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# BIFURCATIONS OF LIMIT CYCLES IN COUPLED NETWORKS OF HAMILTONIAN SYSTEMS

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**Abstract.** This paper is devoted to the analysis of bifurcations of limit cycles in planar polynomial near-Hamiltonian systems. It is motivated by the second part of the sixteenth Hilbert's problem. We introduce a class of Hamiltonian systems which admit a high number of non-degenerate centers that can be arbitrarily located in the plane. We study several perturbations of those Hamiltonian systems, and analyze their effect by using the Melnikov method. One of those perturbations is defined along the gradient of the initial Hamiltonian.

*Keywords :* Hamiltonian system; perturbation; bifurcation; limit cycle; Melnikov function.

## §1 Introduction

The sixteenth Hilbert's problem was stated in 1900 at the International Congress of Mathematicians in Paris [1]. Its second part is concerned with the location and the maximal number  $H(n)$  of limit cycles in planar vector fields of the form

$$\dot{x} = P_n(x, y), \quad \dot{y} = Q_n(x, y),$$

where  $P_n$  and  $Q_n$  are polynomials of degree  $n$ . Before the problem was stated, Poincaré had introduced the concept of limit cycle, and proved that a planar polynomial vector field without saddle connections cannot admit an infinite number of limit cycles [2]. The *finiteness theorem* in the general case was proved separately by Ilyashenko [3, 4] and Écalle [5], but it is still unknown whether  $H(n)$  is finite or not, even for quadratic systems. The sixteenth Hilbert's problem has produced a huge literature, thus it is impossible to give a complete list of the existing works in this field. However, detailed surveys are presented in [4], [6] or [7]. For example, the particular study of Liénard equations has been a rich source of innovative reasonings [8], [9], [10]. Among others, several methods have been developed to give lower bounds for estimating the growth rate of  $H(n)$ . One of those methods is to apply a perturbation to a Hamiltonian system, which leads to the study of differential systems that can be written

$$\dot{x} = \frac{\partial H}{\partial y} + \varepsilon f(x, y), \quad \dot{y} = -\frac{\partial H}{\partial x} + \varepsilon g(x, y),$$

where  $H$ ,  $f$  and  $g$  are polynomials, and  $\varepsilon$  is a real coefficient. A common approach is to make a bifurcation analysis in order to prove that the perturbation can give birth to nests of limit cycles, in a neighborhood of a weak focus or a center. In [11], [12], it was proved that the growth rate of  $H(n)$  is at least of order  $n^2$ . Afterwards, a new lower bound was obtained by Christopher and Lloyd in 1995 [13], and is of order  $n^2 \log n$ . Some mistakes in their paper were corrected in [7] and [14]. Anyway, Lloyd [15] conjectured in 1988 that  $H(n)$  should be of order  $n^3$ . We quote his reasoning below.

*I conjecture that  $H(n) = O(n^3)$ . My reasoning is simply that  $O(n^2)$  critical points can be encircled by limit cycles, and that there are likely to be at most  $O(n)$  limit cycles around each critical point.*

In this paper, we shall introduce a class of polynomial planar Hamiltonian systems, admitting a high number of non-degenerate centers that can be arbitrarily located. These systems have been constructed with a *coupled network* approach which is quite different from the previous works in this field. Indeed, it is known that the effect of coupling oscillators can produce a richness of dynamics, with the emergence of new periodic solutions [16]. In particular, we succeed in constructing a planar polynomial Hamiltonian system of degree  $O(n)$  admitting  $O(n^2)$  non-degenerate centers. We remark that a similar form of Hamiltonian systems is considered in [17] in order to prove that any configuration of limit cycles can be achieved by a polynomial vector field, which comforts our conviction that this class of Hamiltonian systems can be an interesting source of understanding the bifurcations of limit cycles in planar polynomial vectors fields. Those Hamiltonian systems are presented in the first section. In the second section, we study several polynomial perturbations under which a nest of limit cycles appears in the neighborhood of each non-degenerate center. To that aim, we begin by analyzing basic geometric configurations for the location of the centers, involving *lines* and *rings*. Then we propose a perturbation of any Hamiltonian system along its own gradient, and prove how it is efficient to make centers bifurcate into nests of limit cycles. Our theoretical results are systematically illustrated by computer made figures, in order to show the geometrical ingredients that are omnipresent in our reasoning.

## §2 Construction of a class of Hamiltonian systems

### 2.1. Hamiltonian systems admitting arbitrarily located centers

We begin our paper with the construction of Hamiltonian systems admitting centers that can be arbitrarily located. Let us consider a finite family of distinct points  $\mathcal{P} = \{(x_i, y_i), 1 \leq i \leq n\}$  in  $\mathbb{R}^2$ , where  $n$  is a positive integer. For each point  $p = (x_i, y_i)$  in  $\mathcal{P}$ ,  $1 \leq i \leq n$ , we consider the differential system  $(\sigma_i)$  defined by

$$(\sigma_i) \begin{cases} \dot{x} = y - y_i \\ \dot{y} = -(x - x_i), \end{cases} \quad (1)$$

which admits a center at  $p = (x_i, y_i)$ , and is a Hamiltonian system:

$$(\sigma_i) \begin{cases} \dot{x} = \frac{\partial G_i}{\partial y} \\ \dot{y} = -\frac{\partial G_i}{\partial x}, \end{cases}$$

with

$$G_i = \frac{1}{2}[(x - x_i)^2 + (y - y_i)^2]. \quad (2)$$

Next, our goal is to build a differential system which admits a center at every point  $p = (x_i, y_i)$  in  $\mathcal{P}$ , and which behaves like the system (1) in a neighborhood of  $p = (x_i, y_i)$  for  $1 \leq i \leq n$ . To that aim, we introduce the system

$$(\mathcal{R}) \begin{cases} \dot{x} = \sum_{i=1}^n (y - y_i) \prod_{\substack{j=1 \\ j \neq i}}^n [(x - x_j)^2 + (y - y_j)^2] \\ \dot{y} = \sum_{i=1}^n -(x - x_i) \prod_{\substack{j=1 \\ j \neq i}}^n [(x - x_j)^2 + (y - y_j)^2], \end{cases}$$

which is obtained by coupling the systems  $(\sigma_i)$ ,  $1 \leq i \leq n$ , in the same phase space, that is  $\mathbb{R}^2$ . Indeed, for each  $i$ , the coupling term

$$\prod_{\substack{j=1 \\ j \neq i}}^n [(x - x_j)^2 + (y - y_j)^2],$$

defined as a product of squared euclidean distances, controls the effect of the vector field induced by  $(\sigma_i)$  in the neighborhood of the other points  $(x_j, y_j)$  for  $j \neq i$ . Furthermore, we remark that the latter system is indeed a Hamiltonian system:

$$(\mathcal{H}) \quad \begin{cases} \dot{x} = G_y \\ \dot{y} = -G_x, \end{cases} \quad (3)$$

where  $G = G(\mathcal{P})$  is given by

$$G = \frac{1}{2} \prod_{i=1}^n 2G_i = \frac{1}{2} \prod_{i=1}^n [(x - x_i)^2 + (y - y_i)^2], \quad (4)$$

for all  $(x, y) \in \mathbb{R}^2$ . For short, we have used the notations  $G_x = \frac{\partial G}{\partial x}$ ,  $G_y = \frac{\partial G}{\partial y}$ . Before we state our first proposition, we recall a sufficient condition for a critical point to be a non-degenerate center (see [8], Theorem 2 in section 2.14, or [18]).

**Theorem 1.** *Let  $H$  be a smooth function defined in  $\mathbb{R}^2$ . A point  $(x_0, y_0)$  is a non-degenerate center of the Hamiltonian system*

$$\dot{x} = H_y, \quad \dot{y} = -H_x$$

*if the two following properties are satisfied:*

- i.  $H_x(x_0, y_0) = H_y(x_0, y_0) = 0$ ,
- ii.  $H_{xx}(x_0, y_0)H_{yy}(x_0, y_0) - H_{xy}(x_0, y_0)^2 > 0$ .

*In that case, there exists a continuous band of closed orbits  $\gamma_h \subset H^{-1}(\{h\})$  for  $h \in ]0, h_0[$ , with  $h_0 > 0$ , encircling  $(x_0, y_0)$ .*

**Proposition 1.** *The Hamiltonian system (3) is a polynomial planar system of degree  $2n - 1$ . It admits  $n$  non-degenerate centers at each point of the family  $\mathcal{P}$ .*

*Proof.* Since  $G$  is a polynomial of degree  $2n$ , it is clear that (3) is a planar polynomial system of degree  $2n - 1$ . Next, we compute  $G_x$ ,  $G_y$  and  $G_{xx}G_{yy} - G_{xy}^2$  at each point  $(x_k, y_k)$ ,  $1 \leq k \leq n$ . For each  $k \in \{1, \dots, n\}$ , we have

$$\begin{aligned} \frac{\partial G}{\partial x}(x_k, y_k) &= \sum_{i=1}^n (x_k - x_i) \prod_{\substack{j=1 \\ j \neq i}}^n [(x_k - x_j)^2 + (y_k - y_j)^2] \\ &= (x_k - x_k) \prod_{\substack{j=1 \\ j \neq k}}^n [(x_k - x_j)^2 + (y_k - y_j)^2] \\ &\quad + \sum_{\substack{i=1 \\ i \neq k}}^n (x_k - x_i) \prod_{\substack{j=1 \\ j \neq i}}^n [(x_k - x_j)^2 + (y_k - y_j)^2] \\ &= 0. \end{aligned}$$

A similar computation leads to

$$\frac{\partial G}{\partial y}(x_k, y_k) = 0.$$

Next, we compute the derivatives of  $G$  of order 2. We have

$$\frac{\partial^2 G}{\partial x^2} = \sum_{i=1}^n \left\{ 1 \times \prod_{\substack{j=1 \\ j \neq i}}^n [(x - x_j)^2 + (y - y_j)^2] + (x - x_i) \sum_{\substack{j=1 \\ j \neq i}}^n (x - x_j) \prod_{\substack{k=1 \\ k \neq i, k \neq j}}^n [(x - x_k)^2 + (y - y_k)^2] \right\},$$

from which we deduce

$$\frac{\partial^2 G}{\partial x^2}(x_k, y_k) = \prod_{\substack{j=1 \\ j \neq k}}^n [(x_k - x_j)^2 + (y_k - y_j)^2].$$

Analogously, we have

$$\frac{\partial^2 G}{\partial y^2}(x_k, y_k) = \prod_{\substack{j=1 \\ j \neq k}}^n [(x_k - x_j)^2 + (y_k - y_j)^2].$$

Finally, we compute

$$\frac{\partial^2 G}{\partial x \partial y} = \sum_{i=1}^n (x - x_i) \left\{ \sum_{\substack{j=1 \\ j \neq i}}^n (y - y_j) \prod_{\substack{l=1 \\ l \neq i, l \neq j}}^n [(x - x_l)^2 + (y - y_l)^2] \right\},$$

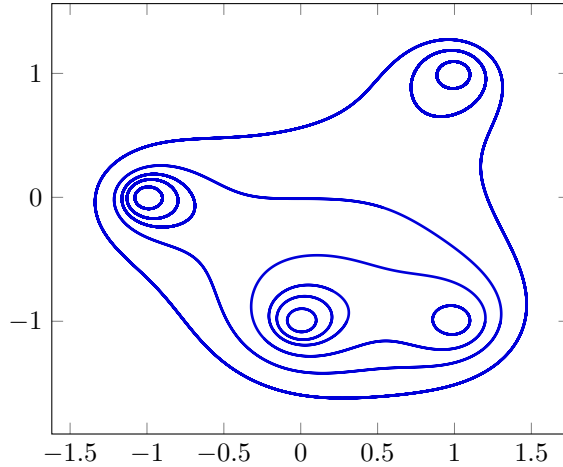
thus

$$\frac{\partial^2 G}{\partial x \partial y}(x_k, y_k) = 0.$$

We obtain

$$\frac{\partial^2 G}{\partial x^2}(x_k, y_k) \frac{\partial^2 G}{\partial y^2}(x_k, y_k) - \left( \frac{\partial^2 G}{\partial x \partial y}(x_k, y_k) \right)^2 = \prod_{\substack{j=1 \\ j \neq k}}^n [(x_k - x_j)^2 + (y_k - y_j)^2]^2 > 0,$$

since the points of  $\mathcal{P}$  are distinct. This achieves the proof.  $\square$



**Figure 1:** Phase portrait of system (3) with  $\mathcal{P} = \{(-1, 0), (0, -1), (1, -1), (1, 1)\}$ , showing continuous bands of closed orbits in the neighborhood of each point  $p \in \mathcal{P}$ . The position of the centers can be arbitrarily chosen.

This first proposition means that the Hamiltonian system (3) answers favorably for what it was expected to, that is, admitting centers at arbitrarily chosen locations. The phase portrait of the system (3) with  $\mathcal{P} = \{(-1, 0), (0, -1), (1, -1), (1, 1)\}$  is depicted in Figure 1. It is worth noting that system (3) can admit other critical points that are saddles, and is likely to exhibit saddles heteroclinic or homoclinic orbits. Anyway, we focus in this paper on the possibility to make each center bifurcate into a nest of limit cycles.

## 2.2. Hamiltonian system for a rectangular grid of centers

Next, our aim is to construct a Hamiltonian system whose degree is of order  $n$ , and which admits  $n^2$  non-degenerate centers. Let us consider two positive integers  $n, m$ . We introduce the Hamiltonian system  $(\mathcal{S}_{n,m})$  given by

$$(\mathcal{S}_{n,m}) \quad \begin{cases} \dot{x} = F_y \\ \dot{y} = -F_x, \end{cases} \quad (5)$$

where  $F$  is defined by

$$F = \sum_{k=0}^{m-1} F_k \beta_k, \quad (6)$$

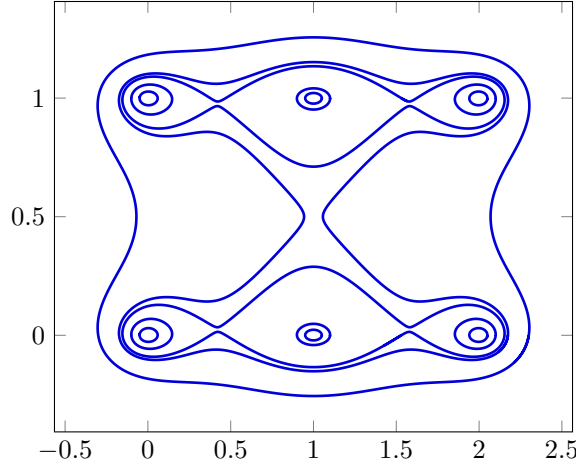
with

$$F_k = \frac{1}{2} \prod_{i=0}^{n-1} [(x-i)^2 + (y-k)^2], \quad \beta_k = \prod_{\substack{l=0 \\ l \neq k}}^{m-1} (y-l)^2.$$

The Hamiltonian functions  $F_k$  are related to the Hamiltonian function  $G$  given by (4) through the relation

$$F_k = G(\mathcal{P}_k), \quad \mathcal{P}_k = \{(0, k), (1, k), \dots, (n-1, k)\},$$

thus the Hamiltonian function  $F$  is obtained by coupling Hamiltonian functions  $F_k$  admitting centers that are located on parallel lines.



**Figure 2:** Phase portrait of system (5) with  $n = 3$  and  $m = 2$ , showing closed orbits in the neighborhood of the points  $(i, k)$  for  $0 \leq i \leq 2$  and  $0 \leq k \leq 1$ , distributed on a rectangular grid.

The next proposition states that the resulting system (5) admits a grid of centers. It can easily be modified so that the distance between each line in the grid is different. The phase portrait of system (5) with  $n = 3$  and  $m = 2$  shown in Figure 2 illustrates the geometric disposal of those centers.

**Proposition 2.** *The Hamiltonian system (5) is a planar polynomial system whose degree is equal to  $2n + 2m - 3$ . Furthermore, it admits  $n \times m$  non-degenerate centers located at the points  $(i, k)$ ,  $0 \leq i \leq n - 1$ ,  $0 \leq k \leq m - 1$ .*

*Proof.* First,  $F$  is a polynomial of degree  $2n + 2(m - 1)$ , thus it is clear that the system (5) is a planar polynomial system of degree  $2n + 2m - 3$ . Next, we note that for each  $k \in \{1, \dots, m\}$ , we

have, by virtue of the Proposition 1:

$$\begin{aligned}\frac{\partial F_k}{\partial x}(i, k) &= \frac{\partial F_k}{\partial y}(i, k) = 0, \\ \frac{\partial^2 F_k}{\partial x^2}(i, k) \frac{\partial^2 F_k}{\partial x^2}(i, k) - \left( \frac{\partial^2 F_k}{\partial x \partial y}(i, k) \right)^2 &> 0,\end{aligned}$$

for all  $i \in \{1, \dots, n\}$ , since  $F_k = G(\mathcal{P}_k)$  with  $\mathcal{P}_k = \{(0, k), (1, k), \dots, (n-1, k)\}$ . We compute the derivatives of orders 1 and 2 of  $F$ . We begin with

$$F_x = \sum_{k=0}^{m-1} \frac{\partial}{\partial x} [F_k(x, y) \beta_k(y)] = \sum_{k=0}^{m-1} \beta_k(y) \frac{\partial}{\partial x} [F_k(x, y)].$$

Now, let  $(i_0, k_0)$  such that  $0 \leq i_0 \leq n-1$  and  $0 \leq k_0 \leq m-1$ . We have

$$\begin{aligned}\frac{\partial F_{k_0}}{\partial x}(i_0, k_0) &= 0, \\ \beta_k(k_0) &= \prod_{\substack{l=0 \\ l \neq k}}^{m-1} (k_0 - l)^2 = 0, \text{ if } k \neq k_0.\end{aligned}$$

It follows that

$$\frac{\partial F_{k_0}}{\partial x}(i_0, k_0) = \beta_{k_0}(k_0) \frac{\partial F_{k_0}}{\partial x}(i_0, k_0) + \sum_{\substack{k=0 \\ k \neq k_0}}^{m-1} \beta_k(k_0) \frac{\partial F_k}{\partial x}(i_0, k_0) = 0.$$

We continue with the derivative with respect to  $y$ :

$$F_y = \sum_{k=0}^{m-1} \frac{\partial}{\partial y} [F_k(x, y) \beta_k(y)] = \sum_{k=0}^{m-1} \left\{ \beta_k(y) \frac{\partial}{\partial y} [F_k(x, y)] + \frac{d\beta_k}{dy}(y) F_k(x, y) \right\}.$$

As previously, we have

$$F_{k_0}(i_0, k_0) = \frac{\partial F_{k_0}}{\partial y}(i_0, k_0) = 0, \quad \beta_k(k_0) = 0 \text{ if } k \neq k_0.$$

Furthermore, we compute

$$\frac{d\beta_k}{dy}(y) = \frac{d}{dy} \left\{ \prod_{\substack{l=0 \\ l \neq k}}^{m-1} (y - l)^2 \right\} = \sum_{\substack{l=0 \\ l \neq k}}^{m-1} 2(y - l) \left\{ \prod_{\substack{q=0 \\ q \neq k, q \neq l}}^{m-1} (y - q)^2 \right\},$$

thus  $\frac{d\beta_k}{dy}(k_0) = 0$  if  $k \neq k_0$ , and we obtain  $\frac{\partial F}{\partial y}(i_0, k_0) = 0$ . After elementary but tedious computations, we prove that

$$(F_{xx}F_{yy} - F_{xy}^2)(i_0, k_0) \geq [\beta_{k_0}(k_0)]^2 \times \left[ \frac{\partial^2 F_{k_0}}{\partial x^2} \frac{\partial^2 F_{k_0}}{\partial y^2} - \left( \frac{\partial^2 F_{k_0}}{\partial x \partial y} \right)^2 \right](i_0, k_0) > 0,$$

and this achieves the proof.  $\square$

For  $m = n$ , the system (5) is a planar polynomial system whose degree is of order  $n$ , admitting  $n^2$  centers. The first step in the conjecture stated by Lloyd in 1988, that we have quoted in our introduction, is reached. Of course, it is possible to compute an instance of the Hamiltonian function

$G$  given by (4) which also admits  $n^2$  centers, but in that case, the resulting system would be of degree  $2n^2 - 1$  (see Proposition 1). Now, the difficult step is to find a suitable perturbation of degree  $O(n)$  of the system (5), so that each center bifurcates into a nest of  $O(n)$  limit cycles. The investigation of such perturbations will be presented in the next section. Before that, we present a geometrical variant of the Hamiltonian system (5) for which the centers are located on concentric rings [19].

### 2.3. Hamiltonian system for coupled rings of centers

Let us introduce the Hamiltonian system defined by

$$(\mathcal{H}_{n,m}) \begin{cases} \dot{x} = E_y \\ \dot{y} = -E_x, \end{cases} \quad (7)$$

with positive integers  $n, m$ , where the Hamiltonian function  $E$  is given by

$$E = \sum_{k=1}^m E_k \delta_k, \quad (8)$$

and the functions  $E_k$  and  $\delta_k$  are defined by for each  $k \in \{1, \dots, m\}$  by

$$E_k = \frac{1}{2} \prod_{i=1}^n [(x - x_{i,k})^2 + (y - y_{i,k})^2], \quad \delta_k = \prod_{\substack{l=1 \\ l \neq k}}^m (x^2 + y^2 - \rho_l^2)^2, \quad (9)$$

with  $x_{i,k} = \rho_k \cos \frac{2i\pi}{n}$ ,  $y_{i,k} = \rho_k \sin \frac{2i\pi}{n}$  and positive radii  $\rho_1 < \rho_2 < \dots < \rho_m$ . For each  $k \in \{1, \dots, m\}$ , the polynomial  $E_k$  is again a particular instance of the Hamiltonian function  $G$  given by (4), that is

$$E_k = G(\mathcal{P}_k), \quad \mathcal{P}_k = \left\{ \left( \rho_k \cos \frac{2i\pi}{n}, \rho_k \sin \frac{2i\pi}{n} \right), 1 \leq i \leq n \right\}.$$

**Proposition 3.** *For any positive integers  $n, m$ , the system (7) is a planar polynomial system, invariant under the rotation of angle  $\frac{2\pi}{n}$ . Furthermore, its degree is equal to  $2n + 4m - 5$ , and it admits  $n \times m$  non-degenerate centers at  $(x_{i,k}, y_{i,k})$ ,  $1 \leq i \leq n$ ,  $1 \leq k \leq m$ .*

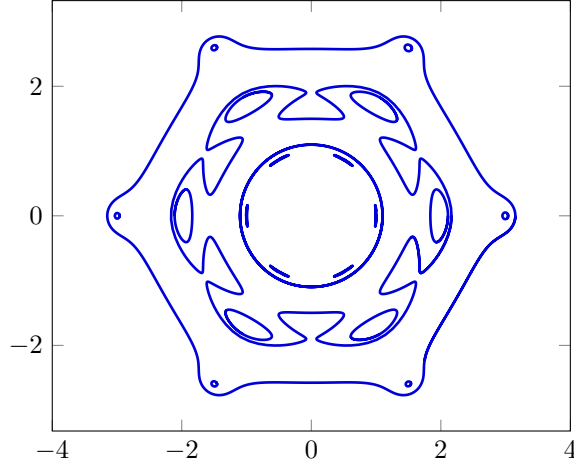
*Proof.* Since  $E$  is a polynomial of degree  $2n + 4(m - 1)$ , it is clear that the system (7) is a planar polynomial system of degree  $2n + 4m - 5$ . By construction, it is invariant under the rotation of angle  $\frac{2\pi}{n}$ . As previously, we check after basic computations that

$$\begin{aligned} E_x(x_{i,k}, y_{i,k}) &= E_y(x_{i,k}, y_{i,k}) = 0, \\ (E_{xx}E_{yy} - E_{xy}^2)(x_{i,k}, y_{i,k}) &> 0, \end{aligned}$$

for  $1 \leq i \leq n$  and  $1 \leq k \leq m$ . □

As expected, the Hamiltonian system (7) presents a geometric structure with  $m$  coupled rings of  $n$  centers. The phase portrait of (7) for  $m = 3$ ,  $n = 6$  and  $\rho_k = k$ ,  $1 \leq k \leq 3$ , is depicted in Figure 3. Applying the same method, it is possible to build a Hamiltonian system admitting centers located on various curves, like cubical curves or hyperbolic curves for instance. This shows that the Hamiltonian system (3) offers a rich palette of geometric patterns. We shall see in the coming section how the location of the centers in both systems (5) and (7), with regular geometric distributions, can facilitate the construction of a perturbation.





**Figure 3:** Phase portrait of the system (7) for  $m = 3$ ,  $n = 6$  and  $\rho_k = k$ ,  $1 \leq k \leq 3$ , showing 3 rings of 6 centers, with continuous bands of closed orbits of small amplitude encircling those centers. The system is invariant under the rotation of angle  $\frac{\pi}{3}$ .

### §3 Construction of polynomial perturbations

In this section, we study various perturbations of the Hamiltonian systems previously introduced. We begin with the presentation of the classical material that we will use in order to construct polynomial perturbations of Hamiltonian systems, and to analyze their effect on the unperturbed system.

#### 3.1. Poincaré map, displacement function and Melnikov integral

Let us consider any perturbed Hamiltonian system of the form

$$\dot{x} = H_y + \varepsilon f(x, y), \quad \dot{y} = -H_x + \varepsilon g(x, y), \quad (10)$$

where  $H$ ,  $f$  and  $g$  are polynomials and  $\varepsilon$  is a real coefficient. Suppose that the unperturbed system, obtained for  $\varepsilon = 0$ , admits a non-degenerate center  $\bar{p} = (\bar{x}, \bar{y})$  with a family of closed orbits  $\gamma_h$  included in the so-called *energy levels*  $H^{-1}(\{h\})$  of the Hamiltonian function  $H$ , continuously depending on  $h \in ]0, h_0[$ , encircling  $\bar{p}$ . The Melnikov integral [20] at  $\bar{p}$ , (also called *Abelian integral* [21], [22]), is defined by

$$M(\bar{p}, h) = \oint_{\gamma_h} f(x, y)dy - g(x, y)dx. \quad (11)$$

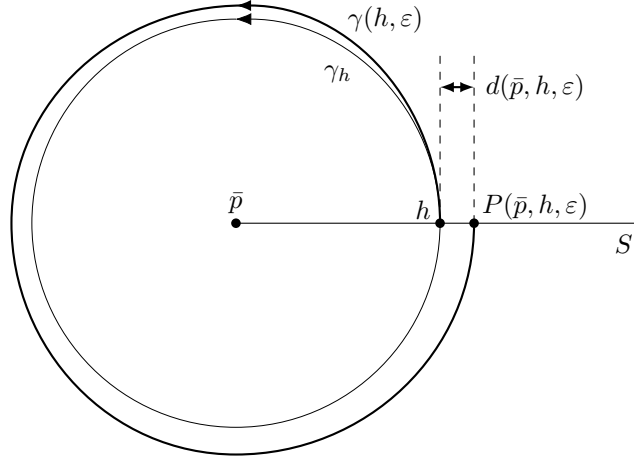
Equivalently, we have

$$\oint_{\gamma_h} f(x, y)dy - g(x, y)dx = \iint_{\Gamma_h} (f_x + g_y)dxdy, \quad (12)$$

where  $\Gamma_h$  denotes the interior of the closed curve  $\gamma_h$ .

Now let us consider a cross section  $S$  of the family of closed orbits  $\gamma_h$ ,  $h \in ]0, h_0[$ , stemming from  $\bar{p}$ , parametrized by  $h$ , and  $\varepsilon$  sufficiently small. It is well known that one can define a Poincaré map (see [8] or [23] for instance), independent of the choice of the cross section  $S$ , that associates to each value of  $h \in ]0, h_0[$ , the value  $P(\bar{p}, h, \varepsilon)$  that corresponds to the first return of the perturbed orbit  $\gamma(h, \varepsilon)$  across  $S$  (see Figure 4). We then define the displacement function

$$d(\bar{p}, h, \varepsilon) = P(\bar{p}, h, \varepsilon) - h, \quad (13)$$



**Figure 4:** Poincaré first return map and displacement function. Under a small perturbation, the closed orbit  $\gamma_h$  changes continuously. Its first return across the section  $S$  is captioned by the Poincaré map. The displacement function is a measure of the variation between this first return and the starting point.

for  $h \in ]0, h_0[$  and  $\varepsilon$  sufficiently small. It is known [8] that the Poincaré map, and consequently the displacement function, are analytic functions in their domains of definition, when  $H$ ,  $f$  and  $g$  are analytic, and *a fortiori* polynomials. The next theorem gives a fundamental relationship between the displacement function and the Melnikov integral. A detailed proof is given in [21] for instance.

**Theorem 2** (Poincaré-Pontryagin). *The displacement function satisfies*

$$d(\bar{p}, h, \varepsilon) = \varepsilon [M(\bar{p}, h) + \varepsilon \varphi(h, \varepsilon)],$$

as  $\varepsilon$  tends to 0, where  $\varphi(h, \varepsilon)$  is analytic and uniformly bounded for  $(h, \varepsilon)$  in a compact region near  $(h, 0)$ , with  $h \in ]0, h_0[$ .

The following lemma establishes the link between the zeros of the Melnikov integral and the existence of limit cycles in the perturbed system (10).

**Lemma 1.** *We suppose that  $M(\bar{p}, h)$  is not identically zero for  $h \in ]0, h_0[$ . Then the following statements hold.*

1. *If there exists  $h_1 \in ]0, h_0[$  such that  $M(\bar{p}, h_1) = 0$  and  $M'(\bar{p}, h_1) \neq 0$ , then the perturbed system (10) admits a unique limit cycle bifurcating from  $\gamma_{h_1}$ . Moreover, this limit cycle is hyperbolic.*
2. *If there exist  $h_1, h_2 \in ]0, h_0[$  such that  $M(\bar{p}, h_1) > 0$  and  $M(\bar{p}, h_2) < 0$ , then the perturbed system (10) admits at least one limit cycle between  $\gamma_{h_1}$  and  $\gamma_{h_2}$ .*

The first statement is proved in detail in [21]. The second statement is presented in [13] without proof. For the sake of completeness, we intend to give a detailed proof in the present paper.

*Proof of the second statement.* By assumption, there exist  $h_1, h_2 \in ]0, h_0[$  such that the Melnikov integral at  $\bar{p}$  satisfies  $M(\bar{p}, h_1) > 0$  and  $M(\bar{p}, h_2) < 0$ . For  $\varepsilon \neq 0$  and  $h \in ]0, h_0[$ , we consider the function  $\delta$  defined by  $\delta(\bar{p}, h, \varepsilon) = \varepsilon^{-1}d(\bar{p}, h, \varepsilon)$ . By virtue of the Poincaré-Pontryagin Theorem, we have

$$\delta(\bar{p}, h, \varepsilon) = M(\bar{p}, h) + \varepsilon \varphi(h, \varepsilon).$$

It follows that

$$\begin{aligned} \delta(\bar{p}, h_1, \varepsilon) &= M(\bar{p}, h_1) + \varepsilon \varphi(h_1, \varepsilon) \geq \frac{1}{2}M(\bar{p}, h_1) > 0, \quad 0 < |\varepsilon| \leq \varepsilon_1, \\ \delta(\bar{p}, h_2, \varepsilon) &= M(\bar{p}, h_2) + \varepsilon \varphi(h_2, \varepsilon) \leq \frac{1}{2}M(\bar{p}, h_2) < 0, \quad 0 < |\varepsilon| \leq \varepsilon_2. \end{aligned}$$

Now let us consider  $\varepsilon_0$  such that  $0 < |\varepsilon_0| < \min(\varepsilon_1, \varepsilon_2)$ . Since  $\delta$  is analytic and satisfies

$$\delta(\bar{p}, h_1, \varepsilon_0) > 0, \quad \delta(\bar{p}, h_2, \varepsilon_0) < 0,$$

the Rolle theorem guarantees that there exists a (not necessarily unique)  $h(\varepsilon_0) \in ]h_1, h_2[$  such that  $\delta(\bar{p}, h(\varepsilon_0), \varepsilon_0) = 0$ . Thus the perturbed system admits at least one limit cycle between  $\gamma_{h_1}$  and  $\gamma_{h_2}$  for any sufficiently small  $\varepsilon$ , and this achieves the proof.  $\square$

It is remarked that the second statement does not guaranty the uniqueness nor the hyperbolicity of the limit cycle appearing in the perturbed system for  $\varepsilon$  sufficiently small. In particular, bifurcations involving multiple limit cycles can occur when  $\varepsilon$  varies in a neighborhood of 0.

### 3.2. Perturbation of a line and a ring of centers

In this section, our aim is to construct a perturbation in the case of basic geometric configurations of the Hamiltonian system (3) introduced in the previous section. In [13], the authors construct a system admitting centers which are located on the orthogonal axes of the plane, that is  $x = 0$  and  $y = 0$ . This disposal of the centers in the unperturbed system is a key ingredient in their paper. We begin with the situation when the points of the family  $\mathcal{P}$  are located on a vertical axis.

**Theorem 3.** *Let  $\mathcal{P} = \{(x_0, y_1), \dots, (x_0, y_n)\}$  be a family of  $n$  points located on the vertical axis  $x = x_0$ , with  $y_1 < y_2 < \dots < y_n$ ,  $r$  be a positive integer, and  $H$  be a polynomial Hamiltonian function such that each point  $p \in \mathcal{P}$  is a non-degenerate center for the Hamiltonian system*

$$\dot{x} = H_y, \quad \dot{y} = -H_x.$$

*Then there exists a polynomial  $f_{x_0}(x)$  of degree  $2r + 1$ , such that the perturbed system*

$$\dot{x} = H_y + \varepsilon f_{x_0}(x), \quad \dot{y} = -H_x$$

*admits at least  $n \times r$  limit cycles for  $\varepsilon$  sufficiently small, that are located by nests of  $r$  limit cycles in a neighborhood of each point  $p \in \mathcal{P}$ . Furthermore,  $f_{x_0}(x)$  is given by*

$$f_{x_0}(x) = \sum_{s=0}^r (-1)^{r-s} \alpha_s (x - x_0)^{2s+1}, \quad (14)$$

*with real coefficients  $\alpha_s$ ,  $0 \leq s \leq r$ , which enjoy the property*

$$\alpha_0 < \alpha_1 < \dots < \alpha_r = 1. \quad (15)$$

*Proof.* Let us suppose that  $x_0 = 0$ . By assumption, each point  $p \in \mathcal{P}$  is a non-degenerate center for the unperturbed system

$$\dot{x} = H_y, \quad \dot{y} = -H_x.$$

By virtue of Theorem 1, for each  $p = (0, y_i) \in \mathcal{P}$ , there exists a continuous band of closed orbits  $\gamma_h^i$  encircling  $p = (0, y_i)$ ,  $1 \leq i \leq n$ . Our aim is to build a polynomial perturbation  $f_{x_0}(x)$  such that the Melnikov integrals at each point of the family  $\mathcal{P}$ , change of sign  $r$  times along the orbits of the latter continuous bands. Let us choose  $n$  orbits  $\gamma_0^1, \gamma_0^2, \dots, \gamma_0^n$  encircling the points  $(0, y_1), (0, y_2), \dots, (0, y_n)$  respectively. We begin by considering the function  $N_0$  defined for any close orbit  $\ell$  by

$$N_0(\ell) = \oint_{\ell} x^{2r+1} dy.$$

In particular, we have

$$N_0(\gamma_0^i) = \oint_{\gamma_0^i} x^{2r+1} dy = \iint_{\Gamma_0^i} (2r+1)x^{2r} dx dy > 0, \quad 1 \leq i \leq n,$$

where  $\Gamma_0^i$  denotes the interior of the closed orbit  $\gamma_0^i$ ,  $1 \leq i \leq n$ . Next, we introduce, for each  $i \in \{1, \dots, n\}$ , the function  $N_1^i$  defined for any close orbit  $\ell$  by

$$N_1^i(\ell) = \oint_{\ell} (x^{2r+1} - \alpha_{r-1}^i x^{2r-1}) dy,$$

where we have chosen  $n$  sufficiently small coefficients  $\alpha_{r-1}^1, \alpha_{r-1}^2, \dots, \alpha_{r-1}^n$  with  $\alpha_{r-1}^i < 1$ , for  $1 \leq i \leq n$ , such that

$$N_1^i(\gamma_0^i) = \oint_{\gamma_0^i} (x^{2r+1} - \alpha_{r-1}^i x^{2r-1}) dy = \iint_{\Gamma_0^i} [(2r+1)x^{2r} - (2r-1)\alpha_{r-1}^i x^{2r}] dx dy > 0,$$

for  $1 \leq i \leq n$ . By introducing  $\alpha_{r-1} = \min(\alpha_{r-1}^i, 1 \leq i \leq n)$ , it follows that the  $i$ -dependence of  $N_1^i$  vanishes. More precisely, we consider the function  $N_1$  defined for any close orbit  $\ell$  by

$$N_1(\ell) = \oint_{\ell} (x^{2r+1} - \alpha_{r-1} x^{2r-1}) dy,$$

and we obtain:

$$N_1(\gamma_0^i) = \oint_{\gamma_0^i} (x^{2r+1} - \alpha_{r-1} x^{2r-1}) dy > 0,$$

for all  $i \in \{1, \dots, n\}$ . Next we choose  $n$  sufficiently small closed orbits  $\gamma_1^1, \gamma_1^2, \dots, \gamma_1^n$  contained in the interior of  $\gamma_0^1, \gamma_0^2, \dots, \gamma_0^n$  and encircling  $(0, y_1), (0, y_2), \dots, (0, y_n)$  respectively, such that

$$N_1(\gamma_1^i) = \oint_{\gamma_1^i} (x^{2r+1} - \alpha_{r-1} x^{2r-1}) dy < 0,$$

for all  $i \in \{1, \dots, n\}$ . At this stage, we have proved, by virtue of Lemma 1, that the perturbed system

$$\dot{x} = H_y + \varepsilon(x^{2r+1} - \alpha_{r-1} x^{2r-1}), \quad \dot{y} = -H_x,$$

admits, for  $\varepsilon$  sufficiently small,  $n$  limit cycles which are located in a neighborhood of each point  $p \in \mathcal{P}$ . Next, we apply the same method and construct a finite sequence  $\alpha_s$ ,  $0 \leq s \leq r$ , with  $\alpha_0 < \alpha_1 < \dots < \alpha_r = 1$ , and  $n$  finite families of closed orbits

$$(\gamma_s^1)_{0 \leq s \leq r}, (\gamma_s^2)_{0 \leq s \leq r}, \dots, (\gamma_s^n)_{0 \leq s \leq r},$$

encircling  $(0, y_1), (0, y_2), \dots, (0, y_n)$  respectively, satisfying the property

$$\gamma_{s+1}^i \subset \gamma_s^i, \quad 0 \leq s \leq r-1, \quad 1 \leq i \leq n,$$

such that the function  $N_r$  defined for any close orbit  $\ell$  by

$$N_r(\ell) = \oint_{\ell} [x^{2r+1} - \alpha_{r-1} x^{2r-1} + \dots + (-1)^r \alpha_0 x] dy$$

admits  $r$  changes of signs along the orbits  $\gamma_s^i$ :

$$N_r(\gamma_0^i) > 0, \quad N_r(\gamma_1^i) < 0, \quad \dots, \quad 1 \leq i \leq n.$$

Finally, we set  $f_{x_0}(x) = x^{2r+1} - \alpha_{r-1} x^{2r-1} + \dots + (-1)^r \alpha_0 x$ . For each closed orbit  $\gamma_s^i$ ,  $0 \leq s \leq r$ ,  $1 \leq i \leq n$ , we have

$$N_r(\gamma_s^i) = M((0, y_i), h_s^i),$$

with  $\gamma_s^i \subset H^{-1}(\{h_s^i\})$ . Thus, the Melnikov integrals at each point  $p \in \mathcal{P}$  change their signs  $r$  times along the orbits contained in the continuous bands encircling  $(0, y_i)$ ,  $1 \leq i \leq n$ . By virtue of the second statement in the Lemma 1, we conclude that the perturbed system admits, for  $\varepsilon$  sufficiently small, at least  $n \times r$  limit cycles that are located by nests of  $r$  limit cycles around each point  $p \in \mathcal{P}$ . This achieves the proof in the case  $x_0 = 0$ .

The case  $x_0 \neq 0$  is easily treated by a change of variable  $x \rightarrow x + x_0$ . □

Obviously, this theorem applies to the Hamiltonian function  $G$  given by (4), when  $\mathcal{P}$  is composed of points that are located on a vertical axis. It also applies to the Hamiltonian function  $F$  given by (6). We remark that a similar result is stated in [13] without proof. In [14], it is proved that co linear centers can be simultaneously perturbed in order to bifurcate into nests of limit cycles, by using another technique involving the focus values [24], [23]. With that method, the limit cycles tend to 0 when  $\varepsilon$  approaches 0. This bifurcation is sometimes called the *multiple Hopf bifurcation* [23]. At the opposite, in our reasoning, the limit cycles tend to the non isolated closed orbits  $\gamma_s^i$ ,  $0 \leq s \leq r$ ,  $1 \leq i \leq n$ , when  $\varepsilon$  decreases to 0. Another important fact has to be emphasized, about the coefficients  $\alpha_s$ ,  $0 \leq s \leq r$ , of the perturbation  $f_{x_0}$ . Indeed, those coefficients depend on the choice of the closed orbits  $\gamma_s^i$ ,  $0 \leq s \leq r$ ,  $1 \leq i \leq n$ . However, we can estimate uniformly the values of the perturbation given by (14).

**Proposition 4.** *Let  $f_{x_0}$  be the polynomial perturbation constructed by the Theorem 3, given by the expression (14), with coefficients  $\alpha_s$ ,  $1 \leq s \leq r$ , satisfying the property (15). Let  $\eta \in ]0, 1]$ ,  $x_1 \in \mathbb{R}$  and  $p_1$  be any point of the axis  $x = x_1$ . Then, there exist positive constants  $k_j$ ,  $1 \leq j \leq 2r + 1$ , which depend only on  $x_0$ ,  $x_1$  and  $r$ , so that for all  $(a, b)$  in the closed ball  $B(p_1, \eta)$ , we have*

$$\left| \frac{d^j f_{x_0}}{dx^j}(a) \right| \leq k_j, \quad 1 \leq j \leq 2r + 1. \quad (16)$$

*Proof.* By virtue of equation (15), we have for any  $b \in \mathbb{R}$ :

$$|f_{x_0}(a)| \leq \sum_{s=0}^r |\alpha_s(a - x_0)|^{2s+1} \leq \sum_{s=0}^r |a - x_0|^{2s+1}.$$

Now, if  $(a, b) \in B(p_1, \eta)$ , we have

$$\begin{aligned} |f_{x_0}(a)| &\leq \sum_{s=0}^r (|a - x_1| + |x_1 - x_0|)^{2s+1} \\ &\leq \sum_{s=0}^r \sum_{j=0}^{2s+1} \binom{2s+1}{j} |a - x_1|^j |x_1 - x_0|^{2s+1-j} \\ &\leq \sum_{s=0}^r \sum_{j=0}^{2s+1} \binom{2s+1}{j} \eta^j |x_1 - x_0|^{2s+1-j} \\ &\leq \sum_{s=0}^r \sum_{j=0}^{2s+1} \binom{2s+1}{j} |x_1 - x_0|^{2s+1-j}, \end{aligned}$$

since  $\eta \leq 1$ . We obtain the expected estimation by setting

$$k_1 = \sum_{s=0}^r \sum_{j=0}^{2s+1} \binom{2s+1}{j} |x_1 - x_0|^{2s+1-j}.$$

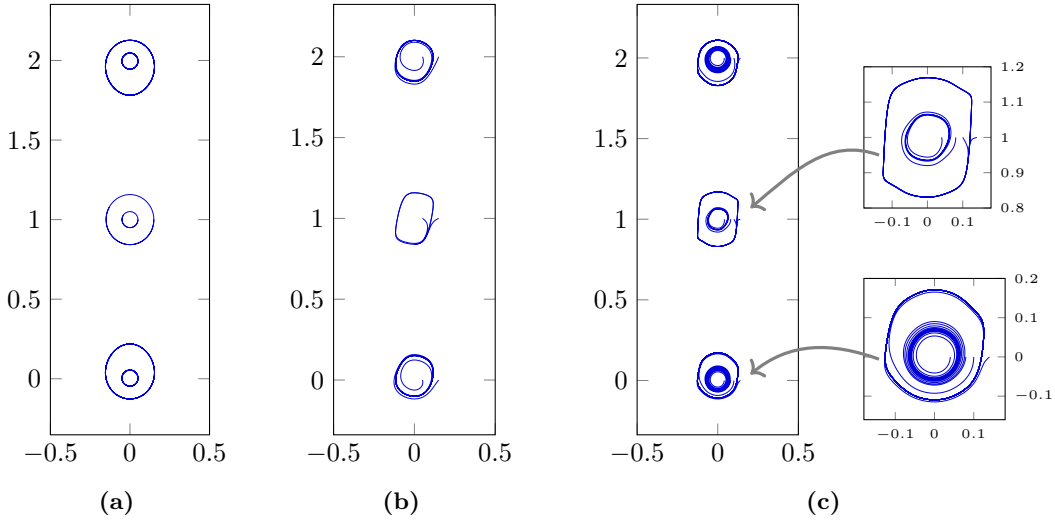
The estimation for  $\left| \frac{d^j f_{x_0}}{dx^j}(a) \right|$  is proved by similar arguments.  $\square$

Those uniform bounds will be used in the last section, in order to build a perturbation of a grid of centers.

**Example 1.** Let us consider the perturbed Hamiltonian system obtained with a family a 3 centers located on the axis  $x = 0$ ,  $\mathcal{P} = \{(0, 0), (0, 1), (0, 2)\}$ , defined by

$$\dot{x} = G_y + \varepsilon f(x), \quad \dot{y} = -G_x. \quad (17)$$

We have computed the orbits of the perturbed system in two cases. The first case is obtained with  $f(x) = x^3 - 0.01x$ , and provokes the birth of one stable limit cycle around each point  $p \in \mathcal{P}$ . The second case is obtained with  $f(x) = x^7 - 0.0233x^5 + 0.00014416x^3 - 2.304 \cdot 10^{-7}x$ , and leads to the emergence of two stable limit cycles encircling one unstable limit cycle around each point  $p \in \mathcal{P}$ . Since the unstable limit cycles are repulsive, the orbits starting in a neighborhood of those limit cycles are pushed towards the stable limit cycles. The corresponding phase portraits are depicted in Figure 5. We remark that the perturbation is oriented along a single direction, thus the limit cycles present a flattening. A numerical approximation of the displacement function for the point  $(0, 0)$  is shown in Figure 6, in order to visualize the link between its zeros and the limit cycles in the perturbed system.  $\triangleleft$



**Figure 5:** Perturbation of a line of 3 centers. (a) The unperturbed system (17) obtained for  $\varepsilon = 0$  admits 3 non-degenerate centers located on the axis  $x = 0$ . (b) Under a suitable perturbation of degree 3, the perturbed system (17) admits 3 stable limit cycles. (c) Under a suitable perturbation of degree 7, the perturbed system (17) admits 6 stable limit cycles and 3 unstable limit cycles.

We continue with the construction of a perturbation that makes centers located on a circle bifurcate into nests of limit cycles. The perturbation is oriented along the radial direction stemming from the origin. Similar perturbations have been studied in [25].

**Theorem 4.** Let  $\mathcal{P} = \{(x_i, y_i), 1 \leq i \leq n\}$  be a family of  $n$  points located on a circle of positive radius  $\rho$ , centered at the origin. Let  $r$  be a positive integer, and  $H$  be a polynomial Hamiltonian function such that each point  $p \in \mathcal{P}$  is a non-degenerate center for the Hamiltonian system

$$\dot{x} = H_y, \quad \dot{y} = -H_x.$$

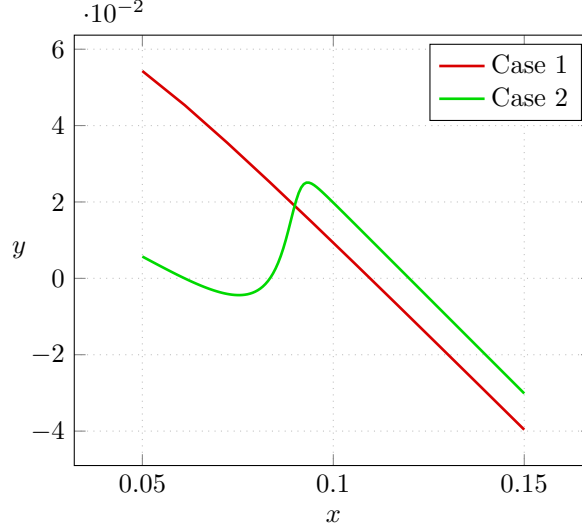
Then there exists a polynomial  $f$  of degree  $2r + 1$ , such that the perturbed system

$$\dot{x} = H_y + \varepsilon x f(x^2 + y^2), \quad \dot{y} = -H_x + \varepsilon y f(x^2 + y^2)$$

admits at least  $n \times r$  limit cycles for  $\varepsilon$  sufficiently small, that are located by nests of  $r$  limit cycles in a neighborhood of each point  $p \in \mathcal{P}$ .

*Proof.* By assumption, each point  $p \in \mathcal{P}$  is a non-degenerate center for the unperturbed system

$$\dot{x} = H_y, \quad \dot{y} = -H_x.$$



**Figure 6:** Displacement functions for a cross section by the axis  $y = 0$ , in the neighborhood of  $(0, 0)$ , for the example 1. In the first case, the perturbation is of degree 3, thus the displacement function admits a unique zero. In the second case, the perturbation is of degree 7, and the displacement function admits 3 zeros. When the derivative of the displacement function at one of those zeros is negative, the corresponding limit cycle is stable. When it is positive, the corresponding limit cycle is unstable.

By virtue of Theorem 1, for each  $p = (x_i, y_i) \in \mathcal{P}$ , there exists a continuous band of closed orbits  $\gamma_h^i$  encircling  $p = (x_i, y_i)$ ,  $1 \leq i \leq n$ . The Melnikov integral at  $p$  is given by

$$\begin{aligned} M(p, h_i) &= \oint_{\gamma_h^i} x f(x^2 + y^2) dy - y f(x^2 + y^2) dx \\ &= \iint_{\Gamma_h^i} \left[ \frac{\partial}{\partial x} (x f(x^2 + y^2)) + \frac{\partial}{\partial y} (y f(x^2 + y^2)) \right] dx dy \\ &= 2 \iint_{\Gamma_h^i} [f(x^2 + y^2) + (x^2 + y^2) f'(x^2 + y^2)] dx dy, \end{aligned}$$

where  $\Gamma_h^i$  denote the interior of the closed orbit  $\gamma_h^i$ . We introduce  $q(u) = f(u) + u f'(u)$ , choose real coefficients  $\beta_s$ ,  $0 \leq s \leq r$ , by the same method as in the proof of the Theorem 3, and we set

$$q(u) = \sum_{s=0}^r (-1)^{r-s} \beta_s u^{2s+1},$$

so that the integrals  $\iint_{\Gamma_h^i} q(x^2 + y^2) dx dy$  change their signs  $r$  times along the orbits encircling each point  $p \in \mathcal{P}$ . Finally, we determine  $f$  by solving the equation  $f(u) + u f'(u) = q(u)$ . It suffices to set  $\alpha_s = \frac{(-1)^{n-s} \beta_s}{1+s}$  for each  $s \in \{0, \dots, r\}$ , and

$$f(u) = \sum_{s=0}^r \alpha_s u^{2s+1}.$$

The polynomial  $f(u)$  satisfies the expected properties. This achieves the proof.  $\square$

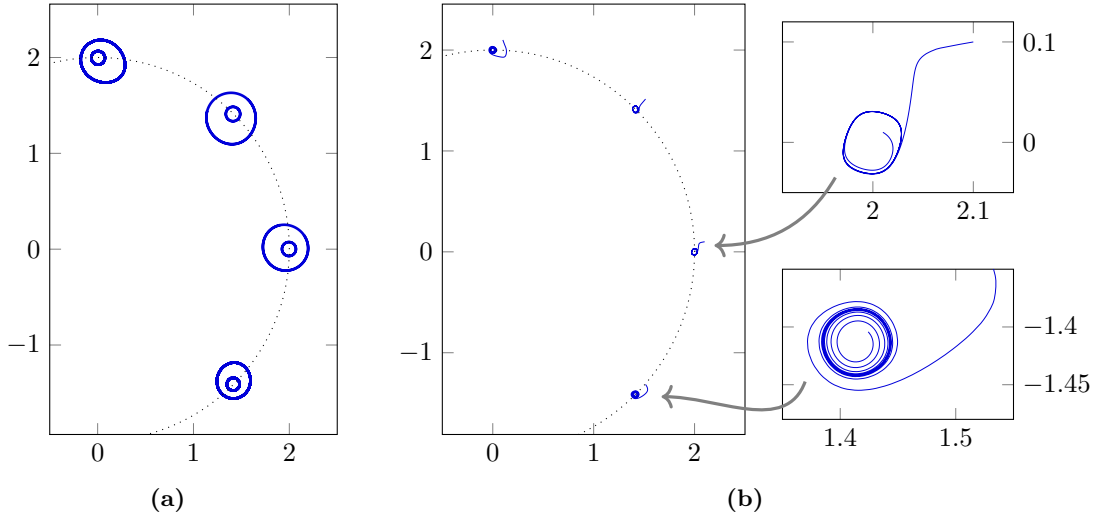
**Example 2.** We have computed the orbits of the perturbed system for

$$\mathcal{P} = \{(2, 0), (\sqrt{2}, \sqrt{2}), (0, 2), (\sqrt{2}, -\sqrt{2})\},$$

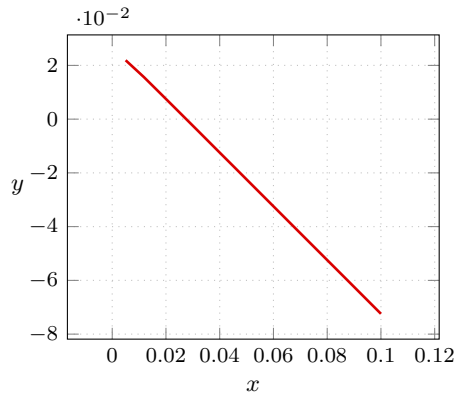
with  $f(r) = r^3 - 0.01r$ ,  $r = x^2 + y^2 - 4$ , and

$$\begin{cases} \dot{x} = H_y + \varepsilon x f(r) \\ \dot{y} = -H_x + \varepsilon y f(r). \end{cases} \quad (18)$$

According to the Theorem 4, each non-degenerate center bifurcates into a limit cycle. The corresponding phase portraits are shown in the Figure 7, as well as a numerical approximation of the displacement function in a neighborhood of the point  $(2, 0)$ , with a cross section by the horizontal axis. We remark once again the typical flattening of the limit cycles, due to the fact that the perturbation is oriented along a single direction.  $\triangleleft$



**Figure 7:** (a) Phase portrait of the system (3) for  $\mathcal{P} = \{(2, 0), (\sqrt{2}, \sqrt{2}), (0, 2), (\sqrt{2}, -\sqrt{2})\}$ , showing 4 non-degenerate centers, encircled by continuous bands of closed orbits, located on a ring. (b) Under a suitable perturbation oriented along the normal of the unit circle, each non-degenerate center bifurcates into a single limit cycle of small amplitude.



**Figure 8:** Displacement function in the neighborhood of the point  $(2, 0)$  in the system (18), obtained for a cross section by the horizontal axis. The zero of the displacement function corresponds to the birth of the limit cycle in the perturbed system (18). The derivative of the displacement function is negative, thus the corresponding limit cycle is stable.



### 3.3. Perturbation of a Hamiltonian system along its gradient

In this section, we prove that any planar polynomial Hamiltonian system admitting centers can be perturbed along its gradient, so that each center bifurcates into a nest of limit cycles.

**Theorem 5.** *Let consider any planar differential system of the form*

$$\dot{x} = H_y + \varepsilon H_x f(H), \quad \dot{y} = -H_x + \varepsilon H_y f(H), \quad (19)$$

where  $H, f$  are polynomials and  $\varepsilon$  is a real coefficient, admitting a finite number of non-degenerate centers  $p_1, \dots, p_n$ . Then  $f$  can be chosen so that each non-degenerate center  $p_i$ ,  $1 \leq i \leq n$ , of the unperturbed system (19) obtained for  $\varepsilon = 0$ , bifurcates into a given number  $r > 0$  of limit cycles for  $\varepsilon$  sufficiently small.

*Proof.* Assume that  $\bar{p} = (\bar{x}, \bar{y})$  is a non-degenerate center for the unperturbed system (19) obtained for  $\varepsilon = 0$ . By virtue of the Theorem 1, there exists a continuous band of closed orbits  $\gamma_{h,\bar{p}} \subset H^{-1}(\{h\})$  encircling  $\bar{p}$ , with  $h \in ]0, h_0]$ , and we have

$$(H_{xx}H_{yy} - H_{xy}^2)(\bar{x}, \bar{y}) > 0.$$

It follows that  $H_{xx}(\bar{x}, \bar{y}) \times H_{yy}(\bar{x}, \bar{y}) > 0$ , thus  $H_{xx}(\bar{x}, \bar{y})$  and  $H_{yy}(\bar{x}, \bar{y})$  have the same sign. Let us suppose for instance that  $H_{xx}(\bar{x}, \bar{y}) > 0$  and  $H_{yy}(\bar{x}, \bar{y}) > 0$ . Then we have

$$H_{xx}(\bar{x}, \bar{y}) + H_{yy}(\bar{x}, \bar{y}) > 0.$$

Now, let us choose a closed neighborhood  $W_{\bar{p}}$  of  $\bar{p} = (\bar{x}, \bar{y})$  and a positive coefficient  $\alpha$ , for which we have

$$H_{xx}(x, y) + H_{yy}(x, y) \geq \alpha > 0, \quad \forall (x, y) \in W_{\bar{p}}.$$

We compute the Melnikov integral at  $\bar{p} = (\bar{x}, \bar{y})$ :

$$\begin{aligned} M(\bar{p}, h) &= \oint_{\gamma_{h,\bar{p}}} H_x f(H) dy - H_y f(H) dx \\ &= f(h) \oint_{\gamma_{h,\bar{p}}} H_x dy - H_y dx, \end{aligned}$$

since  $H = h$  along the orbit  $\gamma_{h,\bar{p}}$ . It follows that

$$M(\bar{p}, h) = f(h) \iint_{\Gamma_{h,\bar{p}}} (H_{xx} + H_{yy}) dx dy,$$

where  $\Gamma_{h,\bar{p}}$  denotes the interior of  $\gamma_{h,\bar{p}}$ . Next, we choose  $f$  so that it admits  $r$  zeros  $h_1, h_2, \dots, h_r$  in the interior of  $]0, h_0]$ . Those zeros can be chosen in such a way that  $f'(h_i) \neq 0$ , and  $\gamma_{h,\bar{p}} \subset W_{\bar{p}}$ ,  $1 \leq i \leq r$ , for all non-degenerate center  $\bar{p}$ . Since  $f(h_i) = 0$ , we can deduce that  $M(\bar{p}, h_i) = 0$ , for  $1 \leq i \leq r$ . Afterwards, we introduce

$$\psi(h) = \iint_{\Gamma_{h,\bar{p}}} (H_{xx} + H_{yy}) dx dy,$$

and we compute the derivative of the Melnikov integral at  $\bar{p}$  with respect to  $h$ :

$$M'(\bar{p}, h) = f'(h)\psi(h) + f(h)\psi'(h).$$

Since  $f(h_i) = 0$ , we have  $M'(\bar{p}, h_i) = f'(h_i)\psi(h_i)$ , for each  $i \in \{1, \dots, r\}$ . Furthermore, it holds that

$$\psi(h_i) = \iint_{\Gamma_{h_i,\bar{p}}} (H_{xx} + H_{yy}) dx dy \geq \alpha |\Gamma_{h_i,\bar{p}}| > 0,$$

where  $|\Gamma_{h_i, \bar{p}}|$  denotes the area of  $\Gamma_{h_i, \bar{p}}$ . It follows that  $M'(\bar{p}, h_i) \neq 0$ . By the same method, we can treat the case when  $H_{xx}(\bar{x}, \bar{y}) < 0$  and  $H_{yy}(\bar{x}, \bar{y}) < 0$ . The first statement of the Lemma 1 applies, and guarantees the existence and uniqueness of a limit cycle bifurcating from  $\gamma_{h_i, \bar{p}}$ , for each  $i \in \{1, \dots, r\}$ , thus a nest of  $r$  limit cycles encircling  $\bar{p}$ . Since the reasoning holds for each non-degenerate center  $\bar{p}$ , we have proved the existence of  $n$  nests of  $r$  limit cycles encircling each non-degenerate center of the system (19). The proof is complete.  $\square$

This Theorem applies to the Hamiltonian systems (3), (5) and (7) introduced previously. In particular, by choosing  $n = r$ , it guarantees the existence of polynomial perturbations under which  $n^2$  centers can bifurcate into  $n^3$  limit cycles, distributed by nests of  $n$  limit cycles. However, it is obvious that the constructed polynomial perturbation  $(H_x f(H), H_y f(H))^T$  has a degree which is not of order  $n$ , but of order  $n^2$ . Thus the necessity to improve the construction of the perturbation.

It is remarkable that the system (19) can be viewed as a Hamiltonian system perturbed by a gradient system. Indeed, it can be rewritten

$$\dot{x} = H_y + \varepsilon K_x, \quad \dot{y} = -H_x + \varepsilon K_y, \quad (20)$$

where  $K = F(H)$ ,  $F$  being a primitive of  $f$ . Many properties of that system can be analyzed additionally, for instance the possibility it offers to make continuous bands of periodic orbits encircling multiple critical points, bifurcate into limit cycles of great amplitude, or polycycles (see example 4 below). For those reasons, we consider that the system (20) is canonical.

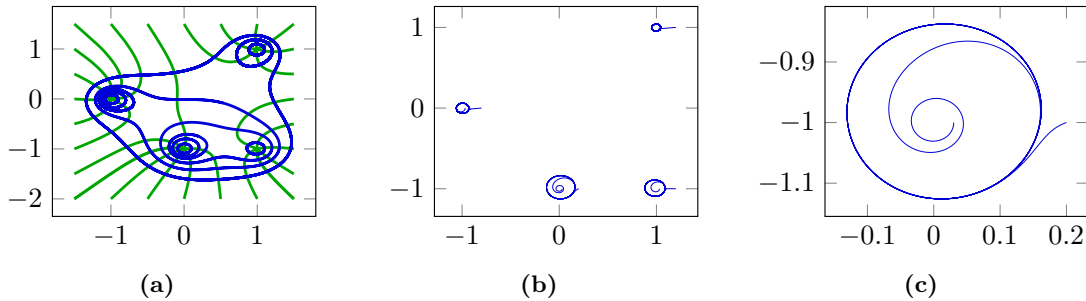
**Example 3.** Let us consider again the Hamiltonian system presented in the first section, defined by

$$\dot{x} = G_y, \quad \dot{y} = -G_x,$$

where  $G = G(\mathcal{P})$ , with  $\mathcal{P} = \{(-1, 0), (0, -1), (1, -1), (1, 1)\}$ . We introduce the polynomial  $f(u) = 0.01u - u^3$ , and perturb the latter system along its gradient by considering the system

$$\dot{x} = G_y + \varepsilon G_x f(G), \quad \dot{y} = -G_x + \varepsilon G_y f(G).$$

According to the Theorem 5, each point in  $\mathcal{P}$  bifurcates into a single limit cycle (see Figure 9). We remark that the 4 limit cycles have different radii. Since they have been obtained by a perturbation along the gradient of the energy levels of the Hamiltonian function  $G$ , they present an homogeneous shape, at the opposite of the limit cycles obtained by a perturbation along a single direction, which present a flattening.  $\triangleleft$

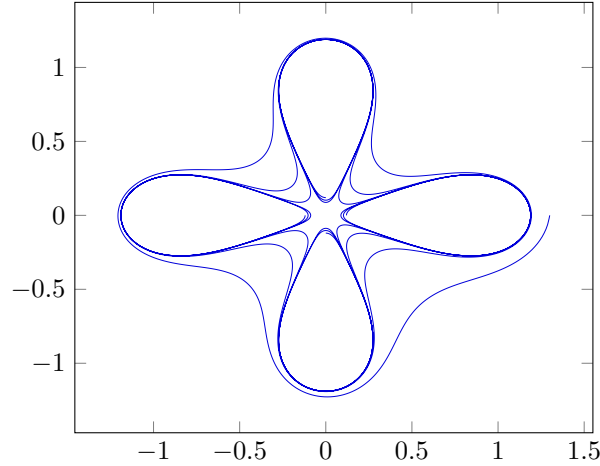


**Figure 9:** (a) Phase portrait of the system (3) with  $\mathcal{P} = \{(-1, 0), (0, -1), (1, -1), (1, 1)\}$ , showing continuous bands of closed orbits in the neighborhood of each point  $p \in \mathcal{P}$ . The green lines correspond to the orbits of its gradient. (b) Under the effect of a suitable perturbation, each non-degenerate center bifurcates into a unique limit cycle. (c) The limit cycles present an homogeneous shape, without flattening.

**Example 4.** Next, let us consider the system defined by

$$\dot{x} = G_y + \varepsilon G_x f(G), \quad \dot{y} = -G_x + \varepsilon G_y f(G), \quad (21)$$

where  $G = G(\mathcal{P})$  with  $\mathcal{P} = \{(-1, 0), (1, 0), (0, -1), (0, 1)\}$ , and  $f(u) = u - \frac{1}{2}$ . The unperturbed system admits 4 non-degenerate centers located at each point of  $\mathcal{P}$ , and a saddle at the origin, with four homoclinic orbits starting and ending at this saddle, corresponding to the energy level  $G = \frac{1}{2}$ . The perturbation function  $f$  is chosen according to this energy level, thus the perturbed system exhibits 5 limit cycles, with 4 limit cycles corresponding to the homoclinic orbits of the unperturbed system, and 1 limit polycycle passing through the saddle, and attracting the orbits which start in the exterior of the homoclinic orbits. (see Figure 10).  $\triangleleft$



**Figure 10:** Phase portrait of the system (21), showing 4 limit cycles corresponding to the homoclinic orbits of the unperturbed system, and 1 limit polycycle passing through the saddle, and attracting the orbits which start in the exterior of the homoclinic orbits.

## Conclusion and perspectives

By introducing a new class of Hamiltonian systems, we have proposed various situations in which a planar polynomial system admits a given set of non-degenerate centers that can be arbitrarily located. We have constructed polynomial perturbations which provoke the bifurcation of those centers into nests of limit cycles. Many geometric distributions have not been studied in this article, thus we believe that our method is likely to produce improved results.

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