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# Moutard transforms for the conductivity equation <sup>\*</sup>

P.G. Grinevich <sup>†</sup>      R.G. Novikov<sup>‡</sup>

## Abstract

We construct Darboux-Moutard type transforms for the two-dimensional conductivity equation. This result continues our recent studies of Darboux-Moutard type transforms for generalized analytic functions. In addition, at least, some of the Darboux-Moutard type transforms of the present work admit direct extension to the conductivity equation in multidimensions.

## 1 Introduction

We consider the two-dimensional isotropic conductivity equation:

$$\operatorname{div}(\sigma(x)\nabla u(x)) = 0, \quad x = (x_1, x_2), \quad x \in D \subseteq \mathbb{R}^2, \quad (1)$$

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<sup>†</sup>L.D. Landau Institute for Theoretical Physics, pr. Akademika Semenova 1a, Chernogolovka, 142432, Russia; Lomonosov Moscow State University, Faculty of Mechanics and Mathematics, Russia, 119991, Moscow, GSP-1, Leninskiye Gory 1, Main Building; Moscow Institute of Physics and Technology, 9 Institutskiy per., Dolgoprudny, Moscow Region, 141700, Russia; e-mail: pgg@landau.ac.ru

<sup>‡</sup>CNRS (UMR 7641), Centre de Mathématiques Appliquées, École Polytechnique, 91128, Palaiseau, France; IEPT RAS, 117997, Moscow, Russia; e-mail: novikov@cmap.polytechnique.fr

where  $D$  is an open domain in  $\mathbb{R}^2$ . This equation arises in different physical context; see, for example, [5], [6]. In particular, in electrical problems  $\sigma(x)$  is the electrical conductivity in  $D$ , and  $u(x)$  is the electric potential. In solid state thermal problems  $\sigma(x)$  is the heat conductivity, and  $u(x)$  is the temperature.

In the present work we show that the conductivity equation (1) admits Moutard-type transforms, going back to [8]. Such transforms were successfully used in studies of integrable systems of mathematical physics and differential geometry, in spectral theory and in complex analysis; see, for example, [9], [16], [14], [12], [13], [10], [7], [2], [3], [4], [11]. In particular, the present article can be considered as a direct continuation of our recent works [2]-[4] on Moutard-type transforms for the generalized analytic functions. In turn, works [2]-[4] were stimulated by [12], [13].

In particular, in the present work we use the fact that equation (1) can be written as a reduction of the following two-dimensional Dirac equation (see [1]):

$$\left[ \begin{pmatrix} \partial_{\bar{z}} & 0 \\ 0 & \partial_z \end{pmatrix} - \begin{pmatrix} 0 & q \\ \bar{q} & 0 \end{pmatrix} \right] \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0 \quad \text{in } D, \quad (2)$$

where

$$\begin{aligned} \partial_z &= \frac{1}{2}(\partial_{x_1} - i\partial_{x_2}), & \partial_{\bar{z}} &= \frac{1}{2}(\partial_{x_1} + i\partial_{x_2}), \\ q &= q(x), & \psi_j &= \psi_j(x), \quad j = 1, 2, \quad x = (x_1, x_2). \end{aligned} \quad (3)$$

We recall that if  $u$  satisfies (1), then

$$\psi_1 = \sigma^{1/2}\partial_z u, \quad \psi_2 = \sigma^{1/2}\partial_{\bar{z}} u, \quad (4)$$

satisfy (2), where

$$q = -\frac{1}{2}\partial_z \log(\sigma), \quad \bar{q} = -\frac{1}{2}\partial_{\bar{z}} \log(\sigma). \quad (5)$$

We use also that (2) is equivalent to the following equation:

$$\partial_{\bar{z}}\psi = q\bar{\psi} \quad \text{in } D, \quad (6)$$

which is the basic equation of the generalized analytic functions theory (see [15]). More precisely:

- (i) if  $\psi_1, \psi_2$  satisfy (2), then  $\psi_+ = \frac{1}{2}(\psi_1 + \overline{\psi_2})$  and  $\psi_- = \frac{1}{2i}(\psi_1 - \overline{\psi_2})$  solve (6);
- (ii) if  $\psi_+, \psi_-$  satisfy (6), then  $\psi_1 = \psi_+ + i\psi_-, \psi_2 = \overline{\psi_+} + i\overline{\psi_-}$  solve (2).

The property that  $\psi_1, \psi_2$  and  $q$  in (2) admit representations (4), (5) implies a non-trivial reduction of equation (2). The compatibility of this reduction with the Moutard-type transforms from [2]-[4] is established in the present article.

The main results of the present work are given in Sections 3 and 5. In Section 3 we construct Moutard-type transforms for the two-dimensional conductivity equation (1). In Section 5 we show that, at least, some of the Moutard-type transforms constructed in Section 3 admit direct extension to the conductivity equation in multidimension.

Finally, it is in order to mention that:

Results of the present work admits an extension to some cases of the Beltrami equation.

Results of the present work can be easily used for constructing explicit exactly solvable examples of equation (1) and related equations.

The presentation of the present work is formal, but it can be realized in a proper analytical framework.

These points will be addressed in subsequent publications.

## 2 Simple Moutard transforms for generalized analytic functions

Following [15], [2]-[4], we consider the pair of conjugate equations of the generalized analytic function theory:

$$\partial_{\bar{z}}\psi = q\bar{\psi} \quad \text{in } D, \tag{7}$$

$$\partial_{\bar{z}}\psi^+ = -\bar{q}\bar{\psi}^+ \quad \text{in } D, \tag{8}$$

where  $\partial_z, \partial_{\bar{z}}$  are defined in (3),  $z = x_1 + ix_2, \bar{z} = x_1 - ix_2, D$  is an open simply connected domain in  $\mathbb{C} \cong \mathbb{R}^2, q = q(z)$  is a given function in  $D$ . In addition, in this article the notation  $f = f(x) = f(z)$  does not mean that  $f(z)$  is holomorphic function in  $z$  unless it is explicitly specified.

Next, as in [15], [2]-[4], we associate with a pair of functions  $\psi, \psi^+$ , satisfying (7), (8), respectively, the following imaginary-valued potential  $\omega_{\psi, \psi^+}$  defined by:

$$\partial_z \omega_{\psi, \psi^+} = \psi \psi^+, \quad \partial_{\bar{z}} \omega_{\psi, \psi^+} = -\overline{\psi \psi^+} \quad \text{in } D, \quad (9)$$

where the pure imaginary integration constant may depend on the particular situation. We recall that the compatibility of (9) follows from (7), (8).

Let  $f, f^+$  be some fixed solutions of equations (7), (8), respectively, with given  $q$ . Then a simple Moutard-type transform  $\mathcal{M} = \mathcal{M}_{q, f, f^+}$  for the pair of conjugate equations (7), (8) is given by the formulas (see [2]-[4]):

$$\tilde{q} = \mathcal{M}q = q + \frac{f \overline{f^+}}{\omega_{f, f^+}}, \quad (10)$$

$$\tilde{\psi} = \mathcal{M}\psi = \psi - \frac{\omega_{\psi, f^+}}{\omega_{f, f^+}} f, \quad \tilde{\psi}^+ = \mathcal{M}\psi^+ = \psi^+ - \frac{\omega_{f, \psi^+}}{\omega_{f, f^+}} f^+, \quad (11)$$

where  $\psi, \psi^+$  are arbitrary solutions of (7) and (8).

The point is that the functions  $\tilde{\psi}, \tilde{\psi}^+$  defined in (11) satisfy the conjugate pair of Moutard-transformed equations (see [2]-[4]):

$$\partial_{\bar{z}} \tilde{\psi} = \tilde{q} \tilde{\psi} \quad \text{in } D, \quad (12)$$

$$\partial_z \tilde{\psi}^+ = -\overline{\tilde{q}} \tilde{\psi}^+ \quad \text{in } D, \quad (13)$$

where  $\tilde{q}$  is defined in (10).

### 3 Simple Moutard transforms for the two-dimensional conductivity equation

In this Section we assume that  $D$  is an open simply connected domain in  $\mathbb{C} \cong \mathbb{R}^2$ .

Let

$$f_R^+ = \sqrt{\sigma(z)}, \quad f_I^+ = \frac{i}{\sqrt{\sigma(z)}}, \quad (14)$$

where  $\sigma$  is the conductivity in equation (1).

Note that  $\psi^+ = f_R^+$  and  $\psi^+ = f_I^+$  are solutions of equation (8), where  $q$  is given by (5) with a regular positive  $\sigma$ .

**Lemma 1** *A regular complex-valued function  $\psi(z)$  satisfies equation (7) with  $q(z)$  given by (5) with a positive  $\sigma(z)$  if and only if there exists a real-valued solution  $u(z)$  of (1) such that*

$$\psi(z) = \sigma^{1/2}(z)\partial_z u(z), \quad \overline{\psi(z)} = \sigma^{1/2}(z)\partial_{\bar{z}} u(z). \quad (15)$$

*In addition,*

$$u = -i\omega_{\psi, f_I^+}, \quad (16)$$

*where  $f_I^+$  is defined in (14),  $\psi$  is defined in (15),  $\omega_{\psi, \psi^+}$  is defined via (9).*

Lemma 1 is proved in Section 4.

Note also that

$$\psi = \sigma^{-1/2}(J_1 + iJ_2), \quad J_1 = \sigma \frac{\partial u}{\partial x_1}, \quad J_2 = \sigma \frac{\partial u}{\partial x_2}, \quad (17)$$

where  $\psi, u$  are the functions of (15), and  $J$  is the current for the conductivity equation (1).

**Theorem 1** *Let  $q(z)$  be given by (5) in  $D$  with a positive regular  $\sigma(z)$ . Let the transform  $q \rightarrow \tilde{q}, \psi \rightarrow \tilde{\psi}$  be defined by:*

$$\tilde{q} = \mathcal{M}q = q + \frac{f\overline{f^+}}{\omega_{f, f^+}}, \quad \tilde{\psi} = \mathcal{M}\psi = \psi - \frac{\omega_{\psi, f^+}}{\omega_{f, f^+}} f, \quad (18)$$

*where  $\psi$  denotes an arbitrary solution of (7),  $f$  is a fixed solution of equation (7),  $f^+ = f_R^+$  or  $f^+ = f_I^+$ , where  $f_R^+$  and  $f_I^+$  are defined in (14).*

*Then  $\tilde{\psi}$  satisfies the Moutard-transformed equation (12), and  $\tilde{q}$  admits the representation*

$$\tilde{q} = -\frac{1}{2}\partial_z \log(\tilde{\sigma}), \quad (19)$$

*where*

$$\tilde{\sigma} = \begin{cases} -\frac{\sigma}{\omega_{f, f_R^+}^2} & \text{if } f^+ = f_R^+, \\ -\sigma\omega_{f, f_I^+}^2 & \text{if } f^+ = f_I^+. \end{cases} \quad (20)$$

*In addition, the following Moutard-transformed conductivity equation holds:*

$$\operatorname{div}(\tilde{\sigma}\nabla\tilde{u}) = 0 \quad \text{in } D, \quad (21)$$

*where*

$$\tilde{u} = -i\omega_{\tilde{\psi}, \hat{f}^+}, \quad \hat{f}^+ = \frac{i}{\sqrt{\tilde{\sigma}}}. \quad (22)$$

The following scheme summarizes the Moutard-type transforms for the conductivity equation (1) given in Theorem 1:

$$\begin{aligned}
\sigma &\xrightarrow[\{f, f^+\}]{(20)} \tilde{\sigma} \xrightarrow{(19)} \tilde{q}, \\
\sigma, u &\xrightarrow{(5), (15)} q, \psi \xrightarrow[\{f, f^+\}]{(18)} \tilde{q}, \tilde{\psi}, \\
\tilde{\sigma}, \tilde{\psi} &\xrightarrow{(22)} \tilde{u}.
\end{aligned} \tag{23}$$

The point is that each step in scheme (23) is given by quadratures.

Theorem 1 is proved in Section 4.

**Remark 1** *In Theorem 1 we have the following two important cases:*

- (i) *If  $\omega_{f, f^+}$  has no zeroes in  $D$ , then  $\tilde{\sigma}$  arising in (20) is a regular positive function in  $D$ . In addition, if  $D$  is bounded, then  $\omega_{f, f^+}$  can be always defined without zeroes by an appropriate choice of integration constant.*
- (ii) *If  $\omega_{f, f^+}$  has zeroes in  $D$ , then  $\tilde{\sigma}$  arising in (20) is non-negative and has either zeros or poles in  $D$ . In these singular cases the standard methods for solving the conductivity equation (21) does not work; but these both singular cases are interesting and relevant for physical problems. The point is that the Moutard-type transform of Theorem 1 generating  $\tilde{\sigma}$  simultaneously provides a method for solving equation (21).*

## 4 Proofs of Lemma 1 and Theorem 1

**Proof of Lemma 1.** If  $\sigma$  is a real-valued regular positive function,  $u$  is a real-valued regular function, and  $q, \psi$  are defined by (5) and (15), respectively, then it is known that  $\psi$  satisfies (7) if and only if  $u$  satisfies (1); see Introduction. This can be also verified by a direct calculation.

Conversely, suppose that  $q$  is defined by (5) with a regular real-valued positive  $\sigma$ , and  $\psi$  satisfies (7). Define  $u$  by (16). It remains to verify that (15) holds. This verification uses (9), (14) and consists of the following:

$$\partial_z u = \partial_z(-i\omega_{\psi, f_I^+}) = -i\psi f_I^+ = -i\psi \frac{i}{\sqrt{\sigma}} = \frac{\psi}{\sqrt{\sigma}}, \tag{24}$$

$$\partial_{\bar{z}} u = \partial_{\bar{z}}(-i\omega_{\psi, f_I^+}) = -i(-\bar{\psi} \overline{f_I^+}) = i\bar{\psi} \frac{-i}{\sqrt{\sigma}} = \frac{\bar{\psi}}{\sqrt{\sigma}}. \tag{25}$$

Lemma 1 is proved.

**Proof of Theorem 1.** The statement that  $\tilde{\psi}$  satisfies Moutard-transformed equation (12) was proved in [2].

If  $q$  is defined by (19) with  $\tilde{\sigma}$  defined by (20), then:

$$\begin{aligned}\tilde{q} &= -\frac{1}{2}\partial_z \log(\tilde{\sigma}) = -\frac{1}{2}\partial_z \log(\sigma) + \partial_z \log(\omega_{f,f_R^+}) = \\ &= q + \frac{f f_R^+}{\omega_{f,f_R^+}} = q + \frac{\overline{f f_R^+}}{\omega_{f,f_R^+}} \quad \text{if } f^+ = f_R^+, \end{aligned} \quad (26)$$

$$\begin{aligned}\tilde{q} &= -\frac{1}{2}\partial_z \log(\tilde{\sigma}) = -\frac{1}{2}\partial_z \log(\sigma) - \partial_z \log(\omega_{f,f_I^+}) = \\ &= q - \frac{f f_I^+}{\omega_{f,f_I^+}} = q + \frac{\overline{f f_I^+}}{\omega_{f,f_I^+}} \quad \text{if } f^+ = f_I^+, \end{aligned} \quad (27)$$

Here, we used that  $f_R^+$  is real-valued and  $f_I^+$  is imaginary-valued.

In fact, calculations (26), (27) prove representations (19), (20) for  $\tilde{q}$  in (18).

Formulas (21), (22) follow directly from the Moutard-transformed equation (12), the representation (19) and Lemma 1.

This completes the proof of Theorem 1 under the assumption that  $\omega_{f,f^+}$  has no zeroes in  $D$ .

**Remark 2** *Formally, the proof of Theorem 1 remains valid if  $\omega_{f,f^+}$  has zeroes in  $D$ , but a proper analytic picture requires a subsequent investigation.*

## 5 Simplest Moutard transforms for the multidimensional conductivity equation

Suppose that  $f^+ = f_I^+$  in Theorem 1 in Section 3. Then the Moutard-type transform of Theorem 1 for the two-dimensional conductivity equation (1) is reduced to the transform

$$\begin{aligned}\sigma &\rightarrow \tilde{\sigma} = \mathcal{M}\sigma = w^2\sigma, \\ u &\rightarrow \tilde{u} = \mathcal{M}u = w^{-1}u, \end{aligned} \quad (28)$$



where  $w = i\omega_{f, f_I^+}$ .

The Moutard-type transform (28) admits a direct extension to the conductivity equation

$$\operatorname{div}(\sigma(x)\nabla u(x)) = 0, \quad x = (x_1, x_2, \dots, x_d), \quad x \in D \subseteq \mathbb{R}^d, \quad (29)$$

in dimension  $d \geq 1$  (and, in particular, in dimension  $d = 3$ ).

**Theorem 2** *Let  $\sigma$  be a real positive regular function in  $D$ , where  $D$  is an open domain in  $\mathbb{R}^d$ ,  $d \geq 1$ . Let the transform  $\sigma \rightarrow \tilde{\sigma}$ ,  $u \rightarrow \tilde{u}$  be defined by*

$$\begin{aligned} \tilde{\sigma} &= \mathcal{M}\sigma = w^2\sigma, \\ \tilde{u} &= \mathcal{M}u = w^{-1}u, \end{aligned} \quad (30)$$

where  $u$  denotes an arbitrary solution of (29), and  $w$  is a fixed solution of (29). Then the following Moutard-transformed conductivity equation holds:

$$\operatorname{div}(\tilde{\sigma}(x)\nabla \tilde{u}(x)) = 0, \quad x \in D \subseteq \mathbb{R}^d. \quad (31)$$

Theorem 2 is proved by direct calculations.

Studies of Moutard-type transforms for the multidimensional conductivity equation will be continued in a subsequent publication.

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