobi	Y	5×10^{-3}	656	0.08	N	n/a	n/a	n/a
L_FGS	Y	2×10^{-3}	239	0.05	N	n/a	n/a	n/a
L_BGS	Y	4×10^{-3}	197	0.04	N	n/a	n/a	n/a
L_2Jacobi	Y	5×10^{-3}	594	2.21	N	n/a	n/a	n/a
L_2FGS	Y	1×10^{-3}	169	0.1	N	n/a	n/a	n/a
L_SGS	Y	2×10^{-3}	76	0.03	Y	4×10^{-1}	757	274
L_LUSGS	Y	1×10^{-3}	77	0.03	Y	1×10^{-1}	897	325
R_Jacobi	Y	4×10^{-3}	660	1.05	N	n/a	n/a	n/a
R_FGS	Y	3×10^{-3}	199	0.05	N	n/a	n/a	n/a
R_BGS	Y	3×10^{-3}	198	0.04	N	n/a	n/a	n/a
R_2Jacobi	Y	4×10^{-3}	600	1.70	N	n/a	n/a	n/a
R_2FGS	Y	3×10^{-3}	155	0.09	N	n/a	n/a	n/a
R_SGS	Y	3×10^{-3}	75	0.03	Y	2×10^{-1}	886	321
R_LUSGS	Y	3×10^{-3}	74	0.03	Y	2×10^{-1}	897	325

Exp 5: Closely-spaced obstacles comparison (Dirichlet)

FSSL order =2 ; Size matrix $= 10^4 \times 10^4$; GMRES stop criteria (10^{-6}, 5000, 400)

Solver	Post-proc (n16)	Rel $\mathbb{H}_{1/2}$ diff	Rel L^2 diff	♯ iter	Preproc. time (s)	Postproc. time (s)	Total (s)
Mumps (n16)	Exact	3×10^{-10}	8×10^{-14}	n/a	242	96.0	338
Mumps (n16)	Inter	3×10^{-10}	9×10^{-6}	n/a	242	36.0	278
Lapack (n1)	Exact	0	0	n/a	80.4	96.0	176
Lapack (n1)	Inter	0	9×10^{-6}	n/a	80.4	37.5	118
R_LUSGS (n1)	Exact	1×10^{-1}	4×10^{-5}	1146	573	95.8	669
R_LUSGS (n1)	Inter	1×10^{-1}	4×10^{-5}	1146	573	36.2	609
R_SGS (n1)	Exact	1×10^{-1}	4×10^{-5}	1151	598	95.8	694
R_SGS (n1)	Inter	1×10^{-1}	4×10^{-5}	1151	598	36.2	635
Scala (n16)	Exact	3×10^{-10}	8×10^{-14}	n/a	34.6	95.6	130
Scala (n16)	Inter	3×10^{-10}	9×10^{-6}	n/a	34.6	36.1	70.9

PW of 90° ; $\kappa = 10.0$; $N_{\text{Obs}} = 2000$; Obs. Rad. $= 0.03$; Obs. Dist. $= 0.30$;

$$\kappa \times (\text{Obs Rad}) = 0.3\ , \quad \frac{\lambda}{\text{Obs. Rad}} \sim 21\ , \quad \frac{\lambda}{\text{Obs Dis}} \sim 2\ , \quad \frac{\text{Obs Dist}}{\text{Obs Rad}} = 10.$$

Exp 6: Far apart obstacles (Dirichlet)

FSSL order 2 ; Size matrix = 10000 × 10000; GMRES stop criteria (10^{-7}, 5000,500).

Solver	**Post-proc** (n16)	**Rel $\mathbb{H}_{1/2}$ diff**	**Rel L^2 diff**	**♯ iter**	**Pre-proc. time** (s)	**Post-proc. time** (s)	**Total** (s)
Mumps (n1)	Exact	0.0	0.0	n/a	251	96.0	347
Mumps (n1)	Inter	0.0	1×10^{-5}	n/a	251	37.5	289
Lapack (n1)	Exact	4×10^{-12}	2×10^{-15}	n/a	79.9	96.0	176
Lapack (n1)	Inter	4×10^{-12}	1×10^{-5}	n/a	79.9	37.5	118
R_LUSGS (n1)	Exact	3×10^{-4}	1×10^{-7}	57	37.5	96.0	134
R_LUSGS (n1)	Inter	3×10^{-4}	1×10^{-5}	57	37.5	37.5	75.3
R_SGS (n1)	Exact	4×10^{-4}	1×10^{-7}	56	37.0	96.0	133
R_SGS (n1)	Inter	4×10^{-4}	1×10^{-5}	56	37.0	37.5	74.6
Scala (n16)	Exact	1×10^{-11}	4×10^{-15}	n/a	34.9	96.0	131
Scala (n16)	Inter	1×10^{-11}	1×10^{-5}	n/a	34.9	37.5	72.5

PW of 90.0°; $\kappa = 10.0$; ♯ obs = 2000; Obs. Rad. = 0.01; Obs. Dist. = 2.00;

$$\kappa \times (\text{Obs Rad}) = 0.1\ , \quad \frac{\lambda}{\text{Obs. Rad.}} \sim 63\ , \quad \frac{\lambda}{\text{Obs Dist.}} \sim 0.3\ , \quad \frac{\text{Obs Dist}}{\text{Obs Rad.}} = 200.$$

Plan

Full waveform inversion

Minimize

$$J := \frac{1}{2}\|\mathcal{H}_{\mathbf{rec}}\, u_{\mathbf{scatt}} - d_{\mathbf{obs}}\|^2 \quad ; \quad \widehat{J}(p) = \frac{1}{2}\|\Phi(p) - d_{\mathbf{obs}}\|^2 .$$

Optimization method : nonlinear conjugate gradient with Polak-Ribière coefficient.

Calculate **gradient** $\nabla_p \widehat{J}$ by adjoint method (with FSSL formulation).

Trace operator at the receptors $\mathcal{H}_{\mathbf{rec}}$: $u|_{\text{receptor}}$, $\partial_n u|_{\text{receptor}}$, etc.

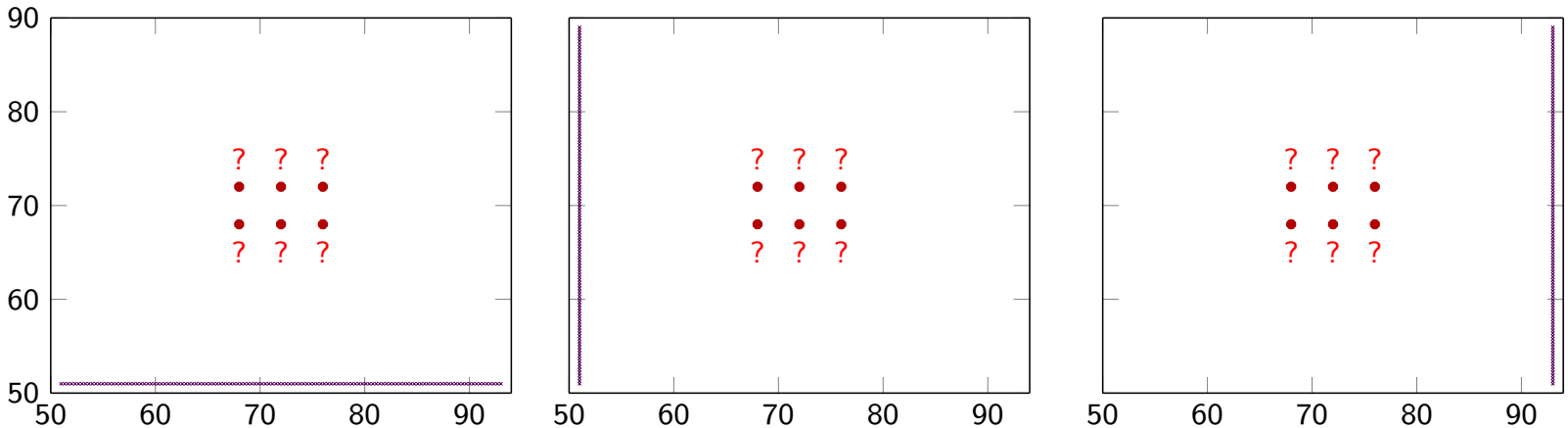
Observed data at receptors : $d_{\mathbf{obs}}$.

Forward map Φ : parameters $\mapsto$ values at receptors.

Acquisition Data

Inversion problem : Retrieve the position of 6 hard-scattering obstacles of radius 0.5 (distanced 3 and 4) placed at

$$(68\,,\,68)\;,\;(68\,,\,72)\;,\;(72\,,\,68)\;,\;(72\,,\,72)\;,\;(76\,,\,68)\;,\;(76\,,\,72)\,.$$

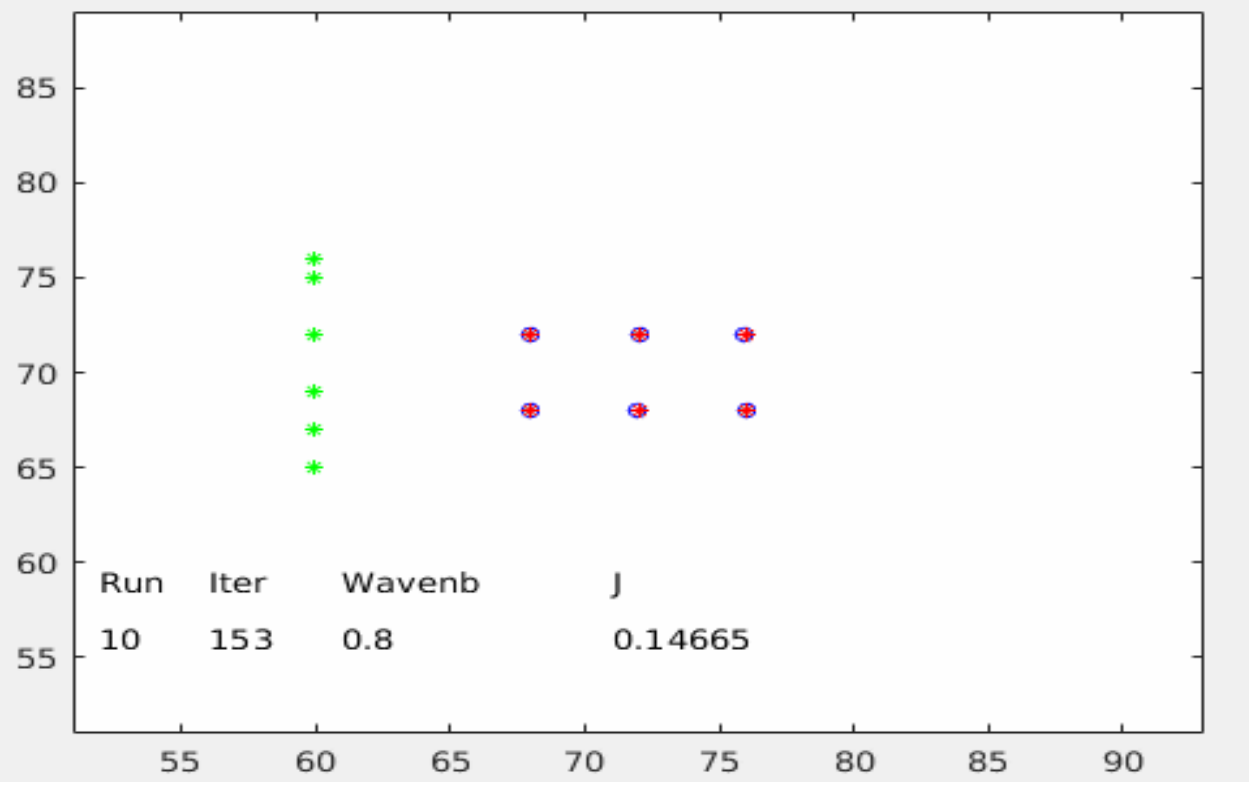


(a) $\alpha_{\text{inc}} = 90^o$ (b) $\alpha_{\text{inc}} = 0^o$ (c) $\alpha_{\text{inc}} = 180^o$

The position of 128 equally spaced receivers vary with the angle of incidence.

Data is produced by FSSL order 12 with solver Lapack.

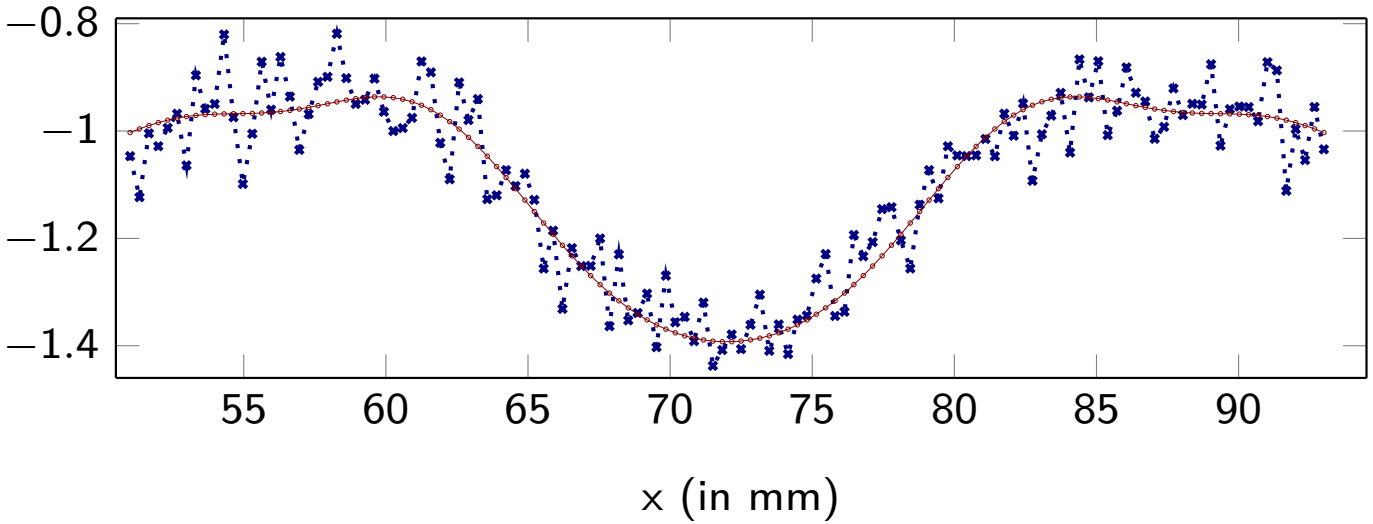
Reconstruction results using data with no noise.



- ⋆ One angle of acquisitions: 90°
- ⋆ FSSL order 3 using Mumps .
- ⋆ Error Tolerance 10^{-5}.
- ⋆ Initial guess rel. error. : 115% (relative to size of domain). Final position's rel. error : 1.3%
- ⋆ Initial J : 0.002 at $\kappa = 0.08$ Final J : 0.002 at $\kappa = 1.5$
- ⋆ Run time : 7.92 secs
- ⋆ Nb max linesearch (LS) 30. Nb LS used in each run ≤ 11.

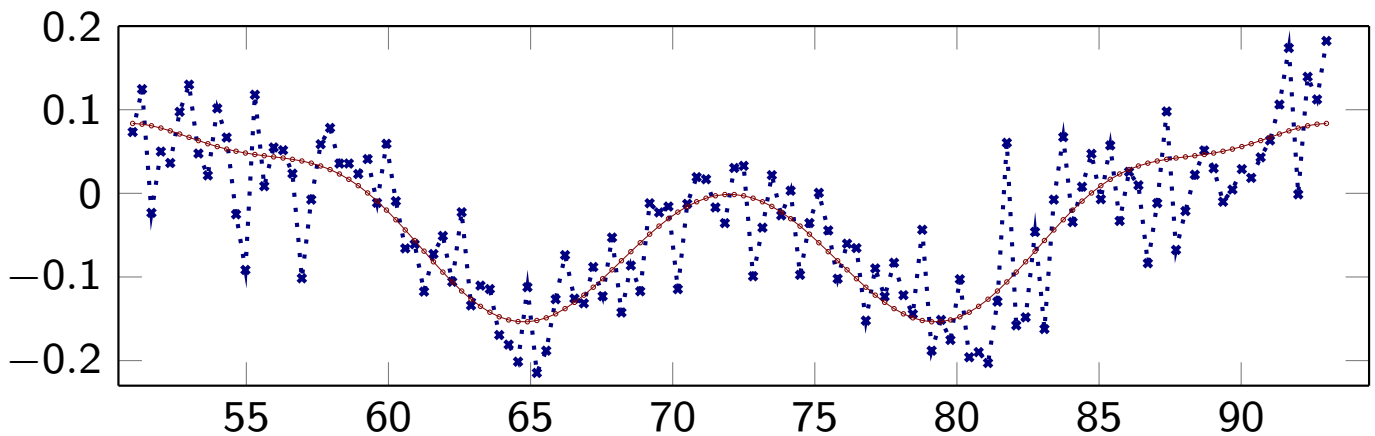
Run	κ	♯ Niter Max	Step size	Run	κ	♯ Niter Max	Step size	Run	κ	♯ Niter Max	Step size
1	0.08	300	8	7	0.5	200	5	13	1.10	200	2
2	0.09	300	8	8	0.6	200	5	14	1.20	200	2
3	0.1	200	8	9	0.7	200	5	15	1.30	200	2
4	0.2	200	8	10	0.8	200	5	16	1.40	200	2
5	0.3	200	7	11	0.9	200	5	17	1.50	100	2
6	0.4	200	7	12	1.0	200	2				

Noise Data at 23dB



(a) Real part of total wave at 128 receivers at $\kappa = 0.8$ with PW 90°

White Gaussian noise is added by using wgn in Matlab.



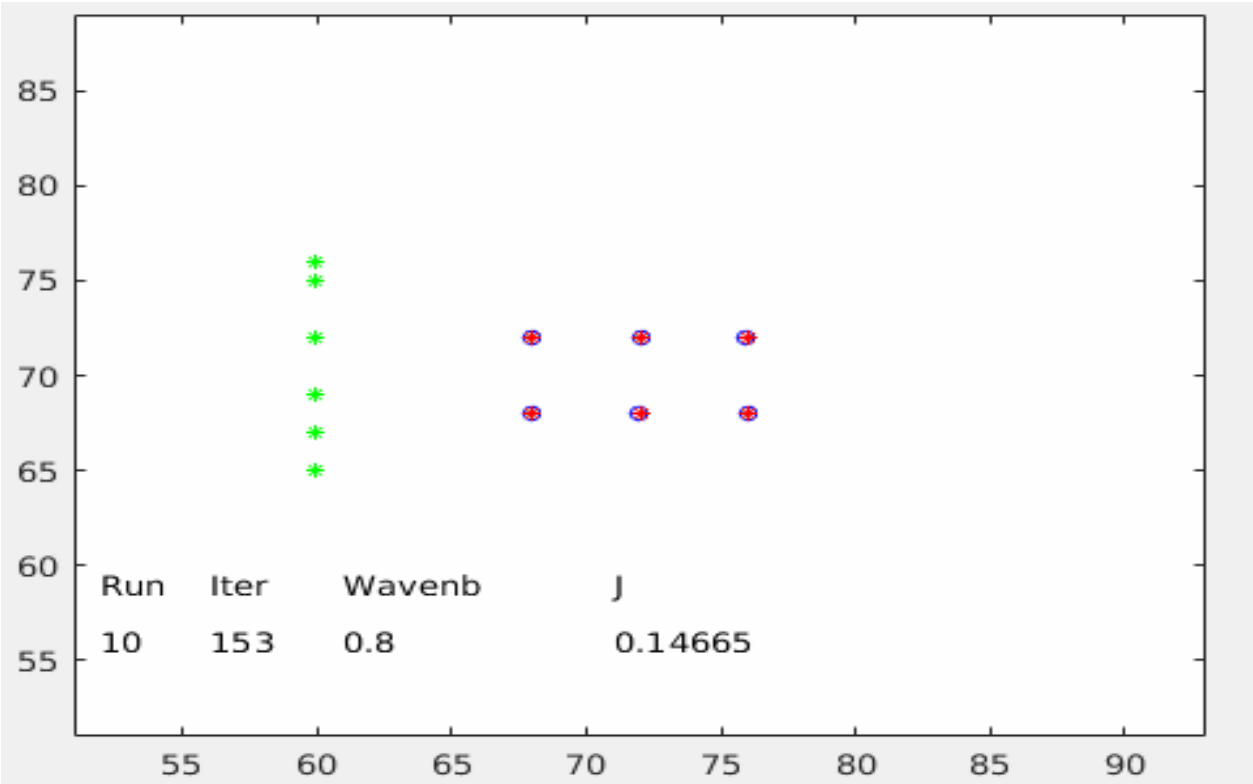
(b) Imaginary part .

Rel. error in norm

$l^2 = 7\%$,

$l^\infty = 18\ \%$

Inversion result for data with 23dB noise



⋆ Three angles of acquisitions: $90°$, $0°$, $180°$

⋆ FSSL order 3 using Mumps .

⋆ Error Tolerance 10^{-5}.

⋆ Initial guess rel. error. : 115% (relative to size of domain).

Final position's rel. error: 0.38%

⋆ Initial J 1.00 at $\kappa = 0.08$

Final J: 0.79 at $\kappa = 0.7$

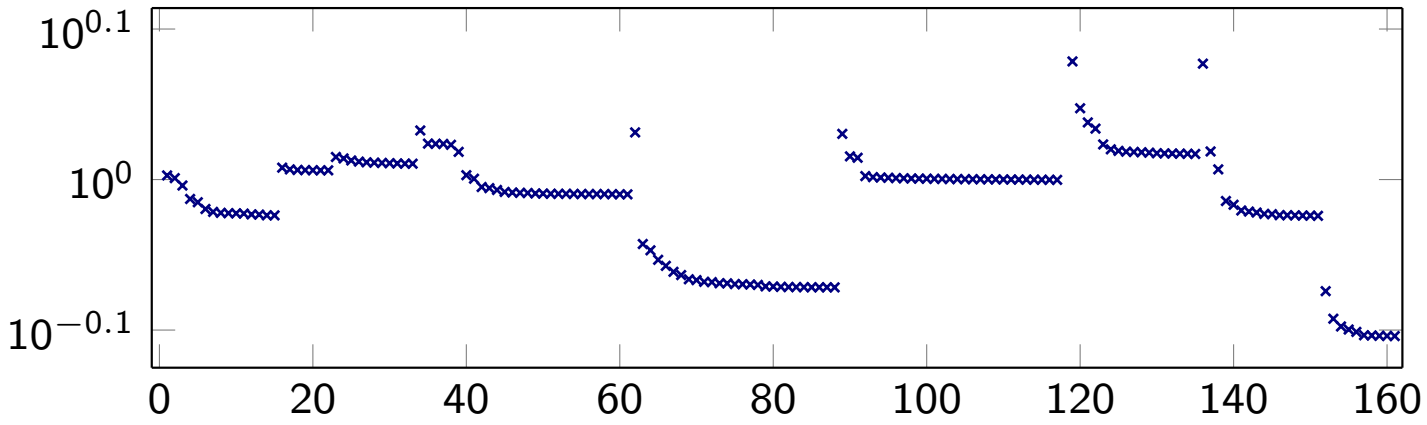
⋆ Run time: 2.79 secs

Run	κ	♯ Niter Max	Step size
1	0.08	300	5
2	0.09	300	4
3	0.1	200	2
4	0.2	200	2
5	0.3	200	2

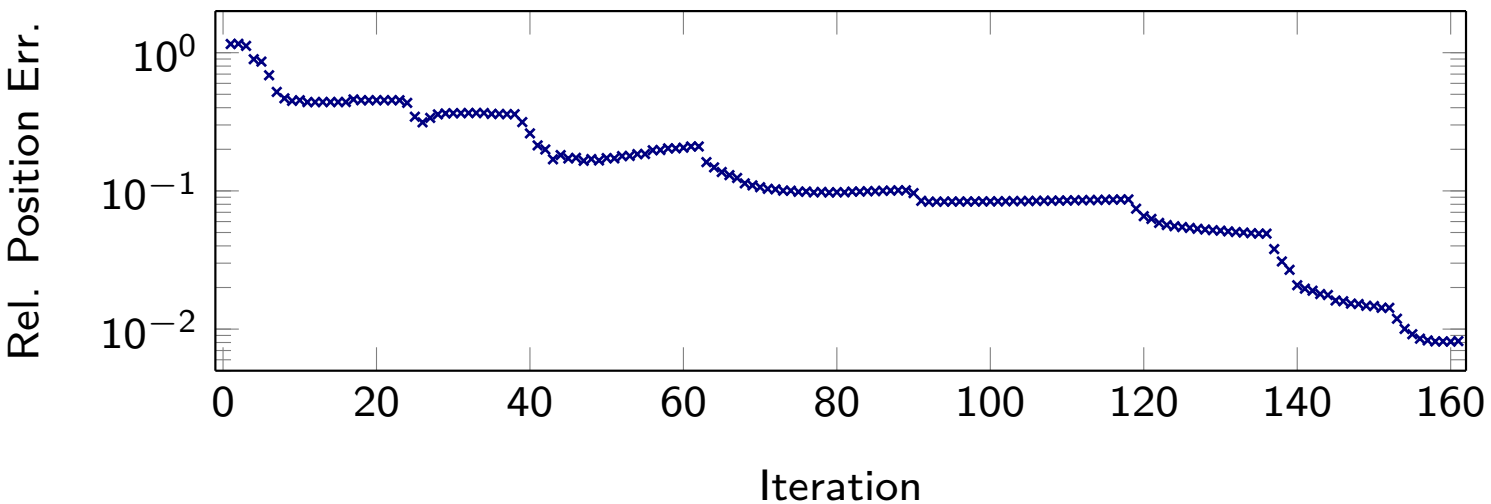
Run	κ	♯ Niter Max	Step size
6	0.4	200	2
7	0.5	200	2
8	0.6	200	2
9	0.7	200	1

Nb max linesearch (LS) 30.
Nb LS used in each run ≤ 10.

Inversion result for data with 23dB noise (cnt)



Initial J =1.0 at $\kappa = 0.08$; Final J = 0.79 at $\kappa = 0.7$



Initial guess rel. err. = 115% (rel. to domain size); Final err. = 0.8%.

⋆ Three angles of acquisitions: 90°, 0°, 180°

⋆ FSSL order 3 using Mumps;

⋆ Err. Tol. $= 10^{-5}$;

⋆ Niter total = 161;

⋆ Use 9 κs: 0.08, 0.09, 0.1, . . . , 0.7

⋆ Run time: 2.79 s

Conclusion

⋆ FSSL is robust in simulating the multi-scattering by small circular obstacles in large homogeneous media.

⋆ Its linear systems have simple expressions ⇒ Easy coding and implementation.

⋆ Direct Solvers (Lapack and Scalapack) are more efficient when the obstacles are close together.

⋆ Iterative solvers are more preferable when the obstacles are far apart.
In particular, GMRES with LUSGS and SGS are faster than Lapack and as fast as Scalapack.

⋆ LUSGS and SGS are the most robust among the preconditioners considered.

⋆ In both settings, Scalapack is fastest.

⋆ Using shared memory architecture, Scalapack can handle the largest Nb of obstacles.

Further advantages of direct solvers, regarding application to inverse problem using Full waveform inversion

- multi-RHS, high precision
- the forward and adjoint problem use the same factorization.

Thank you for your attention