



HAL
open science

Moutard transform for the two-dimensional conductivity equation

Piotr Grinevich, Roman Novikov

► **To cite this version:**

Piotr Grinevich, Roman Novikov. Moutard transform for the two-dimensional conductivity equation . 2017. hal-01673778v1

HAL Id: hal-01673778

<https://hal.science/hal-01673778v1>

Preprint submitted on 31 Dec 2017 (v1), last revised 5 Mar 2019 (v4)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Moutard transform for the two-dimensional conductivity equation ^{*}

P.G. Grinevich [†] R.G. Novikov[‡]

Abstract

We construct a Darboux-Moutard type transform for the two-dimensional conductivity equation. This result continues our recent studies of Darboux-Moutard type transforms for generalized analytic functions.

1 Introduction

We consider the two-dimensional isotropic conductivity equation:

$$\operatorname{div}(\sigma(x)\nabla u(x)) = 0, \quad x = (x_1, x_2), \quad x \in D \subseteq \mathbb{R}^2, \quad (1) \quad \{\text{eq:hc1}\}$$

where D is an open domain in \mathbb{R}^2 . This equation arises in different physical context; see, for example, [5], [6]. In particular, in electrical problems $\sigma(x)$

^{*}The main part of the work was fulfilled during the visit of the first author to the Institut des Hautes Études Scientifiques in November 2017. The first author was partially supported by the Russian Foundation for Basic Research, grant 17-51-150001 “Quasilinear equations, inverse problems, and their applications” and by the program of the Presidium of the Russian Academy of Sciences “Fundamental problems of nonlinear dynamics”. The second author was partially supported by PRC 1545 CNRS/RFBR: “Équations quasi-linéaires, problèmes inverses et leurs applications”.

[†]L.D. Landau Institute for Theoretical Physics, pr. Akademika Semanova 1a, Chernogolovka, 142432, Russia; Lomonosov Moscow State University, Faculty of Mechanics and Mathematics, Russia, 119991, Moscow, GSP-1, Leninskiye Gory 1, Main Building; Moscow Institute of Physics and Technology, 9 Institutskiy per., Dolgoprudny, Moscow Region, 141700, Russia; e-mail: pgg@landau.ac.ru

[‡]CNRS (UMR 7641), Centre de Mathématiques Appliquées, École Polytechnique, 91128, Palaiseau, France; IEPT RAS, 117997, Moscow, Russia; e-mail: novikov@cmap.polytechnique.fr

is the electrical conductivity in D , and $u(x)$ is the electric potential. In solid state thermal problems $\sigma(x)$ is the heat conductivity, and $u(x)$ is the temperature.

In the present work we show that the conductivity equation (1) admits Moutard-type transforms, going back to [8]. Such transforms were successfully used in studies of integrable systems of mathematical physics and differential geometry, in spectral theory and in complex analysis; see, for example, [9], [16], [14], [12], [13], [10], [7], [2], [3], [4], [11]. In particular, the present article can be considered as a direct continuation of our recent works [2]-[4] on Moutard-type transforms for the generalized analytic functions. In turn, works [2]-[4] were stimulated by [12], [13].

In particular, in the present work we use the fact that equation (1) can be written as a reduction of the following two-dimensional Dirac equation (see [1]):

$$\left[\begin{pmatrix} \partial_{\bar{z}} & 0 \\ 0 & \partial_z \end{pmatrix} - \begin{pmatrix} 0 & q \\ \bar{q} & 0 \end{pmatrix} \right] \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0 \quad \text{in } D, \quad (2) \quad \{\text{eq:hc2}\}$$

where

$$\begin{aligned} \partial_z &= \frac{1}{2}(\partial_{x_1} - i\partial_{x_2}), \quad \partial_{\bar{z}} = \frac{1}{2}(\partial_{x_1} + i\partial_{x_2}), \\ q &= q(x), \quad \psi_j = \psi_j(x), \quad j = 1, 2, \quad x = (x_1, x_2). \end{aligned} \quad (3) \quad \{\text{eq:hc3}\}$$

We recall that if u satisfies (1), then

$$\psi_1 = \sigma^{1/2}\partial_z u, \quad \psi_2 = \sigma^{1/2}\partial_{\bar{z}} u, \quad (4) \quad \{\text{eq:hc4}\}$$

satisfy (2), where

$$q = -\frac{1}{2}\partial_z \log(\sigma), \quad \bar{q} = -\frac{1}{2}\partial_{\bar{z}} \log(\sigma). \quad (5) \quad \{\text{eq:hc5}\}$$

We use also that (2) is equivalent to the following equation:

$$\partial_{\bar{z}}\psi = q\bar{\psi} \quad \text{in } D, \quad (6) \quad \{\text{eq:hc6}\}$$

which is the basic equation of the generalized analytic functions theory (see [15]). More precisely:

- (i) if ψ_1, ψ_2 satisfy (2), then $\psi_+ = \frac{1}{2}(\psi_1 + \bar{\psi}_2)$ and $\psi_- = \frac{1}{2i}(\psi_1 - \bar{\psi}_2)$ solve (6);

(ii) if ψ_+, ψ_- satisfy (6), then $\psi_1 = \psi_+ + i\psi_-, \psi_2 = \overline{\psi_+} + i\overline{\psi_-}$ solve (2).

The property that ψ_1, ψ_2 and q in (2) admit representations (4), (5) implies a non-trivial reduction of equation (2). The compatibility of this reduction with the Moutard-type transforms from [2]-[4] is established in the present article.

The main results of the present work are given in Section 3, where we construct Moutard-type transforms for the conductivity equations (1).

Finally, it is in order to mention that:

Results of the present work admits an extension to some cases of the Beltrami equation.

Results of the present work can be easily used for constructing explicit exactly solvable examples of equation (1) and related equations.

The presentation of the present work is formal, but it can be realized in a proper analytical framework.

These points will be addressed in subsequent publications.

2 Simple Moutard transforms for generalized analytic functions

{sec:2}

Following [15], [2]-[4], we consider the pair of conjugate equations of the generalized analytic function theory:

$$\partial_{\bar{z}}\psi = q\bar{\psi} \quad \text{in } D, \quad (7) \quad \{\text{eq:gan1}\}$$

$$\partial_{\bar{z}}\psi^+ = -\bar{q}\bar{\psi}^+ \quad \text{in } D, \quad (8) \quad \{\text{eq:gan2}\}$$

where $\partial_z, \partial_{\bar{z}}$ are defined in (3), $z = x_1 + ix_2, \bar{z} = x_1 - ix_2, D$ is an open simply connected domain in $\mathbb{C} \cong \mathbb{R}^2, q = q(z)$ is a given function in D . In addition, in this article the notation $f = f(x) = f(z)$ does not mean that $f(z)$ is holomorphic function in z unless it is explicitly specified.

Next, as in [15], [2]-[4], we associate with a pair of functions ψ, ψ^+ , satisfying (7), (8), respectively, the following imaginary-valued potential ω_{ψ, ψ^+} defined by:

$$\partial_z\omega_{\psi, \psi^+} = \psi\psi^+, \quad \partial_{\bar{z}}\omega_{\psi, \psi^+} = -\overline{\psi\psi^+} \quad \text{in } D, \quad (9) \quad \{\text{eq:k1}\}$$

where the pure imaginary integration constant may depend on the particular situation. We recall that the compatibility of (9) follows from (7), (8).

Let f, f^+ be some fixed solutions of equations (7), (8), respectively, with given q . Then a simple Moutard-type transform $\mathcal{M} = \mathcal{M}_{q,f,f^+}$ for the pair of conjugate equations (7), (8) is given by the formulas (see [2]-[4]):

$$\tilde{q} = \mathcal{M}q = q + \frac{f\overline{f^+}}{\omega_{f,f^+}}, \quad (10) \quad \{\text{eq:m3}\}$$

$$\tilde{\psi} = \mathcal{M}\psi = \psi - \frac{\omega_{\psi,f^+}}{\omega_{f,f^+}} f, \quad \tilde{\psi}^+ = \mathcal{M}\psi^+ = \psi^+ - \frac{\omega_{f,\psi^+}}{\omega_{f,f^+}} f^+, \quad (11) \quad \{\text{eq:m1}\}$$

where ψ, ψ^+ are arbitrary solutions of (7) and (8).

The point is that the functions $\tilde{\psi}, \tilde{\psi}^+$ defined in (11) satisfy the conjugate pair of Moutard-transformed equations (see [2]-[4]):

$$\partial_{\bar{z}}\tilde{\psi} = \tilde{q}\overline{\tilde{\psi}} \quad \text{in } D, \quad (12) \quad \{\text{eq:gan3}\}$$

$$\partial_z\tilde{\psi}^+ = -\overline{\tilde{q}}\tilde{\psi}^+ \quad \text{in } D, \quad (13) \quad \{\text{eq:gan4}\}$$

where \tilde{q} is defined in (10).

3 Simple Moutard transforms for the conductivity equation

{sec:3}

In this Section we assume that D is an open simply connected domain in $\mathbb{C} \cong \mathbb{R}^2$.

Let

$$f_R^+ = \sqrt{\sigma(z)}, \quad f_I^+ = \frac{i}{\sqrt{\sigma(z)}}, \quad (14) \quad \{\text{eq:psiplus}\}$$

where σ is the conductivity in equation (1).

Note that $\psi^+ = f_R^+$ and $\psi^+ = f_I^+$ are solutions of equation (8), where q is given by (5) with a regular positive σ .

{lem:1}

Lemma 1 *A regular complex-valued function $\psi(z)$ satisfies equation (7) with $q(z)$ given by (5) with a positive $\sigma(z)$ if and only if there exists a real-valued solution $u(z)$ of (1) such that*

$$\psi(z) = \sigma^{1/2}(z)\partial_z u(z), \quad \overline{\psi(z)} = \sigma^{1/2}(z)\partial_{\bar{z}} u(z). \quad (15) \quad \{\text{eq:red1.1}\}$$

In addition,

$$u = -i\omega_{\psi,f_I^+}, \quad (16) \quad \{\text{eq:red1.2}\}$$

where f_I^+ is defined in (14), ψ is defined in (15), ω_{ψ,ψ^+} is defined via (9).

Lemma 1 is proved in Section 4.

Note also that

$$\psi = \sigma^{-1/2}(J_1 + iJ_2), \quad J_1 = \sigma \frac{\partial u}{\partial x_1}, \quad J_2 = \sigma \frac{\partial u}{\partial x_2}, \quad (17)$$

where ψ , u are the functions of (15), and J is the current for the conductivity equation (1).

Theorem 1 *Let $q(z)$ be given by (5) in D with a positive regular $\sigma(z)$. Let the transform $q \rightarrow \tilde{q}$, $\psi \rightarrow \tilde{\psi}$ be defined by:* {thm:1}

$$\tilde{q} = \mathcal{M}q = q + \frac{f\overline{f^+}}{\omega_{f,f^+}}, \quad \tilde{\psi} = \mathcal{M}\psi = \psi - \frac{\omega_{\psi,f^+}}{\omega_{f,f^+}} f, \quad (18) \quad \{\text{eq:moutard1}\}$$

where ψ denotes an arbitrary solution of (7), f is a fixed solution of equation (7), $f^+ = f_R^+$ or $f^+ = f_I^+$, where f_R^+ and f_I^+ are defined in (14).

Then $\tilde{\psi}$ satisfies the Moutard-transformed equation (12), and \tilde{q} admits the representation

$$\tilde{q} = -\frac{1}{2}\partial_z \log(\tilde{\sigma}), \quad (19) \quad \{\text{eq:hc7}\}$$

where

$$\tilde{\sigma} = \begin{cases} -\frac{\sigma}{\omega_{f,f_R^+}^2} & \text{if } f^+ = f_R^+, \\ -\sigma\omega_{f,f_I^+}^2 & \text{if } f^+ = f_I^+. \end{cases} \quad (20) \quad \{\text{eq:hc8}\}$$

In addition, the following Moutard-transformed conductivity equation holds:

$$\text{div}(\tilde{\sigma}\nabla\tilde{u}) = 0 \quad \text{in } D, \quad (21) \quad \{\text{eq:hcm1}\}$$

where

$$\tilde{u} = -i\omega_{\tilde{\psi},\hat{f}^+}, \quad \hat{f}^+ = \frac{i}{\sqrt{\tilde{\sigma}}}. \quad (22) \quad \{\text{eq:moutard2}\}$$

The following scheme summarizes the Moutard-type transforms for the conductivity equation (1) given in Theorem 1:

$$\begin{array}{ccc} \sigma & \xrightarrow[\{f,f^+\}]{(20)} & \tilde{\sigma} \xrightarrow{(19)} \tilde{q}, \\ \sigma, u & \xrightarrow{(5),(15)} & q, \psi \xrightarrow[\{f,f^+\}]{(18)} \tilde{q}, \tilde{\psi}, \\ \tilde{\sigma}, \tilde{\psi} & \xrightarrow{(22)} & \tilde{u}. \end{array} \quad (23) \quad \{\text{eq:pr2:1}\}$$

The point is that each step in scheme (23) is given by quadratures.

Theorem 1 is proved in Section 4.

Remark 1 *In Theorem 1 we have the following two important cases:*

- (i) *If ω_{f,f^+} has no zeroes in D , then $\tilde{\sigma}$ arising in (20) is a regular positive function in D . In addition, if D is bounded, then ω_{f,f^+} can be always defined without zeroes by an appropriate choice of integration constant.*
- (ii) *If ω_{f,f^+} has zeroes in D , then $\tilde{\sigma}$ arising in (20) is non-negative and has either zeros or poles in D . In these singular cases the standard methods for solving the conductivity equation (21) does not work; but these both singular cases are interesting and relevant for physical problems. The point is that the Moutard-type transform of Theorem 1 generating $\tilde{\sigma}$ simultaneously provides a method for solving equation (21).*

4 Proofs of Lemma 1 and Theorem 1

{sec:4}

Proof of Lemma 1. If σ is a real-valued regular positive function, u is a real-valued regular function, and q, ψ are defined by (5) and (15), respectively, then it is known that ψ satisfies (7) if and only if u satisfies (1); see Introduction. This can be also verified by a direct calculation.

Conversely, suppose that q is defined by (5) with a regular real-valued positive σ , and ψ satisfies (7). Define u by (16). It remains to verify that (15) holds. This verification uses (9), (14) and consists of the following:

$$\partial_z u = \partial_z(-i\omega_{\psi, f_I^+}) = -i\psi f_I^+ = -i\psi \frac{i}{\sqrt{\sigma}} = \frac{\psi}{\sqrt{\sigma}}, \quad (24)$$

$$\partial_{\bar{z}} u = \partial_{\bar{z}}(-i\omega_{\psi, f_I^+}) = -i(-\bar{\psi} \overline{f_I^+}) = i\bar{\psi} \frac{-i}{\sqrt{\sigma}} = \frac{\bar{\psi}}{\sqrt{\sigma}}. \quad (25)$$

Lemma 1 is proved.

Proof of Theorem 1. The statement that $\tilde{\psi}$ satisfies Moutard-transformed equation (12) was proved in [2].

If q is defined by (19) with $\tilde{\sigma}$ defined by (20), then:

$$\begin{aligned} \tilde{q} &= -\frac{1}{2}\partial_z \log(\tilde{\sigma}) = -\frac{1}{2}\partial_z \log(\sigma) + \partial_z \log(\omega_{f, f_R^+}) = & (26) \quad \{\text{eq:thm1:1}\} \\ &= q + \frac{f f_R^+}{\omega_{f, f_R^+}} = q + \frac{\overline{f f_R^+}}{\omega_{f, f_R^+}} \quad \text{if } f^+ = f_R^+, \end{aligned}$$

$$\begin{aligned}\tilde{q} &= -\frac{1}{2}\partial_z \log(\tilde{\sigma}) = -\frac{1}{2}\partial_z \log(\sigma) - \partial_z \log(\omega_{f,f_I^+}) = & (27) \quad \{\text{eq:thm1:2}\} \\ &= q - \frac{f f_I^+}{\omega_{f,f_I^+}} = q + \frac{f \overline{f_I^+}}{\overline{\omega_{f,f_I^+}}} & \text{if } f^+ = f_I^+, \end{aligned}$$

Here, we used that f_R^+ is real-valued and f_I^+ is imaginary-valued.

In fact, calculations (26), (27) prove representations (19), (20) for \tilde{q} in (18).

Formulas (21), (22) follow directly from the Moutard-transformed equation (12), the representation (19) and Lemma 1.

This completes the proof of Theorem 1 under the assumption that ω_{f,f^+} has no zeroes in D .

Remark 2 *Formally, the proof of Theorem 1 remains valid if ω_{f,f^+} has zeroes in D , but a proper analytic picture requires a subsequent investigation.*

References

- [1] R.M. Brown, G.A. Uhlmann, “Uniqueness in the inverse conductivity problem for nonsmooth conductivities in two dimensions”, *Communications in Partial Differential Equations*, **22**, Numbers 5-6, (1997), pp. 1009-1027; doi:10.1080/03605309708821292.
- [2] P.G. Grinevich, R.G. Novikov, “Moutard transform for the generalized analytic functions”, *The Journal of Geometric Analysis*, **26** Issue 4 (2016), pp 2984-2995; doi:10.1007/s12220-015-9657-8.
- [3] P.G. Grinevich, R.G. Novikov, “Generalized analytic functions, Moutard-type transforms, and holomorphic maps”, *Funct. Anal. Appl.*, **50**, Issue 2 (2016), pp.150-152; doi:10.1007/s10688-016-0140-5.
- [4] P.G. Grinevich, R.G. Novikov, “Moutard transform approach to generalized analytic functions with contour poles”, *Bulletin des sciences mathematiques*, **140**, Issue 6 (2016), pp. 638-656; doi:10.1016/j.bulsci.2016.01.003.
- [5] L.D. Landau, E.M. Lifshitz (1986). *Theory of Elasticity*. Vol. 7 (3rd ed.). Butterworth-Heinemann.

- [6] L.D. Landau, E.M. Lifshitz, L.P. Pitaevskii (1984). *Electrodynamics of Continuous Media*. Vol. 8 (2nd ed.). Butterworth-Heinemann.
- [7] V.B. Matveev, M.A. Salle, *Darboux transformations and solitons*, Springer Series in Nonlinear Dynamics. Springer-Verlag, Berlin, 1991.
- [8] T.F. Moutard, “Sur la construction des équations de la forme $\frac{1}{z} \frac{\partial^2 z}{\partial x \partial y} = \lambda(x, y)$ qui admettent une intégrale générale explicite”, *J. École Polytechnique*, **45** (1878), pp. 1-11.
- [9] J.J.C. Nimmo, W.K. Schief, “Superposition principles associated with the Moutard transformation: an integrable discretization of a 2+1-dimensional sine-Gordon system”, *Proc. R. Soc. London A*, *453* (1997), pp. 255-279.
- [10] R.G. Novikov, I.A. Taimanov, S.P. Tsarev, “Two-dimensional von Neumann-Wigner potentials with a multiple positive eigenvalue”, *Functional Analysis and Its Applications*, **48**:4 (2014), pp. 295-297.
- [11] R.G. Novikov, I.A. Taimanov, “Moutard type transformation for matrix generalized analytic functions and gauge transformations”, *Russian Mathematical Surveys*, **71** Number 5 (2016), pp. 970-972; doi:10.1070/RM9741.
- [12] I.A. Taimanov, “Blowing up solutions of the modified Novikov-Veselov equation and minimal surfaces”, *Theoretical and Mathematical Physics*, **182**:2 (2015), pp. 173-181.
- [13] I.A. Taimanov, “The Moutard transformation of two-dimensional Dirac operators and Möbius geometry”, *Mathematical Notes*, **97**:1 (2015), pp. 124-135.
- [14] I.A. Taimanov, S.P. Tsarev, “On the Moutard transformation and its applications to spectral theory and Soliton equations”, *Journal of Mathematical Sciences*, **170**:3 (2010), pp. 371-387.
- [15] I.N. Vekua, *Generalized Analytic Functions*, Pergamon Press Ltd. 1962.
- [16] D. Yu, Q.P. Liu, S. Wang, “Darboux transformation for the modified Veselov-Novikov equation”, *Journal of Physics A: Mathematical and General*, **35**:16 (2002), pp. 3779-3786.