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TOPOLOGICAL PHASE TRANSITION V: INTERIOR SEPARATIONS AND CYCLONE FORMATION THEORY

RUIKUAN LIU, TIAN MA, SHOUHONG WANG, AND JIAYAN YANG

ABSTRACT. We establish in this paper a systematic theory for interior separations of fluid flows, and apply it to the formation of tornados and hurricanes. First, we derive the interior separation equations, which fully characterize when, where and how an interior separation occurs. A U-flow separation theory is then established, and is applied to the formation mechanism of tornados and hurricanes in geophysical fluid dynamics.

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1. INTRODUCTION

The atmospheric and oceanic flows exhibit recurrent large-scale patterns. These patterns, while evolving irregularly in time, manifest characteristic frequencies across a large range of temporal scales, from intraseasonal through interdecadal; see among many others [16, 3, 4, 17, 5]. The appropriate modeling and theoretical understanding of such irregular geophysical fluid mechanical patterns remain a great challenge.

It is clear that for a two-dimensional incompressible flow, the only two processes, that lead to more complex or turbulent fluid behavior, are through boundary-layer separations or through interior separations; see among others [2, 6]. Hence the fundamental level understanding of this challenge problem boils down to topological phase transition associated with boundary-layer and interior separations of fluid flows. In the previous paper [15], two of the authors obtained a systematic theory for boundary-layer separations.

The main objectives of this paper are to provide a systematic theory for interior separations, and to apply it to the formation of tornados and hurricanes. This is part of the research program initiated recently by the authors on the theory and applications of topological phase transitions (TPTs), including

- (1) quantum phase transitions [12],
- (2) galactic spiral structure [14],
- (3) sunspots and solar eruptions [13], and
- (4) boundary-layer separations of fluid flows [15].

The main ingredients of this paper are as follows.

First, as a TPT study the change in its topological structure in the physical space of the system, the geometric theory of incompressible flows developed by the authors plays a crucial role for the study of TPTs of fluids, including in particular the boundary-layer separation studied in the previous paper [15] and the interior separation in this paper. The complete account of this geometric theory is given in the authors' research monograph [11]. This theory has been directly used to study the transitions of topological structure associated with the quantum phase transitions of Bose-Einstein condensates (BEC), superfluidity and superconductivity [12].

Second, at the kinematic level, a structural bifurcation theorem was proved in [11]; see also Theorem 2.2.

Third, hurricanes, typhoons, and tornados are typical interior separation phenomena in geophysical fluid dynamics, which are caused by external wind-driven forces and by the non-homogenous temperature

distributions. Therefore, the crucial factors for the formation of interior separations in the atmospheric and oceanic flows are the three ingredients: (1) the initial velocity field, (2) the external force, and (3) the temperature. Hence the dynamical fluid model for interior separations has to incorporate properly the heat effect.

The Boussinesq equations are mainly for convective flows, and are not suitable for studying interior separations, associated in particular with such geophysical processes as hurricanes, typhoons, and tornados. For this purpose, two of the authors Yang and Liu [18] introduced the horizontal heat-driven fluid dynamical equations; see (2.13)-(2.16). This system couples the Navier-Stokes equations and the heat diffusion equation with a new equation of state; see Section 2.2 for details.

Fourth, we derive the interior separation equations with velocity given in the form:

$$(1.1) \quad u(x, t) = \varphi + vt,$$

The separation equations include all physical information about the interior separations of the solution u for the system, in terms of the initial state (φ, T_0) and the external force f .

Fifth, theoretical analysis and observations show that interior separation can only occur when

- one of φ or v in (1.1) is U-shaped, which we call the U-flow, and the other is either a U-flow or a flat flow, as shown in Figure 3.3; and
- φ and v have reversed orientations.

In view of this fact, a precise U-flow theory is then established, including in particular precise interior separation equations, and explicit formulas for the interior separation point (x_0, t_0) .

Sixth, tornados and hurricanes (or typhoons) are among the most important phenomena in the atmospheric sciences, which are topological phase transitions described by interior separations of the atmospheric flows. We refer interested readers to [9, 1, 7, 8, 10] and the references therein for the phenomena. A typical development of a hurricane consists of several stages including an early tropical disturbance, a tropical depression, a tropical storm, and finally a hurricane stage.

Using the U-flow theory, we derive the formation mechanism of tornados and hurricanes, providing precise conditions for their formation and explicit formulas on the time and location where tornados and hurricanes form. Basically, we demonstrate that the early stage of a hurricane is through the horizontal interior flow separations, and we identify

the physical conditions for the formation of the U-flow, corresponding to the tropical disturbance, and the temperature-driven counteracting force needed as the source for tropical depression.

The paper is organized as follows. In Section 2, we introduce the kinematic conditions for interior separation, recall the basic fluid model suitable for interior separation, and derive the interior separation equations. The U-flow theory is established in Section 3, and is applied to the formation of tornados and hurricanes in Section 4.

2. INTERIOR SEPARATION FLUID DYNAMICS

2.1. Interior separations of fluid flows. Interior separation of fluid flows is a common phenomenon in fluid dynamics, especially in geophysical fluid dynamics, such as the formation of hurricanes, typhoons and tornados, and gyres of oceanic flows. In general, the interior separation refers to sudden appearance of a vortex from the interior of a fluid flow. In [11], Ma and Wang have established an exact mathematical theory to determine interior separations for two-dimensional (2D) incompressible vector fields. Based on the Ma-Wang's theory, Yang and Liu [18] introduced a system of horizontal heat-driven fluid dynamical equations, and discussed the predictability of interior separation for the solutions of these equations.

For convenience, we briefly recall the interior separation theory of 2D incompressible vector fields. Let $\Omega \subset \mathbb{R}^2$ be an open set, and $u(x, \lambda)$ be a one-parameter family of divergence-free vector fields defined on Ω .

Definition 2.1. *We say that a family of 2D incompressible vector fields $u(x, \lambda)$ has an interior separation from (x_0, λ_0) with $x_0 \in \Omega$, if $u(x, \lambda)$ is locally topologically equivalent to the structure of Figure 2.1(a) near x_0 for any $\lambda < \lambda_0$, and to the structure of Figure 2.1(b) for $\lambda_0 < \lambda$.*



FIGURE 2.1. A schematic diagram of interior separation, (a) a parallel flow near x_0 , and (b) a vortex flow.

Usually, the parameter λ is taken as the time t . Let $u(x, t)$ be the first-order Taylor expansion with respect to t at $t_0 > 0$:

$$(2.1) \quad \begin{aligned} u(x, t) &= u^0(x) + (t - t_0)u^1(x) + o(|t - t_0|), \\ u^0(x) &= u(x, t_0), \\ u^1(x) &= \left. \frac{\partial u}{\partial t} \right|_{t=t_0}. \end{aligned}$$

Let $x_0 \in \Omega$ be a degenerate singular point of u^0 , and the Jacobian $Du^0(x_0) \neq 0$. Then, there are two unit vectors e_1 and e_2 orthogonal to each other, satisfying that

$$(2.2) \quad \begin{aligned} Du^0(x_0)e_1 &= 0, \\ Du^0(x_0)e_2 &= \alpha e_1 \quad \text{with } \alpha \neq 0. \end{aligned}$$

Theorem 2.2 (Interior Separation [11]). *Let $u(x, t)$ be a one-parameter family of 2D divergence-free vector fields, and have the Taylor expansion (2.1) at $t = t_0$. If $x_0 \in \Omega$ is an isolated singular point if $u^0(x)$ with $Du^0(x_0) \neq 0$, and satisfies that*

$$(2.3) \quad \begin{aligned} \text{ind}(u^0, x_0) &= 0, \\ u^1(x_0) \cdot e_2 &\neq 0, \end{aligned}$$

where e_2 is the vector as in (2.2), then $u(x, t)$ has an interior separation from (x_0, t_0) .

Theorem 2.2 holds also true for $u(x, \lambda)$, where λ is another parameter instead of the time t .

2.2. Horizontal heat-driven fluid dynamical equations. Hurricane, typhoon, and tornado are typical interior separation phenomena in geophysical fluid dynamics, which are caused by external wind-driven forces and by the non-homogenous temperature distributions. Therefore, the crucial factors for the appearance of interior separations in the atmospheric and oceanic flows are the three ingredients: (1) the initial velocity field, (2) the external force, and (3) the temperature. Hence the dynamical fluid model for interior separations has to incorporate properly the heat effect.

The Boussinesq equations are the classical fluid dynamical model coupling the heat conduction equation. However, this model can not be used as the dynamical equations to describe interior separation of fluid flows. To show this point, we recall the Boussinesq equations,

given by

$$(2.4) \quad \begin{aligned} \frac{\partial u}{\partial t} + (u \cdot \nabla)u &= \nu \Delta u - \frac{1}{\rho} \nabla p - (1 - \alpha T)g\vec{k} + f, \\ \frac{\partial T}{\partial t} + (u \cdot \nabla)T &= \kappa \Delta T + Q, \\ \operatorname{div} u &= 0. \end{aligned}$$

We note that the driving force acting on the fluid by heat is the term in the first equation of (2.4) as follows

$$(2.5) \quad f_T = -\alpha T g \vec{k},$$

where g is the acceleration due to gravity, α is the thermal expansion coefficient, T represents the temperature, and $\vec{k} = (0, 0, 1)$ is the unit vector orthogonal to the horizontal plane. By (2.5), it is easy to see that equations (2.4) mainly govern the thermal convective fluid motion in the vertical direction, rather than the horizontal motion such as the interior separation fluid flows.

In [18], a fluid dynamical model coupling heat was established, which can describe the plane fluid flows as the interior separation behaviors. We introduce this model in the following. Consider the Navier-Stokes equations

$$(2.6) \quad \begin{aligned} \frac{\partial u}{\partial t} + (u \cdot \nabla)u &= \nu \Delta u - \frac{1}{\rho} \nabla p + f, \quad x \in \Omega, \\ \operatorname{div}(\rho u) &= 0, \end{aligned}$$

and the coupling heat equation

$$(2.7) \quad \frac{\partial T}{\partial t} + (u \cdot \nabla)T = \kappa \Delta T, \quad x \in \Omega,$$

where $\Omega \subset \mathbb{R}^2$ is an open set, and $u = (u_1, u_2)$ is the velocity governing a horizontal fluid motion.

To make the temperature T into the Navier-Stokes equations (2.6), we introduce the state equations in thermodynamics. For gaseous systems, we have

$$(2.8) \quad pV = nRT,$$

where n is the molar number, and R is the gaseous constant. In addition, the gas density is given by

$$(2.9) \quad \rho = nm/V, \quad m \text{ is the molecular mass.}$$

Thus, (2.8) can be written as

$$(2.10) \quad p = \beta_0 \rho T, \quad \beta_0 = R/m \text{ is the special gas constant.}$$

For liquid systems, the equation of state is expressed as

$$(2.11) \quad V = \alpha_1 T - \alpha_2 p + V_0,$$

where $\alpha_1, \alpha_2 > 0$ are constants. By (2.9), (2.11) is rewritten as

$$(2.12) \quad p = \beta_1 T - \frac{\beta_2}{\rho} + \beta_3,$$

where $\beta_1, \beta_2, \beta_3$ are parameters.

The equations (2.6), (2.7), (2.10), and (2.12) are together to constitute the horizontal heat-driven fluid dynamical equations:

$$(2.13) \quad \frac{\partial u}{\partial t} + (u \cdot \nabla)u = \nu \Delta u - \frac{1}{\rho} \nabla p + f,$$

$$(2.14) \quad \frac{\partial T}{\partial t} + (u \cdot \nabla)T = \kappa \Delta T,$$

$$(2.15) \quad \operatorname{div}(\rho u) = 0,$$

$$(2.16) \quad p = \begin{cases} \beta_0 \rho T & \text{for gaseous systems,} \\ \beta_1 T - \beta_2 \rho^{-1} + \beta_3 & \text{for liquid systems.} \end{cases}$$

Remark 2.3. As $\rho \neq 0$, the orbit of u with $\operatorname{div} u = 0$ are the same as those of v with $\operatorname{div}(\rho v) = 0$. Since the proof of Theorem 2.2 relies only on the topological structure of orbits of divergence-free vector fields, Theorem 2.2 is also valid for the vector fields u satisfying (2.15).

2.3. Interior separation equations. Fluid separation equations are crucial mathematical tools to study vortex generation problem. For boundary-layer separations, separation equations for both the free and the rigid boundary conditions were established in [11, 15]; they play a very important role in both the boundary-layer separation theory and its applications. For interior separation, such separations are clearly needed as well.

To this end, we introduce two physically sound conditions:

1) The gradient of ρ is small:

$$(2.17) \quad \nabla \rho \simeq 0.$$

By (2.15), it implies that

$$(2.18) \quad \operatorname{div} u = -\frac{1}{\rho} \nabla \rho \cdot u \simeq 0.$$

Physically, for large scale motion, the relaxation time for the fluid mobility is much smaller than the relaxation time of compression. Hence for the large scale fluid, the change of density ρ is small, leading to the validity of assumption (2.17).

2) The relaxation time t_0 of an interior separation is very small:

$$(2.19) \quad 0 < t_0 \ll 1,$$

where the relaxation time refers to the time elapsed from the initial time when the separation condition is formed to the time when the separation takes place. It is also called the separation relaxation time.

Based on the model (2.13)–(2.16) and the two physical conditions (2.17) and (2.19), we now derive the separation equation.

Let the initial values of u and T are given by

$$(2.20) \quad u(x, 0) = \varphi(x), \quad T(x, 0) = T_0(x).$$

Inserting (2.16) into (2.13), integrating both sides of the resulting equation with respect to t , and ignoring the terms of $\nabla\rho$ (by (2.17)), we deduce that

$$(2.21) \quad u(x, t) = \varphi(x) + \int_0^t [\nu\Delta u - (u \cdot \nabla)u - \alpha\nabla T + f]dt,$$

for $0 < t < t_0$, where φ is as in (2.20), α is given by

$$(2.22) \quad \alpha = \begin{cases} \beta_0 & \text{for a gaseous system,} \\ \beta_1/\rho & \text{for a liquid system,} \end{cases}$$

$\beta_0 = R/m$ is as in (2.10), and β_1 is as in (2.12). Take the first-order Taylor expansion on time t for the solution $(u(x, t), T(x, t))$ of (2.13)–(2.16):

$$(u(x, t), T(x, t)) = (\varphi(x) + t\tilde{u}(x, t), T_0(x) + t\tilde{T}(x, t)),$$

and insert it into the right side of (2.21). Then we obtain that

$$(2.23) \quad \begin{aligned} u(x, t) = & \varphi(x) + [\nu\Delta\varphi - (\varphi \cdot \nabla)\varphi - \alpha\nabla T_0 + f]t \\ & + \int_0^t [\nu\Delta\tilde{u} - \alpha\nabla\tilde{T} - (\varphi \cdot \nabla)\tilde{u} - (\tilde{u} \cdot \nabla)\varphi]tdt \\ & + \int_0^t (\tilde{u}\nabla\tilde{u}) t^2 dt. \end{aligned}$$

By (2.18), we have

$$\operatorname{div}\varphi \simeq 0, \quad \operatorname{div}[\nu\Delta\varphi - (\varphi \cdot \nabla)\varphi - \alpha\nabla T_0 + f] \simeq 0.$$

Hence we deduce that

$$(2.24) \quad \frac{\partial\varphi_1}{\partial x_2} \frac{\partial\varphi_2}{\partial x_1} + \left(\frac{\partial\varphi_1}{\partial x_1} \right)^2 + \frac{\alpha}{2}\Delta T_0 - \frac{1}{2}\operatorname{div}f \simeq \nu\Delta\operatorname{div}\varphi \simeq 0,$$

where $\varphi = (\varphi_1, \varphi_2)$. It is clear that

$$\int_0^t [\nu\Delta\tilde{u} - \alpha\nabla\tilde{T} - (\varphi \cdot \nabla)\tilde{u} - (\tilde{u} \cdot \nabla)\varphi]tdt = o(t).$$

Hence, by (2.19), the first-order approximation of (2.23) on t ($0 < t < t_0$) and (2.24) lead to the following interior separation equations:

$$(2.25) \quad u(x, t) = \varphi + [\nu\Delta\varphi - (\varphi \cdot \nabla)\varphi - \alpha\nabla T_0 + f]t,$$

$$(2.26) \quad \frac{\partial\varphi_1}{\partial x_2} \frac{\partial\varphi_2}{\partial x_1} + \left(\frac{\partial\varphi_1}{\partial x_1}\right)^2 + \frac{\alpha}{2}\Delta T_0 - \frac{1}{2}\operatorname{div}f \simeq 0,$$

where $\varphi = (\varphi_1, \varphi_2)$ and T_0 are the initial functions as in (2.20).

Remark 2.4. The interior separation equations (2.25) and (2.26) include all physical information about the interior separations of the solution u for system (2.13)–(2.16) given by the initial state (φ, T_0) and the external force f .

3. U-FLOW THEORY

3.1. Interior separation of fluid dynamical equations. Theorem 2.2 provides a sufficient condition for general 2D incompressible vector fields to undergo an interior separation. Now, we apply the separation equation (2.25) to derive the interior separation theorem for solutions of the horizontal heat-driven fluid dynamical equations (2.13)–(2.16). Let

$$(3.1) \quad v(x) = \nu\Delta\varphi - (\varphi \cdot \nabla)\varphi - \alpha\nabla T_0 + f.$$

Then $u(x, t)$ in (2.25) is written as

$$(3.2) \quad u(x, t) = \varphi(x) + tv(x).$$

For such $u(x, t)$, condition (2.3) becomes the following:

- (1) there is a point $x_0 \in V$, with $V \subset \Omega$ being a neighborhood of x_0 , such that

$$(3.3) \quad \begin{aligned} \varphi(x) + tv(x) &\neq 0, & \forall 0 \leq t < t_0, \\ \varphi(x_0) + t_0v(x_0) &= 0, \end{aligned}$$

- (2) the Jacobian of $u^0 = \varphi + t_0v$ at x_0 is nonzero:

$$(3.4) \quad Du^0(x_0) = D(\varphi + t_0v)|_{x_0} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

- (3) for the unit vector e_2 as in (2.2), we have

$$(3.5) \quad v(x_0) \cdot e_2 \neq 0.$$

Namely, Conditions (3.3)–(3.5) for $u(x, t)$ in (3.2) are equivalent to (2.3). Therefore, by Theorem 2.2, we obtain immediately the following interior separation theorem for the solution of equations (2.13)–(2.16).

Theorem 3.1. *Let (φ, T_0) be the initial functions as in (2.20), f be the external force, and v be defined by (3.1). If (φ, T_0, f) satisfies (3.3)–(3.5), then under the conditions (2.17) and (2.19), the solution $u(x, t)$ of (2.13)–(2.16) has an interior separation from (x_0, t_0) .*

3.2. U-flow pattern for interior separation. Theorem 3.1 shows that under conditions (3.3)–(3.5) an interior separation occurs. However, we need to know which expressions of (φ, T_0, f) satisfy (3.3)–(3.5). To solve this problem, we should investigate the topological structure of the two vector fields φ and v in (3.2).

For convenience, we rewrite (3.1) and (3.2) in the form

$$(3.6) \quad \begin{aligned} u(x, t) &= u^0(x) + (t - t_0)v(x), \\ u^0(x) &= \varphi(x) + t_0v(x), \\ v(x) &= \nu\Delta\varphi - (\varphi \cdot \nabla)\varphi - \alpha\nabla T_0 + f. \end{aligned}$$

In [11], Ma and Wang proved that if $u(x, t)$ in the form of (3.6) satisfies (3.3) and (3.4), then $\text{ind}(u^0, x_0) = 0$ and $u^0(x)$ is topologically equivalent to the structure near $x_0 \in \Omega$, as shown in Figure 3.1; see [11, Section 5.6.1].

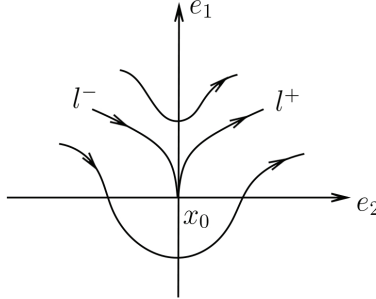


FIGURE 3.1. The flow pattern of the vector field u^0 satisfying $u^0(x_0) = 0$, $\text{ind}(u^0, x_0) = 0$, and $Du^0(x_0) \neq 0$. Here the two orbits l^- and l^+ of u^0 connected to x_0 are tangent to the e_1 -axis at x_0 , e_1, e_2 are as in (2.2).

Thus, it is clear that $u^0 = \varphi + t_0v$ is a superposition of φ and t_0v . It implies that if u^0 has the structure as in Figure 3.1 and $\varphi \neq 0$ in a neighborhood of x_0 , then by the evolution procedure:

$$u(x, 0) = \varphi \xrightarrow{t \rightarrow t_0} u(x, t_0) = \varphi + t_0v,$$

we can see that φ and v have the reversed orientations, and their flow structures are either the U-flow pattern as shown in Figure 3.2 or one of the two patterns as shown in Figure 3.3 (a) and (b). In particular, if φ (or v) is in the pattern of Figure 3.2 then v (or φ) must be in one of the patterns (a) and (b) in Figure 3.3.

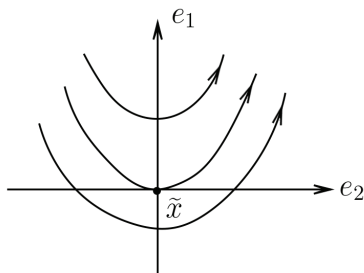


FIGURE 3.2. U-flow pattern of a vector field u with $u \cdot e_2 > 0$ at \tilde{x} .

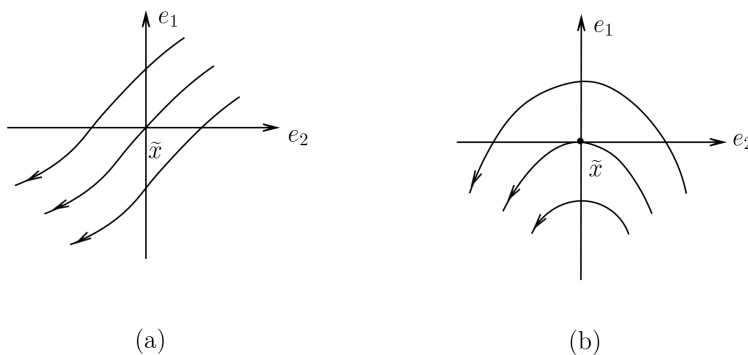


FIGURE 3.3. (a) A flat flow pattern of u with $u \cdot e_2 < 0$ at \tilde{x} , and (b) U-flow pattern of u with $u \cdot e_2 < 0$ at \tilde{x} .

3.3. Physical conclusions of U-flow theory. Mathematically, the three situations that lead to the formation of interior separations are as follows:

- (i) φ is a U-flow and v is a flat flow;
- (ii) φ is a flat flow and v is a U-flow; and
- (iii) φ and v are U-flows.

However, in reality, case (i) appears more often. Hence, here we only consider the case:

$$(3.7) \quad \begin{aligned} &\varphi \text{ is a U-flow,} \\ &v \text{ is a flat flow, and} \\ &\varphi \text{ and } v \text{ have reversed orientations,} \end{aligned}$$

where φ and v are as in (3.6).

Based on (3.7) we now establish the U-flow theory of the interior separation for the solutions of equations (2.13)–(2.16). The main components of this theory are as follows:

1). The physical data (φ, T_0, f) are approximated by their first-order polynomials. In fact, we take the Taylor expansions near \tilde{x} :

$$(3.8) \quad \begin{aligned} \varphi &= \varphi(\tilde{x}) + D\varphi(\tilde{x})(x - \tilde{x}) + \text{h.o.t.}, \\ T_0 &= T(\tilde{x}) + \nabla T(\tilde{x})(x - \tilde{x}) + \text{h.o.t.}, \\ f &= f(\tilde{x}) + Df(\tilde{x})(x - \tilde{x}) + \text{h.o.t.}, \end{aligned}$$

where h.o.t. represents the higher-order terms. Because φ, T_0, f are *nearly independent*, by the second separation equation (2.26), we see that the coefficients of the n -th order terms for $n \geq 2$ are almost zero, and

$$(3.9) \quad \left. \frac{\partial \varphi_1}{\partial x_2} \frac{\partial \varphi_2}{\partial x_1} \right|_{\tilde{x}}, \quad \left. \frac{\partial \varphi_1}{\partial x_1} \right|_{\tilde{x}}, \quad \left. \frac{\partial \varphi_2}{\partial x_2} \right|_{\tilde{x}}, \quad \left. \frac{\partial f_1}{\partial x_1} \right|_{\tilde{x}}, \quad \left. \frac{\partial f_2}{\partial x_2} \right|_{\tilde{x}} \simeq 0.$$

Then we derive the first physical conclusion for the U-flow theory.

Physical Conclusion 3.2. *The physical data φ, T_0 and f are, in essence, the first-order polynomials as*

$$(3.10) \quad \begin{aligned} \varphi &= \varphi(\tilde{x}) + \begin{pmatrix} 0 & \frac{\partial \varphi_1(\tilde{x})}{\partial x_2} \\ \frac{\partial \varphi_2(\tilde{x})}{\partial x_1} & 0 \end{pmatrix} \begin{pmatrix} x_1 - \tilde{x}_1 \\ x_2 - \tilde{x}_2 \end{pmatrix} + \text{h.o.s.t.}, \\ T_0 &= T(\tilde{x}) + \frac{\partial T(\tilde{x})}{\partial x_1}(x_1 - \tilde{x}_1) + \frac{\partial T(\tilde{x})}{\partial x_2}(x_2 - \tilde{x}_2) + \text{h.o.s.t.}, \\ f &= f(\tilde{x}) + \begin{pmatrix} 0 & \frac{\partial f_1(\tilde{x})}{\partial x_2} \\ \frac{\partial f_2(\tilde{x})}{\partial x_1} & 0 \end{pmatrix} \begin{pmatrix} x_1 - \tilde{x}_1 \\ x_2 - \tilde{x}_2 \end{pmatrix} + \text{h.o.s.t.}, \end{aligned}$$

and by (3.9), $\left. \frac{\partial \varphi_1}{\partial x_2} \frac{\partial \varphi_2}{\partial x_1} \right|_{\tilde{x}} \simeq 0$, where $\tilde{x} = (\tilde{x}_1, \tilde{x}_2)$ is a point near the separation point x_0 , and the abbreviation *h.o.s.t.* refers to higher-order terms with small coefficients.

2). For the U-flow in Figure 3.2, we take the coordinate (x_1, x_2) with \tilde{x} as its origin, the x_1 -axis in e_2 direction, and the x_2 -axis in e_1

direction. By (3.1), let φ be the U-flow as in Figure 3.2. By (3.3) and (3.10), $\varphi = (\varphi_1, \varphi_2)$ should be in the form:

$$(3.11) \quad \begin{aligned} \varphi_1 &= a_0 + a_1x_2 + a_2x_2^2 + \text{h.o.s.t.}, \\ \varphi_2 &= b_0x_1 + \text{h.o.s.t.}, \end{aligned}$$

where

$$(3.12) \quad a_0, a_2, b_0 > 0, \quad a_1 \simeq 0, \quad a_0 \text{ is relatively small.}$$

The second-order term $a_2x_2^2$ in φ_1 is specifically retained to ensure the zero of the superposition is an isolated zero, and to ensure the formation of interior separation. We need the condition that

$$(3.13) \quad b_0 \text{ is of the same order as or larger than } a_0,$$

to ensure the flow φ in (3.11) is a U-flow.

Also by (3.10)–(3.12), T_0 and $f = (f_1, f_2)$ are as

$$(3.14) \quad \begin{aligned} T_0 &= \tau_0 + \tau_1x_1 + \tau_2x_2 + \text{h.o.s.t.}, \\ f_1 &= -f_1^0 + f_1^2x_2 + \text{h.o.s.t.}, \\ f_2 &= -f_2^0 + f_2^1x_1 + \text{h.o.s.t.}, \end{aligned}$$

and the vector field $v = (v_1, v_2)$ in (3.6) can be written as

$$(3.15) \quad \begin{aligned} v_1 &= -\alpha\tau_1 - f_1^0 + f_1^2x_2 + \text{h.o.s.t.}, \\ v_2 &= -a_0b_0 - \alpha\tau_2 - f_2^0 + f_2^1x_1 + \text{h.o.s.t.} \end{aligned}$$

Because v is a flat flow as in Figure 3.3(a), and v and φ have reversed orientations, (3.15) is rewritten as

$$(3.16) \quad v = -(\alpha\tau_1 + f_1^0, \alpha\tau_2 + a_0b_0 + f_2^0) + \text{h.o.s.t.},$$

where

$$(3.17) \quad \alpha\tau_1 + f_1^0 > 0.$$

Based on (3.11)–(3.17), we get the second physical conclusion for the U-flow theory.

Physical Conclusion 3.3. *The first separation equation (2.25) for the U-flow φ and flat flow v in (3.7) is expressed as*

$$(3.18) \quad \begin{aligned} u_1 &= (a_0 + a_1x_2 + a_2x_2^2) - t(\alpha\tau_1 + f_1^0) + \text{h.o.s.t.}, \\ u_2 &= b_0x_1 - t(\alpha\tau_2 + a_0b_0 + f_2^0) + \text{h.o.s.t.}, \end{aligned}$$

and the parameters in (3.18) satisfy (3.12), (3.17), and

$$(3.19) \quad 0 < 4a_0a_2 - a_1^2 \ll 4a_2(\alpha\tau_1 + f_1^0).$$

In particular, the separation equation (3.18) satisfying (3.12), (3.17) and (3.19) obeys conditions (3.3)–(3.5); consequently an interior separation takes place from (x_0, t_0) with

$$(3.20) \quad \begin{aligned} x_0 &= \left(\frac{t_0(\alpha\tau_2 + a_0b_0 + f_2^0)}{b_0}, -\frac{a_1}{2a_2} \right), \\ t_0 &= \frac{4a_0a_2 - a_1^2}{4a_2(\alpha\tau_1 + f_1^0)}, \end{aligned}$$

where α is the thermal parameter as in (2.22).

Remark 3.4. The expression (3.18) is the unique form for the U-flow (3.7) to generate a vortex from an interior point $x_0 \in \Omega$. Namely, the initial physical data φ and v in (3.7) must take the form as in (3.18) for the formation of an interior separation at x_0 . In particular, the parameters a_0 and b_0 represent the strength of the U-flow φ , (τ_1, τ_2) is the initial temperature gradient ∇T_0 at \tilde{x} , and (f_1^0, f_2^0) is the main part of the external force f . The six parameters $\{a_0, b_0, \tau_1, \tau_2, f_1^0, f_2^0\}$ determine the energy level and scale of the generated vortex.

In addition, for the cases (ii) and (iii) listed in the beginning of Section 3.3, the associated unique forms of the U-flow for the formation of interior separations are just slight variations of (3.18).

4. TORNADO AND HURRICANE FORMATIONS

4.1. Atmospheric cyclone dynamical model. In nature, tornados and hurricanes (or typhoons) are the most important phenomena in the atmospheric sciences, which are topological phase transitions described by interior separations of the atmospheric flows.

In this section, we shall address the tornado and hurricane formation mechanism by using the U-flow theory established in the last section.

The main factors for the formation of tornados and hurricanes are

- the initial velocity field φ ,
- the initial temperature field T_0 , and
- the Coriolis force f , measuring the earth's rotational effect.

To derive the atmospheric cyclone dynamic model, we adopt the conventional β -plane assumption.

Let the earth's hemisphere be approximatively given as a rectangle:

$$(4.1) \quad \Omega = (0, 2\pi r_0) \times (0, \pi r_0/2),$$

where r_0 is the earth's radius. The Coriolis force is taken as

$$(4.2) \quad f = -\beta x_2 \vec{k} \times u = (\beta x_2 u_2, -\beta x_2 u_1),$$

where $\beta = 2\omega/r_0$, and ω is the the angular velocity of the earth's rotation. For the domain Ω , the coordinate system is (x_1, x_2) with x_1 -axis being the longitude coordinate and x_2 -axis for the latitude. (4.1) and (4.2) are called the β -plane assumption.

With the above Coriolis force f , the system (2.4) serves as the basic atmospheric cyclone dynamical model for the initial stages of the formation of hurricanes.

With the Coriolis force (4.2), the interior separation equations (2.25) and (2.26) become

$$(4.3) \quad \begin{cases} u(x, t) = \varphi + tv(x), \\ v(x) = \nu\Delta\varphi - (\varphi \cdot \nabla)\varphi - \alpha\nabla T_0 - \beta x_2 \vec{k} \times \varphi, \end{cases}$$

$$(4.4) \quad \frac{\partial\varphi_1}{\partial x_2} \frac{\partial\varphi_2}{\partial x_1} + \left(\frac{\partial\varphi_1}{\partial x_1}\right)^2 + \frac{\alpha}{2}\Delta T_0 + \frac{\beta x_2}{2} \left(\frac{\partial\varphi_1}{\partial x_2} - \frac{\partial\varphi_2}{\partial x_1}\right) + \frac{\beta}{2}\varphi_1 \simeq 0.$$

4.2. Atmospheric cyclone generations. We now discuss tornado and hurricane in the northern hemisphere based on the U-flow theory. We proceed in a few steps as follows.

1). *Initial physical data φ , T_0 and the Coriolis force f .* In the northern hemisphere, the atmospheric flows near the earth surface often appear in the form of the U-flows as shown in Figure 4.1, in which $x_2 = 0$ represents the equator.

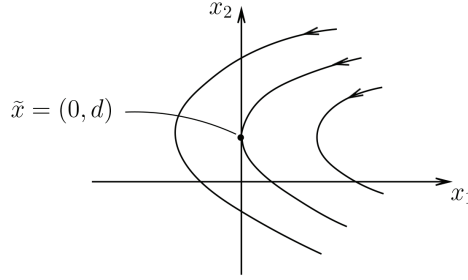


FIGURE 4.1. A U-flow in the northern hemisphere, $x_2 = 0$ represents the equator, and $\tilde{x} = (0, d)$ with $d > 0$ is the point as in (3.10).

Corresponding to (3.11) and (3.14), as the initial velocity field $\varphi = (\varphi_1, \varphi_2)$, the U-flow in Figure 4.1 should be rewritten in the following form

- the initial flow:

$$(4.5) \quad \begin{aligned} \varphi_1 &= -b_0(x_2 - d) + \text{h.o.s.t.}, \\ \varphi_2 &= -(a_0 + a_1x_1 + a_2x_1^2) + \text{h.o.s.t.}, \end{aligned}$$

- the initial temperature T_0 :

$$(4.6) \quad T_0 = \tau_1(\theta - x_2) + \text{h.o.s.t.} \quad (\theta > 0 \text{ is a constant}),$$

- the Coriolis force $f = (f_1, f_2)$:

$$(4.7) \quad \begin{aligned} f_1 &= \beta x_2 \varphi_2 = -a_0 \beta x_2 + \text{h.o.s.t.}, \\ f_2 &= -\beta x_2 \varphi_1 = b_0 \beta x_2 (x_2 - d) + \text{h.o.s.t.}, \end{aligned}$$

where β is as in (4.2), and a_i ($0 \leq i \leq 2$), b_0 and τ_1 satisfy (3.12) and (3.17) with $f_1^0 = 0$.

2). *Interior separation equation.* By (4.3), it follows from (4.5)–(4.7) that the interior separation equation for the atmospheric cyclone is

$$(4.8) \quad \begin{aligned} u_1 &= -b_0(x_2 - d) - t[a_0 b_0 + a_0 \beta x_2] + \text{h.o.s.t.}, \\ u_2 &= -(a_0 + a_1 x_1 + a_2 x_1^2) + t[\alpha \tau_1 + b_0 \beta x_2 (x_2 - d)] + \text{h.o.s.t.}, \end{aligned}$$

3). *Conditions for the formation of tornados.* Tornadoses form in higher latitudes; hence $d > 0$ is relatively large. Let $x = x_1$, $y = x_2 - d$ in (4.8). Then the separation point (x_0, y_0, t_0) satisfy the equation

$$(4.9) \quad \begin{aligned} -b_0 y_0 - t_0 a_0 (b_0 + \beta d + \beta y_0) &= 0, \\ a_0 + a_1 x_0 + a_2 x_0^2 - t_0 (\alpha \tau_1 + b_0 \beta d y_0 + b_0 \beta y_0^2) &= 0. \end{aligned}$$

The solution of (4.9) satisfying (2.19), (3.12) and (3.17) is

$$(4.10) \quad \begin{aligned} (x_0, y_0) &= \left(-\frac{a_1}{2a_2}, -\frac{a_0(b_0 + \beta d)}{b_0 + \beta a_0 t_0} t_0 \right), \\ t_0 &= \frac{4a_0 a_2 - a_1^2}{4a_2(\alpha \tau_1 + b_0 \beta d y_0 + b_0 \beta y_0^2)} \simeq \frac{4a_0 a_2 - a_1^2}{4\alpha a_2 \tau_1}. \end{aligned}$$

By $0 < t_0 \ll 1$ and $y_0 \simeq 0$, (4.10) implies that

$$(4.11) \quad 0 < \frac{1}{\alpha \tau_1} \left(a_0 - \frac{a_1^2}{4a_2} \right) \ll 1, \quad \frac{a_0(b_0 + \beta d)}{b_0} \sim o(1).$$

These are exactly the conditions required for the formation of tornados.

4). *Conditions for the formation of hurricanes.* A hurricane can form near the equator, i.e. $d \simeq 0$. Let $x = x_1$, $y = x_2$, and $d = 0$ in (4.8). Then the location and time (x_0, y_0, t_0) , where a hurricane forms, satisfies

$$(4.12) \quad \begin{aligned} -b_0 y_0 - t_0 a_0 (b_0 + \beta y_0) &= 0, \\ a_0 + a_1 x_0 + a_2 x_0^2 - \alpha \tau_1 t_0 &= 0, \end{aligned}$$

whose solution is

$$(4.13) \quad (x_0, y_0) \simeq \left(-\frac{a_1}{2a_2}, -a_0 t_0 \right), \quad t_0 = \frac{4a_0 a_2 - a_1^2}{4\alpha \tau_1 a_2}.$$

In summary, from the above discussions, we see that the initial velocity field (4.5) of a U-flow, the initial temperature distribution (4.6) and the Coriolis force (4.7) determine the necessary conditions for the formation of tornadoes and hurricanes.

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(Liu) DEPARTMENT OF MATHEMATICS, SICHUAN UNIVERSITY, CHENGDU, P. R. CHINA

E-mail address: liuruikuan2008@163.com

(Ma) DEPARTMENT OF MATHEMATICS, SICHUAN UNIVERSITY, CHENGDU, P. R. CHINA

E-mail address: matian56@sina.com

(Wang) DEPARTMENT OF MATHEMATICS, INDIANA UNIVERSITY, BLOOMINGTON, IN 47405

E-mail address: showang@indiana.edu

(Yang) DEPARTMENT OF MEDICAL INFORMATICS AND ENGINEERING, SOUTHWEST MEDICAL UNIVERSITY, LUZHOU, P. R. CHINA

E-mail address: jiayan_1985@163.com