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Rankin-Cohen deformations of the algebra of Jacobi forms

YOUNGJU CHOIE, FRANÇOIS DUMAS, FRANÇOIS MARTIN, AND EMMANUEL ROYER

ABSTRACT. The aim of this work is to emphasize the arithmetical and algebraic aspects of the Rankin-Cohen brackets in order to extend them to several natural number-theoretical situations. We build an analytically consistent derivation on the algebra $\widetilde{\mathcal{T}}_{\text{ev},*}$ of weak Jacobi forms. From this derivation, we obtain a sequence of bilinear forms on $\widetilde{\mathcal{T}}_{\text{ev},*}$ that is a formal deformation and whose restriction to the algebra \mathcal{M}_* of modular forms is an analogue of Rankin-Cohen brackets associated to the Serre derivative. Using a classification of all admissible Poisson brackets, we generalize this construction to build a family of Rankin-Cohen deformations of $\widetilde{\mathcal{T}}_{\text{ev},*}$. The algebra $\widetilde{\mathcal{T}}_{\text{ev},*}$ is a polynomial algebra in four generators. We consider some localization $\mathcal{K}_{\text{ev},*}$ of $\widetilde{\mathcal{T}}_{\text{ev},*}$ with respect to one of the generators. We construct Rankin-Cohen deformations on $\mathcal{K}_{\text{ev},*}$. We study their restriction to $\widetilde{\mathcal{T}}_{\text{ev},*}$ and to some subalgebra of $\mathcal{K}_{\text{ev},*}$ naturally isomorphic to the algebra of quasimodular forms.

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CONTENTS

1. Introduction	3
1.1. Rankin-Cohen brackets	3
1.2. A prototype	4
1.3. Main results	4
2. Framework	6
2.1. Jacobi forms	6
2.1.1. The notion of weak Jacobi form	7
2.1.2. Generators	8
2.1.3. Formal algebraic point of view	10
2.2. Formal deformations and Rankin-Cohen brackets	11
2.2.1. Definition and first properties	11
2.2.2. Isomorphic formal deformations	11
2.2.3. Connes & Moscovici's Theorem	12
2.2.4. Examples: Rankin-Cohen brackets on modular forms	12
2.2.5. Examples: formal deformations on quasimodular forms	13
2.3. A derivation on Jacobi weak forms	14
2.3.1. Two intermediate functions	15
2.3.2. Oberdieck's derivation	18
2.3.3. Oberdieck-Rankin-Cohen brackets	20
3. Formal deformations for Jacobi forms	20
3.1. Admissible Poisson brackets on weak Jacobi forms	20
3.1.1. Determination of admissible Poisson brackets	20
3.1.2. Admissible Poisson brackets having the shape of a Rankin-Cohen bracket	23
3.2. A family of formal deformations for Jacobi forms	24
3.2.1. Construction	24
3.2.2. Classification	25
4. formal deformations for a localization of the algebra of Jacobi forms	26
4.1. Relation with quasimodular forms	26
4.1.1. Extension of the Serre derivation, associated Poisson brackets	26
4.1.2. Admissible derivations of localized Jacobi forms	27
4.2. Relation with modular forms	30
5. Graphical abstract	33
References	34

1. INTRODUCTION

1.1. Rankin-Cohen brackets. Rankin-Cohen brackets for modular forms have been widely studied. Rankin [Ran85, Ran57, Ran56] determined (quite complicated) necessary conditions that a polynomial has to satisfy so that its evaluation at modular forms and their derivatives is still a modular form. Cohen [Coh75] gave an explicit construction of such differential polynomials in two variables. These bilinear operators have been named *Rankin-Cohen* brackets by Zagier in [Zag94]. In this work, Zagier introduced the notion of Rankin-Cohen algebra as a graded vector space with bilinear operations that satisfy all the algebraic identities satisfied by Rankin-Cohen brackets. In [CMZ97], Cohen, Manin & Zagier continued the description of a conceptual framework for Rankin-Cohen brackets with the eyes of noncommutative geometry. In order to do that, they define a lifting to some invariant pseudo-differential operators and prove that suitable combinations of Rankin-Cohen brackets correspond by this lifting to noncommutative products of invariant operators. The Hecke operators on pseudodifferential operators are further investigated in [CL07] and [Cho98a]. In [OR03], Ovsienko & Redou develop in the context of differential geometry the vision of Rankin-Cohen brackets as a projective version of the transvectants of the classical invariant theory, following the work of Gordan in 19. century [Gor87, Olv99]. The works of Pevzner & van Dijk [vDP07], Pevzner & Kobayashi [KP16] and El Gradechi [EG06] emphasize the Lie-theoretic nature of the Rankin-Cohen brackets whereas Beliaevski, Tang & Yao [BTY07] deal with quantization theory. Without pretending to be exhaustive on such a vast and diversified literature, we mention finally the major work by Connes & Moscovici [CM04].

A reason why Rankin-Cohen brackets are interesting is that they combine derivatives of modular forms whereas the derivative of a modular form is generally not a modular form. This lack of stability of the algebra of modular forms by derivation is the *raison d'être* of quasimodular forms [Zag08, Section 5] or [MR05, Roy12] since the derivative of a quasimodular form is still a quasimodular form. The question of a definition of Rankin-Cohen brackets for quasimodular forms is then natural. A first answer was given by Martin & Royer in [MR09] (Zagier informed us after the publication of the paper that he did the same construction in an unpublished note). In this work, maps are build that have the shape of Rankin-Cohen brackets and send a pair of quasimodular forms of respective depths s and t to a quasimodular form of depth $s + t$. Here, the focus is put to the shape of the brackets and the minimisation of the depth, at the cost of the lost of the algebraic structure. The brackets indeed do not lead anymore to a formal deformation. Changing the shape of the brackets (more precisely the shape of the derivation involved in the definition of the brackets), Dumas & Royer [DR14] build formal deformations of the algebra of quasimodular forms. See also [CL17].

In the following, we focus on the construction of Rankin-Cohen brackets for the algebra of weak Jacobi forms. This study has been initiated by Choie and Choie & Ehlozer [Cho97, Cho98b, CE98]. Their brackets rest on the heat operator this involves second order derivatives. For this reason, they are not a formal deformation since the first bracket is not a Poisson bracket. In [CE01], Choie & Ehlozer defined a notion of generalized Rankin-Cohen algebra for the bigraded algebra of Jacobi forms. Since the definition involves the composition of two derivations, their structure is not a formal

deformation. In the following, we concentrate on the construction of bilinear maps that extend the Rankin-Cohen brackets from modular forms to Jacobi forms and provide the algebra of Jacobi forms the structure of a formal deformation.

1.2. A prototype. Let J_1 and J_2 be the two functions defined by

$$\forall \tau \in \mathcal{H}, \forall z \in \mathbb{C}, z \notin \mathbb{Z} + \tau\mathbb{Z} \quad J_1(\tau, z) = \frac{1}{2\pi i} \zeta(\tau, z) + \frac{\pi i}{6} z E_2(\tau)$$

where $\mathcal{H} = \{z \in \mathbb{C} : \text{Im} z > 0\}$ is the Poincaré upper half plane, ζ is the Weierstraß zeta function, E_2 is the Eisenstein series of weight 2 and

$$J_2 = D_z J_1 - \frac{1}{12} E_2 + J_1^2$$

where $D_z = \frac{\partial}{2\pi i \partial z}$ (see below (2.13) and (2.3) for the definitions of ζ and E_2).

We define a derivation on the algebra bigraded $\tilde{\mathcal{T}}_{\text{ev},*}$ of weak Jacobi forms (see § 2.1.1) on $\text{SL}(2, \mathbb{Z})$ by

$$\text{Ob}(f) = D_\tau(f) - \frac{k}{12} E_2 f - J_1 D_z(f) + p J_2 f$$

for any f in the space $\tilde{\mathcal{T}}_{k,p}$ of weak Jacobi forms of weight k and index p , where $D_\tau = \frac{\partial}{2\pi i \partial \tau}$ (we shall say that an element of the algebra $\tilde{\mathcal{T}}_{\text{ev},*}$ that belongs to some vector space $\tilde{\mathcal{T}}_{k,p}$ is homogeneous, the vector space being called a homogeneous component).

Let $\mu \in \mathbb{C}$. The sequence $(\text{ORC}_n^\mu)_{n \in \mathbb{Z}_{\geq 0}}$ of bilinear forms on $\tilde{\mathcal{T}}_{\text{ev},*}$ defined by

$$\text{ORC}_n^\mu(f, g) = \sum_{r=0}^n (-1)^r \binom{k + \mu p + n - 1}{n-r} \binom{\ell + \mu q + n - 1}{r} \text{Ob}^r(f) \text{Ob}^{n-r}(g)$$

for all homogeneous forms $(f, g) \in \tilde{\mathcal{T}}_{k,p} \times \tilde{\mathcal{T}}_{\ell,q}$ is a formal deformation of $\tilde{\mathcal{T}}_{\text{ev},*}$ that extends the formal deformation of the Serre-Rankin-Cohen brackets on modular forms, see (2.11). We shall call such a formal deformation a Rankin-Cohen deformation.

The aim of this work is to generalize this result to provide a systematic method of construction of similar Rankin-Cohen deformations on $\tilde{\mathcal{T}}_{\text{ev},*}$ and recover Rankin-Cohen deformations on the algebra $\mathcal{M}_*^{\leq \infty}$ of quasimodular forms on $\text{SL}(2, \mathbb{Z})$.

1.3. Main results. The algebra \mathcal{M}_* of modular forms on $\text{SL}(2, \mathbb{Z})$ is a polynomial algebra over \mathbb{C} with generators the two algebraically independent Eisenstein series E_4 and E_6 defined in (2.2). The algebra of weak Jacobi forms, $\tilde{\mathcal{T}}_{\text{ev},*}$, is a polynomial extension of the algebra \mathcal{M}_* by the two algebraically independent functions A and B defined in (2.4). The generators E_4, E_6, A and B have a weight and an index as in Table 1 that describe the bigraduation of $\tilde{\mathcal{T}}_{\text{ev},*}$. Let $(a, b) \in \mathbb{C}^2$, we define a derivation $\text{Se}_{a,b}$ on $\tilde{\mathcal{T}}_{\text{ev},*}$, that extends Serre derivation Se on \mathcal{M}_* , by $\text{Se}_{a,b}(A) = aB$ and $\text{Se}_{a,b}(B) = bE_4A$ (the definition of Serre derivation is given in (2.10)). We use this derivation to build, for any nonnegative integer $n \in \mathbb{Z}_{\geq 0}$ and any $c \in \mathbb{C}$, the bilinear map $\tilde{\mathcal{T}}_{\text{ev},*} \times \tilde{\mathcal{T}}_{\text{ev},*} \rightarrow \tilde{\mathcal{T}}_{\text{ev},*}$ defined by bilinear extension of

$$\forall (f, g) \in \tilde{\mathcal{T}}_{k,p} \times \tilde{\mathcal{T}}_{\ell,q} \quad \{f, g\}_n^{[a,b,c]} = \sum_{r=0}^n (-1)^r \binom{k + cp + n - 1}{n-r} \binom{\ell + cq + n - 1}{r} \text{Se}_{a,b}^r(f) \text{Se}_{a,b}^{n-r}(g)$$

for all $(k, \ell) \in \mathbb{Z}^2$ and $(p, q) \in \mathbb{Z}_{\geq 0}^2$. We prove in the following Theorem that these brackets are formal deformations and classify them up to modular isomorphism (see Definition 14).

Theorem A— For all $(a, b, c) \in \mathbb{C}^3$,

- (1) the sequence $\left(\{\cdot, \cdot\}_n^{[a, b, c]}\right)_{n \in \mathbb{Z}_{\geq 0}}$ is a formal deformation of $\tilde{\mathcal{T}}_{\text{ev},*}$,
- (2) $\{\tilde{\mathcal{T}}_{k,p}, \tilde{\mathcal{T}}_{\ell,q}\}_n^{[a, b, c]} \subset \tilde{\mathcal{T}}_{k+\ell+2n, p+q}$ for all (k, p, ℓ, q, n) ,
- (3) the subalgebra \mathcal{M}_* is stable by $\left(\{\cdot, \cdot\}_n^{[a, b, c]}\right)_{n \in \mathbb{Z}_{\geq 0}}$ and the induced formal deformation is given by the Serre-Rankin-Cohen brackets,
- (4) the formal deformation $(\{\cdot, \cdot\}_n^{[a, b, c]})_{n \in \mathbb{Z}_{\geq 0}}$ of $\tilde{\mathcal{T}}_{\text{ev},*}$ is modular-isomorphic to one of the following formal deformations:
 - (i) the formal deformation $(\{\cdot, \cdot\}_n^{[1, b', c']})_{n \in \mathbb{Z}_{\geq 0}}$ for some $(b', c') \in \mathbb{C}^2$;
 - (ii) the formal deformation $(\{\cdot, \cdot\}_n^{[0, 1, c']})_{n \in \mathbb{Z}_{\geq 0}}$ for some $c' \in \mathbb{C}$;
 - (iii) the formal deformation $(\{\cdot, \cdot\}_n^{[0, 0, c']})_{n \in \mathbb{Z}_{\geq 0}}$ for some $c' \in \mathbb{C}$
 that are pairwise non modular-isomorphic for different values of the parameters.

Recall that the algebra $\mathcal{M}_*^{\leq \infty}$ of quasimodular forms is the polynomial extension of the algebra \mathcal{M}_* by E_2 . In order to compare our results with the ones obtained for quasimodular forms in [DR14], we localize the algebra $\tilde{\mathcal{T}}_{\text{ev},*}$ with respect to A , setting $\mathcal{K}_{\text{ev},*} = \tilde{\mathcal{T}}_{\text{ev},*}[A^{-1}] = \mathcal{M}_*[F_2, A^{\pm 1}]$ where $F_2 = BA^{-1}$ has weight 2 and index 0 (note that, up to a scalar, F_2 is the Weierstraß \wp function). The algebra $\mathcal{K}_{\text{ev},*}$ is bigraded by extension of the bigraduation of $\tilde{\mathcal{T}}_{\text{ev},*}$.

For $(\alpha, \beta) \in \mathbb{C}^2$, let d_α and δ_β the two derivations of $\mathcal{K}_{\text{ev},*}$ defined by $d_\alpha(f) = \text{Se}(f) + \alpha k F_2 f$ and $\delta_\beta(f) = \text{Se}(f) + \beta k F_2 f$ if f is a modular form of weight k and

$$d_\alpha(A) = -2\alpha A F_2, \quad d_\alpha(F_2) = -\frac{1}{12} E_4 + 2\alpha F_2^2,$$

$$\delta_\beta(A) = -2\beta A F_2, \quad \delta_\beta(F_2) = 2\beta F_2^2.$$

We prove the following Proposition.

Proposition B— For any complex parameters α, β , and c , let the sequences $([\cdot, \cdot]_n^{\alpha, c})_{n \in \mathbb{Z}_{\geq 0}}$ and $(\langle \cdot, \cdot \rangle_n^{\beta, c})_{n \in \mathbb{Z}_{\geq 0}}$ of maps $\mathcal{K}_{\text{ev},*} \times \mathcal{K}_{\text{ev},*} \rightarrow \mathcal{K}_{\text{ev},*}$ be defined by bilinear extension of the formulas:

$$[f, g]_n^{\alpha, c} = \sum_{i=0}^n (-1)^i \binom{k+cp+n-1}{n-i} \binom{\ell+cq+n-1}{i} d_\alpha^i(f) d_\alpha^{n-i}(g),$$

$$\langle f, g \rangle_n^{\beta, c} = \sum_{i=0}^n (-1)^i \binom{k+cp+n-1}{n-i} \binom{\ell+cq+n-1}{i} \delta_\beta^i(f) \delta_\beta^{n-i}(g)$$

for all homogeneous $f \in \mathcal{K}_{k,p}$ and $g \in \mathcal{K}_{\ell,q}$. Then,

- (1) the sequences $([\cdot, \cdot]_n^{\alpha, c})_{n \in \mathbb{Z}_{\geq 0}}$ and $(\langle \cdot, \cdot \rangle_n^{\alpha, c})_{n \in \mathbb{Z}_{\geq 0}}$ are formal deformations of $\mathcal{K}_{\text{ev},*}$,
- (2) $[\mathcal{K}_{k,p}, \mathcal{K}_{\ell,q}]_n^{\alpha, c} \subset \mathcal{K}_{k+\ell+2n, p+q}$ and $\langle \mathcal{K}_{k,p}, \mathcal{K}_{\ell,q} \rangle_n^{\beta, c} \subset \mathcal{K}_{k+\ell+2n, p+q}$,

- (3) the subalgebra $\mathcal{Q}_* = \mathcal{M}_*[F_2]$ is stable by $([\cdot, \cdot]_n^{\alpha, c})_{n \in \mathbb{Z}_{\geq 0}}$ and $(\langle \cdot, \cdot \rangle_n^{\beta, c})_{n \in \mathbb{Z}_{\geq 0}}$ and the induced formal deformations of \mathcal{Q}_* are isomorphic to formal deformations on the algebra of quasimodular forms by the extension of the identity on modular forms by $F_2 \mapsto E_2$,
- (4) the subalgebra $\widetilde{\mathcal{T}}_{\text{ev},*}$ is stable by $([\cdot, \cdot]_n^{\alpha, c})_{n \in \mathbb{Z}_{\geq 0}}$ if and only if $\alpha = 0$ and by $(\langle \cdot, \cdot \rangle_n^{\alpha, c})_{n \in \mathbb{Z}_{\geq 0}}$ if and only if $\beta = 0$.

Point (3) of Proposition B shows that our construction is a consistent extension of the brackets constructed in [DR14].

Finally, we extend directly the usual Rankin-Cohen brackets on modular forms into formal deformations of $\mathcal{K}_{\text{ev},*}$. For $u \in \mathbb{C}$, let ∂_u be the derivation of $\mathcal{K}_{\text{ev},*}$ defined by

$$\begin{aligned} \partial_u(E_4) &= -\frac{1}{3}(E_6 - E_4 F_2) & \partial_u(E_6) &= \frac{1}{2}(E_4^2 - E_6 F_2) \\ \partial_u(A) &= u B & \partial_u(B) &= \left(u + \frac{1}{12}\right) B F_2 - \frac{1}{12} E_4 A. \end{aligned}$$

We prove the following Theorem.

Theorem C– For any complex parameters u and v , let $([\cdot, \cdot]_n^{u, v})_{n \in \mathbb{Z}_{\geq 0}}$ be the sequence of maps $\mathcal{K}_{\text{ev},*} \times \mathcal{K}_{\text{ev},*} \rightarrow \mathcal{K}_{\text{ev},*}$ defined by bilinear extension of

$$[f, g]_n^{u, v} = \sum_{r=0}^n (-1)^r \binom{k + vp + n - 1}{n - r} \binom{\ell + vq + n - 1}{r} \partial_u^r(f) \partial_u^{n-r}(g),$$

for all homogeneous $f \in \mathcal{K}_{k,p}$ and $g \in \mathcal{K}_{\ell,q}$. Then, for all $(u, v) \in \mathbb{C}^2$,

- (1) the sequence $([\cdot, \cdot]_n^{u, v})_{n \in \mathbb{Z}_{\geq 0}}$ is a formal deformation of $\mathcal{K}_{\text{ev},*}$,
- (2) $[\mathcal{K}_{k,p}, \mathcal{K}_{\ell,q}]_n^{u, v} \subset \mathcal{K}_{k+\ell+2n, p+q}$,
- (3) the sequence $([\cdot, \cdot]_n^{u, v})_{n \in \mathbb{Z}_{\geq 0}}$ restricts to the formal deformation of the algebra \mathcal{M}_* given by the usual Rankin-Cohen brackets.

We prove that $[\cdot, \cdot]_1^{u, v}$ defines a Poisson bracket of $\widetilde{\mathcal{T}}_{\text{ev},*}$ if and only if $v = 12u + 1$ and conjecture that the sequence $([\cdot, \cdot]_n^{u, v})_{n \in \mathbb{Z}_{\geq 0}}$ restricts to a formal deformation of the algebra $\widetilde{\mathcal{T}}_{\text{ev},*}$ if and only if $v = 12u + 1$.

The following diagram summarizes the ways we follow to build formal deformations of the algebras of Jacobi forms and quasimodular forms.

$$\begin{array}{ccc} \widetilde{\mathcal{T}}_{\text{ev},*} = \mathbb{C}[E_4, E_6, A, B] & \hookrightarrow & \mathcal{K}_{\text{ev},*} = \mathbb{C}[E_4, E_6, A^{\pm 1}, B] \\ \uparrow & \nearrow & \uparrow \\ \mathcal{M}_* = \mathbb{C}[E_4, E_6] & \hookrightarrow & \mathcal{Q}_* = \mathbb{C}[E_4, E_6, F_2] \simeq \mathcal{M}_*^{\leq \infty} \end{array}$$

2. FRAMEWORK

2.1. Jacobi forms. The aim of this part is to collect the notions we shall need on weak Jacobi forms. The main reference is [EZ85].

2.1.1. *The notion of weak Jacobi form.* Let \mathcal{H} be the upper half plane, k an integer and m a nonnegative integer. The multiplicative group $\mathrm{SL}(2, \mathbb{Z})$ acts on \mathbb{Z}^2 by right multiplication. The semidirect product of $\mathrm{SL}(2, \mathbb{Z})$ and \mathbb{Z}^2 with respect to this action is the Jacobi group: $\mathrm{SL}(2, \mathbb{Z})^J = \mathrm{SL}(2, \mathbb{Z}) \ltimes \mathbb{Z}^2$. Let \mathbf{F} be the set of functions from $\mathcal{H} \times \mathbb{C}$ to \mathbb{C} . Let k and m be two integers. We have the following actions of $\mathrm{SL}(2, \mathbb{Z})$ and \mathbb{Z}^2 on \mathbf{F} . Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$, let $(\lambda, \mu) \in \mathbb{Z}^2$, let $\Phi \in \mathbf{F}$, then

$$\begin{aligned} \Phi|_{k,m} \begin{pmatrix} a & b \\ c & d \end{pmatrix} (\tau, z) &= (c\tau + d)^{-k} e^{-2i\pi \frac{m\tau^2}{c\tau+d}} \Phi \left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d} \right) \\ \Phi|_m (\lambda, \mu) (\tau, z) &= e^{2i\pi m(\lambda^2 \tau + 2\lambda z)} \Phi(\tau, z + \lambda\tau + \mu) \end{aligned}$$

for all $(\tau, z) \in \mathcal{H} \times \mathbb{C}$. These two actions induce an action of $\mathrm{SL}(2, \mathbb{Z})^J$ on \mathbf{F} the following way: if $(\gamma, (\lambda, \mu)) \in \mathrm{SL}(2, \mathbb{Z})^J$, if $\Phi \in \mathbf{F}$, then we define

$$\Phi|_{k,m} (\gamma, (\lambda, \mu)) = (\Phi|_{k,m} \gamma)|_m (\lambda, \mu).$$

Explicitly, if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$ and $(\lambda, \mu) \in \mathbb{Z}^2$, then

$$\begin{aligned} f|_{k,m} (\gamma, (\lambda, \mu)) (\tau, z) &= \\ (c\tau + d)^{-k} \exp \left(2\pi i m \left(-\frac{c(z + \lambda\tau + \mu)^2}{c\tau + d} + \lambda^2 \tau + 2\lambda z \right) \right) f \left(\frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \mu}{c\tau + d} \right) \end{aligned}$$

for all $(\tau, z) \in \mathcal{H} \times \mathbb{C}$. A function is invariant by the action of $\mathrm{SL}(2, \mathbb{Z})^J$ if and only if it is invariant by both the action of $\mathrm{SL}(2, \mathbb{Z})$ and the action of \mathbb{Z}^2 .

A Jacobi form of weight k and index m is a holomorphic function $\Phi: \mathcal{H} \times \mathbb{C} \rightarrow \mathbb{C}$ that is invariant by the action of the Jacobi group and that has a Fourier expansion of the form

$$\Phi(\tau, z) = \sum_{n=0}^{+\infty} \sum_{\substack{r \in \mathbb{Z} \\ r^2 \leq 4nm}} c(n, r) e^{2\pi i(n\tau + rz)}. \quad (2.1)$$

The vector space $\mathcal{J}_{k,m}$ of such functions is finite dimensional. We identify functions on $\mathcal{H} \times \mathbb{C}$ that are not depending on the second variable with functions on \mathcal{H} and define

$$\mathcal{J}_{k,0} = \mathcal{M}_k.$$

The space \mathcal{M}_k is the space of holomorphic modular forms of weight k on $\mathrm{SL}(2, \mathbb{Z})$ and we have

$$\mathcal{M}_* = \bigoplus_{\substack{k \in 2\mathbb{Z}_{\geq 0} \\ k \neq 2}} \mathcal{M}_k.$$

The action $|_{k,0}$ of $\mathrm{SL}(2, \mathbb{Z})^J$ on $\mathcal{J}_{k,0}$ induces an action of $\mathrm{SL}(2, \mathbb{Z})$ on \mathcal{M}_k . This action is $|_{k,0}$ and we shall simply write $|_k$.

The bigraded algebra

$$\mathcal{J}_{*,*} = \bigoplus_{k,m} \mathcal{J}_{k,m}$$

is *not* finitely generated and hence we introduce the notion of weak Jacobi form.

A weak Jacobi form of weight k and index m is a function invariant by the action of the Jacobi group but with a Fourier expansion of the form

$$\Phi(\tau, z) = \sum_{n=0}^{+\infty} \sum_{\substack{r \in \mathbb{Z} \\ r^2 \leq 4nm + m^2}} c(n, r) e^{2i\pi(n\tau + rz)}$$

instead of the one given in (2.1). For any given integer $n \geq 0$, the fact that the sum over r is limited to $r^2 \leq 4nm + m^2$ is a consequence of some periodicity of the coefficients [EZ85, p. 105]. The vector space $\tilde{\mathcal{J}}_{k,m}$ of such functions is still finite dimensional [EZ85, Theorem 9.2]. As a consequence, we obtain that

$$\tilde{\mathcal{J}}_{k,0} = \mathcal{M}_k.$$

The principal object of our study is the bigraded algebra

$$\tilde{\mathcal{J}}_{\text{ev},*} = \bigoplus_{\substack{k \in 2\mathbb{Z} \\ m \in \mathbb{Z}_{\geq 0}}} \tilde{\mathcal{J}}_{k,m}.$$

2.1.2. Generators. The algebra $\tilde{\mathcal{J}}_{\text{ev},*}$ is a polynomial algebra on two generators over the algebra \mathcal{M}_* of modular forms. We describe these two generators.

Let $\mathbf{1}$ be the constant function taking value 1 everywhere (of one or two variables, depending on the context). The subgroup of the modular group $\text{SL}(2, \mathbb{Z})$ of elements γ with $\mathbf{1}|_k \gamma = \mathbf{1}$ is

$$\text{SL}(2, \mathbb{Z})_{\infty} = \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}.$$

The Eisenstein series of weight $k \in \mathbb{Z}_{\geq 4}$ is

$$E_k(\tau) = \sum_{\gamma \in \text{SL}(2, \mathbb{Z})_{\infty} \setminus \text{SL}(2, \mathbb{Z})} \mathbf{1}|_k \gamma(\tau) = \frac{1}{2} \sum_{\substack{(c,d) \in \mathbb{Z}^2 \\ (c,d)=1}} (c\tau + d)^{-k}. \quad (2.2)$$

Its Fourier expansion is given in terms of the divisor functions

$$\forall u \in \mathbb{C} \quad \forall n \in \mathbb{Z}_{\geq 0}^* \quad \sigma_u(n) = \sum_{d|n} d^u$$

by

$$\forall \tau \in \mathcal{H} \quad E_k(\tau) = 1 - \frac{2k}{B_k} \sum_{n=1}^{+\infty} \sigma_{k-1}(n) q^n$$

where $q = \exp(2\pi i \tau)$ and B_k is the Bernoulli number of order k . We use this Fourier expansion to define an Eisenstein series of weight two:

$$E_2(\tau) = 1 - 24 \sum_{n=1}^{+\infty} \sigma_1(n) q^n. \quad (2.3)$$

For all even $k \geq 2$, we shall sometimes use another normalisation:

$$G_k = -\frac{(2\pi i)^k}{k!} B_k E_k.$$

If $m \neq 0$, the subgroup of the Jacobi group $\mathrm{SL}(2, \mathbb{Z})^J$ of elements $\alpha \in \mathrm{SL}(2, \mathbb{Z})^J$ with $\mathbf{1} \parallel_{k,m} \alpha = \mathbf{1}$ is

$$\mathrm{SL}(2, \mathbb{Z})_\infty^J = \left\{ \left(\pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, (0, \mu) \right) : n, \mu \in \mathbb{Z} \right\}.$$

The Eisenstein series of weight $k \geq 4$ and index m is

$$\begin{aligned} E_{k,m}(\tau, z) &= \sum_{\alpha \in \mathrm{SL}(2, \mathbb{Z})_\infty^J \setminus \mathrm{SL}(2, \mathbb{Z})^J} \mathbf{1} \parallel_{k,m} \alpha(\tau, z) \\ &= \frac{1}{2} \sum_{\substack{(c,d) \in \mathbb{Z}^2 \\ (c,d)=1}} \sum_{\lambda \in \mathbb{Z}} (c\tau + d)^{-k} \exp \left(2i\pi m \left(\lambda^2 \frac{a\tau + b}{c\tau + d} + 2 \frac{\lambda z - cz^2}{c\tau + d} \right) \right). \end{aligned}$$

The Eisenstein series E_4 and E_6 generate the algebra of modular forms:

$$\mathcal{M}_* = \mathbb{C}[E_4, E_6].$$

Let us define

$$\Phi_{10,1} = \frac{1}{144} (E_6 E_{4,1} - E_4 E_{6,1}) \in \widetilde{\mathcal{J}}_{10,1},$$

$$\Phi_{12,1} = \frac{1}{144} (E_4^2 E_{4,1} - E_6 E_{6,1}) \in \widetilde{\mathcal{J}}_{12,1},$$

and

$$\Delta = \frac{1}{1728} (E_4^3 - E_6^2) \in \mathcal{M}_{12}.$$

The two generators of $\widetilde{\mathcal{J}}_{\mathrm{ev},*}$ over \mathcal{M}_* are

$$A = \frac{\Phi_{10,1}}{\Delta} \in \widetilde{\mathcal{J}}_{-2,1} \quad \text{and} \quad B = \frac{\Phi_{12,1}}{\Delta} \in \widetilde{\mathcal{J}}_{0,1}. \quad (2.4)$$

It follows that

$$\widetilde{\mathcal{J}}_{\mathrm{ev},*} = \mathbb{C}[E_4, E_6, A, B]$$

[EZ85, Theorem 9.3].

Using the algorithm proved in [EZ85, p. 39], we can compute the Fourier expansion of $\Phi_{10,1}$ and $\Phi_{12,1}$ and deduce the ones of A and B . We obtain

$$A(\tau, z) = (\xi^{1/2} - \xi^{-1/2})^2 - 2(\xi^{1/2} - \xi^{-1/2})^4 q + (\xi^{1/2} - \xi^{-1/2})^4 (\xi - 8 + \xi^{-1}) q^2 + O(q^3)$$

and

$$\begin{aligned} B(\tau, z) &= (\xi + 10 + \xi^{-1}) + 2(\xi^{1/2} - \xi^{-1/2})^2 (5\xi - 22 + 5\xi^{-1}) q \\ &\quad + (\xi^{1/2} - \xi^{-1/2})^2 (\xi^2 + 110\xi - 294 + 110\xi^{-1} + \xi^{-2}) q^2 + O(q^3) \end{aligned}$$

where $\xi = \exp(2\pi i z)$, $z \in \mathbb{C}$.

	E_4	E_6	A	B	F_2
weight	4	6	-2	0	2
index	0	0	1	1	0

TABLE 1. Weights and indices of the generators

2.1.3. *Formal algebraic point of view.* We will work with the two commutative polynomial algebras

$$\mathcal{M}_* = \bigoplus_{k \in 2\mathbb{Z}_{\geq 0}, k \neq 2} \mathcal{M}_k = \mathbb{C}[E_4, E_6]$$

and

$$\widetilde{\mathcal{J}}_{\text{ev},*} = \bigoplus_{k \in 2\mathbb{Z}, m \in \mathbb{Z}_{\geq 0}} \widetilde{\mathcal{J}}_{k,m} = \mathbb{C}[E_4, E_6, A, B] = \mathcal{M}_*[A, B].$$

The algebra \mathcal{M}_* is graded by the weight, and the algebra $\widetilde{\mathcal{J}}_{\text{ev},*}$ is bigraded by the weight and the index.

We introduce the algebra

$$\mathcal{K}_{\text{ev},*} = \mathbb{C}[E_4, E_6, A^{\pm 1}, B] \supset \widetilde{\mathcal{J}}_{\text{ev},*}.$$

This is the localization of $\widetilde{\mathcal{J}}_{\text{ev},*}$ with respect to the powers of A . The notions of weight and index naturally extend to $\mathcal{K}_{\text{ev},*}$ defining a bigraduation

$$\mathcal{K}_{\text{ev},*} = \bigoplus_{\substack{k \in 2\mathbb{Z} \\ m \in \mathbb{Z}}} \mathcal{K}_{k,m}.$$

We set:

$$F_2 = BA^{-1}.$$

This function has a number-theoretic meaning since

$$F_2 = -\frac{3}{\pi^2} \wp \tag{2.5}$$

where \wp is the Weierstraß function [EZ85, Theorem 3.6]. Since

$$\mathcal{K}_{\text{ev},*} = \mathbb{C}[E_4, E_6, F_2, A^{\pm 1}] = \mathbb{C}[E_4, E_6, F_2][A^{\pm 1}]$$

we are led to introduce the subalgebra

$$\mathcal{Q}_* = \mathbb{C}[E_4, E_6, F_2].$$

The elements of \mathcal{Q}_* appear as the elements in $\mathcal{K}_{\text{ev},*}$ of index zero. From a number-theoretical point of view, it follows from (2.5) that \mathcal{Q}_* is the subalgebra generated by modular forms and the Weierstraß function

$$\mathcal{Q}_* = \mathcal{M}_*[\wp]. \tag{2.6}$$

Table 1 summarizes the weights and indices attached to the generators.

Another arithmetical point of view consists in seeing \mathcal{Q}_* as a formal analogue to the algebra $\mathcal{M}_*^{\leq \infty} = \mathcal{M}_*[E_2]$ of quasimodular forms. This algebra is graded by the weight. We have

$$\mathcal{M}_* \subset \mathcal{M}_*^{\leq \infty} \simeq \mathcal{Q}_* \subset \mathcal{K}_{\text{ev},*}.$$

The algebra isomorphism involved is

$$\begin{aligned} \omega : \quad \mathcal{Q}_* &\rightarrow \mathcal{M}_*^{\leq \infty} \\ P(E_4, E_6, F_2) &\mapsto P(E_4, E_6, E_2). \end{aligned}$$

The degree related to F_2 of any $f \in \mathcal{Q}_*$ is the depth of the quasimodular form $\omega(f)$. The isomorphism (2.1.3) and (2.6) emphasize that, from an algebraic point of view, the Weierstraß \wp function is similar to the Eisenstein series E_2 .

2.2. Formal deformations and Rankin-Cohen brackets. In this section we remind the basic properties of formal deformations and their isomorphisms. Our primary reference for this is [LGPV13, Chapter 13]. We exhibit Connes & Moscovici result that provides a general method to construct formal deformations.

2.2.1. Definition and first properties. For any commutative \mathbb{C} -algebra \mathcal{R} , let $\mathcal{R}[[\hbar]]$ be the commutative algebra of formal power series in one variable \hbar with coefficients in \mathcal{R} . A *formal deformation* of \mathcal{R} is a family $(\mu_j)_{j \in \mathbb{Z}_{\geq 0}}$ of bilinear maps $\mu_j : \mathcal{R} \times \mathcal{R} \rightarrow \mathcal{R}$ such that μ_0 is the product of \mathcal{R} and such that the (non commutative) product on $\mathcal{R}[[\hbar]]$ defined by extension of

$$\forall (f, g) \in \mathcal{R}^2 \quad f \star g = \sum_{j \geq 0} \mu_j(f, g) \hbar^j$$

is associative. This associativity translates to

$$\forall n \in \mathbb{Z}_{\geq 0} \quad \forall (f, g, h) \in \mathcal{R}^3 \quad \sum_{r=0}^n \mu_{n-r}(\mu_r(f, g), h) = \sum_{r=0}^n \mu_{n-r}(f, \mu_r(g, h))$$

If $(\mu_j)_{j \in \mathbb{Z}_{\geq 0}}$ is a formal deformation of \mathcal{R} , if μ_1 is skew-symmetric and if μ_2 is symmetric, then μ_1 is a Poisson bracket on \mathcal{R} .

2.2.2. Isomorphic formal deformations. Let $(\mu_j)_{j \in \mathbb{Z}_{\geq 0}}$ and $(\mu'_j)_{j \in \mathbb{Z}_{\geq 0}}$ be two formal deformations of \mathcal{R} . They are *isomorphic* if there exists a \mathbb{C} -linear bijective map $\phi : \mathcal{R} \rightarrow \mathcal{R}$ such that

$$\forall j \in \mathbb{Z}_{\geq 0} \quad \forall (f, g) \in \mathcal{R}^2 \quad \phi(\mu_j(f, g)) = \mu'_j(\phi(f), \phi(g)). \quad (2.7)$$

Assume that μ_1 is skew-symmetric and μ_2 is symmetric. Formula (2.7) for $j = 0$ and $j = 1$ implies, in particular, that ϕ is an automorphism of the Poisson algebra (\mathcal{R}, μ_1) . We denote by \star and $\#$ the products on $\mathcal{R}[[\hbar]]$ respectively associated to the formal deformations $(\mu_j)_{j \in \mathbb{Z}_{\geq 0}}$ and $(\mu'_j)_{j \in \mathbb{Z}_{\geq 0}}$. The $\mathbb{C}[[\hbar]]$ -linear extension $\phi : \mathcal{R}[[\hbar]] \rightarrow \mathcal{R}[[\hbar]]$ satisfies

$$\forall (f, g) \in \mathcal{R}^2 \quad \phi(f \star g) = \phi(f) \# \phi(g).$$

2.2.3. *Connes & Moscovici's Theorem.* If F is a derivation on an algebra \mathcal{R} and $m \in \mathbb{Z}_{\geq 0}$, we define the m -th Pochhammer symbol of F by

$$F^{<m>} = \begin{cases} \text{Id} & \text{if } m = 0 \\ F^{<m-1>} \circ (F + (m-1)\text{Id}) & \text{otherwise} \end{cases}$$

where Id is the identity of \mathcal{R} .

The following Proposition is a special case of a Theorem due to Connes & Moscovici. See [CM04, eq. (1.5)].

Proposition 1— *Let V and W be two derivations on \mathcal{R} . Assume that $W \circ V - V \circ W = 2V$. For all $n \in \mathbb{Z}_{\geq 0}$, let $\chi_n: \mathcal{R} \times \mathcal{R} \rightarrow \mathcal{R}$ be defined by*

$$\chi_n(f, g) = \sum_{r=0}^n \frac{(-1)^r}{r!(n-r)!} \left(V^r \circ (W + r\text{Id})^{<n-r>} \right)(f) \cdot \left(V^{n-r} \circ (W + (n-r)\text{Id})^{<r>} \right)(g)$$

for all $(f, g) \in \mathcal{R}^2$. Then $(\chi_n)_{n \in \mathbb{Z}_{\geq 0}}$ is a formal deformation of \mathcal{R} .

In this work, the algebra \mathcal{R} is a double graded algebra over \mathbb{C} . The derivation V will have degree $(2, 0)$: if f belongs to a homogeneous component $\mathcal{R}_{k,p}$ then $V(f)$ lies in the homogeneous component $\mathcal{R}_{k+2,p}$. The derivation W will be a weighted Euler derivation: there exists a function $\kappa: \mathbb{Z}^2 \rightarrow \mathbb{C}$ such that, for all $(k, p) \in \mathbb{Z}^2$ and for all $f \in \mathcal{R}_{k,p}$ we have $W(f) = \kappa(k, p)f$. Since W is a derivation, κ is additive. There exists $(\lambda, \mu) \in \mathbb{C}^2$ such that $\kappa(k, p) = \lambda k + \mu p$. It follows that $W \circ V - V \circ W = 2\lambda V$ and hence $\lambda = 1$. Finally,

$$\forall (k, p) \in \mathbb{Z}^2, \quad \forall f \in \mathcal{R}_{k,p} \quad W(f) = (k + \mu p)f.$$

In this setting, for all $\mu \in \mathbb{C}$, we obtain from Proposition 1 a formal deformation $(\chi_n^{[\mu]})_{n \in \mathbb{Z}_{\geq 0}}$ defined on the homogeneous components by

$$\chi_n^{[\mu]}(f, g) = \sum_{r=0}^n (-1)^r \binom{k + \mu p + n - 1}{n-r} \binom{\ell + \mu q + n - 1}{r} V^r(f) V^{n-r}(g) \quad (2.8)$$

for all f in $\mathcal{R}_{k,p}$ and g in $\mathcal{R}_{\ell,q}$.

2.2.4. *Examples: Rankin-Cohen brackets on modular forms.*

We consider $\mathcal{R} = \mathcal{M}_*$.

The classical Rankin-Cohen brackets sequence $(\text{RC}_n)_{n \in \mathbb{Z}_{\geq 0}}$ is defined by:

$$\forall (f, g) \in \mathcal{M}_k \times \mathcal{M}_\ell \quad \text{RC}_n(f, g) = \sum_{i=0}^n (-1)^i \binom{k+n-1}{n-i} \binom{\ell+n-1}{i} D_\tau^{(i)}(f) D_\tau^{(n-i)}(g) \quad (2.9)$$

(see [CS17, §5.3.4]). Rankin-Cohen brackets satisfy

$$\forall n \in \mathbb{Z}_{\geq 0} \quad \text{RC}_n(\mathcal{M}_k, \mathcal{M}_\ell) \subset \mathcal{M}_{k+\ell+2n}.$$

The sequence $(\text{RC}_n)_{n \in \mathbb{Z}_{\geq 0}}$ is a formal deformation. The product on $\mathcal{M}_*[\hbar]$ defined by

$$\forall (f, g) \in \mathcal{M}_k \times \mathcal{M}_\ell \quad f \# g = \sum_{n \geq 0} \text{RC}_n(f, g) \hbar^n$$

corresponds to so called Elhozer product.

The complex derivation D_τ does not stabilize \mathcal{M}_* , this is not a derivation of \mathcal{M}_* . Serre derivation is the derivation of \mathcal{M}_* defined by

$$\forall f \in \mathcal{M}_k \quad \text{Se}(f) = D_\tau(f) - \frac{k}{12} E_2 f. \quad (2.10)$$

We replace D_τ in (2.9) by the derivation Se and obtain the Serre-Rankin-Cohen brackets:

$$\forall (f, g) \in \mathcal{M}_k \times \mathcal{M}_\ell \quad \text{SRC}_n(f, g) = \sum_{i=0}^n (-1)^i \binom{k+n-1}{n-i} \binom{\ell+n-1}{i} \text{Se}^i(f) \text{Se}^{n-i}(g) \quad (2.11)$$

for any $n \in \mathbb{Z}_{\geq 0}$. By application of Zagier's construction [Zag94, Page 67] or by Proposition 1, the sequence $(\text{SRC}_n)_{n \in \mathbb{Z}_{\geq 0}}$ is a formal deformation of \mathcal{M}_* . It also satisfies

$$\forall n \in \mathbb{Z}_{\geq 0} \quad \text{SRC}_n(\mathcal{M}_k, \mathcal{M}_\ell) \subset \mathcal{M}_{k+\ell+2n}.$$

Let us precise the relationship between the Serre-Rankin-Cohen brackets (2.11) and the usual Rankin-Cohen brackets (2.9). Using the values

$$\text{Se}(E_2) = -\frac{1}{12}(E_2^2 + E_4), \quad \text{Se}(E_4) = -\frac{1}{3}E_6, \quad \text{Se}(E_6) = -\frac{1}{2}E_4^2,$$

we express

$$\text{Se}^2(f) = D_\tau^2 f - \frac{k+1}{6} E_2 D_\tau f + \frac{k}{144} ((k+1)E_2^2 + E_4) f,$$

and by iteration

$$\forall f \in \mathcal{M}_k \quad \text{Se}^i(f) = D_\tau^i(f) + \sum_{j=0}^{i-1} F_{i,j}(k) D_\tau^j(f),$$

where $F_{i,j}(k)$ is a quasimodular forms of weight $2(i-j)$. We deduce that

$$\forall (f, g) \in \mathcal{M}_k \times \mathcal{M}_\ell \quad \text{SRC}_1(f, g) = \text{RC}_1(f, g)$$

and for instance

$$\forall (f, g) \in \mathcal{M}_k \times \mathcal{M}_\ell \quad \text{SRC}_2(f, g) = \text{RC}_2(f, g) + \frac{1}{288} k\ell(k+\ell+2)fgE_4.$$

2.2.5. Examples: formal deformations on quasimodular forms. The aim of the work in [DR14] was to build deformations of $\mathcal{R} = \mathcal{M}_*^{\leq \infty}$ having the shape of $(\text{RC}_n)_{n \in \mathbb{Z}_{\geq 0}}$, extending $(\text{SRC}_n)_{n \in \mathbb{Z}_{\geq 0}}$ and preserving the depth. Since we shall recover some of them, we recall the construction of two families of such extensions.

(1) For any $a \in \mathbb{C}$, let v_a be the derivation defined by

$$\begin{aligned} v_a(E_2) &= -\frac{1}{12}E_4 + 2aE_2^2 \\ v_a(E_4) &= -\frac{1}{3}E_6 + 4aE_4E_2, \quad v_a(E_6) = -\frac{1}{2}E_4^2 + 6aE_6E_2. \end{aligned} \quad (2.12)$$

We consider the brackets defined for any integer $n \geq 0$ by

$$\forall (f, g) \in \mathcal{M}_k^{\leq \infty} \times \mathcal{M}_\ell^{\leq \infty} \quad [f, g]_{v_a, n} = \sum_{r=0}^n (-1)^r \binom{k+n-1}{n-r} \binom{\ell+n-1}{r} v_a^r(f) v_a^{n-r}(g).$$

Then,

i) for all weights k and ℓ , we have

$$[\mathcal{M}_k^{\leq \infty}, \mathcal{M}_\ell^{\leq \infty}]_{v_a, n} \subset \mathcal{M}_{k+\ell+2n}^{\leq \infty}$$

ii) the sequence $([\ , \]_{v_a, n})_{n \in \mathbb{Z}_{\geq 0}}$ is a formal deformation of $\mathcal{M}_*^{\leq \infty}$

iii) for all weights k and ℓ , for all depths s and t , we have

$$[\mathcal{M}_k^{\leq s}, \mathcal{M}_\ell^{\leq t}]_{v_a, n} \subset \mathcal{M}_{k+\ell+2n}^{\leq s+t}$$

if and only if $a = 0$.

(2) For any complex numbers α and b , let $w_{\alpha, b}$ be the derivation on $\mathcal{M}_*^{\leq \infty}$ defined by

$$w_{\alpha, b}(E_2) = -3b\alpha E_2^2, \quad w_{\alpha, b}(E_4) = -\frac{1}{3}E_6 + 4bE_4E_2, \quad w_{\alpha, b}(E_6) = -\frac{1}{2}E_4^2 + 6bE_6E_2.$$

We consider the brackets defined for any integer $n \geq 0$ by

$$[f, g]_{w_{\alpha, b}, n}^\alpha = \sum_{r=0}^n (-1)^r \binom{k-(3\alpha+2)s+n-1}{n-r} \binom{\ell-(3\alpha+2)t+n-1}{r} w_{\alpha, b}^r(f) w_{\alpha, b}^{n-r}(g)$$

for any $f \in \mathcal{M}_{k-2s} E_2^s$ and $g \in \mathcal{M}_{\ell-2t} E_2^t$. Then,

i) for all weights k and ℓ , we have

$$[\mathcal{M}_k^{\leq \infty}, \mathcal{M}_\ell^{\leq \infty}]_{w_{\alpha, b}, n}^\alpha \subset \mathcal{M}_{k+\ell+2n}^{\leq \infty}$$

ii) the sequence $([\ , \]_{w_{\alpha, b}, n}^\alpha)_{n \in \mathbb{Z}_{\geq 0}}$ is a formal deformation of $\mathcal{M}_*^{\leq \infty}$

iii) for all weights k and ℓ , for all depths s and t , we have

$$[\mathcal{M}_k^{\leq s}, \mathcal{M}_\ell^{\leq t}]_{w_{\alpha, b}, n}^\alpha \subset \mathcal{M}_{k+\ell+2n}^{\leq s+t}$$

if and only if $b = 0$.

These results are proved in [DR14], Theorems B and D respectively.

2.3. A derivation on Jacobi weak forms. The aim of this part is to build a natural derivation on Jacobi forms that extends Serre derivation. Our construction has been influenced by a construction of some differential operator by Oberdieck in [Obe14] and hence we shall call this derivation the Oberdieck derivation (see also [DLM00, GK09, MTZ08]). References for the Weierstraß \wp and ζ functions are [Lan87, Ch. 18], [Sil94, Ch. 1] and [CS17, Ch. 2].

2.3.1. *Two intermediate functions.* For all $\tau \in \mathcal{H}$, let $\Lambda_\tau = \mathbb{Z} \oplus \tau\mathbb{Z}$. The ζ function associated to Λ_τ is defined by

$$\forall z \in \mathbb{C} - \Lambda_\tau \quad \zeta(\tau, z) = \frac{1}{z} + \sum_{\substack{\omega \in \Lambda_\tau \\ \omega \neq 0}} \left(\frac{1}{z - \omega} + \frac{1}{\omega} + \frac{z}{\omega^2} \right). \quad (2.13)$$

Sometimes, we shall use the notation $\zeta(\Lambda_\tau, z)$ instead of $\zeta(\tau, z)$. The function $z \mapsto \zeta(z, \tau)$ is meromorphic over \mathbb{C} . Its poles are the points of Λ_τ and they are simple.

We define J_1 by

$$\forall \tau \in \mathcal{H}, \forall z \in \mathbb{C} - \Lambda_\tau \quad J_1(\tau, z) = \frac{1}{2\pi i} \zeta(\tau, z) + \frac{\pi i}{6} z E_2(\tau).$$

To describe the transformation relations satisfied by J_1 , we define a function $X(M)$, for any $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$ by

$$\begin{aligned} X(M) : \mathcal{H} \times \mathbb{C} &\rightarrow \mathbb{C} \\ (\tau, z) &\mapsto \frac{cz}{c\tau + d}. \end{aligned}$$

It satisfies

$$\forall (M, N) \in \mathrm{SL}(2, \mathbb{Z})^2 \quad X(M)|_{1,0} N = X(MN) - X(N).$$

Lemma 2– *The function J_1 satisfies the following transformation properties:*

$$\begin{aligned} \forall (\lambda, \mu) \in \mathbb{Z}^2 \quad J_1|_0(\lambda, \mu) &= J_1 - \lambda \\ \forall M \in \mathrm{SL}(2, \mathbb{Z}) \quad J_1|_{1,0} M &= J_1 + X(M). \end{aligned}$$

The Fourier expansion of J_1 is

$$J_1(\tau, z) = -\frac{1}{2} + \frac{\xi}{\xi - 1} - \sum_{n=1}^{+\infty} \left(\sum_{d|n} (\xi^d - \xi^{-d}) \right) q^n$$

valid if $\xi \neq 1$ and $|q| < |\xi| < |q|^{-1}$.

Its Laurent expansion around 0 is

$$J_1(\tau, z) = \frac{1}{2\pi i z} - \frac{1}{2\pi i} \sum_{n=0}^{+\infty} G_{2n+2}(\tau) z^{2n+1}$$

valid for all $\tau \in \mathcal{H}$ and z in any punctured neighborhood of 0 containing no point of Λ_τ .

Proof. We prove the transformation property by the action of \mathbb{Z}^2 . We have

$$J_1(\tau, z + \lambda\tau + \mu) - J_1(\tau, z) = \frac{1}{2\pi i} (\zeta(\tau, z + \lambda\tau + \mu) - \zeta(\tau, z)) + \frac{\pi i}{6} (\lambda\tau + \mu) E_2(\tau).$$

Let η be the quasi-period map associated to Λ_τ . Then,

$$\zeta(\tau, z + \lambda\tau + \mu) - \zeta(\tau, z) = \eta(\lambda\tau + \mu).$$

The map η is a homomorphism of the group Λ_τ and hence

$$\eta(\lambda\tau + \mu) = \lambda\eta(\tau) + \mu\eta(1).$$

The Legendre relation implies that $\tau\eta(1) - \eta(\tau) = 2\pi i$ so that

$$\eta(\lambda\tau + \mu) = (\lambda\tau + \mu)\eta(1) - 2\pi i\lambda.$$

We have also

$$\eta(1) = -\frac{(2\pi i)^2}{12} E_2(\tau).$$

We deduce

$$\frac{1}{2\pi i} (\zeta(\tau, z + \lambda\tau + \mu) - \zeta(\tau, z)) = -\frac{\pi i}{6} (\lambda\tau + \mu) E_2(\tau) - \lambda$$

and

$$J_1(\tau, z + \lambda\tau + \mu) - J_1(\tau, z) = -\lambda.$$

We prove the transformation property by the action of $SL(2, \mathbb{Z})$. First, note that if $z \notin \Lambda_\tau$, then $\frac{z}{c\tau+d} \notin \Lambda_{M\tau}$. Let us show that it is sufficient to prove the result for $M \in \{S, T\}$. Let M and N be such that

$$J_1|_{1,0} M = J_1 + X(M) \quad \text{and} \quad J_1|_{1,0} N = J_1 + X(N).$$

Then,

$$\begin{aligned} J_1|_{1,0} MN &= (J_1|_{1,0} M)|_{1,0} N = (J_1 + X(M))|_{1,0} N = J_1 + X(N) + X(MN) - X(N) \\ &= J_1 + X(MN). \end{aligned}$$

The multiplicative group $SL(2, \mathbb{Z})$ is generated by

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

We deduce that if $J_1|_{1,0} S = J_1 + X(S)$ and $J_1|_{1,0} T = J_1$ then $J_1|_{1,0} M = J_1 + X(M)$ for all $M \in SL(2, \mathbb{Z})$.

Let us prove that $J_1|_{1,0} T = J_1$. We have

$$\begin{aligned} J_1(\tau + 1, z) &= \frac{1}{2\pi i} \zeta(\Lambda_{\tau+1}, z) + \frac{\pi i}{6} z E_2(\tau + 1) \\ &= \frac{1}{2\pi i} \zeta(\Lambda_\tau, z) + \frac{\pi i}{6} z E_2(\tau) = J_1(\tau, z) \end{aligned}$$

since $\Lambda_{\tau+1} = \Lambda_\tau$ and E_2 is periodic of period 1.

Finally, let us prove $J_1|_{1,0} S = J_1 + X(S)$. We have

$$J_1\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = \frac{1}{2\pi i} \zeta\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) + \frac{\pi i}{6} \frac{z}{\tau} E_2\left(-\frac{1}{\tau}\right).$$

We compute

$$\begin{aligned} \zeta\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) &= \zeta(\tau^{-1}\Lambda_\tau, \tau^{-1}z) \quad \text{since } \Lambda_{-1/\tau} = \tau^{-1}\Lambda_\tau \\ &= \tau \zeta(\Lambda_\tau, z) \quad \text{by homogeneity} \\ &= \tau \zeta(\tau, z) \end{aligned}$$

and recall that

$$\tau^{-2} E_2\left(-\frac{1}{\tau}\right) = E_2(\tau) + \frac{6}{\pi i} \frac{1}{\tau}.$$

Finally,

$$\tau^{-1} J_1 \left(-\frac{1}{\tau}, \frac{z}{\tau} \right) = \frac{1}{2\pi i} \zeta(z, \tau) + \frac{\pi i}{6} z E_2(\tau) + \frac{z}{\tau}$$

or, equivalently,

$$J_1|_{1,0} S = J_1 + X(S).$$

The Fourier expansion of J_1 is a consequence of the following expansion for ζ :

$$\frac{1}{2\pi i} \zeta(\tau, z) = \sum_{n \geq 1} \left(\frac{\xi^{-1}}{1 - \xi^{-1} q^n} - \frac{\xi}{1 - \xi q^n} \right) q^n - \frac{i\pi}{6} z E_2(\tau) - \frac{1}{2} - \frac{\xi}{1 - \xi}.$$

The Laurent expansion of J_1 is a consequence of the following expansion for ζ :

$$\zeta(\tau, z) = \frac{1}{z} - \sum_{k=1}^{+\infty} G_{2k+2}(\tau) z^{2k+1}.$$

□

We define the J_2 function by

$$J_2 = D_z J_1 - \frac{1}{12} E_2 + J_1^2.$$

Lemma 3– The function J_2 satisfies the following transformations properties:

$$\forall (\lambda, \mu) \in \mathbb{Z}^2 \quad J_2|_0(\lambda, \mu) = J_2 - 2\lambda J_1 + \lambda^2$$

$$\forall M \in \mathrm{SL}(2, \mathbb{Z}) \quad J_2|_{2,0} M = J_2 + 2J_1 X(M) + X(M)^2.$$

The Fourier expansion of J_2 is

$$J_2(\tau, z) = \frac{1}{6} - 2 \sum_{n=1}^{+\infty} \left(\sum_{d|n} \frac{n}{d} (\xi^d - \xi^{-d}) \right) q^n$$

valid if $|q| < |\xi| < |q|^{-1}$.

Its Laurent expansion around 0 is

$$J_2(\tau, z) = -\frac{2}{(2\pi i)^2} G_2(\tau) - \sum_{n=0}^{+\infty} \frac{1}{n+1} D_\tau(G_{2n+2})(\tau) z^{2n+2}$$

valid for all $\tau \in \mathcal{H}$ and z in any punctured neighborhood of 0 containing no point of Λ_τ .

Proof. To prove the transformation properties, we apply D_z to the transformation relations satisfied by J_1 and get

$$D_z(J_1)|_{2,0} M = D_z(J_1) + \frac{1}{2\pi i z} X(M)$$

and

$$D_z(J_1)|_1(\lambda, \mu) = D_z(J_1).$$

The relations for J_2 follow from these equalities and the definition.

From the definition of J_2 and the Laurent expansion of J_1 , we have

$$(2\pi i)^2 J_2(\tau, z) = -2 G_2(\tau) + \sum_{k \geq 0} \left[-(2k+5) G_{2k+4}(\tau) + \sum_{a+b=k} G_{2a+2}(\tau) G_{2b+2}(\tau) \right] z^{2k+2}.$$

The Laurent expansion of J_2 follows then from an equality due to Ramanujan (see [Sko93, Eq. (1)]).

As a corollary of the Laurent expansions of J_1 and J_2 , we have that $D_z(J_2) = 2D_\tau(J_1)$. We get from the Fourier expansion of J_1 the following

$$D_z(J_2)(\tau, z) = -2 \sum_{n \geq 1} n \sum_{d|n} (\xi^d - \xi^{-d}) q^n = -2 D_z \left(\sum_{n \geq 1} \sum_{d|n} \frac{n}{d} (\xi^d + \xi^{-d}) q^n \right).$$

We deduce that a function H exists such that

$$J_2(\tau, z) = -2 \sum_{n \geq 1} \sum_{d|n} \frac{n}{d} (\xi^d + \xi^{-d}) q^n + H(\tau).$$

We have

$$J_2(\tau, 0) = H(\tau) - 4 \sum_{n \geq 1} \sum_{d|n} \frac{n}{d} q^n = H(\tau) + \frac{1}{6} (E_2(\tau) - 1)$$

and hence

$$J_2(\tau, 0) = H(\tau) - \frac{1}{6} - \frac{2}{(2\pi i)^2} G_2(\tau).$$

The Laurent expansion of J_2 implies

$$J_2(\tau, 0) = -\frac{2}{(2\pi i)^2} G_2(\tau).$$

We deduce $H(\tau) = 1/6$. □

2.3.2. Oberdieck's derivation. Let $(k, p) \in 2\mathbb{Z} \times \mathbb{Z}_{\geq 0}$. For $f \in \widetilde{\mathcal{J}}_{k,p}$, let

$$\text{Ob}(f) = D_\tau(f) - \frac{k}{12} f E_2 - J_1 D_z(f) + p J_2 f.$$

Proposition 4– For $(k, p) \in 2\mathbb{Z} \times \mathbb{Z}_{\geq 0}$, the map Ob is linear from $\widetilde{\mathcal{J}}_{k,p}$ to $\widetilde{\mathcal{J}}_{k+2,p}$. Moreover, if $(\ell, q) \in 2\mathbb{Z} \times \mathbb{Z}_{\geq 0}$ and $(f, g) \in \widetilde{\mathcal{J}}_{k,p} \times \widetilde{\mathcal{J}}_{\ell,q}$ then

$$\text{Ob}(fg) = \text{Ob}(f)g + f \text{Ob}(g).$$

Remark 5– This proposition shows that, after extension by linearity, Ob is a derivation on $\widetilde{\mathcal{J}}_{\text{ev},*}$. Since $\widetilde{\mathcal{J}}_{0,1} = \mathbb{C}B$ and $\widetilde{\mathcal{J}}_{2,1} = \mathbb{C}E_4A$, the comparison of the Fourier expansions implies that Ob is characterized by its following values on the generators:

$$\text{Ob}(E_4) = -\frac{1}{3} E_6, \quad \text{Ob}(E_6) = -\frac{1}{2} E_4^2, \quad \text{Ob}(A) = -\frac{1}{6} B, \quad \text{Ob}(B) = -\frac{1}{3} E_4 A.$$

The restriction of Ob to the algebra of modular forms is the Serre derivative.

Proof. The computation of $\text{Ob}(fg)$ is left to the reader. Let $f \in \widetilde{\mathcal{J}}_{k,p}$ and $M \in \text{SL}(2, \mathbb{Z})$. We have

$$D_\tau(f|_{k,p} M) = \left(p X(M)^2 - \frac{k}{2\pi i z} X(M) \right) f|_{k,p} M - X(M) (D_z(f)|_{k+1,p} M) + D_\tau(f)|_{k+2,p} M$$

and

$$D_z(f|_{k,p} M) = -2p X(M) (f|_{k,p} M) + D_z(f)|_{k+1,p} M.$$

Since $f|_{k,p} M = f$ we deduce

$$D_\tau(f)|_{k+2,p} M = D_\tau(f) + \left(\frac{k}{2\pi i z} X(M) - p X(M)^2 \right) f + X(M) (D_z(f)|_{k+1,p} M)$$

and

$$D_z(f)|_{k+1,p} M = D_z(f) + 2p X(M) f.$$

In particular,

$$D_\tau(f)|_{k+2,p} M = D_\tau(f) + \left(D_z(f) + \frac{k}{2\pi i z} f \right) X(M) + p f X(M)^2. \quad (2.14)$$

From,

$$(J_1 D_z(f))|_{k+2,p} M = (J_1|_{1,0} M) (D_z(f)|_{k+1,p} M)$$

we get

$$(J_1 D_z(f))|_{k+2,p} M = J_1 D_z(f) + (D_z(f) + 2p J_1 f) X(M) + 2p f X(M)^2. \quad (2.15)$$

Similarly,

$$\left(-\frac{k}{12} E_2 f \right)|_{k+2,p} M = -\frac{k}{12} E_2 f - \frac{k}{2\pi i z} f X(M) \quad (2.16)$$

and

$$(p J_2 f)|_{k+2,p} M = p J_2 f + 2p J_1 f X(M) + p f X(M)^2. \quad (2.17)$$

Equations (2.14), (2.15), (2.16) and (2.17) lead to

$$\text{Ob}(f)|_{k+2,p} M = f.$$

Let $(\lambda, \mu) \in \mathbb{Z}^2$. Then

$$D_z(f)|_p(\lambda, \mu) = D_z(f) - 2p f \lambda$$

and

$$D_\tau(f)|_p(\lambda, \mu) = D_\tau(f) - D_z(f) \lambda + p f \lambda^2 \quad (2.18)$$

and so

$$(-J_1 D_z(f))|_p(\lambda, \mu) = -J_1 D_z(f) + (D_z(f) + 2p f J_1) \lambda - 2p f \lambda^2.$$

We also have

$$(p J_2 f)|_p(\lambda, \mu) = p J_2 f - 2p J_1 f \lambda + p f \lambda^2. \quad (2.19)$$

Equations (2.18)–(2.19) lead to

$$\text{Ob}(f)|_p(\lambda, \mu) = f.$$

Finally, let $\tau \in \mathcal{H}$. We prove that $\text{Ob}_\tau : z \mapsto \text{Ob}(f)(\tau, z)$ is holomorphic. By invariance by the action of \mathbb{Z}^2 , it is sufficient to prove that Ob_τ has no pole in $\mathcal{F}_\tau = \{a + b\tau : (a, b) \in [0, 1]^2\}$. The invariance of f by the action of $\text{SL}(2, \mathbb{Z})$ implies that the Laurent expansion of f around 0 is

$$f(\tau, z) = \sum_{\nu=0}^{+\infty} Q_{2\nu}(\tau) z^{2\nu}$$

where $Q_{2\nu}$ is a quasimodular form of weight $k + 2\nu$ and depth less than or equal to ν (see [Roy12], [MR05] or [Zag08]). The lack of odd powers in z is a consequence of the

non existence of odd weight quasimodular form. The only pole of ζ in \mathcal{F}_τ is 0 and so J_1 has no other pole than 0 in \mathcal{F}_τ . The Laurent expansion of J_1 implies that the Laurent expansion of $J_1 D_z f$ around $z = 0$ is bounded and hence $J_1 D_z f$ has no pole in \mathcal{F}_τ . The function J_2 has no other pole in \mathcal{F}_τ than 0 as it can be seen from its definition. The Laurent expansion of J_2 implies than 0 is not a pole. Finally, Ob_τ is holomorphic. \square

2.3.3. Oberdieck-Rankin-Cohen brackets. From Oberdieck's derivation we build a sequence $(\text{ORC}_n)_{n \in \mathbb{Z}_{\geq 0}}$, called Oberdieck-Rankin-Cohen brackets, which is a formal deformation of the algebra $\widetilde{\mathcal{T}}_{\text{ev},*}$.

For any $\mu \in \mathbb{C}$, the general method described § 2.2.3 provides a formal deformation $(\text{ORC}_n^\mu)_{n \in \mathbb{Z}_{\geq 0}}$ of $\widetilde{\mathcal{T}}_{\text{ev},*}$. We take $V = \text{Ob}$ and get

$$\forall (f, g) \in \widetilde{\mathcal{T}}_{k,p} \times \widetilde{\mathcal{T}}_{\ell,q}$$

$$\text{ORC}_n^\mu(f, g) = \sum_{r=0}^n (-1)^r \binom{k + \mu p + n - 1}{n - r} \binom{\ell + \mu q + n - 1}{r} \text{Ob}^r(f) \text{Ob}^{n-r}(g)$$

for all $(k, \ell, p, q) \in (2\mathbb{Z})^2 \times \mathbb{Z}_{\geq 0}^2$. The bracket ORC_1^μ gives $\widetilde{\mathcal{T}}_{\text{ev},*}$ the structure of a Poisson algebra. Since it is a Poisson bracket, it is characterized by its values on the generators

$$\begin{aligned} \text{ORC}_1^\mu(E_4, E_6) &= 2(E_6^2 - E_4^3) \\ \text{ORC}_1^\mu(E_4, A) &= -\frac{2}{3}E_4 B + \frac{\mu - 2}{3}E_6 A & \text{ORC}_1^\mu(E_4, B) &= \frac{\mu}{3}E_6 B - \frac{4}{3}E_4^2 A \\ \text{ORC}_1^\mu(E_6, A) &= -E_6 B + \frac{\mu - 2}{2}E_4^2 A & \text{ORC}_1^\mu(E_6, B) &= \frac{\mu}{2}E_4^2 B - 2E_4 E_6 A \\ \text{ORC}_1^\mu(A, B) &= \frac{\mu}{6}B^2 + \frac{2 - \mu}{3}E_4 A^2. \end{aligned}$$

The restriction of $(\text{ORC}_n^\mu)_{n \in \mathbb{Z}_{\geq 0}}$ to the algebra of modular forms is $(\text{SRC}_n)_{n \in \mathbb{Z}_{\geq 0}}$ defined in (2.11).

3. FORMAL DEFORMATIONS FOR JACOBI FORMS

The aim of this section is to construct a family of Rankin-Cohen brackets that generalizes the brackets built from Oberdieck's derivation. The method is purely algebraic. It begins with the determination of all possible first brackets (Poisson brackets) that enter our level of specialization (*i.e.* that comes from arithmetical consideration). We shall find seven families of Poisson brackets. We prove that only one can be extended, with our method, to Rankin-Cohen brackets.

3.1. Admissible Poisson brackets on weak Jacobi forms.

3.1.1. Determination of admissible Poisson brackets.

Definition 6— A Poisson bracket $\{\cdot, \cdot\}$ on $\widetilde{\mathcal{T}}_{\text{ev},*}$ is admissible if

$$(1) \quad \forall (f, g) \in \mathcal{M}_*^2 \quad \{f, g\} = \text{RC}_1(f, g)$$

(2)

$$\forall (k, \ell, p, q) \in 2\mathbb{Z}_{\geq 0} \times \mathbb{Z} \times 2\mathbb{Z}_{\geq 0} \times \mathbb{Z} \quad \{\tilde{\mathcal{J}}_{k,p}, \tilde{\mathcal{J}}_{\ell,q}\} \subset \tilde{\mathcal{J}}_{k+\ell+2,p+q}.$$

A Poisson bracket $\{\cdot, \cdot\}$ is admissible if and only if

$$(C1) \quad \{E_4, E_6\} = RC_1(E_4, E_6) = -2E_4^3 + 2E_6^2,$$

(C2) There exist two linear maps σ_1, δ_1 on \mathcal{M}_* such that

$$\forall k \in 2\mathbb{Z}_{\geq 0} \quad \sigma_1(\mathcal{M}_k) \subset \mathcal{M}_{k+2} \text{ and } \delta_1(\mathcal{M}_k) \subset \mathcal{M}_k,$$

$$\forall f \in \mathcal{M}_* \quad \{A, f\} = \sigma_1(f)A + \delta_1(f)B,$$

(C3) There exist two linear maps σ_2, δ_2 on \mathcal{M}_* such that

$$\forall k \in 2\mathbb{Z}_{\geq 0} \quad \sigma_2(\mathcal{M}_k) \subset \mathcal{M}_{k+4} \text{ and } \delta_2(\mathcal{M}_k) \subset \mathcal{M}_{k+2},$$

$$\forall f \in \mathcal{M}_* \quad \{B, f\} = \sigma_2(f)A + \delta_2(f)B,$$

(C4) $\{A, B\} = \xi E_4 A^2 + \eta B^2$, with $\xi, \eta \in \mathbb{C}$.

Proposition 7— The admissible Poisson brackets on $\tilde{\mathcal{J}}_{ev,*}$ are defined by the following values on the generators:

$$\begin{aligned} \{E_4, E_6\} &= -2E_4^3 + 2E_6^2 \\ \{A, E_4\} &= \alpha E_6 A + \gamma E_4 B & \{A, E_6\} &= \beta E_4^2 A + \delta E_6 B \\ \{B, E_4\} &= \lambda E_4^2 A + \varepsilon E_6 B & \{B, E_6\} &= \mu E_4 E_6 A + \theta E_4^2 B \\ \{A, B\} &= \xi E_4 A^2 + \eta B^2 \end{aligned} \quad (3.1)$$

where the ten complex parameters $\alpha, \beta, \gamma, \delta, \lambda, \mu, \theta, \varepsilon, \xi, \eta$ belong to one of the following families:

	α	β	γ	δ	λ	μ	θ	ε	ξ	η
\mathcal{A}	ε	$\frac{3}{2}\varepsilon + 1$	$\gamma \neq 0$	γ	$\frac{4}{\gamma}$	$\frac{8}{\gamma}$	$\frac{3}{2}\varepsilon - 1$	ε	$\frac{4}{\gamma}\varepsilon$	$(-\frac{3}{4}\varepsilon + \frac{1}{2})\gamma$
\mathcal{B}	$\varepsilon + \frac{2}{3}$	$\frac{3}{2}\varepsilon + 1$	γ	$\frac{3}{2}\gamma$	λ	$\frac{3}{2}\lambda$	$\frac{3}{2}\varepsilon$	ε	$(\frac{3}{4}\varepsilon + \frac{1}{2})\lambda$	$-\frac{3}{4}\varepsilon\gamma$
\mathcal{C}_1	4	6	$\gamma \neq 0$	$-\gamma$	0	0	0	0	0	0
\mathcal{C}_2	0	0	0	0	$\lambda \neq 0$	-2λ	6	4	0	0
\mathcal{D}	ε	$\frac{3}{2}\varepsilon$	0	0	0	0	$\frac{3}{2}\varepsilon$	ε	0	η
\mathcal{E}	$\alpha \neq \varepsilon + \frac{2}{3}$	$\frac{3}{2}\alpha$	0	0	0	0	$\frac{3}{2}\varepsilon$	ε	0	0

Proof. Sei-Qwon Oh [Oh06] described a method to extend the Poisson structure of a Poisson algebra \mathcal{R} to the algebra of polynomials in one variable with coefficients in \mathcal{R} . Our proof rests on a generalization of this method.

An admissible Poisson bracket on $\mathcal{M}_*[A, B]$ extending the Rankin-Cohen bracket RC_1 on \mathcal{M}_* has the particular form:

$$\forall f \in \mathcal{M}_* \quad \{A, f\} = \sigma_1(f)A + \delta_1(f)B, \quad \{B, f\} = \sigma_2(f)A + \delta_2(f)B$$

and

$$\{A, B\} = pA^2 + qB^2, \text{ for } p = \xi E_4 \text{ and } q = \eta \text{ where } \xi, \eta \text{ are fixed complex numbers.}$$

The extended bracket is bilinear and skewsymmetric if and only if the four maps $\sigma_1, \sigma_2, \delta_1, \delta_2: \mathcal{M}_* \rightarrow \mathcal{M}_*$ are linear. The extended bracket is a bi-derivation (*i.e.* satisfies Leibniz relation with respect to each variable) if and only if the four linear maps $\sigma_1, \sigma_2, \delta_1, \delta_2$ are derivations of \mathcal{M}_* . The extended bracket satisfies Jacobi condition if and only if $\sigma_1, \sigma_2, \delta_1, \delta_2$ satisfy:

$$\begin{aligned} \{A, \{f, g\}\} + \{f, \{g, A\}\} + \{g, \{A, f\}\} &= 0 \\ \{B, \{f, g\}\} + \{f, \{g, B\}\} + \{g, \{B, f\}\} &= 0 \\ \{f, \{A, B\}\} + \{A, \{B, f\}\} + \{B, \{f, A\}\} &= 0 \end{aligned} \quad (3.2)$$

for all $(f, g) \in \mathcal{M}_*^2$. The first and second relations respectively translate into

$$\sigma_1(\{f, g\}) = \{f, \sigma_1(g)\} + \{\sigma_1(f), g\} + \delta_1(f)\sigma_2(g) - \sigma_2(f)\delta_1(g) \quad (3.3)$$

$$\delta_1(\{f, g\}) = \{f, \delta_1(g)\} + \{\delta_1(f), g\} + \delta_1(f)\delta_2(g) - \delta_2(f)\delta_1(g) + \sigma_1(f)\delta_1(g) - \delta_1(f)\sigma_1(g)$$

$$\sigma_2(\{f, g\}) = \{f, \sigma_2(g)\} + \{\sigma_2(f), g\} + \sigma_2(f)\sigma_1(g) - \sigma_1(f)\sigma_2(g) + \delta_2(f)\sigma_2(g) - \sigma_2(f)\delta_2(g)$$

$$\delta_2(\{f, g\}) = \{f, \delta_2(g)\} + \{\delta_2(f), g\} + \sigma_2(f)\delta_1(g) - \delta_1(f)\sigma_2(g) \quad (3.4)$$

for all $(f, g) \in \mathcal{M}_*^2$, where $\{f, g\} = \text{RC}_1(f, g)$. The third relation in (3.2) translates into

$$\begin{aligned} \sigma_1\sigma_2 - \sigma_2\sigma_1 &= p\sigma_1 - p\delta_2 + \{p, \cdot\} \\ \delta_1\delta_2 - \delta_2\delta_1 &= -q\sigma_1 + q\delta_2 + \{q, \cdot\} \\ \sigma_1\delta_2 - \delta_2\sigma_1 + \delta_1\sigma_2 - \sigma_2\delta_1 &= 2p\delta_1 + 2q\sigma_2. \end{aligned} \quad (3.5)$$

The four derivations $\sigma_1, \sigma_2, \delta_1, \delta_2$ of \mathcal{M}_* are defined by their values on the generators E_4, E_6 . The assumptions on the weight in conditions (C2) and (C3) of the definition of an admissible Poisson bracket imply that

$$\begin{aligned} \sigma_1(E_4) &= \alpha E_6, & \sigma_2(E_4) &= \lambda E_4^2, & \delta_1(E_4) &= \gamma E_4, & \delta_2(E_4) &= \varepsilon E_6, \\ \sigma_1(E_6) &= \beta E_4^2, & \sigma_2(E_6) &= \mu E_4 E_6, & \delta_1(E_6) &= \delta E_6, & \delta_2(E_6) &= \theta E_4^2. \end{aligned}$$

for some $\alpha, \beta, \gamma, \delta, \lambda, \mu, \varepsilon, \theta$ in \mathbb{C} .

Then applying the identities (3.3), (3.4), (3.5) to E_4 and E_6 respectively, we obtain the following algebraic relations between the ten complex numbers $\alpha, \beta, \gamma, \delta, \lambda, \mu, \varepsilon, \theta, \xi, \eta$

- relations (3.3) are equivalent to

$$\mu\gamma - \lambda\delta = 4\beta - 6\alpha, \quad \gamma\theta - \beta\gamma = 2\delta - 4\gamma, \quad \alpha\delta - \delta\varepsilon = 2\delta - 2\gamma,$$

- relations (3.4) are equivalent to

$$\beta\lambda - \lambda\theta = 2\mu - 2\lambda, \quad \mu\varepsilon - \mu\alpha = 2\mu - 4\lambda, \quad \mu\gamma - \lambda\delta = 6\varepsilon - 4\theta,$$

- the first relation of (3.5) is equivalent to

$$\alpha(2\lambda - \mu) = \xi(\alpha - \varepsilon), \quad \xi\beta - \xi\theta - 2\xi = \mu\beta - 2\lambda\beta, \quad \alpha\mu = 2\xi,$$

- the second relation of (3.5) is equivalent to

$$\varepsilon(\delta - \gamma) = \eta(\varepsilon - \alpha), \quad 2\gamma\theta - \theta\delta = \eta(\theta - \beta),$$

- the third relation of (3.5) is equivalent to

$$\varepsilon\beta - \alpha\theta + \lambda\gamma = 2\xi\gamma + 2\eta\lambda, \quad 2\alpha\theta - 2\beta\varepsilon + \mu\gamma = 2\xi\delta + 2\eta\mu.$$

When the four parameters $\gamma, \delta, \lambda, \mu$ are nonzero, we deduce easily from the above relations that the quotients $s = \delta/\gamma$ and $t = \mu/\lambda$ satisfy $2s - 4 = 2 - 2t$ and $2 - \frac{2}{s} = \frac{4}{t} - 2$. This implies $s + t = 3$ and $(2s - 3)(s - 1) = 0$. Then, either $\delta = \gamma$ and $\mu = 2\lambda$ (this is case *A*), or $2\delta = 3\gamma$ and $2\mu = 3\lambda$ (this is case *B*). Straightforward calculations lead to the calculation of others parameters and to other cases of the above table. \square

3.1.2. Admissible Poisson brackets having the shape of a Rankin-Cohen bracket.

Definition 8—A derivation d of $\tilde{\mathcal{J}}_{\text{ev},*}$ is admissible if d preserves the index and increases the weight by two.

Our goal in this part is to obtain a differential expression of the admissible brackets on $\tilde{\mathcal{J}}_{\text{ev},*}$ similar to the one of first usual Rankin-Cohen bracket. More precisely we find, when this is possible, an admissible derivation d of $\tilde{\mathcal{J}}_{\text{ev},*}$ such that $\{f, g\} = \kappa(f)fd(g) - \kappa(g)gd(f)$ for any f and g in homogeneous components of $\tilde{\mathcal{J}}_{\text{ev},*}$, where $\kappa(f)$ is some scalar depending only of the weight k and the index p of f . Therefore we denote $\kappa(k, p)$ instead of $\kappa(f)$. Since the bracket is a biderivation, κ must be additive: there exists complex numbers u and v such that $\kappa(k, p) = uk + vp$ for any $f \in \tilde{\mathcal{J}}_{k,p}$.

Remark 9—We have

$$\forall (k, p) \in 2\mathbb{Z} \times \mathbb{Z}_{\geq 0} \quad \forall f \in \tilde{\mathcal{J}}_{k,p} \quad \kappa(f) = \frac{\kappa(4, 0)}{4}k + \kappa(0, 1)p.$$

Proposition 10—Let $\{\cdot, \cdot\}$ be an admissible Poisson bracket on $\tilde{\mathcal{J}}_{\text{ev},*}$. The two following assertions are equivalent.

- 1) There exist a nonzero admissible derivation d of $\tilde{\mathcal{J}}_{\text{ev},*}$ and two complex numbers u and v such that

$$\forall (f, g) \in \tilde{\mathcal{J}}_{k,p} \times \tilde{\mathcal{J}}_{\ell,q} \quad \{f, g\} = \kappa(f)fd(g) - \kappa(g)gd(f),$$

where κ is defined by

$$\forall f \in \tilde{\mathcal{J}}_{k,p} \quad \kappa(f) = \kappa(k, p) = uk + vp.$$

- 2) The bracket $\{\cdot, \cdot\}$ is the admissible bracket corresponding to the case *B* of the classification in Proposition 7 (depending on three complex parameters $\gamma, \lambda, \varepsilon$), with a function κ defined by

$$\kappa(k, p) = u(k - 3\varepsilon p) \quad \text{where } u \neq 0 \text{ is an arbitrary complex parameter} \quad (3.6)$$

and a derivation d defined by

$$d(E_4) = -\frac{1}{3u}E_6, \quad d(E_6) = -\frac{1}{2u}E_4^2, \quad d(A) = -\frac{\gamma}{4u}B, \quad d(B) = -\frac{\lambda}{4u}E_4A. \quad (3.7)$$

Remark 11—If we want to emphasize on the parameters for the bracket described in 2), we shall note $\{\cdot, \cdot\} = \{\cdot, \cdot\}_{(u; \gamma, \lambda, \varepsilon)}$.

Proof. It is clear that 2) implies 1). Assume 1) is satisfied. A Poisson bracket on a finitely generated algebra is characterized by its values on the generators. Moreover, using Remark 9, we know that, if u and v are defined by $u = \kappa(4, 0)/4$ and $v = \kappa(0, 1)$,

then $\kappa(6, 0) = 6u$ and $\kappa(-2, 1) = -2u + v$. Moreover, d is admissible: there exists complex numbers x, y, z and t such that

$$d(E_4) = xE_6, \quad d(E_6) = yE_4^2, \quad d(A) = zB, \quad d(B) = tE_4A. \quad (3.8)$$

We write all the values of the bracket on the generators and compare with (3.1): the complex number u is necessarily non zero and

$$x = -\frac{1}{3u}, \quad y = -\frac{1}{2u}, \quad z = -\frac{\gamma}{4u}, \quad t = -\frac{\lambda}{4u}.$$

Moreover, $v = \varepsilon/x = -3\varepsilon u$. We deduce that κ is given by (3.6) and that the values of d on the generators are given by (3.7). It remains to prove that $\{\cdot, \cdot\}$ belongs to the family \mathcal{B} . We use $\alpha = (-2u + v)x$ and $\varepsilon = vx$ to get $\alpha = \varepsilon + 2/3$. From $\beta = (-2u + v)y$ and $\varepsilon = vx$ we deduce $\beta = 1 + 3\varepsilon/2$. Then, $\delta = -6uz$ leads to $\delta = 3\gamma/2$. From $\mu = -6ut$, we get $\mu = 2\lambda/2$ and from $\theta = vy$, we have $\theta = 3\varepsilon/2$. Finally, $\xi = (-2u + v)t$ and $\eta = -vz$ lead respectively to $\xi = (3\varepsilon/2)\lambda/4$ and $\eta = 3\varepsilon\gamma/4$. We end the proof in computing $\kappa(f)fdg - \kappa(g)gdf$ for f and g in $\{E_4, E_6, A, B\}$ and obtaining each time $\{f, g\}$. \square

3.2. A family of formal deformations for Jacobi forms.

3.2.1. Construction. We recall that the Serre derivation Se is the restriction of the Oberdieck derivation to the algebra of modular forms. We generalize Oberdieck's derivation in defining an admissible derivation on Jacobi forms $\text{Se}_{a,b}$ for any complex numbers a and b by

$$\text{Se}_{a,b}(E_4) = -\frac{1}{3}E_6, \quad \text{Se}_{a,b}(E_6) = -\frac{1}{2}E_4^2, \quad \text{Se}_{a,b}(A) = aB, \quad \text{Se}_{a,b}(B) = bE_4A. \quad (3.9)$$

We have $\text{Ob} = \text{Se}_{-1/6, -1/3}$. Moreover, for any (a, b) , Se is still the restriction of $\text{Se}_{a,b}$ to the algebra of modular forms.

For all $(a, b, c) \in \mathbb{C}^2$, for any $n \in \mathbb{Z}_{\geq 0}$, let $\{\cdot, \cdot\}_n^{[a,b,c]}$ be the bilinear map from $\widetilde{\mathcal{J}}_{\text{ev},*} \times \widetilde{\mathcal{J}}_{\text{ev},*}$ to $\widetilde{\mathcal{J}}_{\text{ev},*}$ defined by bilinear extension of

$$\{f, g\}_n^{[a,b,c]} = \sum_{r=0}^n (-1)^r \binom{k+cp+n-1}{n-r} \binom{\ell+cq+n-1}{r} \text{Se}_{a,b}^r(f) \text{Se}_{a,b}^{n-r}(g) \quad (3.10)$$

for all homogeneous $f \in \widetilde{\mathcal{J}}_{k,p}$ and $g \in \widetilde{\mathcal{J}}_{\ell,q}$.

Let $\{\cdot, \cdot\} = \{\cdot, \cdot\}_{(u;\gamma,\lambda,\varepsilon)}$ be a Poisson bracket as in Proposition 10. Then,

$$\{\cdot, \cdot\}_{(u;\gamma,\lambda,\varepsilon)} = \{\cdot, \cdot\}_1^{[-\gamma/(4u), -\lambda/(4u), -3\varepsilon]}. \quad (3.11)$$

Reciprocally, for any $(a, b, c) \in \mathbb{C}^3$, we have

$$\{\cdot, \cdot\}_1^{[a,b,c]} = \{\cdot, \cdot\}_{(1;-4a, -4b, -c/3)}.$$

Remark 12- The subalgebra of modular forms \mathcal{M}_* is stable by $\{\cdot, \cdot\}_n^{[a,b,c]}$ and that its restriction to \mathcal{M}_* is SRC_n . Note also that we have $\text{ORC}_n^\mu = \{\cdot, \cdot\}_n^{[-1/6, -1/3, \mu]}$.

Theorem 13— For all $(a, b, c) \in \mathbb{C}^3$, the sequence $\left(\{\cdot, \cdot\}_n^{[a, b, c]}\right)_{n \in \mathbb{Z}_{\geq 0}}$ is a formal deformation of $\tilde{\mathcal{T}}_{\text{ev},*}$ that satisfies

$$\{\tilde{\mathcal{T}}_{k,p}, \tilde{\mathcal{T}}_{\ell,q}\}_n^{[a, b, c]} \subset \tilde{\mathcal{T}}_{k+\ell+2n, p+q}$$

for all $(k, p, \ell, q, n) \in 2\mathbb{Z} \times \mathbb{Z}_{\geq 0} \times 2\mathbb{Z} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$.

Proof. The derivation $\text{Se}_{a,b}$ is clearly of degree $(2, 0)$. The Theorem is then a consequence of (2.8) since $\{\cdot, \cdot\}_n^{[a, b, c]} = \chi_n^{[c]}$ for all $n \in \mathbb{Z}_{\geq 0}$. \square

3.2.2. Classification. The definition of formal deformations $(\{\cdot, \cdot\}_n^{[a, b, c]})_{n \in \mathbb{Z}_{\geq 0}}$ depends on three parameters. Can we classify them up to isomorphism? The question can be considered at different levels of specialization of the definition of isomorphic formal deformations with respect to the arithmetical context studied here. We give here a complete answer for the following notion of isomorphism.

Definition 14— Two formal deformations $\left(\{\cdot, \cdot\}_n^{[a, b, c]}\right)_{n \in \mathbb{Z}_{\geq 0}}$ and $\left(\{\cdot, \cdot\}_n^{[a', b', c']}\right)_{n \in \mathbb{Z}_{\geq 0}}$ of $\tilde{\mathcal{T}}_{\text{ev},*}$ are modular-isomorphic if there exists a \mathbb{C} -linear bijective map $\phi: \tilde{\mathcal{T}}_{\text{ev},*} \rightarrow \tilde{\mathcal{T}}_{\text{ev},*}$ such that

- (1) ϕ preserves the index and the weight of homogeneous Jacobi forms
- (2) $\phi(\{f, g\}_j^{[a, b, c]}) = \{\phi(f), \phi(g)\}_j^{[a', b', c']}$ for all $j \in \mathbb{Z}_{\geq 0}$ and $f, g \in \tilde{\mathcal{T}}_{\text{ev},*}$.

In particular ϕ is an \mathbb{C} -algebra automorphism of $\tilde{\mathcal{T}}_{\text{ev},*}$ and a Poisson isomorphism from $(\tilde{\mathcal{T}}_{\text{ev},*}, \{\cdot, \cdot\}_1^{[a, b, c]})$ to $(\tilde{\mathcal{T}}_{\text{ev},*}, \{\cdot, \cdot\}_1^{[a', b', c']})$.

Lemma 15— If two formal deformations $(\{\cdot, \cdot\}_n^{[a, b, c]})_{n \in \mathbb{Z}_{\geq 0}}$ and $(\{\cdot, \cdot\}_n^{[a', b', c']})_{n \in \mathbb{Z}_{\geq 0}}$ are modular-isomorphic, then $c = c'$, and there exists $\xi \in \mathbb{C}^*$ such that $a' = \xi a$ and $b' = \xi^{-1} b$.

Proof. Let $\phi: \tilde{\mathcal{T}}_{\text{ev},*} \rightarrow \tilde{\mathcal{T}}$ be as in Definition 14. By (1), there exists $(\alpha, \beta) \in \mathbb{C}^{*2}$ such that $\phi(E_4) = \alpha E_4$ and $\phi(E_6) = \beta E_6$. By (2), we know that $\phi\left(\{E_4, E_6\}_1^{[a, b, c]}\right) = \alpha\beta\{E_4, E_6\}_1^{[a', b', c']}$, i.e. $-2\alpha^3 E_4^3 + 2\beta^2 E_6^2 = -2\alpha\beta E_4^3 + 2\alpha\beta E_6^2$. We deduce that $\alpha = \beta = 1$: the restriction of ϕ to \mathcal{M}_* is the identity.

Let $f \in \tilde{\mathcal{T}}_{k,p}$. Then $\phi(f) \in \tilde{\mathcal{T}}_{k,p}$. The restriction of $\text{Se}_{a,b}$ to \mathcal{M}_* is Se . The kernel of Se is $\mathbb{C}[\Delta]$ (see, for example, [DR14, Proposition 8]) and $\phi(\Delta) = \Delta$. We deduce that

$$\phi\left(\{f, \Delta\}_1^{[a, b, c]}\right) = -12\Delta\phi(\text{Se}_{a,b}(f)) \quad \text{and} \quad \{\phi(f), \phi(g)\}_1^{[a', b', c']} = -12\Delta\text{Se}_{a', b'}(\phi(f))$$

and hence

$$\phi \circ \text{Se}_{a,b} = \text{Se}_{a', b'} \circ \phi. \tag{3.12}$$

It follows that, for all $f \in \tilde{\mathcal{T}}_{k,p}$ and $g \in \tilde{\mathcal{T}}_{\ell,q}$, we have

$$\{\phi(f), \phi(g)\}_j^{[a', b', c']} = \phi((k + c'p)f \text{Se}_{a,b}(g) - (\ell + c'q)g \text{Se}_{a,b}(f))$$

and (2) leads to

$$(c' - c)(pf \text{Se}_{a,b}(g) - qg \text{Se}_{a,b}(f)) = 0.$$

We apply this equality to $f = A E_4$ and $g = E_6$ to obtain $c' = c$. Moreover, (3.12) applied to A gives $a\mu = a'\lambda$ and (3.12) applied to B gives $b'\mu = b\lambda$. We obtain $a' = \xi a$ and $b' = \xi^{-1}b$ with $\xi = \mu/\lambda$. \square

Theorem 16— Let $(a', b', c') \in \mathbb{C}^3$. The formal deformation $(\{\cdot, \cdot\}_n^{[a', b', c']})_{n \in \mathbb{Z}_{\geq 0}}$ of $\widetilde{\mathcal{T}}_{\text{ev},*}$ is modular-isomorphic to one of the following formal deformations:

- 1) the formal deformation $(\{\cdot, \cdot\}_n^{[1, b, c]})_{n \in \mathbb{Z}_{\geq 0}}$ for some $(b, c) \in \mathbb{C}^2$
- 2) the formal deformation $(\{\cdot, \cdot\}_n^{[0, 1, c]})_{n \in \mathbb{Z}_{\geq 0}}$ for some $c \in \mathbb{C}$
- 3) the formal deformation $(\{\cdot, \cdot\}_n^{[0, 0, c]})_{n \in \mathbb{Z}_{\geq 0}}$ for some $c \in \mathbb{C}$.

These deformations are pairwise non modular-isomorphic for different values of the parameters.

Proof. For any $(\lambda, \mu) \in \mathbb{C}^{*2}$, let us denote by $\phi_{\lambda, \mu}$ the \mathbb{C} -algebra automorphism of $\widetilde{\mathcal{T}}_{\text{ev},*}$ that fixes E_4 and E_6 and such that $\phi_{\lambda, \mu}(A) = \lambda A$ and $\phi_{\lambda, \mu}(B) = \mu B$. We compare the images of any monomial in $\widetilde{\mathcal{T}}_{\text{ev},*}$. It shows that for any $(a, b, a', b') \in \mathbb{C}^4$,

$$\phi_{\lambda, \mu} \circ \text{Se}_{a', b'} = \text{Se}_{a, b} \circ \phi_{\lambda, \mu} \text{ if and only if } a'\mu = a\lambda \text{ and } b'\lambda = b\mu. \quad (3.13)$$

It is clear by the definition, see (3.10), that, if this condition is satisfied, then the formal deformations $\{\cdot, \cdot\}_n^{[a', b', c']}$ and $\{\cdot, \cdot\}_n^{[a, b, c]}$ are isomorphic. Since it follows from (3.13) that

$$\phi_{a', 1} \circ \text{Se}_{a', b'} = \text{Se}_{1, a' b'} \circ \phi_{a', 1} \text{ for any } a' \neq 0, \quad \text{and} \quad \phi_{1, b'} \circ \text{Se}_{0, b'} = \text{Se}_{0, 1} \circ \phi_{1, b'} \text{ for any } b' \neq 0,$$

the proof that $(\{\cdot, \cdot\}_n^{[a', b', c']})_{n \in \mathbb{Z}_{\geq 0}}$ is modular-isomorphic to one of given formal deformations is complete. The separation of the different cases up to modular isomorphism follows from a direct application of Lemma 15. \square

4. FORMAL DEFORMATIONS FOR A LOCALIZATION OF THE ALGEBRA OF JACOBI FORMS

Recall that we have introduced the algebra $\mathcal{K}_{\text{ev},*} = \mathbb{C}[E_4, E_6, A^{\pm 1}, B] \supset \widetilde{\mathcal{T}}_{\text{ev},*}$ and set $F_2 = BA^{-1}$.

4.1. Relation with quasimodular forms. From the deformations we have built on the algebra of Jacobi forms, we want to produce deformations on the algebra of quasimodular forms. In order to do so, we extend the deformation from $\widetilde{\mathcal{T}}_{\text{ev},*}$ to $\mathcal{K}_{\text{ev},*}$ and then restrict this extension to $\mathcal{Q}_* = \mathbb{C}[E_4, E_6, F_2]$.

4.1.1. Extension of the Serre derivation, associated Poisson brackets. For any $a, b \in \mathbb{C}$, the derivation $\text{Se}_{a, b}$ extends canonically to $\mathcal{K}_{\text{ev},*}$ by

$$\text{Se}_{a, b}(A^{-1}) = -A^{-2} \text{Se}_{a, b}(A) = -a A^{-2} B.$$

This implies

$$\text{Se}_{a, b}(F_2) = b E_4 - a F_2^2.$$

It follows that the algebra \mathcal{Q}_* is stable by $\text{Se}_{a, b}$ and hence for $\{\cdot, \cdot\}_n^{[a, b, c]}$ for all $n \in \mathbb{Z}_{\geq 0}$. Therefore, the Poisson bracket $\{\cdot, \cdot\}_1^{[a, b, c]}$ provides \mathcal{Q}_* the structure of a Poisson algebra.

This bracket does not depend on c since the functions of \mathcal{Q}_* have index 0. It is characterized by

$$\begin{aligned} \{E_4, E_6\}_1^{[a,b]} &= -2E_4^3 + 2E_6^2, \\ \{E_4, F_2\}_1^{[a,b]} &= 4bE_4^2 + \frac{2}{3}E_6F_2 - 4aE_4F_2^2, \quad \{E_6, F_2\}_1^{[a,b]} = 6bE_4E_6 + E_4^2F_2 - 6aE_6F_2^2. \end{aligned}$$

We consider three cases:

- (1) If $a = 0$ and $b \neq 0$, the algebra isomorphism ω is a Poisson isomorphism between $(\mathcal{Q}_*, \{\cdot, \cdot\}_1^{[0,b]})$ and $(\mathcal{M}_*^{\leq \infty}, \{\cdot, \cdot\}_{-12b})$ (see [DR14, Proposition A]);
- (2) If $a = b = 0$, the algebra isomorphism ω is a Poisson isomorphism between $(\mathcal{Q}_*, \{\cdot, \cdot\}_1^{[0,0]})$ and $(\mathcal{M}_*^{\leq \infty}, (\cdot, \cdot)_{-2/3})$ (see [DR14, Proposition C]);
- (3) If $a \neq 0$, the Poisson bracket of $\mathcal{M}_*^{\leq \infty}$ obtained through the isomorphism ω increases the depth too much (for example, the depth of the evaluation of this bracket at (E_4, E_2) is 2 whereas it should be less than or equal to 1) and hence $\{\cdot, \cdot\}_1^{[0,b]}$ does not correspond to any bracket defined in [DR14].

4.1.2. Admissible derivations of localized Jacobi forms. Let d be an admissible derivation of $\widetilde{\mathcal{T}}_{\text{ev},*}$. We have seen that the restriction of such a derivation to \mathcal{M}_* is of the form $d(E_4) = xE_6$ and $d(E_6) = yE_4^2$, see (3.8). We have proven in Proposition 10 that the admissible derivations of $\widetilde{\mathcal{T}}_{\text{ev},*}$ that give our Poisson brackets the shape of a first Rankin-Cohen bracket are the derivations $\text{Se}_{a,b}$. Their restriction to \mathcal{M}_* are the Serre derivation Se .

The situation is a bit different for $\mathcal{K}_{\text{ev},*}$. An admissible derivation d of $\mathcal{K}_{\text{ev},*}$ (that is a derivation preserving the index and increasing the weight by 2) acts on the generators by

$$d(E_4) = xE_6 + x'E_4F_2, \quad d(E_6) = yE_4^2 + y'E_4F_2, \quad d(A) = zF_2A, \quad d(F_2) = tE_4 + t'F_2^2, \quad (4.1)$$

for some complex numbers x, x', y, y', z, t, t' . Therefore we introduce naturally the linear map $\pi: \mathcal{K}_{\text{ev},*} \rightarrow \mathcal{K}_{\text{ev},*}$ defined by:

$$\forall f \in \mathcal{K}_{k,m} \quad \pi(f) = kfF_2. \quad (4.2)$$

It is clear that π is a derivation of $\mathcal{K}_{\text{ev},*}$ of degree $(2, 1)$.

We shall extend Theorem 13 to $\mathcal{K}_{\text{ev},*}$ replacing $\text{Se}_{a,b}$ by a linear combination of $\text{Se}_{a,b}$ and π . The restriction of this construction to the subalgebra \mathcal{Q}_* of $\mathcal{K}_{\text{ev},*}$ leads to the Rankin-Cohen brackets $[\cdot, \cdot]_{v_{a,n}}$, $[\cdot, \cdot]_{w_{a,b,n}}^\alpha$ studied in [DR14] (see § 2.2.5). We provide explicit details on this point in the following two Propositions.

Let us denote by $\text{Se}_\mathcal{K}^\#$ the derivation of $\mathcal{K}_{\text{ev},*}$ defined by

$$\text{Se}_\mathcal{K}^\#(E_4) = -\frac{1}{3}E_6, \quad \text{Se}_\mathcal{K}^\#(E_6) = -\frac{1}{2}E_4^2, \quad \text{Se}_\mathcal{K}^\#(F_2) = -\frac{1}{12}E_4, \quad \text{Se}_\mathcal{K}^\#(A) = \text{Se}_\mathcal{K}^\#(A^{-1}) = 0.$$

We note that, by (3.9), the restriction of $\text{Se}_\mathcal{K}^\#$ to $\widetilde{\mathcal{T}}_{\text{ev},*}$ is $\text{Se}_{0,-1/12}$.

For any complex number α , we introduce the derivation $d_\alpha = \text{Se}_\mathcal{K}^\# + \alpha\pi$ of $\mathcal{K}_{\text{ev},*}$, where π is the derivation of $\mathcal{K}_{\text{ev},*}$ defined in (4.2). Then $d_\alpha(A) = -2\alpha A F_2$ and

$$d_\alpha(F_2) = -\frac{1}{12}E_4 + 2\alpha F_2^2, \quad d_\alpha(E_4) = -\frac{1}{3}E_6 + 4\alpha E_4 F_2, \quad d_\alpha(E_6) = -\frac{1}{2}E_4^2 + 6\alpha E_6 F_2. \quad (4.3)$$

This derivation is used in the following Proposition to prove that the deformation of $\tilde{\mathcal{T}}_{\text{ev},*}$ defined in Theorem 13 in the case $a = 0, b \neq 0$ extends into a deformation of $\mathcal{K}_{\text{ev},*}$.

Proposition 17— *For any complex parameters α, c , we consider the sequence $([\cdot, \cdot]_n^{\alpha, c})_{n \in \mathbb{Z}_{\geq 0}}$ of maps $\mathcal{K}_{\text{ev},*} \times \mathcal{K}_{\text{ev},*} \rightarrow \mathcal{K}_{\text{ev},*}$ defined by bilinear extension of the formula:*

$$[f, g]_n^{\alpha, c} = \sum_{i=0}^n (-1)^i \binom{k+cp+n-1}{n-i} \binom{\ell+cq+n-1}{i} d_\alpha^i(f) d_\alpha^{n-i}(g),$$

for all homogeneous $f \in \mathcal{K}_{k,p}$ and $g \in \mathcal{K}_{\ell,q}$.

Then

- (i) The sequence $([\cdot, \cdot]_n^{\alpha, c})_{n \in \mathbb{Z}_{\geq 0}}$ is a formal deformation of $\mathcal{K}_{\text{ev},*}$,
- (ii) $[\mathcal{K}_{k,p}, \mathcal{K}_{\ell,q}]_n^{\alpha, c} \subset \mathcal{K}_{k+\ell+2n, p+q}$,
- (iii) The subalgebra \mathcal{Q}_* is stable by $([\cdot, \cdot]_n^{\alpha, c})_{n \in \mathbb{Z}_{\geq 0}}$, and the formal deformation $(\mathcal{Q}_*, ([\cdot, \cdot]_n^{\alpha, c})_n)$ is isomorphic to the formal deformation $(\mathcal{M}_*^{\leq \infty}, ([\cdot, \cdot]_{v_{\alpha, n}})_n)$,
- (iv) The subalgebra $\tilde{\mathcal{T}}_{\text{ev},*}$ is stable by $([\cdot, \cdot]_n^{\alpha, c})_{n \in \mathbb{Z}_{\geq 0}}$ if and only if $\alpha = 0$. The restriction of $([\cdot, \cdot]_n^{\alpha, c})_{n \in \mathbb{Z}_{\geq 0}}$ to $\tilde{\mathcal{T}}_{\text{ev},*}$ is the deformation $([\cdot, \cdot]_n^{[0, b, c]})_{n \in \mathbb{Z}_{\geq 0}}$ of $\tilde{\mathcal{T}}_{\text{ev},*}$ determined in Theorem 13 for $b = -\frac{1}{12}$ (and then up to equivalence for any $b \in \mathbb{C}^\times$).

Proof. That the sequence $([\cdot, \cdot]_n^{\alpha, c})_{n \in \mathbb{Z}_{\geq 0}}$ is a formal deformation of $\mathcal{K}_{\text{ev},*}$ follows from the general settings described § 2.2.3. Then, $[\mathcal{K}_{k,p}, \mathcal{K}_{\ell,q}]_n^{\alpha, c} \subset \mathcal{K}_{k+\ell+2n, p+q}$ is a consequence of the admissibility of d_α .

The restriction of d_α to \mathcal{Q}_* is a derivation of \mathcal{Q}_* . This implies that $(\mathcal{Q}_*, ([\cdot, \cdot]_n^{\alpha, c})_n)$ is a formal deformation. It is described by

$$[f, g]_n^{\alpha, c} = \sum_{r=0}^n (-1)^r \binom{k+n-1}{n-r} \binom{\ell+n-1}{r} d_\alpha^r(f) d_\alpha^{n-r}(g)$$

for all homogeneous f and g in \mathcal{Q}_* of respective weights k and ℓ . A comparison with § 2.2.5, and in particular the comparison between (4.3) and (2.12), implies it is isomorphic to $(\mathcal{M}_*^{\leq \infty}, ([\cdot, \cdot]_{v_{\alpha, n}})_n)$.

If $\alpha \neq 0$, it is clear by (4.3) that d_α does not restrict into a derivation of $\tilde{\mathcal{T}}_{\text{ev},*}$. We compute

$$[B, E_4]_1^{\alpha, c} = -\frac{c}{3} B E_6 + \frac{1}{3} A E_4^2 + 4\alpha c B E_4 F_2$$

and hence $[B, E_4]_1^{\alpha, c} \notin \tilde{\mathcal{T}}_{\text{ev},*}$ if $\alpha \neq 0$ and $c \neq 0$. If $\alpha \neq 0$ and $c = 0$, we compute

$$[E_4, E_6]_2^{\alpha, 0} = (1 - 12\alpha) E_4^2 E_6 + 144\alpha^2 E_4 E_6 F_2^2$$

and conclude that $[E_4, E_6]_2^{\alpha, 0} \notin \tilde{\mathcal{T}}_{\text{ev},*}$.

Suppose that $\alpha = 0$. Then

$$d_\alpha(E_4) = -\frac{1}{3}E_6, \quad d_\alpha(E_6) = -\frac{1}{2}E_4^2, \quad d_\alpha(A) = 0, \quad d_\alpha(B) = -\frac{1}{12}E_4A.$$

Then the proof is complete by (3.9), Theorem 13 and Theorem 16. \square

Definition 18– Two formal deformations $(\mu_n)_{n \in \mathbb{Z}_{\geq 0}}$ and $(\nu_n)_{n \in \mathbb{Z}_{\geq 0}}$ of $\mathcal{K}_{\text{ev},*}$ are modular-isomorphic if there exists a \mathbb{C} -linear bijective map $\phi: \mathcal{K}_{\text{ev},*} \rightarrow \mathcal{K}_{\text{ev},*}$ such that

- (i) ϕ preserves the index and the weight of homogeneous elements of $\mathcal{K}_{\text{ev},*}$,
- (ii) $\phi(\mu_n(f, g)) = \nu_n(\phi(f), \phi(g))$ for all $n \in \mathbb{Z}_{\geq 0}$ and $f, g \in \mathcal{K}_{\text{ev},*}$.

Note that a modular-isomorphism between formal deformations is a Poisson isomorphism between the induced Poisson algebras.

Proposition 19– The formal deformations $([\cdot, \cdot]_n^{\alpha, c})_{n \in \mathbb{Z}_{\geq 0}}$ and $([\cdot, \cdot]_n^{\alpha', c'})_{n \in \mathbb{Z}_{\geq 0}}$ of $\mathcal{K}_{\text{ev},*}$ are modular-isomorphic if and only if $(\alpha, c) = (\alpha', c')$.

Proof. We assume that ϕ is a modular isomorphism between $([\cdot, \cdot]_n^{\alpha, c})_{n \in \mathbb{Z}_{\geq 0}}$ and $([\cdot, \cdot]_n^{\alpha', c'})_{n \in \mathbb{Z}_{\geq 0}}$. By preservation of index and weight, let $\lambda, \mu, \gamma, \nu, \eta, \zeta \in \mathbb{C}$ such that $\phi(F_2) = \gamma F_2$, $\phi(E_4) = \lambda E_4 + \nu F_2^2$ and $\phi(E_6) = \mu E_6 + \eta E_4 F_2 + \zeta F_2^3$. We have $\gamma \neq 0$, $(\lambda, \nu) \neq (0, 0)$ and $(\mu, \eta, \zeta) \neq (0, 0, 0)$. We compute $\phi([E_4, F_2]_1^{\alpha, c})$ and compare it with $[\phi(E_4), \phi(F_2)]_1^{\alpha', c'}$. Replacing E_4 with E_6 , we get

- if $\lambda = 0$ and $\nu = 4\gamma^2$, $\mu = \eta = 0$. This leads to $\phi(4F_2^2 - E_4) = 0$ which is impossible
- hence $\lambda \neq 0$, $\mu = \gamma = \lambda = 1$ and $\eta = \zeta = 0$ and hence the restriction of ϕ to \mathcal{Q}_* is the identity.

Then, we write $\phi(A) = \theta A$ and compare $\phi([E_4, A]_1^{\alpha, c})$ with $[\phi(E_4), \phi(A)]_1^{\alpha', c'}$. This leads to $c = c'$ and $\alpha = \alpha'$. \square

Remark 20– The proof of Proposition 19 shows that the formal deformations $([\cdot, \cdot]_n^{\alpha, c})_{n \in \mathbb{Z}_{\geq 0}}$ and $([\cdot, \cdot]_n^{\alpha', c'})_{n \in \mathbb{Z}_{\geq 0}}$ of $\mathcal{K}_{\text{ev},*}$ are modular-isomorphic if and only if the Poisson algebra $(\mathcal{K}_{\text{ev},*}, [\cdot, \cdot]_1^{\alpha, c})$ and $(\mathcal{K}_{\text{ev},*}, [\cdot, \cdot]_1^{\alpha', c'})$ are Poisson modular-isomorphic.

Let us denote by $\text{Se}_{\mathcal{K}}^b$ the derivation of $\mathcal{K}_{\text{ev},*}$ defined by

$$\text{Se}_{\mathcal{K}}^b(E_4) = -\frac{1}{3}E_6, \quad \text{Se}_{\mathcal{K}}^b(E_6) = -\frac{1}{2}E_4^2, \quad \text{Se}_{\mathcal{K}}^b(F_2) = 0, \quad \text{Se}_{\mathcal{K}}^b(A) = \text{Se}_{\mathcal{K}}^b(A^{-1}) = 0.$$

For any complex number β , we introduce the derivation $\delta_\beta = \text{Se}_{\mathcal{K}}^b + \beta\pi$ of $\mathcal{K}_{\text{ev},*}$, where π is the derivation of $\mathcal{K}_{\text{ev},*}$ defined in (4.2). Then $\delta_\beta(A) = -2\beta A F_2$ and

$$\delta_\beta(E_4) = -\frac{1}{3}E_6 + 4\beta E_4 F_2, \quad \delta_\beta(E_6) = -\frac{1}{2}E_4^2 + 6\beta E_6 F_2, \quad \delta_\beta(F_2) = 2\beta F_2^2.$$

This derivation is used in the following Proposition to prove that the deformation of $\tilde{\mathcal{J}}_{\text{ev},*}$ defined in Theorem 13 in the case $a = b = 0$ extends into a deformation of $\mathcal{K}_{\text{ev},*}$. The proof is similar to the one of Proposition 17.

Proposition 21– For any complex parameters β, c , we consider the sequence $(\langle \cdot, \cdot \rangle_n^{\beta, c})_{n \in \mathbb{Z}_{\geq 0}}$ of maps $\mathcal{K}_{\text{ev},*} \times \mathcal{K}_{\text{ev},*} \rightarrow \mathcal{K}_{\text{ev},*}$ defined by bilinear extension of the formula:

$$\langle f, g \rangle_n^{\beta, c} = \sum_{i=0}^n (-1)^i \binom{k+cp+n-1}{n-i} \binom{\ell+cq+n-1}{i} \delta_\beta^i(f) \delta_\beta^{n-i}(g),$$

for all homogeneous $f \in \mathcal{K}_{k,p}$ and $g \in \mathcal{K}_{\ell,q}$.

Then,

- (i) The sequence $(\langle \cdot, \cdot \rangle_n^{\beta, c})_{n \in \mathbb{Z}_{\geq 0}}$ is a formal deformation of $\mathcal{K}_{\text{ev},*}$,
- (ii) $\langle \mathcal{K}_{k,p}, \mathcal{K}_{\ell,q} \rangle_n^{\beta, c} \subset \mathcal{K}_{k+\ell+2n, p+q}$,
- (iii) The subalgebra \mathcal{Q}_* is stable by $(\langle \cdot, \cdot \rangle_n^{\beta, c})_{n \in \mathbb{Z}_{\geq 0}}$, and the formal deformation $(\mathcal{Q}_*, (\langle \cdot, \cdot \rangle_n^{\beta, c})_n)$ is isomorphic to the formal deformation $(\mathcal{M}_*^{\leq \infty}, ([\cdot, \cdot]_{w_{-2/3,0,n}}^{-2/3})_n)$,
- (iv) The subalgebra $\tilde{\mathcal{J}}_{\text{ev},*}$ is stable by $(\langle \cdot, \cdot \rangle_n^{\beta, c})_{n \in \mathbb{Z}_{\geq 0}}$ if and only if $\beta = 0$. The restriction of $(\langle \cdot, \cdot \rangle_n^{0, c})_{n \in \mathbb{Z}_{\geq 0}}$ to $\tilde{\mathcal{J}}_{\text{ev},*}$ is the deformation $(\langle \cdot, \cdot \rangle_n^{[0,0,c]})_{n \in \mathbb{Z}_{\geq 0}}$ of $\tilde{\mathcal{J}}_{\text{ev},*}$ determined in Theorem 13.

The same way we proved Proposition 19 we can prove the following classification Proposition.

Proposition 22– The formal deformations $(\langle \cdot, \cdot \rangle_n^{\beta, c})_{n \in \mathbb{Z}_{\geq 0}}$ and $(\langle \cdot, \cdot \rangle_n^{\beta', c'})_{n \in \mathbb{Z}_{\geq 0}}$ of $\mathcal{K}_{\text{ev},*}$ are modular-isomorphic if and only if $c = c'$, and $(\beta, \beta') = (0, 0)$ or $(\beta, \beta') \in \mathbb{C}^* \times \mathbb{C}^*$.

4.2. Relation with modular forms. In this section, we build a formal deformation of $\mathcal{K}_{\text{ev},*}$ that restricts to the formal deformation on Rankin-Cohen brackets on \mathcal{M}_* .

Recall that $\omega: \mathcal{Q}_* \rightarrow \mathcal{M}_*^{\leq \infty}$ is the algebra isomorphism that sends (E_4, E_6, F_2) to (E_4, E_6, E_2) . The usual complex derivative D_z defines a derivation on the algebra $\mathcal{M}_*^{\leq \infty}$ of quasimodular forms. We define a derivation on \mathcal{Q}_* by $\partial = \omega^{-1} D_z \omega$. Ramanujan equations become

$$\partial(E_4) = -\frac{1}{3}(E_6 - E_4 F_2), \quad \partial(E_6) = \frac{1}{2}(E_4^2 - E_6 F_2), \quad \partial(F_2) = -\frac{1}{12}(E_4 - F_2^2). \quad (4.4)$$

By (4.1) the unique way to extend ∂ into an admissible derivation ∂_u of $\mathcal{K}_{\text{ev},*}$ is to set

$$\forall f \in \mathcal{Q}_* \quad \partial_u(f) = \partial(f) \text{ and } \partial_u(A) = u A F_2 \quad (4.5)$$

for some $u \in \mathbb{C}$. We compute

$$\partial_u(B) = \partial_u(A F_2) = (u + \frac{1}{12}) B F_2 - \frac{1}{12} E_4 A. \quad (4.6)$$

It is clear that

1. the derivation ∂_u does not restrict into a derivation of \mathcal{M}_* ,
2. the derivation ∂_u restricts into the derivation ∂ of \mathcal{Q}_* ,
3. the derivation ∂_u does not restrict into a derivation of $\tilde{\mathcal{J}}_{\text{ev},*}$

for any $u \in \mathbb{C}$.

The following Theorem can be proved the same way as Proposition 17.

Theorem 23— For any complex parameters u and v , let $(\llbracket \cdot, \cdot \rrbracket_n^{u,v})_{n \in \mathbb{Z}_{\geq 0}}$ be the sequence of maps $\mathcal{K}_{\text{ev},*} \times \mathcal{K}_{\text{ev},*} \rightarrow \mathcal{K}_{\text{ev},*}$ defined by bilinear extension of

$$\llbracket f, g \rrbracket_n^{u,v} = \sum_{r=0}^n (-1)^r \binom{k+vp+n-1}{n-r} \binom{\ell+vq+n-1}{r} \partial_u^r(f) \partial_u^{n-r}(g),$$

for all homogeneous $f \in \mathcal{K}_{k,p}$ and $g \in \mathcal{K}_{\ell,q}$. Then, for all $(u, v) \in \mathbb{C}^2$,

- (i) the sequence $(\llbracket \cdot, \cdot \rrbracket_n^{u,v})_{n \in \mathbb{Z}_{\geq 0}}$ is a formal deformation of $\mathcal{K}_{\text{ev},*}$,
- (ii) $\llbracket \mathcal{K}_{k+,p}, \mathcal{K}_{\ell,q} \rrbracket_n^{u,v} \subset \mathcal{K}_{k+\ell+2n,p+q}$,
- (iii) the sequence $(\llbracket \cdot, \cdot \rrbracket_n^{u,v})_{n \in \mathbb{Z}_{\geq 0}}$ restricts to the formal deformation of the algebra \mathcal{M}_* of modular forms given by the usual Rankin-Cohen brackets.

The same way we proved Proposition 19 we can prove the following classification Proposition.

Proposition 24— The formal deformations $(\llbracket \cdot, \cdot \rrbracket_n^{u,v})_{n \in \mathbb{Z}_{\geq 0}}$ and $(\llbracket \cdot, \cdot \rrbracket_n^{u',v'})_{n \in \mathbb{Z}_{\geq 0}}$ of $\mathcal{K}_{\text{ev},*}$ are modular-isomorphic if and only if $(u, v) = (u', v')$.

Remark 25— It is clear that the subalgebra \mathcal{Q}_* is stable by $(\llbracket \cdot, \cdot \rrbracket_n^{u,v})_{n \in \mathbb{Z}_{\geq 0}}$. However, their restrictions to \mathcal{Q}_* do not preserve the degree in F_2 in the sense that they do not satisfy

$$\forall (f, g) \in \mathcal{M}_* \quad \forall (s, t) \in \mathbb{Z}_{\geq 0}^2 \quad \deg_{F_2}(\llbracket f F_2^s, g F_2^t \rrbracket_n^{u,v}) \leq s + t.$$

Up to the isomorphism ω , they do not preserve the depth of quasimodular forms. For this reason, the restrictions of $(\llbracket \cdot, \cdot \rrbracket_n^{u,v})_{n \in \mathbb{Z}_{\geq 0}}$ to the subalgebra \mathcal{Q}_* can not coincide with the brackets previously studied in [DR14].

Remark 26— Similar computations prove that the deformations $(\llbracket \cdot, \cdot \rrbracket_n^{\alpha,c})_{n \in \mathbb{Z}_{\geq 0}}$, $(\langle \cdot, \cdot \rangle_n^{\alpha,c})_{n \in \mathbb{Z}_{\geq 0}}$ and $(\llbracket \cdot, \cdot \rrbracket_n^{u,v})_{n \in \mathbb{Z}_{\geq 0}}$ are never pairwise modular-isomorphic.

Although the subalgebra $\widetilde{\mathcal{T}}_{\text{ev},*}$ is not stable by the derivation ∂_u , the question arises whether $\widetilde{\mathcal{T}}_{\text{ev},*}$ can be stable by $(\llbracket \cdot, \cdot \rrbracket_n^{u,v})_{n \in \mathbb{Z}_{\geq 0}}$ for some values of the parameters u and v . The following lemma gives a (very) partial answer for $n = 1$.

Lemma 27— The algebra $\widetilde{\mathcal{T}}_{\text{ev},*}$ is stable by the Poisson bracket $\llbracket \cdot, \cdot \rrbracket_1^{u,v}$ if and only if $v - 1 = 12u$. The Poisson bracket $\llbracket \cdot, \cdot \rrbracket_1^{u,v}$ coincides with the Poisson bracket $\{\cdot, \cdot\}_1^{[1/12, -1/12, -(12u+1)/3]}$ of Theorem 13.

Proof. By Theorem 23, we have

$$\forall (f, g) \in \mathcal{K}_{k,p} \times \mathcal{K}_{\ell,q} \quad \llbracket f, g \rrbracket_1^{u,v} = (k+vp)f\partial_u(g) - (\ell+vq)\partial_u(f)g.$$

With (4.4), (4.5) and (4.6), we compute

$$\begin{aligned}
\llbracket A, E_4 \rrbracket_1^{u,v} &= \frac{1}{3}(-12u + v - 2)E_4 B - \frac{1}{3}(v - 2)A E_6 \\
\llbracket A, E_6 \rrbracket_1^{u,v} &= \frac{1}{2}(-12u + v - 2)E_6 B - \frac{1}{2}(v - 2)A E_4^2 \\
\llbracket B, E_4 \rrbracket_1^{u,v} &= \frac{1}{3}(-12u + v - 1)B E_4 F_2 - \frac{1}{3}v E_6 B + \frac{1}{3}E_4^2 A \\
\llbracket B, E_6 \rrbracket_1^{u,v} &= \frac{1}{2}(-12u + v - 1)B E_6 F_2 - \frac{1}{2}v E_4^2 B + \frac{1}{2}E_4 E_6 A \\
\llbracket A, B \rrbracket_1^{u,v} &= \frac{1}{12}(-24u + v - 2)B^2 - \frac{1}{12}(v - 2)E_4 A^2.
\end{aligned}$$

If $\widetilde{\mathcal{T}}_{\text{ev},*}$ is stable by the Poisson brackets $\llbracket \cdot, \cdot \rrbracket_1^{u,v}$, it follows from the third and fourth relations that $12u = v - 1$. Then we have

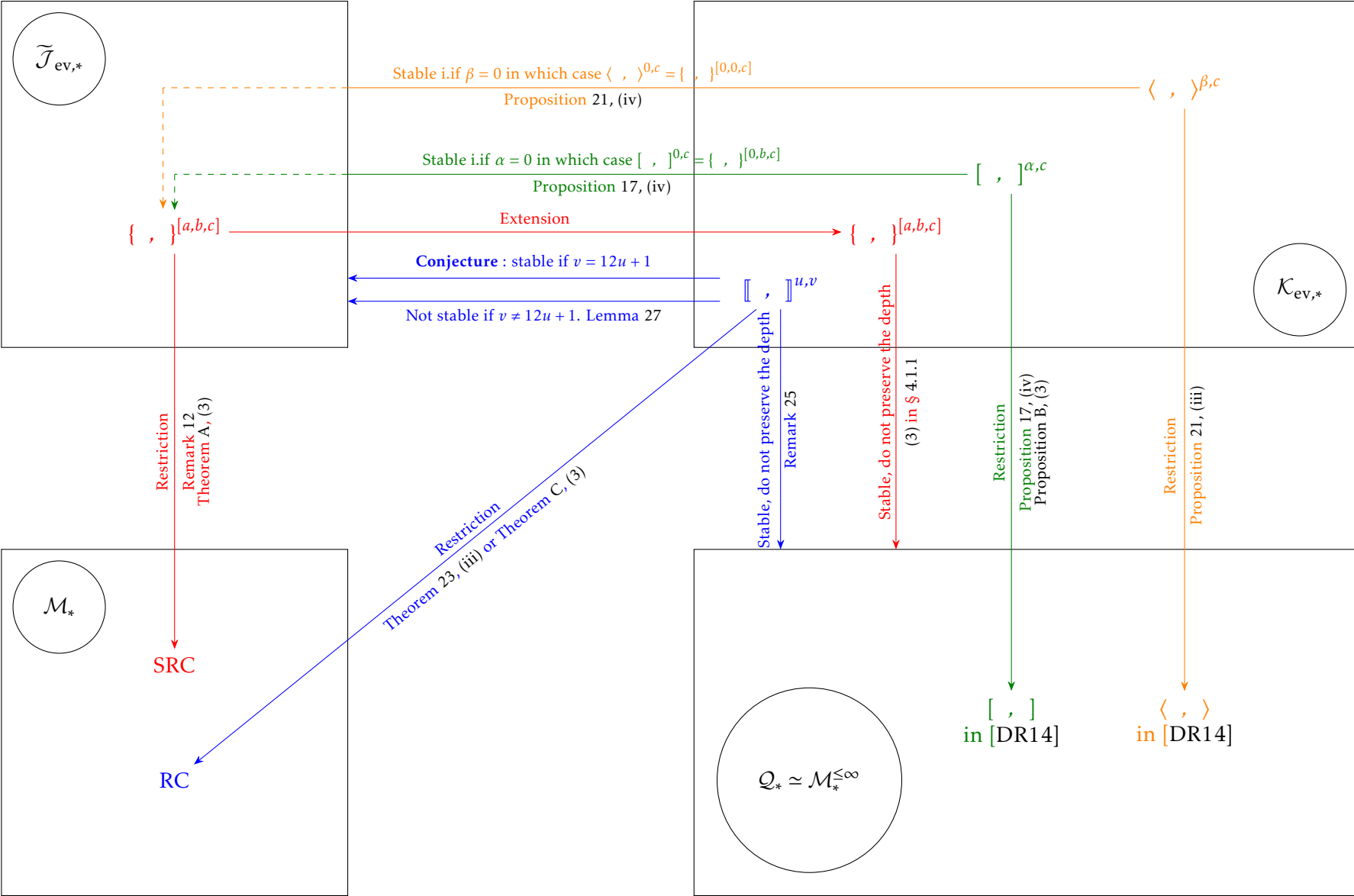
$$\begin{aligned}
\llbracket A, E_4 \rrbracket_1^{u,v} &= \frac{1}{3}(2 - v)E_6 A - \frac{1}{3}E_4 B \\
\llbracket A, E_6 \rrbracket_1^{u,v} &= \frac{1}{2}(2 - v)E_4^2 A - \frac{1}{2}E_6 B \\
\llbracket B, E_4 \rrbracket_1^{u,v} &= \frac{1}{3}E_4^2 A - \frac{1}{3}v E_6 B \\
\llbracket B, E_6 \rrbracket_1^{u,v} &= \frac{1}{2}E_4 E_6 A - \frac{1}{2}v E_4^2 B \\
\llbracket A, B \rrbracket_1^{u,v} &= \frac{1}{12}(2 - v)E_4 A^2 - \frac{1}{12}v B^2.
\end{aligned}$$

Hence the Poisson bracket $\llbracket \cdot, \cdot \rrbracket_1^{u, 12u+1}$ corresponds to the case \mathcal{B} in Proposition 7 for $\gamma = -\frac{1}{3}$, $\lambda = \frac{1}{3}$ and $\varepsilon = -\frac{1}{3}v$. Comparing with (3.11), we conclude that $\llbracket \cdot, \cdot \rrbracket_1^{u, 12u+1}$ is no more than the Poisson bracket $\{\cdot, \cdot\}_1^{[a,b,c]}$ for $a = \frac{1}{12}$, $b = -\frac{1}{12}$ and $c = -\frac{v}{3}$. \square

Convinced by extensive computations with `pari-gp` [The17], we make the following conjecture.

Conjecture— For any complex number u , the sequence $\left(\llbracket \cdot, \cdot \rrbracket_n^{u, 12u+1}\right)_{n \in \mathbb{Z}_{\geq 0}}$ defines by restriction a formal deformation of the algebra $\widetilde{\mathcal{T}}_{\text{ev},*}$ of weak Jacobi forms.

5. GRAPHICAL ABSTRACT



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