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The magnetization ripple: a nonlocal stochastic PDE perspective

Radu Ignat* Felix Otto†

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Abstract

The magnetization ripple is a microstructure formed by the magnetization in a thin-film ferromagnet. It is triggered by the random orientation of the grains in the poly-crystalline material. In an approximation of the micromagnetic model, which is sketched in this paper, this leads to a nonlocal (and strongly anisotropic) elliptic equation in two dimensions with white noise as a right hand side. However, like in singular Stochastic PDE, this right hand side is too rough for the non-linearity in the equation. In order to develop a small-date well-posedness theory, we take inspiration of the Da Prato-Debussche approach to singular SPDE. To this aim, we develop a Schauder theory for the non-standard symbol $|k_1|^3 + k_2^2$.

Résumé

L'ondulation de l'aimantation est une microstructure formée par l'aimantation dans un film ferromagnétique mince. Elle est générée par l'orientation aléatoire des grains poly-cristallins du matériau. Pour analyser cette microstructure, nous introduisons un modèle micromagnétique réduit qui consiste dans une équation elliptique nonlocale, nonlinéaire et anisotrope en dimension deux avec un bruit blanc. Similaire aux autres EDP stochastiques singulières, le bruit blanc est trop rugueux pour la nonlinéarité de l'équation. Afin de montrer que l'équation est bien posée pour des données petites, nous nous inspirons de l'approche de Da Prato-Debussche pour les EDP stochastiques singulières. Plus précisément, nous développons une théorie de type Schauder pour le symbol nonstandard $|k_1|^3 + k_2^2$.

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1 Introduction

The magnetization ripple in a ferromagnetic thin-film sample is the response to the polycrystallinity of the sample. In experiments, it manifests itself as an in-plane oscillation of the magnetization, predominantly in direction of the main (in-plane) magnetization direction. The fact that the sample is made up of (comparatively small) randomly oriented grains leads to an easy axis for the magnetization that is a random field and thus acts like quenched noise. The *anisotropic* response of the magnetization m to this isotropic quenched noise is a consequence of the non-local interaction of the magnetization given by the stray-field energy.

Starting from the three-dimensional micromagnetic (variational) model, we heuristically derive a reduced model (see Section 2) that zooms in on the (different) longitudinal and transversal characteristic length scales of the ripple. The reduced model is a two-dimensional, nonlocal variational model formulated in terms of the transversal magnetization component m_2 . On these scales, the quenched noise acts like random

transversal field of white-noise character (because the grains are smaller than the characteristic scales). We argue that this derivation is self-consistent (see also [29, Section V]).

There are two main challenges of the derived model that we heuristically point out in Section 3. The first challenge can already be seen on the level of the simplification that gets rid of the anharmonic term. The ensuing linear Euler-Lagrange equation can be explicitly solved in Fourier space — however the highest-order term in the energy, the exchange contribution, diverges (see Section 7.1 in Appendix). Hence the ripple should rather be analyzed on the level of the Euler-Lagrange equation than by the direct method of the calculus of variations.

The second challenge is more subtle and more serious: The nonlinearity in the Euler-Lagrange equation is too singular for its right-hand side (RHS) given by white noise ξ . More dramatically, one of the quadratic terms in the Euler-Lagrange equation cannot be given an unambiguous sense even when one plugs in the solution of the linear Euler-Lagrange equation, which is well-characterized. This situation is similar to certain classes of stochastic partial differential equation (SPDE), i.e., time-dependent nonlinear parabolic equations driven by space-time white noise. While noise in SPDEs typically models *thermal* noise and our noise is of *quenched* nature, and while these SPDEs are typically parabolic and our Euler-Lagrange equation is of (non-local) elliptic character, the mathematical challenges are identical.

In fact, the issue is to make sense of the product of a function and a distribution. This can be done in an unambiguous sense provided the function is more regular than the distribution is irregular. In order to make use of this, regularity has to be measured in a way that is consistent with the (leading-order) linear part of the equation. While in the parabolic case, this requires spaces that respect the fact that the time derivative is worth two space derivatives, in our case we have the relationship that two x_2 -derivates are worth three x_1 -derivatives. In the case of *stationary* (i.e. shift-invariant) driving noise, like is the case of white noise, there is no loss in using the scale of Hölder spaces C^α with respect to (w.r.t.) a Carnot-Carathéodory metric that respects the above scaling. On this scale, the crucial product turns out to be border-line singular: The function in this product is slightly worse than $C^{\frac{3}{4}}$ while the distribution is slightly worse than $C^{-\frac{3}{4}}$.

This situation is reminiscent of a fundamental problem in the theory of stochastic (ordinary) differential equations (SDE): The theory requires at a minimum to give a (distributional) sense of the product of (multi-dimensional) Brownian motion and of its derivative, i.e. white noise. Brownian motion is known to be slightly worse than $C^{\frac{1}{2}}$ and thus white noise slightly worse than $C^{-\frac{1}{2}}$. Stochastic analysis has found two ways out of this border-line singular situation: Ito calculus and more recently Lyons' rough path theory (see [23]). While Ito calculus uses the Martingale structure of Brownian motion and thereby the time direction and thus is not easily amenable to a treatment

of irregular spatial noise, rough path theory is oblivious to this structure. In fact, the challenge of our problem is very close to the one tackled by Da Prato and Debussche in their seminal paper [6] on the ϕ_2^4 -model, which is the jargon for the Allen-Cahn equation with the standard double-well potential driven by white noise in two space dimensions. Very much like here, there is a nonlinear (cubic) term that is borderline not-well defined. We follow the strategy laid out in [6] in many ways: By considering the difference between the solution to the non-linear and the linear problem [6, (1.5)], by using that some of the ensuing products are classically defined [6, Proposition 2.1] on the Besov scale (which includes the Hölder scale considered here), by using that the remaining singular product operations can be (Wick) renormalized [6, Lemma 3.2] (which also in its proof is quite similar to our more elementary Lemma 8), and by appealing to a contraction-mapping argument. Next to [6], our work shares the following additional feature of [5, Proposition 3.1] on the Navier-Stokes equation driven by space-time white noise in two space dimensions: Because of a symmetry, despite the borderline singular scaling, the renormalization does not show up in the final formulation. The main difference of our work w.r.t. to the previous ones lies in the non-locality of the operator (and its anisotropic scaling).

This approach of separating an “off-line” stochastic argument to renormalize certain products and a completely deterministic step of perturbative nature is substantially refined through the concepts of controlled rough paths [12] and in particular of regularity structures [14]. While the present work does not require these more subtle tools, it is certainly inspired by the underlying philosophy.

More specifically, our approach consists of two parts: The first part consists in giving an off-line definition to the singular product in the PDE (for u with RHS given by white noise ξ) when the solution v of the linear (constant-coefficient) equation is plugged in. In this case, the singular product F is the product of two *Gaussian* fields and can be characterized by Gaussian calculus: Thanks to stochastic cancellations, an *almost-sure* unambiguous (distributional) sense can be given to this product that is stable under regularization of white noise by convolution (see Section 6.2 below).

The second part consists in setting up a completely *deterministic* (i.e. path-wise) fixed-point problem in $w := u - v$ with a RHS given by the distribution $F \in C^{-\frac{3}{4}-\varepsilon}$ for every $\varepsilon > 0$. All the further non-linearities in the PDE are regular. For this second part, we have to show that:

- 1) $\mathcal{L}^{-1}C^{\alpha-2} \subset C^\alpha$ for our (anisotropic and nonlocal) linear operator \mathcal{L} (see (12));
- 2) $C^\alpha \cdot C^\beta \subset C^\beta$ for $\beta < 0 < \alpha$ with $\alpha + \beta > 0$ (i.e. the regular case).

On the technical side, we heavily use a semi-group adapted to the linear operator to characterize Hölder spaces both of positive (Remark 1) and negative order (Lemma 10), for an efficient approach to Schauder theory (Lemma 5), and for estimating the regular product of a function and a distribution (Lemma 6). This is close

in spirit to [1] (the purpose of that paper is to provide a semi-group-based approach to the para-controlled rough path approach of [13]). More precisely, [1, Proposition 2.5] is analogous to Remark 1, [1, Proposition 3.3] to Lemma 6, [1, Proposition 3.8] to Lemma 5. On the one hand, the approach in [1] is much more general, since it does not require translation invariance and only mild regularity of the geometry, in the tradition of Nash and De Giorgi. On the other hand, it does not (directly) apply to our case, since it is based on Gaussian decay estimates of the semi-group kernel, which fail in our case due to the non-locality of the generator given by the (non-smooth) symbol $|k_1|^3 + k_2^2$. When it comes to this use of the semi-group, rather than generalizing [1], we follow the less far-reaching but more elementary [27]; again the main difference comes from the non-locality of the generator and thus the restricted range of finite moments for the kernel. In fact, the Schauder theory we develop here by elementary means is not (yet) covered by the rapidly developing regularity theory for non-local elliptic operators, mainly because of the (scaling-wise) anisotropic nature of the symbol.

2 Derivation of the model

In this section, we heuristically derive the model we shall analyze. This section is logically independent and may be skipped. The ripple was first explained in [16, 15], see in particular [15, Section C] and also [2] for a valid criticism of the approximation in [16]. While this explanation was carried out on the level of the Euler-Lagrange equation and ignored non-linear terms, we work here on the level of the energy and identify the leading-order non-linear term. We essentially follow [29, Sections I, V], which in turn is based on [28], with a more detailed discussion of the regime of validity, but restricted to the simpler case of a sample in form of an infinitely extended thin-film element instead of an elongated thin-film element.

The purpose of this section is to convince the reader that the (leading-order) non-locality of our model is natural. Here is the executive summary: It is the magnetic field (more precisely, the demagnetizing or stray field) that provides the non-local interaction, typically of repulsive type. From the mathematical point of view of regularity, the corresponding energy contribution is of order 0 as opposed to the short-range attractive energy (exchange energy), which is of order 1. While this is the right counting in bulk samples, in thin-film samples, the stray-field energy effectively turns into a contribution of order $\frac{1}{2}$; and for the highly anisotropic ripple, this does provide the leading order in the longitudinal direction.

The important while subtle role of the stray-field energy in thin-film ferromagnets was recognized in a variety of rigorous (deterministic) contributions of the last two decades (see [9] for a review): In [8, 4], the above-mentioned order $\frac{1}{2}$ of the stray-field energy and its leading-order nature for certain configurations were rigorously derived

by Γ -convergence. In [24, 10], its leading-order influence on the subtle transition layers (symmetric Néel walls) and their interaction [18, 19] was captured. Despite the non-locality of the stray-field energy, the effective wall energy actually acts like a local wall energy [7, 17], and provides compactness [20, 21].

Let us point out that there is a standard connection between ferromagnetism and SPDE: Especially in small samples, thermal fluctuations play a major role and lead to (unwanted) switching of the magnetization. Therefore, the Landau-Lifshitz-Gilbert equation, which is a combination of conservative and dissipative dynamics for the energy, is being driven by space-time white noise. The computational physics community is well-aware of the fact that this requires a renormalization in dimensions two and larger [11]. There is a rigorous (numerical) analysis in case of a spatially colored noise, see [3]; space-time white noise may now be in reach for a rigorous treatment thanks to the recent progress in singular SPDE. Independently of this SPDE issue, the rate functional in a large deviation theory has been (numerically) treated to infer the most likely switching paths, see [22]. This being said, the noise in our model is of quenched, not of thermal origin, and thus mathematically rather unrelated.

The starting point is the micromagnetic model; in its stationary version, it predicts the magnetization m , which locally indicates the orientation of the elementary magnets on a mesoscopic level, of a ferromagnetic sample as the ground state or at least local minimizer of a variational problem. We present a version that is partially non-dimensionalized in the sense that the magnetization, the fields and the energy density are non-dimensional, but length is still dimensional. The energy E is the sum of the following four terms:

- The exchange contribution $d^2 \int |\nabla m|^2 dx$ models a short-range attractive interaction of quantum mechanical origin, where the “exchange length” d is a material parameter, typically in the nanometer range. Since we are below the Curie temperature, physically speaking, this term comes together with a spatially constant, non-vanishing length of the magnetization, which in our non-dimensionalization turns into the unit-length (and thus non-convex) constraint $|m|^2 = 1$.
- The second contribution is the energy $\int |h|^2 dx$ of the stray field h , which is determined through m by the static Maxwell equations $\nabla \cdot (h + m) = 0$ and $\nabla \times h = 0$. For later purpose, it is convenient to think of h as a field to minimize subject to the sole constraint $\nabla \cdot (h + m) = 0$.
- The third contribution is the Zeeman term $-2 \int H_{ext} \cdot m dx$ that models the interaction of the magnetization with the external field H_{ext} .
- The fourth term is the crystalline anisotropy $-Q \int (e \cdot m)^2 dx$, where the “quality factor” Q is a non-dimensional material parameter; for $Q > 0$ it favors the

alignment of the magnetization with the “easy axis” e (e is a unit vector); a ferromagnet is called soft when $Q > 0$ is small.

We are interested in a sample that comes in form of a thin film of thickness t , typically in the range of tens or hundreds of nanometers. For simplicity, we think of an infinitely extended ferromagnetic film and thus disregard boundary effects. Clearly, the exchange, Zeeman, and anisotropy contributions are restricted to the sample, whereas the stray-field energy still is to be taken over the entire space. Also the constraint $\nabla \cdot (h + m) = 0$ has to be imposed in all of space, with m **extended by zero outside the sample**, and thus has to be interpreted in the distributional sense:

$$\int_{\text{space}} \nabla \zeta \cdot h \, dx = - \int_{\text{sample}} \nabla \zeta \cdot m \, dx \quad \text{for all test functions } \zeta.$$

In line with this geometry, we think of an in-plane external field H_{ext} . We choose the coordinate system such that the x_3 -axis is the thickness direction and the x_1 -axis the direction of the external field, which thus assumes the form $H_{ext} = (h_{ext}, 0, 0)$ with $h_{ext} > 0$.

We are further interested in a polycrystalline sample: The sample is formed by single-crystal grains of a given easy axis, we assume that the grains’ orientation is independently and uniformly distributed, which transmits to the easy axis. In other words, e is a random field. In the absence of the anisotropy, the minimizer would be given by $m = (1, 0, 0)$ and $h = 0$ (recall our simplified setting of an infinitely extended ferromagnetic film)¹ which annihilates exchange, stray field, and Zeeman contributions. The heterogeneous anisotropy energy creates a torque perturbing this magnetization.

We now start with a couple of model reductions. The thin-film geometry allows for two simplifications: In conjunction with the exchange energy, one may assume that m only depends on the in-plane variables $x' = (x_1, x_2)$, i.e., $m = m(x')$. **In the following, the prime ' always denotes an in-plane quantity.** In conjunction with the stray-field energy, one may assume that the m_3 -component, which generates a surface “charge” density at the bottom and top surfaces and thus generates a stray field, is suppressed, i.e., $m_3 = 0$. Hence the energy reduces to

$$d^2 t \int |\nabla' m'|^2 \, dx' + \int |h|^2 \, dx - Qt \int m' \cdot \left(\int_0^t e' \otimes e' \, dx_3 \right) m' \, dx' - 2h_{ext} t \int m_1 \, dx'$$

under the constraints $|m'|^2 = 1$ (where $\int_0^t = \frac{1}{t} \int_0^t$) and

$$\int \nabla \zeta \cdot h \, dx = t \int \zeta(\cdot, x_3 = 0) \nabla' \cdot m' \, dx' \quad \text{for all } \zeta.$$

¹In order to speak of a global minimizer, it is convenient to pass to a periodic setting in both in-plane directions.

We now can already explain the anisotropic response of the magnetization to the heterogeneous anisotropy energy by considering oscillations of the (in-plane unit-length) magnetization $m' \approx (1, 0)$ with wave number k' . If k' is aligned with the k_1 -axis, such an oscillation is divergence-free to leading order; in all other cases, it generates a substantial stray-field and thus is penalized not only by exchange but also the stray-field energy. This is indeed confirmed by experiments: The ripple is predominantly an oscillation in direction of the average magnetization (and thus helps to visualize the latter in Kerr microscopy).

Based on this discussion, we make the assumption that the typical x_1 -scale ℓ_1 of the ripple is much smaller than the typical x_2 -scale ℓ_2 , which will be seen to be self-consistent in a relevant regime. This assumption means that $\partial_2 m'$ is dominated by $\partial_1 m'$ and that h_2 is more strongly suppressed than h_1 ; hence the exchange energy simplifies to $d^2 t \int |\partial_1 m'|^2 dx'$ and the stray field energy to $\int h_1^2 + h_3^2 dx$ under the constraint

$$\int \partial_1 \zeta h_1 + \partial_3 \zeta h_3 dx = t \int \zeta(\cdot, x_3 = 0) \nabla' \cdot m' dx' \quad \text{for all } \zeta.$$

Note that the x_2 variable appears just as a parameter when passing from $\nabla' \cdot m'$ to h ; taking the Fourier transform in x_1 one sees that the stray field energy takes the form of $\frac{1}{2} t^2 \int (|\partial_1|^{-\frac{1}{2}} \nabla' \cdot m')^2 dx'$, where the fractional inverse derivative $|\partial_1|^{-\frac{1}{2}}$ is defined as coming from the Fourier multiplier $|k_1|^{-\frac{1}{2}}$.

A further reduction is based on the assumption that the amplitude of the ripple is small in the sense of $|m' - (1, 0)| \ll 1$,² which also will be seen to be self-consistent. Based on this assumption, and because of the constraint $|m'|^2 = 1$, we use $m_1 \approx 1 - \frac{1}{2} m_2^2$ in the stray-field and Zeeman contributions and $m_1 \approx 1$ in the exchange and anisotropy contributions which leads to

$$\begin{aligned} & d^2 t \int (\partial_1 m_2)^2 dx' + \frac{1}{2} t^2 \int (|\partial_1|^{-\frac{1}{2}} (\partial_2 m_2 - \partial_1 \frac{1}{2} m_2^2))^2 dx' \\ & - 2Qt \int (\int_0^t e_1 e_2 dx_3) m_2 dx' + h_{ext} t \int m_2^2 dx' \end{aligned} \quad (1)$$

where we neglected an additive constant independent of the configuration. From (1) we learn that the crystalline anisotropy acts like the random transversal external field $h_{ran}(x') := Q \int_0^t (e_1 e_2)(x) dx_3$ in x_2 -direction.

We finally turn to an (asymptotic) stochastic characterization of this random field h_{ran} . Since the orientation of the grains is uniform, we have

$$\langle h_{ran}(x') \rangle = 0, \quad (2)$$

²We denote in this section $a \ll b$ if $a/b \rightarrow 0$.

where $\langle \cdot \rangle$ stands for the expectation, or ensemble average in the physics jargon. Assuming that the (average) size ℓ of the grains is not much smaller than the thickness t of the sample (in the real samples we have in mind, ℓ is also in the range of tens of nanometers), taking the vertical average in h_{ran} does not lead to full cancellation so that we have

$$\langle h_{ran}^2(x') \rangle \sim Q^2. \quad (3)$$

Since the orientation is independent from grain to grain we have for the covariance (which in view of (2) reduces to a second moment)

$$\langle h_{ran}(x')h_{ran}(y') \rangle = 0 \quad \text{for } |x' - y'| \gg \ell. \quad (4)$$

Under the assumption that the grain size is small compared to the typical length scale of the ripple, that is, under the assumption $\ell \ll \ell_1, \ell_2$, the random field h_{ran} acts on the magnetization m_2 as if (3) and (4) were consolidated into

$$\langle h_{ran}(x')h_{ran}(y') \rangle \sim Q^2 \ell^2 \delta(x' - y'), \quad (5)$$

where here, \sim means that the Dirac distribution δ is multiplied by a fixed constant that is of order $Q^2 \ell^2$, where ℓ^2 comes from the (average) area of the grains. Now (2) and (5) means that

$$h_{ran} \approx Q\ell\xi \quad \text{with } \xi \text{ white noise;}$$

for simplicity, we will set the above constant to unity in the sequel. Hence (1) turns into our final form

$$\begin{aligned} & t \left(d^2 \int (\partial_1 m_2)^2 dx' + \frac{1}{2} t \int (|\partial_1|^{-\frac{1}{2}} (\partial_2 m_2 - \partial_1 \frac{1}{2} m_2^2))^2 dx' \right. \\ & \left. - 2Q\ell \int \xi m_2 dx' + h_{ext} \int m_2^2 dx' \right). \end{aligned} \quad (6)$$

We now pass to suitable reduced units in (6) (which also amounts to a non-dimensionalization of length) which has the merit of:

- 1) getting rid of the various parameters in (6);
- 2) revealing in which regime our assumptions $\ell \ll \ell_1 \ll \ell_2$ and $|m_2| \ll 1$ are self-consistent.

We start with length and, in line with our interpretation of ℓ_1 and ℓ_2 , we make the (anisotropic) Ansatz $x_1 = \ell_1 \hat{x}_1$ and $x_2 = \ell_2 \hat{x}_2$. We'd like to choose ℓ_1 and ℓ_2 such that the energy densities of exchange, Zeeman, and the harmonic part of the stray field contributions, that is,

$$d^2 (\partial_1 m_2)^2, \quad \frac{t}{2} (|\partial_1|^{-\frac{1}{2}} \partial_2 m_2)^2, \quad h_{ext} m_2^2,$$

balance. This is achieved for

$$\ell_1 = h_{ext}^{-\frac{1}{2}}d, \quad \ell_2 = h_{ext}^{-\frac{3}{4}}(td)^{\frac{1}{2}},$$

which is consistent with $\ell \ll \ell_1 \ll \ell_2$ provided the stabilizing external field h_{ext} that sets the predominant direction is sufficiently small. We now turn to the reduced transverse magnetization m_2 , which we choose such as to balance the two terms in the stray-field energy $\partial_2 m_2 - \partial_1 \frac{1}{2} m_2^2 = \ell_2^{-1} \hat{\partial}_2 m_2 - \ell_1^{-1} \hat{\partial}_1 \frac{1}{2} m_2^2$, which is achieved for $m_2 = \frac{\ell_1}{\ell_2} \hat{m}_2$. Again, this is self-consistent with our small-amplitude assumption as soon as $\ell_1 \ll \ell_2$. Taking $h_{ext}^{\frac{3}{2}} \frac{d}{t}$ as the reduced unit for the energy (area) density, we end up with

$$\int (\hat{\partial}_1 \hat{m}_2)^2 d\hat{x}' + \int (|\hat{\partial}_1|^{-\frac{1}{2}} (\hat{\partial}_2 \hat{m}_2 - \hat{\partial}_1 \frac{1}{2} \hat{m}_2^2))^2 d\hat{x}' - 2\sigma \int \hat{\xi} \hat{m}_2 d\hat{x}' + \int \hat{m}_2^2 d\hat{x}' \quad (7)$$

with $\sigma := h_{ext}^{-5/8} Q d^{-5/4} \ell t^{1/4}$ is the renormalized strength of the transverse field of white-noise character $\hat{\xi}$. Here we used that the distribution of white noise is invariant under $\xi = \frac{1}{\sqrt{\ell_1 \ell_2}} \hat{\xi}$ (since the Dirac distribution that characterizes its covariance scales with $\frac{1}{\ell_1 \ell_2}$). We note that both conditions $\ell \ll \ell_1 \ll \ell_2$ and $\sigma \ll 1$ are satisfied for a wide range of stabilizing external fields for typical material parameters.³ (The model (7) was rigorously deduced via Γ -convergence in [4], see also [26, 29]).

Our reduced (non-dimensionalized) model. We drop the hats in (7) and make one last, purely mathematically motivated, simplification. As we shall see, the main challenge in the model lies in the fact that the white noise triggers *small* scales of m_2 . The zero-order term $\int m_2^2 dx'$ coming from the Zeeman contribution does not much affect the small scales. Rather, this term penalizes scales of m_2 much larger than one. For mathematical convenience, we replace this mechanism by another mechanism of the same effect, namely periodic boundary conditions with period one, that is,

$$m_2(x_1 + 1, x_2) = m_2(x_1, x_2 + 1) = m_2(x_1, x_2), \quad (8)$$

and drop the last term

$$\int_{[0,1]^2} (\partial_1 m_2)^2 dx' + \int_{[0,1]^2} (|\partial_1|^{-\frac{1}{2}} (\partial_2 m_2 - \partial_1 \frac{1}{2} m_2^2))^2 dx' - 2\sigma \int_{[0,1]^2} \xi m_2 dx'. \quad (9)$$

This being said, our results would remain valid, and the proofs become only slightly more involved, when keeping the zero-order term $\int m_2^2 dx'$ alongside the periodic boundary conditions. However, it would require new arguments to get rid of the (artificial) periodic boundary conditions. This is a well-known effect when dealing

³For Permalloy thin films, we have $d = 5nm$, $Q = 2.5 \times 10^{-4}$ with a typical thickness of $t = 100nm$ and a grain size $\ell = 20nm$, so that the conditions $\ell \ll \ell_1 \ll \ell_2$ and $\sigma \ll 1$ are equivalent with a choice of $Q^{8/5}(\ell/d)^{8/5}(t/d)^{2/5} \sim 5 \times 10^{-5} \ll h_{ext} \ll 6 \times 10^{-2} \sim \min\{(t/d)^2, (d/\ell)^2\}$.

with (quenched or thermal) noise: even if scaled by a small constant, it will be almost surely somewhere too large in the infinite plane for our arguments.

The periodic boundary conditions (8) in conjunction with the (reduced) stray field contribution in (9) lead to a new constraint. The stray field contribution is only finite provided the expression under the inverse fractional operator $|\partial_1|^{-\frac{1}{2}}$ has vanishing average in x_1 for all x_2 . Because of the periodic boundary conditions, the second contribution $\partial_1^{\frac{1}{2}}m_2^2$ has vanishing average in x_1 so that we need ∂_2m_2 to have vanishing average in x_1 , which means that $\int_0^1 m_2 dx_1$ does not depend in x_2 . We impose something slightly stronger, namely

$$\int_0^1 m_2 dx_1 = 0 \quad \text{for all } x_2.$$

3 Main results

For the sake of a simple notation, we replace $x' = (x_1, x_2)$ by x and m_2 by u , so that the configurations u are 1-periodic functions in both variables and $\int_0^1 u(x_1, x_2) dx_1 = 0$ for all $x_2 \in (0, 1)$. Besides the Schwartz test functions (that are defined on the full space \mathbb{R}^2), all other functions and distributions are periodic w.r.t. the two-dimensional torus $[0, 1)^2$.

We will focus on the (formal) Euler-Lagrange equation of functional (9):

$$(-\partial_1^2 - |\partial_1|^{-1}\partial_2^2)u + P(u\partial_2Ru) + \frac{1}{2}\partial_2Ru^2 - \frac{1}{2}P(u\partial_1Ru^2) = \sigma P\xi, \quad (10)$$

with (periodic) white noise ξ and a small constant $\sigma > 0$, where P is the L^2 -orthogonal projection onto the set of functions of vanishing average in x_1 (extended in the natural way to periodic distributions) and R is the Hilbert transform acting on 1-periodic functions f in x_1 direction as

$$R := \frac{\partial_1}{|\partial_1|}, \quad \text{i.e.,} \quad Rf(k_1) = \begin{cases} i \operatorname{sgn}(k_1)f(k_1) & k_1 \in 2\pi\mathbb{Z} \setminus \{0\}, \\ 0 & k_1 = 0, \end{cases}$$

where the Fourier coefficients of f are $f(k_1) = \int_0^1 e^{-ik_1x_1} f(x_1) dx_1$ for $k_1 \in 2\pi\mathbb{Z}$ and sgn is the signum function. In particular, $RP = PR = R$.

The functional framework is given by anisotropic Hölder spaces. More precisely, the leading-order operator $-\partial_1^2 - |\partial_1|^{-1}\partial_2^2$ in (10) suggests to endow space $[0, 1)^2$ with a (Carnot-Carathéodory -) metric that is homogeneous w.r.t. the scaling $(x_1, x_2) = (\ell\hat{x}_1, \ell^{\frac{3}{2}}\hat{x}_2)$. The simplest expression is given by

$$d(x, y) := |x_1 - y_1| + |x_2 - y_2|^{\frac{2}{3}}, \quad x, y \in [0, 1)^2,$$

which in particular means that we take the x_1 -variable as a reference. We now introduce the scale of Hölder semi-norms based on the distance function d , where we restrict ourselves to the range $\alpha \in (0, \frac{3}{2})$ needed in this work.

Definition 1. For a periodic function f , we denote by $\|f\| = \sup_x |f(x)|$ the supremum norm of f . For an exponent $\alpha \in (0, \frac{3}{2})$, we define

$$[f]_\alpha := \sup_{x \neq y} \frac{1}{d^\alpha(y, x)} \left\{ \begin{array}{ll} |f(y) - f(x)| & \text{for } \alpha \in (0, 1], \\ |f(y) - f(x) - \partial_1 f(x)(y_1 - x_1)| & \text{for } \alpha \in (1, \frac{3}{2}) \end{array} \right\}.$$

We denote by C^α the space of periodic functions f with $[f]_\alpha < \infty$.

Our main result, Theorem 2, starts from the Euler-Lagrange equation (10) with periodic white noise ξ replaced by its convolution $\xi_\ell := \phi_\ell * \xi$, where

$$\phi_\ell(x_1, x_2) := \frac{1}{\ell^{\frac{5}{2}}} \phi\left(\frac{x_1}{\ell}, \frac{x_2}{\ell^{\frac{3}{2}}}\right), \quad x \in \mathbb{R}^2, \quad (11)$$

and ϕ is some symmetric Schwartz function with $\int_{\mathbb{R}^2} \phi dx = 1$. This approximation is natural in view of the heuristic derivation of the equation. Provided $\sigma > 0$ is sufficiently small, Theorem 2 ascertains a small solution u^ℓ and monitors its distance to the solution of the linear problem. Moreover, the latter is given by σv_ℓ , where $v_\ell := \phi_\ell * v$ is the mollification of the solution v of vanishing average in x_1 of the linearized equation:

$$\mathcal{L}v := (-\partial_1^2 - |\partial_1|^{-1} \partial_2^2)v = P\xi \quad \text{in a distributional sense.} \quad (12)$$

Theorem 2. Fix an $\varepsilon \in (0, \frac{1}{4})$, which we think of being small. Let ξ be distributed like white noise on the torus $[0, 1]^2$ under the expectation $\langle \cdot \rangle$. Then there exists a deterministic constant $C < \infty$ and a random constant $\sigma_0 > 0$ with $\langle \sigma_0^{-p} \rangle < \infty$ for all $1 \leq p < \infty$ which is a threshold in the following sense: Provided $\sigma \in [0, \sigma_0]$ and for every $0 < \ell \leq 1$, there exists a unique smooth and periodic u^ℓ of vanishing average in x_1 such that

$$\mathcal{L}u^\ell + P(u^\ell \partial_2 R u^\ell) + \frac{1}{2} \partial_2 R (u^\ell)^2 - \frac{1}{2} P(u^\ell \partial_1 R (u^\ell)^2) = \sigma P \xi_\ell$$

and that is small enough in the sense of

$$[u^\ell - \sigma v_\ell]_{\frac{5}{4} - \varepsilon} \leq C \left(\frac{\sigma}{\sigma_0} \right)^2.$$

Moreover, as $\ell \downarrow 0$, u^ℓ converges in $C^{3/4 - \varepsilon}$ to a limit that is independent of the choice of the (symmetric) convolution kernel ϕ .

The last sentence of Theorem 2 is the most important: The limit of (small) solutions with regularized noise is independent of the regularization. In fact, the limit can be characterized as a solution of the limiting Euler-Lagrange equation as we shall do in Theorem 9. However, giving a rigorous meaning to this equation requires some effort as we shall explain now, also putting light behind the exponents $\frac{3}{4}$ and $\frac{5}{4}$. As is usual, the amount of irregularity of white noise as a distribution depends on the space dimension; the rule of thumb is that the order is just below $-\frac{1}{2} \times$ effective dimension. In view of our anisotropic metric d , the effective dimension is $1 + \frac{3}{2} = \frac{5}{2}$. Hence we expect ξ to be a distribution of order just below $-\frac{5}{4}$, which is indeed true on the level of the following Hölder spaces of negative exponents; we will restrict to the range we require in this work, namely $\beta \in (-\frac{3}{2}, 0)$.

Definition 3. *Let f be a periodic distribution. In case of $\beta \in (-1, 0)$ we set*

$$[f]_\beta := \inf\{|c| + [g]_{\beta+1} + [h]_{\beta+\frac{3}{2}} : f = c + \partial_1 g + \partial_2 h\}$$

and in case of $\beta \in (-\frac{3}{2}, -1]$ we set

$$[f]_\beta := \inf\{|c| + [g]_{\beta+2} + [h]_{\beta+\frac{3}{2}} : f = c + \partial_1^2 g + \partial_2 h\}.$$

In both cases, the expressions are interpreted as $+\infty$ if the distribution f does not allow for a representation in terms of two periodic functions g and h and a constant c . We denote by C^β the space of periodic distributions f with $[f]_\beta < \infty$. We now state the regularity of the white noise. This result is standard in isotropic media (e.g., the torus $[0, 1]^2$ endowed with the euclidian distance); for the convenience of the reader, we prove it here in the anisotropic setting (i.e., $[0, 1]^2$ endowed with the Carnot-Carathéodory distance d defined above).

Lemma 4. *We have for all $1 \leq p < \infty$ and $0 < \varepsilon < \frac{1}{4}$,*⁴

$$\langle \sup_{0 < \ell \leq 1} [P\xi_\ell]_{-\frac{5}{4}-\varepsilon}^p \rangle \lesssim 1 \quad \text{and} \quad \langle \sup_{0 < \ell \leq \ell_0} [\xi_\ell - \xi]_{-\frac{5}{4}-\varepsilon}^p \rangle^{1/p} \lesssim \ell_0^{\varepsilon/2} \quad \text{for } \ell_0 \leq 1.$$

The above estimate holds also true for the white noise ξ (instead of the projected distribution $P\xi$); since we only need $P\xi$ in the sequel, we will restrict to the estimate in Lemma 4.

Another rule of thumb is that the elliptic operator $\mathcal{L} = -\partial_1^2 - |\partial_1|^{-1}\partial_2^2$ increases regularity by two increments (in our anisotropic metric where the first component in the “numeraire” or unit of reference). While this does not fall into the realm of standard Schauder theory because of the non-locality of the elliptic operator, it is indeed true on our Hölder scale:

⁴Here and in the sequel, $a \lesssim b$ means $a \leq Cb$ with a generic constant $C > 0$ that depends on the exponents in the statement of the respective result, e.g., p and ε (but not ℓ_0) in case of Lemma 4.

Lemma 5. *Let $\alpha \in (\frac{1}{2}, \frac{3}{2}) \setminus \{1\}$. For any periodic function f with vanishing average in x_1 we have*

$$[Rf]_\alpha, [f]_\alpha \lesssim [(-\partial_1^2 - |\partial_1|^{-1}\partial_2^2)f]_{\alpha-2}.$$

Hence for the solution v of the linearized equation (12), we obtain from Lemmas 4 and 5 that

$$\langle \sup_{0 < \ell \leq 1} [v_\ell]_{\frac{3}{4}-\varepsilon}^p \rangle < \infty \quad \text{for all } 1 \leq p < \infty. \quad (13)$$

In particular, we have almost surely $[v]_{\frac{3}{4}-\varepsilon} < \infty$, but expect no regularity beyond $\frac{3}{4}$. This is reminiscent of Brownian motion, which almost surely is Hölder continuous with exponent $\frac{1}{2} - \varepsilon$, but almost surely not Hölder continuous with exponent $\frac{1}{2}$ (even not on small intervals). We are now in the position to explain the difficulty with (10): One of the three non-linear terms cannot be given a sense when u is substituted by v (and thus we expect the same problem for u itself). At first sight, both non-divergence form terms

$$v\partial_2 Rv, \quad \frac{1}{2}v\partial_1 Rv^2 \quad (14)$$

look difficult, since neither $\partial_2 Rv$ nor even $\partial_1 Rv$ exist classically. However, like is trivially the case for the third term $\frac{1}{2}\partial_2 Rv^2$, one of the terms in (14) can be given a distributional sense thanks to the following result on the product of a function u and a distribution f . Loosely speaking, uf can be given a canonical sense as a distribution (not better than f) if u is more regular than f is irregular. Results like these are classical and we need a variant compatible with the anisotropic scaling dictated by the anisotropic metric d .

Lemma 6. *Let $\alpha \in (0, \frac{3}{2})$ and $\beta \in (-\frac{3}{2}, 0) \setminus \{-1, -\frac{1}{2}\}$ with $\alpha + \beta > 0$. If u is a periodic function with $[u]_\alpha < \infty$ and f is a periodic distribution with $[f]_\beta < \infty$, then there exists a distribution denoted by uf such that for all convolution scales $\ell \leq 1$:*

$$\ell^{\alpha+\beta}[u]_\alpha[f]_\beta \gtrsim \begin{cases} \|[u, (\cdot)_\ell]f\| & \text{if } \alpha \in (0, 1], \\ \|([\![u, (\cdot)_\ell] - \partial_1 u[x_1, (\cdot)_\ell]\!]])f\| & \text{if } \alpha \in (1, \frac{3}{2}), \end{cases} \quad (15)$$

where we denoted the commutator-convolution:

$$[g, (\cdot)_\ell]f := gf_\ell - (gf)_\ell$$

for the function $g \in \{u, x_1\}$.⁵ This property characterizes uf uniquely, even independently of the (Schwartz symmetric) convolution kernel ϕ . Moreover, provided u has vanishing average in x_1 ,

$$[uf]_\beta \lesssim [u]_\alpha[f]_\beta. \quad (16)$$

⁵If $g(x) = x_1$, then gf has the (standard) meaning of a product between a C^∞ function and a distribution.

Finally, while the usual Hilbert transform is bounded on Hölder spaces, our one-dimensional Hilbert transform R is not; there is a logarithmic loss in the order:

Lemma 7. *Let α and $\varepsilon > 0$ with $\alpha, \alpha - \varepsilon \in (0, \frac{3}{2})$. Then for any function f of vanishing average in x_1 :*

$$[Rf]_{\alpha-\varepsilon} \lesssim [f]_{\alpha}. \quad (17)$$

From Lemma 6 and Lemma 7 we learn that the cubic term in (14) poses no fundamental problem (because of $(\frac{3}{4} - \varepsilon) - (\frac{1}{4} - 2\varepsilon) > 0$ for $\varepsilon \ll 1$). It can be given a sense and estimated as a distribution of order slightly below $-\frac{1}{4}$:

$$\left[\frac{1}{2}v\partial_1 Rv^2\right]_{-\frac{1}{4}-2\varepsilon} \stackrel{(16)}{\lesssim} [v]_{\frac{3}{4}-\varepsilon}[\partial_1 Rv^2]_{-\frac{1}{4}-2\varepsilon} \stackrel{Def\ 3}{\leq} [v]_{\frac{3}{4}-\varepsilon}[Rv^2]_{\frac{3}{4}-2\varepsilon} \stackrel{(17)}{\lesssim} [v]_{\frac{3}{4}-\varepsilon}[v^2]_{\frac{3}{4}-\varepsilon} \lesssim [v]_{\frac{3}{4}-\varepsilon}^3$$

and thus by (13)

$$\left\langle \left[\frac{1}{2}v\partial_1 Rv^2\right]_{-\frac{1}{4}-2\varepsilon}^p \right\rangle < \infty \quad \text{for all } p < \infty.$$

The real issue comes from the quadratic term in (14): Since we expect v to have regularity α slightly below $\frac{3}{4}$ and thus, $\partial_2 v$ (also, $\partial_2 Rv$) to have regularity β slightly below $\frac{3}{4} - \frac{3}{2} = -\frac{3}{4}$, we just miss the condition $\alpha + \beta > 0$ required by Lemma 6. Hence we need an “off-line” stochastic treatment of the term $v_\ell \partial_2 Rv_\ell$. The idea of the proof is reminiscent from [6, Lemma 3.2] where singular products are treated by renormalization via Wick calculus; the novelty here is to treat singular products in the anisotropic set $[0, 1]^2$ endowed with the distance d .

Lemma 8. *Consider $F^\ell := P(v_\ell \partial_2 Rv_\ell)$. We have for all $1 \leq p < \infty$ and $0 < \varepsilon < \frac{1}{4}$:*

$$\begin{aligned} \left\langle \sup_{\ell \leq 1} [F^\ell]_{-\frac{3}{4}-\varepsilon}^p \right\rangle &\lesssim 1, \\ \left\langle \sup_{\ell, \ell' \leq \ell_0} [F^\ell - F^{\ell'}]_{-\frac{3}{4}-\varepsilon}^p \right\rangle^{1/p} &\lesssim \ell_0^{\varepsilon/2} \quad \text{for } \ell_0 \leq 1. \end{aligned}$$

In particular, almost surely, $\{F^\ell\}_{\ell \downarrow 0}$ is a Cauchy “sequence” in the Banach space defined through $[\cdot]_{-\frac{3}{4}-\varepsilon}$ and thus has a limit F such that $F = PF$. Moreover, almost surely, F does not depend on the (Schwartz symmetric) convolution kernel ϕ .

Equipped with v and F , we now may characterize the limit $\lim_{\ell \downarrow 0} u^\ell$ in Theorem 2. To this purpose, we (formally) rewrite (10) in terms of $w = u - \sigma v$ and substitute $P(v\partial_2 Rv)$ by F .

Theorem 9. *This is a continuation of Theorem 2. We have*

$$\lim_{\ell \downarrow 0} [u^\ell - \sigma v_\ell - w]_{\frac{5}{4}-\varepsilon} = 0, \quad \lim_{\ell \downarrow 0} [v_\ell - v]_{\frac{3}{4}-\varepsilon} = 0, \quad (18)$$

where w is the unique periodic function with vanishing average in x_1 that satisfies (in a distributional sense)

$$\begin{aligned}
& (-\partial_1^2 - |\partial_1|^{-1}\partial_2^2)w \\
& + P(\sigma^2 F + \sigma v \partial_2 R w + \sigma w \partial_2 R v + w \partial_2 R w) \\
& + \frac{1}{2} \partial_2 R (w + \sigma v)^2 - \frac{1}{2} P((w + \sigma v) \partial_1 R (w + \sigma v)^2) = 0
\end{aligned} \tag{19}$$

and that is small in the sense of

$$[w]_{\frac{5}{4}-\varepsilon} \leq C \left(\frac{\sigma}{\sigma_0} \right)^2. \tag{20}$$

Let us comment on the distributional interpretation of the non-linear terms in the equation (19): There is no issue with $\frac{1}{2} \partial_2 R (w + \sigma v)^2$ and of course none with $\sigma^2 P F$. Based on Lemmas 6 and 7, we have given the argument for the cubic term $\frac{1}{2} P((w + \sigma v) \partial_1 R (w + \sigma v)^2)$, which just relied on $[w + \sigma v]_{\frac{3}{4}-\varepsilon} < \infty$. The three terms $v \partial_2 R w$, $w \partial_2 R v$, and (in particular) $w \partial_2 R w$ are regular in the sense of Lemma 6: v has regularity $\frac{3}{4} - \varepsilon$, w has regularity $\frac{5}{4} - \varepsilon$ and ∂_2 reduces the regularity by $\frac{3}{2}$, which still gives a positive sum $\frac{3}{4} - \varepsilon + \frac{5}{4} - \varepsilon - \frac{3}{2} > 0$ for $\varepsilon \ll 1$. In fact, we can motivate the exponent $\frac{5}{4} - \varepsilon$ in (20) as follows: The worst distributions in the two last lines of (19) are of the order $-\frac{3}{4} - \varepsilon$; hence by Lemma 5, w is expected to be of order $-\frac{3}{4} - \varepsilon + 2 = \frac{5}{4} - \varepsilon$.

We will establish Theorems 2 and 9 by formulating (19) as a fixed point problem in the ball described by (20). Alongside the limiting fixed point problem, we will also consider the one where F and v in (19) are replaced by F^ℓ and v_ℓ , respectively. The convergence of the fixed points will then follow from the convergences in Lemmas 4 and 8.

Outline of the paper. In Section 4, we characterize the Hölder spaces C^β , $\beta < 0$ introduced in Definition 3; more precisely, we give an equivalent norm for distributions in C^β with $\beta \in (-\frac{3}{2}, 0) \setminus \{-1, -\frac{1}{2}\}$ in Lemma 10 and we prove Lemmas 5, 6 and 7. In Section 5, we prove our main results in Theorems 2 and 9. In Section 6, we prove the estimates of the stochastic terms in Lemmas 4 and 8. Finally, in the appendix, we prove in Section 7.1 that the linearized energy functional does not admit (with a positive probability) critical points of finite energy, while in Section 7.2 we recall some standard results for the anisotropic Hölder spaces.

4 Anisotropic Hölder spaces. Proof of Lemmas 5, 6, and 7

The proof of Lemma 6 on products of functions and distributions relies on an equivalent characterization of the Hölder norms for a negative exponent defined in Definition 3, which is stated in Lemma 10 below and the proof of which relies on Schauder theory for the nonlocal elliptic operator

$$\mathcal{A} := |\partial_1|^3 - \partial_2^2,$$

an operator which clearly is in line with the scaling properties of the distance d . Hence we prove Lemma 5 on Schauder theory for

$$\mathcal{L} = -\partial_1^2 - |\partial_1|^{-1}\partial_2 = |\partial_1|^{-1}\mathcal{A}$$

alongside with Lemma 10. The equivalent characterization of the negative exponent Hölder norms relies on the “heat kernel” of \mathcal{A} used as a convolution family; the Fourier transform of $\{\psi_T\}_{T>0}$ is evidently given by

$$\psi_T(k) = \exp(-T(|k_1|^3 + k_2^2)), \quad \forall k \in \mathbb{R}^2, \quad (21)$$

and has scaling properties in line with d , namely

$$\psi_T(x_1, x_2) = \frac{1}{(T^{1/3})^{1+\frac{3}{2}}}\psi\left(\frac{x_1}{T^{1/3}}, \frac{x_2}{(T^{1/3})^{\frac{3}{2}}}\right), \quad \forall x \in \mathbb{R}^2, \quad (22)$$

where for simplicity we write $\psi := \psi_1$. For a periodic distribution f , we denote by f_T its convolution with ψ_T , i.e., $f_T = \psi_T * f$, which yields a smooth periodic function; the semi-group property

$$(f_t)_T = f_{t+T} \quad \text{for all } t, T > 0 \quad (23)$$

will be very convenient.

Before embarking on the proofs, a remark on periodic distributions is in place (“periodic” always means periodic of period 1 in the two variables x_1 and x_2). By the space of periodic distributions f we understand the (topological) dual of the space of C^∞ functions u on the torus (endowed with the family of semi norms $\{\|\partial_1^j \partial_2^\ell u\|\}_{j,\ell \geq 0}$). As such, the spatial average $\int_{[0,1]^2} f$ and, more generally, the Fourier coefficients $f(k) = \int_{[0,1]^2} \exp(-ik \cdot x) f(x) dx$ for $k \in (2\pi\mathbb{Z})^2$ are well-defined. For a C^∞ -function ψ with integrable derivatives, i. e. $\int_{\mathbb{R}^2} |\partial_1^j \partial_2^\ell \psi| < \infty$ for all $j, \ell \geq 0$, we can also give a sense to $\int_{\mathbb{R}^2} f \psi$ that is consistent with the classical case, and which is needed to give a sense to the convolution $\psi * f$ as a periodic C^∞ function (“convolution” always means convolution on \mathbb{R}^2): Indeed, for such functions ψ , the periodization $u := \sum_{z \in \mathbb{Z}^2} \psi(\cdot - z)$

is well-defined and in C^∞ , so that we may set $\int_{\mathbb{R}^2} f\psi := \int_{[0,1]^2} fu$. We note that this definition implies on the level of Fourier coefficients $(\psi * f)(k) = \psi(k)f(k)$ for all $k \in (2\pi\mathbb{Z})^2$, where $\psi(k) := \int_{\mathbb{R}^2} \exp(-ik \cdot x)\psi(x)dx$ is the Fourier transform of ψ . Indeed, with the above periodization u of ψ we have $(\psi * f)(k) = u(k)f(k)$, so that the statement reduces to the elementary relation $u(k) = \psi(k)$ between the Fourier series of the periodization u and the Fourier transform of the original function ψ .

Note that if f is a periodic distribution with $[f]_\beta < \infty$ for some $\beta \in (-\frac{3}{2}, 0)$, then the constant c in the decomposition of f in Definition 3 is unique and represents the average $\int_{[0,1]^2} f dx$ of the periodic distribution f . Therefore, $[f]_\beta = \left| \int_{[0,1]^2} f dx \right| + \left[f - \int_{[0,1]^2} f dx \right]_\beta$ if $\beta \in (-3/2, 0)$.

4.1 An equivalent C^β -norm, $\beta < 0$. Proof of Lemma 5

We are now in the position to state Lemma 10:

Lemma 10. *Let f be a periodic distribution in $[0, 1]^2$.* ⁶

i) For $\beta \in (-\frac{3}{2}, -1) \cup (-1, -\frac{1}{2}) \cup (-\frac{1}{2}, 0)$, we have

$$[f]_\beta \sim \sup_{T \in (0,1]} (T^{1/3})^{-\beta} \|f_T\|, \quad (24)$$

where we recall that $\|\cdot\|$ denotes the supremum norm, while $a \sim b$ means $a \lesssim b$ and $b \lesssim a$.

ii) For $\beta \in (-\frac{3}{2}, 0)$, then

$$\sup_{\ell \in (0,1]} \ell^{-\beta} \|f_\ell\| \lesssim [f]_\beta. \quad (25)$$

Compared to [27], where similar tools are used, the main difference is that the mask $\psi = \psi(x)$ defining the semi-group convolution family is not a Schwartz function: While being smooth, it only has mild decay due to the limited smoothness of $\psi(k)$ in $k_1 = 0$, cf. (21).

Proofs of Lemmas 5 and 10. Step 1. Moment bounds on the kernel. We claim that for all orders of derivative $j, l \geq 0$ and exponents $\alpha \geq 0$

$$\begin{aligned} \int_{\mathbb{R}^2} |\partial_1^j \partial_2^l \psi| d(x, 0)^\alpha dx &< \infty \quad \text{provided } \alpha \leq j + 2, \\ \int_{\mathbb{R}^2} |\partial_1^j \partial_2^l |\partial_1 \psi| d(x, 0)^\alpha dx &< \infty \quad \text{provided } \alpha \leq j. \end{aligned}$$

⁶In the case of periodic distributions f of vanishing average on $[0, 1]^2$, one can consider the sup over all $T > 0$ in (24) (respectively, over all $\ell > 0$ in (25)).

In view of its definition on the Fourier level $\psi(k) = \exp(-|k_1|^3 - k_2^2)$, cf. (21), the kernel tensorizes into a Gaussian in x_2 and a kernel $\varphi(x_1)$. Hence the above statements reduce to

$$\int_{\mathbb{R}} |\partial_1^j \varphi| (|x_1| + 1)^{j+2} dx_1 < \infty \quad \text{and} \quad \int_{\mathbb{R}} |\partial_1^j |\partial_1 \varphi| (|x_1| + 1)^j dx_1 < \infty.$$

By Cauchy-Schwarz and $\int_{\mathbb{R}} (|x_1| + 1)^{-2} dx_1 < \infty$, these statements in turn reduce to

$$\int_{\mathbb{R}} |\partial_1^j \varphi|^2 (|x_1| + 1)^{2(j+3)} dx_1, \quad \int_{\mathbb{R}} |\partial_1^j |\partial_1 \varphi|^2 (|x_1| + 1)^{2(j+1)} dx_1 < \infty.$$

By Plancherel, this can be expressed as

$$\int_{\mathbb{R}} |\partial_{k_1}^{j+3} (k_1^j \varphi)|^2 dk_1, \quad \int_{\mathbb{R}} |\partial_{k_1}^{j+1} (k_1^j |k_1| \varphi)|^2 dk_1 < \infty.$$

These statements hold since near $k_1 = 0$, $k_1^j \varphi = k_1^j \exp(-|k_1|^3)$ has a bounded $(j+3)$ -th derivative and $k_1^j |k_1| \varphi = k_1^j |k_1| \exp(-|k_1|^3)$ has a bounded $(j+1)$ -th derivative.

Step 2. Scaling. We claim that for all orders of derivative $j, l \geq 0$, exponents $\alpha \geq 0$ and convolution parameters $T > 0$,

$$\int_{\mathbb{R}^2} |\partial_1^j \partial_2^l \psi_T| d(x, 0)^\alpha dx \lesssim (T^{1/3})^{-j - \frac{3}{2}l + \alpha}, \quad \text{provided } \alpha \leq j + 2, \quad (26)$$

$$\int_{\mathbb{R}^2} |\partial_1^j \partial_2^l |\partial_1 \psi_T| d(x, 0)^\alpha dx \lesssim (T^{1/3})^{-j - \frac{3}{2}l - 1 + \alpha}, \quad \text{provided } \alpha \leq j. \quad (27)$$

This follows from Step 1 via the (anisotropic) change of variables $x_1 = T^{1/3} \hat{x}_1$, $x_2 = (T^{1/3})^{\frac{3}{2}} \hat{x}_2$, which of course implies $\partial_1 = (T^{1/3})^{-1} \hat{\partial}_1$ and $\partial_2 = (T^{1/3})^{-\frac{3}{2}} \hat{\partial}_2$. Furthermore, ψ_T is just defined such that $\psi_T(x) dx = \psi(\hat{x}) d\hat{x}$; likewise, d is defined such that $d(x, 0) = T^{1/3} d(\hat{x}, 0)$.

Step 3. Hölder norms of positive exponent. For $\alpha \in (0, 1) \cup (1, \frac{3}{2})$ and any periodic distribution f we claim

$$[f]_\alpha \lesssim \sup_{T>0} (T^{1/3})^{-\alpha} \|T \mathcal{A} f_T\|,$$

with the (implicit) understanding that f is a continuous function (even continuously differentiable in x_1 in case of $\alpha \in (1, \frac{3}{2})$) if the RHS is finite. Here comes the argument: By homogeneity we may assume $\sup_{T>0} (T^{1/3})^{-\alpha} \|T \mathcal{A} f_T\| \leq 1$; by the semi-group property (23) in form of $\partial_1^j \partial_2^l f_T = \partial_1^j \partial_2^l \psi_{\frac{T}{2}} * f_{\frac{T}{2}}$ for all integers $j, l \geq 0$ and Step 2 we may upgrade our assumption to

$$\|\partial_1^j \partial_2^l \mathcal{A} f_T\| \lesssim \frac{1}{T} (T^{1/3})^{-j - \frac{3}{2}l + \alpha} \quad \text{for all } T \in (0, \infty). \quad (28)$$

Reasoning via $\partial_1^j \partial_2^l f_T = \partial_1^j \partial_2^l \psi_{T-1} * f_1$ and using the finiteness of $\|f_1\|$, we have

$$\lim_{T \uparrow \infty} \|\partial_1^j \partial_2^l f_T\| = 0 \quad \text{provided } j + l > 0, \quad (29)$$

which we need as a purely qualitative ingredient. From the form $\psi_T(k) = \exp(-T(|k_1|^3 + k_2^2))$, cf. (21), we learn that $(0, \infty) \times \mathbb{R}^2 \ni (T, x) \mapsto \psi_T(x)$ is a smooth solution of $(\partial_T + \mathcal{A})\psi_T = 0$. Since by Step 2, $x \mapsto \psi_T(x)$ and all its derivatives are integrable, also all its derivatives in T are integrable in x . Hence for our periodic distribution f , also $(0, \infty) \times \mathbb{R}^2 \ni (T, x) \mapsto f_T(x)$ is a smooth solution of $(\partial_T + \mathcal{A})f_T = 0$, so that we have the representation $\partial_1^j \partial_2^l (f_t - f_T) = \int_t^T \partial_1^j \partial_2^l \mathcal{A} f_s ds$ and thus by (28) the estimate

$$\|\partial_1^j \partial_2^l (f_t - f_T)\| \lesssim \int_t^T (s^{1/3})^{-j - \frac{3}{2}l + \alpha} \frac{ds}{s} \quad \text{for all } 0 < t < T < \infty.$$

We use this estimate in two ways: On the one hand,

$$\|\partial_1^j \partial_2^l (f_\tau - f_T)\| \lesssim (T^{1/3})^{-j - \frac{3}{2}l + \alpha} \quad \text{provided } \alpha > j + \frac{3}{2}l \quad (30)$$

for all $T > \tau$. In particular, for $j = l = 0$, by passing to the limit $\tau < T \rightarrow 0$, we deduce that f is a continuous function and the inequality (30) holds if one replaces f_τ by f . On the other hand, appealing to (29),

$$\|\partial_1^j \partial_2^l f_T\| \lesssim (T^{1/3})^{-j - \frac{3}{2}l + \alpha} \quad \text{provided } \alpha < j + \frac{3}{2}l. \quad (31)$$

Equipped with (30) & (31), we are in the position to conclude. We first deal with the case of $\alpha \in (0, 1)$; let two points $x \neq y$ be given. From $f_T(y) - f_T(x) = \int_0^1 (y - x)_1 \partial_1 f_T(sy + (1-s)x) ds + \int_0^1 (y - x)_2 \partial_2 f_T(sy + (1-s)x) ds$ we obtain by definition of the metric d that $|f_T(y) - f_T(x)| \leq \|\partial_1 f_T\| d(y, x) + \|\partial_2 f_T\| d(y, x)^{\frac{3}{2}}$ and thus by the triangle inequality

$$|f(y) - f(x)| \leq 2\|f - f_T\| + \|\partial_1 f_T\| d(y, x) + \|\partial_2 f_T\| d(y, x)^{\frac{3}{2}} \quad \text{for } x, y \in (0, 1]^2. \quad (32)$$

Inserting (30) for $(j, l) = (0, 0)$ (where we need $\alpha > 0$) and (31) for $(j, l) = (1, 0), (0, 1)$ (where we need $\alpha < 1$) we obtain

$$|f(y) - f(x)| \lesssim (T^{1/3})^\alpha + (T^{1/3})^{-1+\alpha} d(y, x) + (T^{1/3})^{-\frac{3}{2}+\alpha} d(y, x)^{\frac{3}{2}} \quad \text{for } x, y \in (0, 1]^2.$$

Optimizing in T through the choice of $T^{1/3} = d(y, x)$ we obtain $|f(y) - f(x)| \lesssim d(y, x)^\alpha$ and thus $f \in C^\alpha$.

The argument for $\alpha \in (1, \frac{3}{2})$ is similar but slightly more involved; we just point out the changes w.r.t. the previous case. First, passing to the limit $\tau < T \rightarrow 0$ in (30) for $j = 1, l = 0$, one deduces that $\partial_1 f$ is a continuous function. Then, we use the identity

$$\begin{aligned} & f_T(y) - f_T(x) - (y - x)_1 \partial_1 f_T(x) \\ &= \int_0^1 (y_2 - x_2) \partial_2 f_T(y_1, sy_2 + (1-s)x_2) ds \\ &+ \int_0^1 (y_1 - x_1)^2 \partial_1^2 f_T(sy_1 + (1-s)x_1, x_2) (1-s) ds, \end{aligned}$$

yielding the inequality

$$\begin{aligned} & |f(y) - f(x) - (y - x)_1 \partial_1 f(x)| \\ &\leq 2\|f - f_T\| + \|\partial_1(f - f_T)\|d(y, x) + \|\partial_2 f_T\|d(y, x)^{\frac{3}{2}} + \frac{1}{2}\|\partial_1^2 f_T\|d(y, x)^2. \end{aligned}$$

One then appeals to (30) for $(j, l) = (0, 0), (1, 0)$ for the first two RHS terms (where one needs $\alpha > 1$) and to (31) for $(j, l) = (0, 1), (2, 0)$ for the last two terms (where one needs $\alpha < \frac{3}{2}$).

Step 4. Hölder norms of negative exponent. For $\beta \in (-\frac{3}{2}, 0)$ and any periodic distribution f of spatial vanishing average we claim

$$\sup_{T>0} (T^{1/3})^{-\beta} \|f_T\| \lesssim [f]_\beta.$$

The same proof also concludes point ii) of Lemma 10 when the convolution is carried by a general (symmetric) mask ϕ instead of the heat kernel ψ . (A different argument is given at footnote 7 below). We will also prove that the inequality remains true for periodic distributions f of *arbitrary spatial average* provided that sup is restricted over $T \in (0, 1]$ (respectively, over $\ell \in (0, 1]$ in (25)).

Assume also for the moment that f is of vanishing average on $(0, 1]^2$. By Definition 3 of $[\cdot]_\beta$, the desired inequality is a consequence of the following: For any $\alpha \in (0, \frac{3}{2})$, any integers $j, l \geq 0$ and any periodic function u we have

$$\|\partial_1^j \partial_2^l u_T\| \lesssim (T^{1/3})^{-j - \frac{3}{2}l + \alpha} [u]_\alpha \quad \text{provided } \alpha \leq j + \frac{3}{2}l. \quad (33)$$

In fact, in case of $\beta \in (-1, 0)$, we need (33) for $(j, l, \alpha) = (1, 0, \beta + 1), (0, 1, \beta + \frac{3}{2})$; in case of $\beta \in (-\frac{3}{2}, -1]$, we need (33) for $(j, l, \alpha) = (2, 0, \beta + 2), (0, 1, \beta + \frac{3}{2})$. Statement (33) is an immediate consequence of Step 2 enriched by the obvious cancellations (which follow from integrations by parts)

$$\begin{aligned} & \int_{\mathbb{R}^2} \partial_1^j \partial_2^l \psi_T = 0 \quad \text{provided } (j, l) \neq (0, 0), \\ & \int_{\mathbb{R}^2} x_1 \partial_1^j \partial_2^l \psi_T = 0 \quad \text{provided } (j, l) \neq (0, 0), (1, 0). \end{aligned}$$

Indeed, these allow to write in the first and second case, respectively,

$$\begin{aligned}\partial_1^j \partial_2^l u_T(x) &= \int_{\mathbb{R}^2} \partial_1^j \partial_2^l \psi_T(x-y)(u(y) - u(x)) dy, \\ \partial_1^j \partial_2^l u_T(x) &= \int_{\mathbb{R}^2} \partial_1^j \partial_2^l \psi_T(x-y)(u(y) - u(x) - (y-x)_1 \partial_1 u(x)) dy,\end{aligned}$$

which we use for $\alpha \in (0, 1]$ and $\alpha \in (1, \frac{3}{2})$, respectively.

Finally, the case of f of nonvanishing average comes by decomposing

$$f = \int_{[0,1)^2} f dx + \left(f - \int_{[0,1)^2} f dx \right)$$

so that the desired inequality in Step 4 with sup restricted to $T \in (0, 1]$ follows by using that $\beta \leq 0$ and $T \leq 1$.

Step 5. Proof of Lemma 5. For a periodic function u and a periodic distribution f related by $\mathcal{L}u = Pf$ we claim the following Schauder estimate: $[u]_\alpha \lesssim [f]_{\alpha-2}$ for $\alpha \in (\frac{1}{2}, 1) \cup (1, \frac{3}{2})$. Here comes the argument: Without loss of generality (w.l.o.g.), we may assume that f has vanishing spatial average, so that by Step 4 we have

$$\sup_{T>0} (T^{1/3})^{2-\alpha} \|f_T\| \lesssim [f]_{\alpha-2};$$

Using the semi-group property in form of $|\partial_1|f_T = (|\partial_1|\psi_{\frac{T}{2}}) * f_{\frac{T}{2}}$ and appealing to Step 2, we upgrade this to

$$\sup_{T>0} (T^{1/3})^{3-\alpha} \||\partial_1|f_T\| \lesssim [f]_{\alpha-2}.$$

We may rewrite the relation between u and f as $\mathcal{A}u = |\partial_1|f$ (in a distributional sense) so that the above takes the form of

$$\sup_{T>0} (T^{1/3})^{-\alpha} \|T\mathcal{A}u_T\| \lesssim [f]_{\alpha-2}.$$

Therefore the claim follows from Step 3 where it is essential to have $\alpha \neq 1$. The same argument also leads to $[Ru]_\alpha \lesssim [f]_{\alpha-2}$ by redoing the above estimates for $\partial_1 f$ and $\mathcal{A}Ru = \partial_1 f$.

Step 6. Second half of equivalence for negative exponents. For $\beta \in (-\frac{3}{2}, -1) \cup (-1, -\frac{1}{2}) \cup (-\frac{1}{2}, 0)$ and any periodic distribution f of vanishing average we claim

$$[f]_\beta \lesssim \sup_{T>0} (T^{1/3})^{-\beta} \|f_T\|.$$

With help of Fourier series, we see that there exists a periodic distribution u of vanishing average such that $\mathcal{A}u = f$ distributionally. W.l.o.g., we assume that $f \neq 0$. By homogeneity we may assume $\sup_{T>0} (T^{1/3})^{-\beta} \|f_T\| = 1$, so that we have

$$\sup_{T>0} (T^{1/3})^{-\beta} \|\mathcal{A}u_T\| \leq 1,$$

which by the semi-group property and Step 2 implies for all $j, \ell \geq 0$:

$$\sup_{T>0} (T^{1/3})^{j+\frac{3}{2}\ell-\beta-3} \|T\mathcal{A}\partial_1^j \partial_2^\ell u_T\| \lesssim 1, \quad (34)$$

$$\sup_{T>0} (T^{1/3})^{j+\frac{3}{2}\ell+1-\beta-3} \|T\mathcal{A}\partial_1^j \partial_2^\ell |\partial_1| u_T\| \lesssim 1. \quad (35)$$

As $f = \mathcal{A}u = \partial_1(\partial_1|\partial_1|u) + \partial_2(-\partial_2u)$, by definition of $[\cdot]_\beta$, for having $[f]_\beta \lesssim 1$ it suffices to show

$$[\partial_1|\partial_1|u]_{\beta+1} \lesssim 1 \quad \text{for } \beta > -1, \quad (36)$$

$$[|\partial_1|u]_{\beta+2} \lesssim 1 \quad \text{for } \beta < -1, \quad (37)$$

$$[\partial_2u]_{\beta+\frac{3}{2}} \lesssim 1. \quad (38)$$

Estimate (38) follows from (34) with $(j, \ell) = (0, 1)$ by Step 3 with $\alpha = \beta + \frac{3}{2}$; estimates (36) and (37) follow from (35) with $(j, \ell) = (1, 0)$ and $(j, \ell) = (0, 0)$, respectively.

Step 7. Proof of the norm equivalence (24). The difference with Steps 4 and 6 is twofold: On the one side, the range of T is restricted to $(0, 1]$; on the other side, the norm equivalence is claimed for periodic f 's without vanishing average. To deal with the latter, we decompose $f = \int_{[0,1)^2} f dx + \left(f - \int_{[0,1)^2} f dx\right)$ and note that obviously,

$$\left[\int_{[0,1)^2} f dx \right]_\beta, \sup_{T \geq 1} (T^{1/3})^{-\beta} \left| \int_{[0,1)^2} f_T dx \right| \leq \left| \int_{[0,1)^2} f dx \right| \leq [f]_\beta, \sup_{T \geq 1} (T^{1/3})^{-\beta} \|f_T\|.$$

We now argue that for a periodic distribution f (of period 1) of vanishing average we have

$$\|f_T\| \lesssim \exp(-T) \|f_1\| \quad \text{for all } T \geq 1.$$

By the semi-group property it suffices to show that for any periodic function f of vanishing average

$$\|f_T\| \lesssim \exp(-T) \|f\| \quad \text{for all } T \geq 0.$$

Since by Step 2, $\|f_T\| \lesssim \|f\|$ for any $T \geq 0$, it is enough to focus on $T \geq 1$. Using the explicit form $\psi_T(k) = \exp(-T(|k_1|^3 + k_2^2))$ of the convolution kernel, cf. (21), we obtain because of the vanishing spatial average of f in form of $f(k=0) = 0$ that $|f_T(k)| = \exp(-T(|k_1|^3 + k_2^2)) |f(k)| \leq \exp(1-T) \exp(-(|k_1|^3 + k_2^2)) |f(k)|$, where in the

second step we used $T \geq 1$ and $|k_1|^3 + k_2^2 \geq 1$ for every $k \in (2\pi\mathbb{Z})^2 \setminus \{(0,0)\}$. Hence we obtain by the Fourier series representation of f_T , Cauchy-Schwarz in frequency space, and Plancherel that

$$\begin{aligned} \|f_T\| &\leq \sum_k |f_T(k)| \leq \exp(1-T) \sum_k \exp(-(|k_1|^3 + k_2^2)) |f(k)| \\ &\leq \exp(1-T) \left(\sum_k \exp(-2(|k_1|^3 + k_2^2)) \sum_k |f(k)|^2 \right)^{\frac{1}{2}} \\ &\lesssim \exp(-T) \left(\int_{[0,1]^2} f^2 \right)^{\frac{1}{2}} \leq \exp(-T) \|f\|. \end{aligned}$$

□

Remark 1. *The arguments presented above yield also the following equivalences:*

1. *If $\alpha \in (0, \frac{3}{2}) \setminus \{1\}$ and f is a periodic function in $[0,1]^2$, then*

$$[f]_\alpha \sim \sup_{T>0} (T^{1/3})^{-\alpha} \|T\mathcal{A}f_T\|;$$

2. *If $\beta \in (-\frac{3}{2}, 0) \setminus \{-1, -\frac{1}{2}\}$ and f is a periodic distribution in $[0,1]^2$ with vanishing average, then*

$$[f]_\beta \sim \sup_{T>0} (T^{1/3})^{-\beta} \|T\mathcal{A}f_T\|;$$

the equivalence still holds if the above sup is restricted to $T \in (0, 1]$.

Remark 2. *We have the following inclusion of our Hölder spaces of periodic functions:*

$$C^\alpha \subset C^\beta \text{ for every } -\frac{3}{2} < \beta < \alpha < \frac{3}{2}, \alpha, \beta \neq 0.$$

Indeed, if $\beta > 0$, we know by Lemma 12 in Appendix that $[\cdot]_\beta \lesssim [\cdot]_\alpha$. If $\alpha < 0$ and $\alpha, \beta \notin \{-1, -\frac{1}{2}\}$, then within the characterization (24) of a periodic distribution f one has $T^{-\beta/3} \|f_T\| \leq T^{-\alpha/3} \|f_T\|$ for all $T \in (0, 1]$ so that $[f]_\beta \lesssim [f]_\alpha$. If $\beta < 0 < \alpha$, $\beta \neq -1, -\frac{1}{2}$ and $f \in C^\alpha$, then we have by Lemma 12 and characterization (24): $[f]_\alpha \geq \|f - \int_{[0,1]^2} f dx\| \geq \sup_{T \in (0,1]} \|f_T - \int_{[0,1]^2} f dx\| \geq \sup_{T \in (0,1]} T^{-\beta/3} \|f_T - \int_{[0,1]^2} f dx\| \gtrsim [f - \int_{[0,1]^2} f dx]_\beta$; in conclusion, $[f]_\alpha + \|f\| \gtrsim [f]_\beta$. It remains to treat the critical cases $\alpha, \beta \in \{-1, -\frac{1}{2}\}$. For that, we only treat here the case $\beta = -\frac{1}{2}$ and $\alpha \in (\beta, 0)$; by Definition 3, we consider an arbitrary decomposition $f = \int_{[0,1]^2} f dx + \partial_1 g + \partial_2 h$ with $g \in C^{\alpha+1}$, $h \in C^{\alpha+3/2}$ of vanishing average. By Lemma 12, we know that $[g]_{\alpha+1} \gtrsim [g]_{\beta+1}$ as well as $[h]_{\alpha+3/2} \gtrsim [h]_{\beta+3/2}$. Passing to infimum over all these decompositions, we deduce $[f]_\alpha \gtrsim [f]_\beta$.

4.2 Product of $C^\alpha \cdot C^\beta$ if $\alpha + \beta > 0$. Proof of Lemma 6

Proof of Lemma 6. Let $u \in C^\alpha$ and $f \in C^\beta$ with $\alpha > 0 > \beta$, $\alpha + \beta > 0$, $\beta \neq -1, -\frac{1}{2}$.

In the first part, we will prove (15) for our semigroup (23), i.e., there exists a distribution denoted by $uf \in C^\beta$ with

$$[uf]_\beta \lesssim (\|u\| + [u]_\alpha)[f]_\beta \quad (39)$$

such that for every $T \in (0, 1]$

$$(T^{1/3})^{\alpha+\beta}[u]_\alpha[f]_\beta \gtrsim \begin{cases} \|[\![u, (\cdot)_T]\!]f\| & \text{if } \alpha \in (0, 1], \\ \|([\![u, (\cdot)_T]\!] - \partial_1 u[x_1, (\cdot)_T])f\| & \text{if } \alpha \in (1, \frac{3}{2}). \end{cases} \quad (40)$$

In the second part, we will show how to extend (40) to a general (symmetric) Schwartz mask ϕ in order to obtain (15).

In order to prove (40), without loss of generality (w.l.o.g.), we may assume that f is **of vanishing average**, i.e., $\int_{[0,1]^2} f dx = 0$ and that $[f]_\beta = 1$; indeed, first, if $f = 0$ the conclusion is obvious, so by homogeneity, we can assume $[f]_\beta = 1$. Second, we can replace f by $f - \int_{[0,1]^2} f dx$ and use Step 1 to check that $\|[\![u, (\cdot)_t]\!]1\| \lesssim [u]_\alpha t^{\alpha/3}$ provided that $\alpha \in (0, 1]$ together with $t^{\alpha/3} \leq [f]_\beta (t^{1/3})^{\alpha+\beta}$ for $t \in (0, 1]$ as $\beta \leq 0$ and $[f]_\beta = 1$ (the case $\alpha \in (1, \frac{3}{2})$ is treated by the same argument).

Step 1. For $\alpha \in (0, \frac{3}{2})$, and any two periodic functions u and f we have that

$$\left. \begin{array}{l} \text{for } \alpha \leq 1 : \|[\![u, (\cdot)_T]\!]f\| \\ \text{for } \alpha > 1 : \|([\![u, (\cdot)_T]\!] - \partial_1 u[x_1, (\cdot)_T])f\| \end{array} \right\} \lesssim (T^{1/3})^\alpha [u]_\alpha \|f\|, \text{ for all } T > 0. \quad (41)$$

Indeed, this follows from (26) via the representations

$$-([\![u, (\cdot)_T]\!]f)(x) = \int_{\mathbb{R}^2} \psi_T(x-y)(u(y) - u(x))f(y)dy, \quad (42)$$

$$-([\![u, (\cdot)_T]\!] - \partial_1 u[x_1, (\cdot)_T])f(x) = \int_{\mathbb{R}^2} \psi_T(x-y)(u(y) - u(x) - (y-x)_1 \partial_1 u(x))f(y)dy \quad (43)$$

and the definition of $[\cdot]_\alpha$.

Step 2. For $\beta \in (-\frac{3}{2}, 0) \setminus \{-1, -\frac{1}{2}\}$ and any periodic distribution f of spatial vanishing average we have

$$\|[\![x_1, (\cdot)_T]\!]f\| \lesssim [f]_\beta (T^{1/3})^{1+\beta} \text{ for all } T > 0. \quad (44)$$

Indeed,

$$([\mathbf{x}_1, (\cdot)_T]f)(x) = T^{1/3} \int_{\mathbb{R}^2} \tilde{\psi}_T(x-y)f(y)dy = T^{1/3}\tilde{\psi}_T * f(x), \quad (45)$$

where $\tilde{\psi}(x) = x_1\psi(x)$ and $\tilde{\psi}_T$ is related to $\tilde{\psi}$ analogously to (22). Note that $|\tilde{\psi}(x)| \leq |\psi(x)|d(x,0)$ so that using (26) for $(j, l, \alpha) = (0, 0, 1)$ and $T = 1$ we deduce that $\tilde{\psi} \in L^1(\mathbb{R}^2)$. Moreover, by the scaling of $\tilde{\psi}_T$, it follows that $\tilde{\psi}_T$ and $\tilde{\psi}$ have the same L^1 -norm. We first argue that

$$\tilde{\psi}_{2T} = 2^{2/3}\tilde{\psi}_T * \psi_T. \quad (46)$$

For that, one writes, using that $*$ is Abelian,

$$\begin{aligned} 2\tilde{\psi}_T * \psi_T(x) &= \int_{\mathbb{R}^2} \frac{x_1 - y_1}{T^{1/3}} \psi_T(x-y)\psi_T(y) dy + \int_{\mathbb{R}^2} \frac{y_1}{T^{1/3}} \psi_T(y)\psi_T(x-y) dy \\ &= \frac{x_1}{T^{1/3}} \psi_{2T}(x) = 2^{1/3}\tilde{\psi}_{2T}(x). \end{aligned}$$

In view of (46) we may rewrite (45) as

$$|([\mathbf{x}_1, (\cdot)_{2T}]f)(x)| = \left| 2^{2/3}(2T)^{1/3} \int_{\mathbb{R}^2} \tilde{\psi}_T(x-y)f_T(y)dy \right| \leq 2T^{1/3}\|f_T\| \int_{\mathbb{R}^2} |\tilde{\psi}| dx.$$

Hence,

$$\|[\mathbf{x}_1, (\cdot)_{2T}]f\| \lesssim T^{1/3}\|f_T\|.$$

Now (44) follows from Step 4 in the proof of Lemma 10.

Step 3. For any $\tau, T > 0$, we have

$$[u]_\alpha[f]_\beta(T^{1/3})^\alpha(\tau^{1/3})^\beta \gtrsim \begin{cases} \|[\mathbf{u}, (\cdot)_T]f_\tau\| & \text{if } \alpha \in (0, 1], \\ \|([\mathbf{u}, (\cdot)_T] - \partial_1\mathbf{u}[\mathbf{x}_1, (\cdot)_T])f_\tau\| & \text{if } \alpha \in (1, \frac{3}{2}). \end{cases}$$

In particular, if $\tau = T > 0$, then the above RHS is bounded by $[u]_\alpha[f]_\beta(T^{1/3})^{\alpha+\beta}$.

Indeed, this is a direct consequence of Step 1 and Lemma 10.

Step 4. For every $0 < t < T$, we have

$$[u]_\alpha[f]_\beta(T^{1/3})^{\alpha+\beta} \gtrsim \begin{cases} \|[\mathbf{u}, (\cdot)_{T-t}]f_t\| & \text{if } \alpha \in (0, 1], \\ \|([\mathbf{u}, (\cdot)_{T-t}]f_t - \partial_1\mathbf{u}[\mathbf{x}_1, (\cdot)_T]f + (\partial_1\mathbf{u}[\mathbf{x}_1, (\cdot)_t]f)_{T-t}\| & \text{if } \alpha \in (1, \frac{3}{2}). \end{cases}$$

To prove that, we distinguish two cases:

The dyadic case. We start with the case of t and T being dyadically related (i.e., $t = T/2^n$). By the semigroup property (23), we have that $([u, (\cdot)_\tau]f_\tau)_{T-2\tau} = (uf_{2\tau} - (uf_\tau)_\tau)_{T-2\tau} = (uf_{2\tau})_{T-2\tau} - (uf_\tau)_{T-\tau}$ which leads to a telescopic sum:

$$[u, (\cdot)_{T-t}]f_t = uf_T - (uf_t)_{T-t} = \sum_{\tau=T/2^k, k=1, \dots, n} ([u, (\cdot)_\tau]f_\tau)_{T-2\tau}. \quad (47)$$

- If $\alpha \in (0, 1]$, then by Step 3,

$$\begin{aligned} \|[u, (\cdot)_{T-t}]f_t\| &\leq \sum_{\tau=T/2^k, k=1, \dots, n} \|[u, (\cdot)_\tau]f_\tau\| \\ &\lesssim [u]_\alpha [f]_\beta \sum_{\tau=T/2^k, k=1, \dots, n} (\tau^{1/3})^{\alpha+\beta} \\ &\lesssim [u]_\alpha [f]_\beta (T^{1/3})^{\alpha+\beta} \sum_{k \geq 1} \left(\frac{1}{2^{(\alpha+\beta)/3}}\right)^k \stackrel{\alpha+\beta > 0}{\lesssim} [u]_\alpha [f]_\beta (T^{1/3})^{\alpha+\beta}. \end{aligned}$$

- If $\alpha \in (1, \frac{3}{2})$, then simple algebra yields

$$-\partial_1 u[x_1, (\cdot)_{2\tau}]f + (\partial_1 u[x_1, (\cdot)_\tau]f)_\tau = -\partial_1 u[x_1, (\cdot)_\tau]f_\tau - [\partial_1 u, (\cdot)_\tau][x_1, (\cdot)_\tau]f \quad (48)$$

which when convoluted with $\psi_{T-2\tau}$ leads to another telescopic sum:

$$\begin{aligned} &-\partial_1 u[x_1, (\cdot)_T]f + (\partial_1 u[x_1, (\cdot)_t]f)_{T-t} \\ &= - \sum_{\tau=T/2^k, k=1, \dots, n} (\partial_1 u[x_1, (\cdot)_\tau]f_\tau + [\partial_1 u, (\cdot)_\tau][x_1, (\cdot)_\tau]f)_{T-2\tau}. \end{aligned}$$

Adding this to (47), we obtain:

$$\begin{aligned} &[u, (\cdot)_{T-t}]f_t - \partial_1 u[x_1, (\cdot)_T]f + (\partial_1 u[x_1, (\cdot)_t]f)_{T-t} \\ &= \sum_{\tau=T/2^k, k=1, \dots, n} ([u, (\cdot)_\tau]f_\tau - \partial_1 u[x_1, (\cdot)_\tau]f_\tau - [\partial_1 u, (\cdot)_\tau][x_1, (\cdot)_\tau]f)_{T-2\tau}. \end{aligned}$$

The first contribution to the summand is estimated by Step 3:

$$\|([u, (\cdot)_\tau] - \partial_1 u[x_1, (\cdot)_\tau])f_\tau\| \lesssim [u]_\alpha [f]_\beta (\tau^{1/3})^{\alpha+\beta};$$

the second contribution is estimated by Steps 1-2 and Lemma 12:

$$\|[\partial_1 u, (\cdot)_\tau][x_1, (\cdot)_t]f\| \lesssim [\partial_1 u]_{\alpha-1} \tau^{\frac{\alpha-1}{3}} \|[x_1, (\cdot)_t]f\| \lesssim [f]_\beta [u]_\alpha \tau^{\frac{\alpha-1}{3}} t^{\frac{\beta+1}{3}}, \quad (49)$$

for every $t, \tau > 0$. The desired estimate follows as in the case $\alpha \in (0, 1]$.

The nondyadic case. In the general case of t not dyadically related to T , we choose $\tilde{T} \in [T/2, T)$ that is dyadically related to t (in particular, $\tilde{T} \geq t$).

- If $\alpha \in (0, 1]$, then we have

$$\lceil u, (\cdot)_{T-t} \rceil f_t = (\lceil u, (\cdot)_{\tilde{T}-t} \rceil f_t)_{T-\tilde{T}} + \lceil u, (\cdot)_{T-\tilde{T}} \rceil f_{\tilde{T}} \quad (50)$$

so that by the dyadic case and Step 3, we conclude:

$$\|\lceil u, (\cdot)_{T-t} \rceil f_t\| \lesssim \|\lceil u, (\cdot)_{\tilde{T}-t} \rceil f_t\| + \|\lceil u, (\cdot)_{T-\tilde{T}} \rceil f_{\tilde{T}}\| \lesssim [u]_\alpha [f]_\beta (T^{1/3})^{\alpha+\beta}.$$

- If $\alpha \in (1, \frac{3}{2})$, starting from (50), we have

$$\begin{aligned} & \lceil u, (\cdot)_{T-t} \rceil f_t - \partial_1 u[x_1, (\cdot)_T] f + (\partial_1 u[x_1, (\cdot)_t] f)_{T-t} \\ &= \left(\lceil u, (\cdot)_{\tilde{T}-t} \rceil f_t - \partial_1 u[x_1, (\cdot)_{\tilde{T}}] f + (\partial_1 u[x_1, (\cdot)_t] f)_{\tilde{T}-t} \right)_{T-\tilde{T}} \end{aligned} \quad (51)$$

$$+ \lceil u, (\cdot)_{T-\tilde{T}} \rceil f_{\tilde{T}} - \partial_1 u[x_1, (\cdot)_T] f + (\partial_1 u[x_1, (\cdot)_{\tilde{T}}] f)_{T-\tilde{T}}. \quad (52)$$

The dyadic case (applied to t and \tilde{T}) yields the estimate of the first term (51) by $\lesssim [u]_\alpha [f]_\beta (T^{1/3})^{\alpha+\beta}$, while the second (52) is estimated using an identity similar to the computation (48):

$$\begin{aligned} & \|\lceil u, (\cdot)_{T-\tilde{T}} \rceil f_{\tilde{T}} - \partial_1 u[x_1, (\cdot)_T] f + (\partial_1 u[x_1, (\cdot)_{\tilde{T}}] f)_{T-\tilde{T}}\| \\ & \leq \|\lceil u, (\cdot)_{T-\tilde{T}} \rceil f_{\tilde{T}} - \partial_1 u[x_1, (\cdot)_{T-\tilde{T}}] f_{\tilde{T}}\| + \|\lceil \partial_1 u, (\cdot)_{T-\tilde{T}} \rceil [x_1, (\cdot)_{\tilde{T}}] f\| \\ & \lesssim [u]_\alpha [f]_\beta ((T - \tilde{T})^{\alpha/3} \tilde{T}^{\beta/3} + (T - \tilde{T})^{(\alpha-1)/3} \tilde{T}^{(\beta+1)/3}) \lesssim [u]_\alpha [f]_\beta T^{(\alpha+\beta)/3} \end{aligned}$$

where we used Step 3 and (49).

Step 5. For a subsequence,

$$\begin{cases} \{u f_{\frac{1}{2^n}}\}_{n \uparrow \infty} & \text{if } \alpha \in (0, 1], \\ \{u f_{\frac{1}{2^n}} - \partial_1 u[x_1, (\cdot)_{\frac{1}{2^n}}] f\}_{n \uparrow \infty} & \text{if } \alpha \in (1, 3/2), \end{cases}$$

converges (in a distributional sense) to a distribution denoted uf that belongs to C^β such that (39) and (40) hold.

- If $\alpha \in (0, 1]$, we set $t = 1/2^n$. We want to prove that $[u f_t]_\beta \lesssim ([u]_\alpha + \|u\|)[f]_\beta$ which by Lemma 10 (as we assumed $\beta \neq -\frac{1}{2}, -1$), it is equivalent to checking that

$$\|(u f_t)_T\| \lesssim ([u]_\alpha + \|u\|)[f]_\beta T^{\beta/3}, \quad \text{for all } T \in (0, 1]. \quad (53)$$

If $T \in (0, t]$, then by Step 3 and Lemma 10, we have

$$\begin{aligned} \|(u f_t)_T\| & \leq \|(u f_t)_T - u f_{T+t}\| + \|u f_{T+t}\| \\ & \lesssim [u]_\alpha [f]_\beta (T^{1/3})^\alpha (t^{1/3})^\beta + \|u\| [f]_\beta (T+t)^{\beta/3} \lesssim (\|u\| + [u]_\alpha) [f]_\beta (T^{1/3})^\beta \end{aligned}$$

because of $\alpha > 0 > \beta$, $(T^{1/3})^\alpha \leq 1$ and $(t^{1/3})^\beta, (T+t)^{\beta/3} \leq (T^{1/3})^\beta$.

If $T \in (t, 1]$, then we have by Step 4 and Lemma 10,

$$\begin{aligned} \|(uf_t)_T\| &= \|((uf_t)_{T-t})_t\| \lesssim \|(uf_t)_{T-t}\| \leq \|(uf_t)_{T-t} - uf_T\| + \|uf_T\| \\ &\lesssim [u]_\alpha [f]_\beta (T^{1/3})^{\alpha+\beta} + \|u\| [f]_\beta T^{\beta/3} \lesssim ([u]_\alpha + \|u\|) [f]_\beta T^{\beta/3}. \end{aligned}$$

Therefore, (53) holds. By Lemma 13 in Appendix, for a subsequence $n \rightarrow \infty$ with $t = 1/2^n$, we have uf_t converges to a distribution that we denote uf which belongs to C^β and $[uf]_\beta \lesssim ([u]_\alpha + \|u\|) [f]_\beta$, so (39) holds. Moreover, for every $T > 0$ we have $\psi_{T-t} \rightarrow \psi_T$ in the sense of Schwartz functions as $t \rightarrow 0$ so that $(uf_t)_{T-t} \rightarrow (uf)_T$ uniformly. By Step 4, passing at the limit $t \rightarrow 0$, we conclude $\|(uf)_T - uf_T\| \lesssim [u]_\alpha [f]_\beta (T^{1/3})^{\alpha+\beta}$ which is (40).

• If $\alpha \in (1, 3/2)$, we want to show

$$\|(uf_t - \partial_1 u[x_1, (\cdot)_t] f)_T\| \lesssim ([u]_\alpha + \|u\|) [f]_\beta T^{\beta/3}, \quad \text{for all } T \in (0, 1],$$

for every $t = 1/2^n$, $n \in \mathbb{N}$.

If $T \in (0, t]$, then by Step 3, (44) and Lemma 10 we have

$$\begin{aligned} \|(uf_t - \partial_1 u[x_1, (\cdot)_t] f)_T\| &= \|(uf_t)_T - uf_{T+t} + uf_{T+t} - (\partial_1 u[x_1, (\cdot)_t] f)_T\| \\ &\lesssim \|[u, (\cdot)_T] f_t - \partial_1 u[x_1, (\cdot)_T] f_t\| + \|\partial_1 u[x_1, (\cdot)_T] f_t\| + \|uf_{T+t}\| + \|\partial_1 u[x_1, (\cdot)_t] f\| \\ &\lesssim [u]_\alpha [f]_\beta (T^{1/3})^\alpha (t^{1/3})^\beta + \|\partial_1 u\| [f]_\beta T^{1/3} (T+t)^{\beta/3} + \|u\| [f]_\beta (T+t)^{\beta/3} + \|\partial_1 u\| [f]_\beta t^{(\beta+1)/3} \\ &\lesssim ([u]_\alpha + \|u\|) [f]_\beta (T^{1/3})^\beta \end{aligned}$$

where we used $\|\partial_1 u\| \lesssim [u]_\alpha$ (by Lemma 12) and a slightly different version of (44):

$$\|[x_1, (\cdot)_T] f_t\| \lesssim [f]_\beta T^{1/3} (T+t)^{\beta/3}.$$

In fact, for the latter estimate, we use the same strategy as in Step 2: by (45)-(46) and Lemma 10, we have that $\|[x_1, (\cdot)_{2T}] f_t\| \lesssim T^{1/3} \|f_{t+T}\| \lesssim [f]_\beta T^{1/3} (T+t)^{\beta/3}$.

If $T \in (t, 1]$, then we have by Step 4, Lemma 10 and (44),

$$\begin{aligned} \|(uf_t - \partial_1 u[x_1, (\cdot)_t] f)_T\| &\lesssim \|(uf_t)_{T-t} - (\partial_1 u[x_1, (\cdot)_t] f)_{T-t}\| \\ &\lesssim \|[u, (\cdot)_{T-t}] f_t - \partial_1 u[x_1, (\cdot)_T] f + (\partial_1 u[x_1, (\cdot)_t] f)_{T-t}\| + \|\partial_1 u[x_1, (\cdot)_T] f\| + \|uf_T\| \\ &\lesssim [u]_\alpha [f]_\beta (T^{1/3})^{\alpha+\beta} + [u]_\alpha [f]_\beta T^{(\beta+1)/3} + \|u\| [f]_\beta T^{\beta/3} \\ &\lesssim ([u]_\alpha + \|u\|) [f]_\beta T^{\beta/3}. \end{aligned}$$

where we used again $\|\partial_1 u\| \lesssim [u]_\alpha$. As we assumed $\beta \neq -\frac{1}{2}, -1$, by Lemma 10 we have that $[uf_t - \partial_1 u[x_1, (\cdot)_t] f]_\beta \lesssim ([u]_\alpha + \|u\|) [f]_\beta$.

By Lemma 13 in Appendix, for a subsequence $n \rightarrow \infty$ with $t = 1/2^n$, we have $uf_t - \partial_1 u[x_1, (\cdot)_t] f \rightarrow uf$ for some distribution denoted by uf that belongs to C^β and $[uf]_\beta \lesssim ([u]_\alpha + \|u\|) [f]_\beta$, so (39) holds. Moreover, for every $T > 0$ we have $\psi_{T-t} \rightarrow \psi_T$

in the sense of Schwartz functions as $t \rightarrow 0$ so that $(uf_t)_{T-t} - (\partial_1 u[x_1, (\cdot)_t]f)_{T-t} \rightarrow (uf)_T$ uniformly. By Step 4, passing at the limit $t \rightarrow 0$, we conclude (40), i.e.,

$$\| \underbrace{(uf)_T - uf_T}_{= -[u, (\cdot)_T]f} + \partial_1 u[x_1, (\cdot)_T]f \| \lesssim [u]_\alpha [f]_\beta (T^{\frac{1}{3}})^{\alpha+\beta}.$$

Step 6. We prove (15) for a general (symmetric) Schwartz kernel ϕ . Indeed, this relies on the following representation of ϕ in terms of ψ_t via a family $\{\omega^t\}_{t \in (0,1]}$ of smooth functions of sufficient decay:

$$\phi = \int_0^1 \omega^t * \psi_t dt \quad \text{and} \quad \int_{\mathbb{R}^2} (1 + d^2(x, 0)) |\omega^t(x)| dx \lesssim 1, \quad t \in (0, 1], \quad (54)$$

where $\omega^t := (1 + (1-t)\mathcal{A})\phi$. The formula in (54) follows via $\phi = \int_0^1 \frac{d}{dt}((t-1)\phi_t) dt$ and the definition $\phi_t = \phi * \psi_t$ from the characterization of the kernel ψ_t in form of $\partial_t \psi_t = -\mathcal{A}\psi_t$. Since ϕ is a Schwartz function, the only issue with the estimate in (54) is the non-local term $\int (1+x_1^2) |\partial_1|^3 \phi dx_1$, which is controlled as in Step 1 in the proof of Lemma 10 by the convergent $(\int (1+x_1^2)^{-1} dx_1)^{\frac{1}{2}}$ times $(\int (1+x_1^6) (|\partial_1|^3 \phi)^2 dx_1)^{\frac{1}{2}}$. By Plancherel, the latter can be rewritten as $\sum_{l=0,3} (\int |\partial_{k_1}^l |k_1|^3 \phi|^2 dk_1)^{\frac{1}{2}}$, which itself is dominated by $\sum_{l=0,3} \sum_{n=0, \dots, l} (\int |k_1^{3-n} \partial_{k_1}^n \phi|^2 dk_1)^{\frac{1}{2}}$. Since ϕ is Schwartz, the integral over x_2 of this expression is bounded. Moreover, using the scaling (11) (in particular, $(\psi_t)_\ell = \psi_{t\ell^3}$), the formula in (54) yields

$$\phi_\ell = \int_0^1 \omega_\ell^t * \psi_{t\ell^3} dt, \quad \text{with} \quad \omega_\ell^t(x_1, x_2) = \ell^{-\frac{5}{2}} \omega^t(\ell^{-1}x_1, \ell^{-\frac{3}{2}}x_2),$$

where the subscript ℓ in ω_ℓ^t denotes the rescaling like for ϕ_ℓ .⁷

Coming back to the proof of (15), as we will use several mollifiers, in order to avoid confusion with $[u, (\cdot)_\ell \text{ (or } t)]$, we introduce the following notation for the commutator-convolution:

$$[u, \phi *]f := u\phi * f - \phi * (uf).$$

Then one checks (by simple algebra) the following:

$$[u, \omega * \psi *]f = \omega * [u, \psi *]f + [u, \omega *](\psi * f). \quad (55)$$

⁷ The representation (54) gives a new proof of point ii) in Lemma 10 by passing from the estimate (24) on $\|f_T\|$ to the desired estimate (25) on $\|f_\ell\|$. Indeed, we convert convolution with ϕ_ℓ into convolution with ψ_t and deduce for $\ell \leq 1$:

$$\|f_\ell\| = \|f * \int_0^1 \omega_\ell^t * \psi_{t\ell^3} dt\| \leq \int_0^1 \|\omega_\ell^t * f_{t\ell^3}\| dt \leq \int_0^1 \|f_{t\ell^3}\| \|\omega_\ell^t\|_{L^1} dt \lesssim [f]_\beta \int_0^1 (t^{\frac{1}{3}} \ell)^\beta dt \lesssim [f]_\beta \ell^\beta,$$

as $\beta > -3$.

Combined with (54), this allows us to convert convolution with ϕ_ℓ into convolution with ψ_t :

$$\begin{aligned} [u, \phi_\ell^*]f &= uf_\ell - (uf)_\ell = \int_0^1 u\omega_\ell^t * f_{t\ell^3} - \omega_\ell^t * (uf)_{t\ell^3} dt \\ &= \int_0^1 [u, \omega_\ell^t * \psi_{t\ell^3}^*]f dt \stackrel{(55)}{=} \int_0^1 \left(\omega_\ell^t * [u, \psi_{t\ell^3}^*]f + [u, \omega_\ell^t *]f_{t\ell^3} \right) dt. \end{aligned} \quad (56)$$

- If $\alpha \in (0, 1]$, by the same arguments, we estimate

$$\|[u, \phi_\ell^*]f\| \stackrel{(56)}{\lesssim} \int_0^1 \left(\|[u, \psi_{t\ell^3}^*]f\| + [u]_\alpha \ell^\alpha \|f_{t\ell^3}\| \right) dt.$$

We now appeal to (40) and Lemma 10 to obtain

$$\|[u, \phi_\ell^*]f\| = \|uf_\ell - (uf)_\ell\| \lesssim [u]_\alpha [f]_\beta \int_0^1 \left((t^{\frac{1}{3}}\ell)^{\alpha+\beta} + \ell^\alpha (t^{\frac{1}{3}}\ell)^\beta \right) dt, \quad \forall \ell \in (0, 1].$$

Because in particular $\beta > -3$, this implies (15). It remains to prove that $uf_\ell \rightarrow uf$ in \mathcal{D}' where uf is the distribution defined by (39), which in particular shows the uniqueness of the limit uf independently of the symmetric mask ϕ . Indeed, for every smooth periodic test function ζ , we have that

$$\begin{aligned} \int_{[0,1]^2} uf_\ell \zeta dx &= \int_{[0,1]^2} (uf_\ell - (uf)_\ell) \zeta dx + \int_{[0,1]^2} (uf)_\ell \zeta dx \\ &\stackrel{(15)}{\lesssim} [u]_\alpha [f]_\beta \ell^{\alpha+\beta} \|\zeta\|_{L^1} + \langle uf, \zeta * \phi_\ell \rangle_{\mathcal{D}', \mathcal{D}} \longrightarrow \langle uf, \zeta \rangle_{\mathcal{D}', \mathcal{D}} \quad \text{as } \ell \rightarrow 0, \end{aligned}$$

where we used that ϕ is symmetric.

- If $\alpha \in (1, 3/2)$, starting from (56), we have by (54):

$$\begin{aligned} [u, \phi_\ell^*]f - \partial_1 u[x_1, \phi_\ell^*]f &= \int_0^1 \left(\omega_\ell^t * [u, \psi_{t\ell^3}^*]f + [u, \omega_\ell^t *]f_{t\ell^3} - \partial_1 u[x_1, \omega_\ell^t * \psi_{t\ell^3}^*]f \right) dt \\ &\stackrel{(55)}{=} \int_0^1 \left(\omega_\ell^t * ([u, \psi_{t\ell^3}^*]f - \partial_1 u[x_1, \psi_{t\ell^3}^*]f) \right. \\ &\quad \left. + ([u, \omega_\ell^t *]f_{t\ell^3} - \partial_1 u[x_1, \omega_\ell^t *]f_{t\ell^3}) - [\partial_1 u, \omega_\ell^t *][x_1, \psi_{t\ell^3}^*]f \right) dt. \end{aligned}$$

Now, by the same arguments using (43) and (49), we estimate

$$\begin{aligned} &\|([u, \phi_\ell^*] - \partial_1 u[x_1, \phi_\ell^*])f\| \\ &\lesssim \int_0^1 \left(\|[u, \psi_{t\ell^3}^*] - \partial_1 u[x_1, \psi_{t\ell^3}^*]f\| + [u]_\alpha \ell^\alpha \|f_{t\ell^3}\| + [u]_\alpha \ell^{\alpha-1} \|[x_1, \psi_{t\ell^3}^*]f\| \right) dt. \end{aligned}$$

We now appeal to (40), (44) and Lemma 10 to obtain

$$\begin{aligned} \|([u, \phi_{\ell^*}] - \partial_1 u[x_1, \phi_{\ell^*}])f\| &\lesssim [u]_{\alpha}[f]_{\beta} \int_0^1 ((t^{\frac{1}{3}}\ell)^{\alpha+\beta} + \ell^{\alpha}(t^{\frac{1}{3}}\ell)^{\beta} + \ell^{\alpha-1}(t^{\frac{1}{3}}\ell)^{\beta+1}) dt \\ &\lesssim [u]_{\alpha}[f]_{\beta} \ell^{\alpha+\beta}, \end{aligned}$$

that is (15). It follows that $uf_{\ell} - \partial_1 u[x_1, \phi_{\ell^*}]f \rightarrow uf$ distributionally as in the case $\alpha \in (0, 1]$. \square

4.3 Regularity of the Hilbert transform. Proof of Lemma 7.

Since the Hilbert transform R acts only on the x_1 -variable, it does not map C^{α} into C^{α} on the two-dimensional torus $[0, 1]^2$, but in a slightly larger Hölder space (corresponding to a smaller exponent $\alpha - \varepsilon$ for any $\varepsilon > 0$). This is the result in Lemma 7.

Proof of Lemma 7. For the reader's convenience, we start by proving the boundedness of the Hilbert transform R over the single-variable Hölder space $C_{x_1}^{\alpha}$. Even if the result is standard, we want to highlight that the method based on the “heat kernel” ψ_T in proving Lemma 5 can be adapted here by using a different “heat kernel”. More precisely, we introduce $G_T(x_1) = \frac{1}{T}G(\frac{x_1}{T})$ where the semigroup kernel G is given in Fourier space:

$$G(k_1) = e^{-|k_1|} \text{ for } k_1 \in \mathbb{R},$$

so that G_T is the “heat kernel” of the semigroup generated by $|\partial_1|$. As for the semigroup ψ_T in (21), while $G(x_1)$ is a smooth (since $G(k_1)$ decays exponentially), its decay is moderate (since $G(k_1)$ is only Lipschitz). More precisely, we claim for $\alpha \in (0, \frac{3}{2})$:

$$\int_{\mathbb{R}} |G_T| dx_1 \lesssim 1, \int_{\mathbb{R}} |\partial_1 G_T| dx_1 \lesssim T^{-1}, \int_{\mathbb{R}} |x_1|^{\alpha} |\partial_1 G_T| dx_1 \lesssim T^{\alpha-1}, \int_{\mathbb{R}} ||\partial_1|G_T| dx_1 \lesssim T^{-1}. \quad (57)$$

By scaling, it is enough to consider $T = 1$. By Cauchy-Schwarz, the above reduces to

$$\int_{\mathbb{R}} (x_1^2 + 1)G^2 dx_1, \int_{\mathbb{R}} (x_1^4 + 1)(\partial_1 G)^2 dx_1, \int_{\mathbb{R}} (x_1^2 + 1)(|\partial_1|G)^2 dx_1 \lesssim 1,$$

which by Plancherel is equivalent to

$$\int_{\mathbb{R}} (\partial_{k_1} G)^2 + G^2 dk_1, \int_{\mathbb{R}} (\partial_{k_1}^2 (k_1 G))^2 + (k_1 G)^2 dk_1, \int_{\mathbb{R}} (\partial_{k_1} (|k_1|G))^2 + (k_1 G)^2 dk_1 \lesssim 1,$$

which holds since $k_1 \mapsto G(k_1), |k_1|G(k_1), k_1 G(k_1), \partial_{k_1}(k_1 G(k_1))$ are Lipschitz and the boundedness of the above quantities follows in combination with the exponential decay of G .

Step 1. If $\alpha \in (0, \frac{3}{2}) \setminus \{1\}$, then the Hilbert transform on the one-dimensional torus $[0, 1)$ satisfies

$$[Ru]_\alpha \lesssim [u]_\alpha \quad (58)$$

for any periodic function $u = u(x_1)$ of vanishing average.

• If $\alpha \in (0, 1)$, let $u = u(x_1) \in C_{x_1}^\alpha$ with $\int_0^1 u \, dx_1 = 0$ and $f := Ru$. We want to prove that $f \in C_{x_1}^\alpha$ with $[f]_\alpha \lesssim [u]_\alpha$. We will follow the main lines in Step 3 of the proof of Lemma 10. More precisely, let $u_T = G_T * u$ and $f_T = G_T * f$.⁸ Note that

$$\partial_T f_T = -|\partial_1| f_T = -\partial_1 u_T. \quad (59)$$

As $\int_{\mathbb{R}} \partial_1 G_T \, dx_1 = 0$, we have

$$\begin{aligned} \partial_1 u_T(x_1) &= \int_{\mathbb{R}} \partial_1 G_T(y_1)(u(x_1 - y_1) - u(x_1)) \, dy_1 \\ \Rightarrow \|\partial_1 u_T\| &\leq [u]_\alpha \int_{\mathbb{R}} |y_1|^\alpha |\partial_1 G_T| \, dy_1 \stackrel{(57)}{\lesssim} T^{\alpha-1} [u]_\alpha \end{aligned}$$

as well as $\|\partial_1^2 u_T\| = \|\partial_1 u_{T/2} * \partial_1 G_{T/2}\| \leq \|\partial_1 u_{T/2}\| \int_{\mathbb{R}} |\partial_1 G_{T/2}| \, dx_1 \stackrel{(57)}{\lesssim} T^{\alpha-2} [u]_\alpha$. Since $\alpha > 0$, it follows for every $0 < \tau < T$:

$$\|f_T - f_\tau\| = \left\| \int_\tau^T \partial_s f_s \, ds \right\| \stackrel{(59)}{\leq} \int_\tau^T \|\partial_1 u_s\| \, ds \lesssim T^\alpha [u]_\alpha \quad (60)$$

which in particular proves by passing to the limit $\tau < T \rightarrow 0$ that f is a continuous function. Moreover, by (57), $\|\partial_1 f_T\| = \|u * |\partial_1| G_T\| \leq \|u\| \int_{\mathbb{R}} |\partial_1| G_T \, dx_1 \rightarrow 0$ as $T \rightarrow \infty$, so that thanks to $\alpha < 1$

$$\|\partial_1 f_T\| = \left\| \int_T^\infty \frac{d}{ds} (\partial_1 f)_s \, ds \right\| \stackrel{(59)}{\leq} \int_T^\infty \|\partial_1^2 u_s\| \, ds \lesssim \frac{1}{T^{1-\alpha}} [u]_\alpha. \quad (61)$$

We now show (58) by the following argument: for $x_1, x'_1 \in [0, 1)$, we write

$$f(x'_1) - f(x_1) = (f - f_T)(x'_1) + (f_T - f)(x_1) + \int_0^1 (x'_1 - x_1) \partial_1 f_T(sx'_1 + (1-s)x_1) \, ds.$$

Hence $|f(x'_1) - f(x_1)| \leq 2\|f - f_T\| + \|\partial_1 f_T\| |x'_1 - x_1| \stackrel{(60),(61)}{\lesssim} (T^\alpha + T^{\alpha-1} |x_1 - x'_1|) [u]_\alpha$. Choosing T such that $T = |x_1 - x'_1|$ yields $[f]_\alpha \lesssim [u]_\alpha$.

• If $\alpha \in (1, \frac{3}{2})$, let $u \in C^\alpha$ with $\int_0^1 u \, dx_1 = 0$ and $f = Ru$. Then by the above argument, $\partial_1 f = R(\partial_1 u) \in C^{\alpha-1}$ and $[\partial_1 f]_{\alpha-1} \lesssim [\partial_1 u]_{\alpha-1} \lesssim [u]_\alpha$ (by Lemma 12).

⁸Do not confound with the notation (23) that uses the semigroup ψ_T defined in (22). We use the semigroup G_T only at this step.

Therefore, $|f(x'_1) - f(x_1) - \partial_1 f(x_1)(x'_1 - x_1)| \leq |x'_1 - x_1| \int_0^1 |\partial_1 f(x_1 + s(x'_1 - x_1)) - \partial_1 f(x_1)| ds \lesssim [\partial_1 f]_{\alpha-1} |x'_1 - x_1|^\alpha$, and we conclude that $[f]_\alpha \lesssim [u]_\alpha$.

Step 2. For $0 < \beta < \alpha < \frac{3}{2}$, the Hilbert transform R satisfies $[Ru]_\beta \lesssim [u]_\alpha$ for all periodic functions u of vanishing average in x_1 , a space we denote by \mathcal{S}_0 for abbreviation.

• If $\alpha \in (0, 1]$, let $u \in C^\alpha \cap \mathcal{S}_0$ and set $f := Ru$. As $\beta < \alpha$, we may w.l.o.g. assume that $\alpha < 1$ (otherwise, replace α by $\tilde{\alpha} := (\alpha + \beta)/2 \in (0, 1)$ and use that $[u]_\alpha \gtrsim [u]_{\tilde{\alpha}}$).

By Step 1 and Lemma 12, we know that for every $x_2 \in [0, 1)$, $f(\cdot, x_2) \in C^\alpha([0, 1]) \cap \mathcal{S}_0$ with $[f(\cdot, x_2)]_\beta \lesssim [f(\cdot, x_2)]_\alpha \lesssim [u(\cdot, x_2)]_\alpha \lesssim [u]_\alpha$ as $\beta \in (0, \alpha)$; moreover, we have thanks to the vanishing average in x_1 : $\|f(\cdot, x_2) - f(\cdot, y_2)\| \lesssim [f(\cdot, x_2) - f(\cdot, y_2)]_{\alpha-\beta} \lesssim [u(\cdot, x_2) - u(\cdot, y_2)]_{\alpha-\beta}$, for every $x_2, y_2 \in (0, 1)$. To conclude, it is enough to bound the latter RHS by $|x_2 - y_2|^{2\beta/3}$. Indeed, by Definition 1, we have for every $x_1, x_2, y_1, y_2 \in [0, 1)$:⁹

$$\begin{aligned} \left| (u(x_1, x_2) - u(x_1, y_2)) - (u(y_1, x_2) - u(y_1, y_2)) \right| &\leq 2[u]_\alpha \min\{|x_1 - y_1|^\alpha, |x_2 - y_2|^{2\alpha/3}\} \\ &\leq 2[u]_\alpha |x_1 - y_1|^{\alpha-\beta} |x_2 - y_2|^{2\beta/3} \end{aligned} \quad (62)$$

yielding $[u(\cdot, x_2) - u(\cdot, y_2)]_{C^{\alpha-\beta}(I)} \lesssim [u]_\alpha |x_2 - y_2|^{2\beta/3}$; thus, $[f]_\beta \lesssim [u]_\alpha$.

• If $\alpha \in (1, \frac{3}{2})$, let $u \in C^\alpha \cap \mathcal{S}_0$ and $f := Ru$. By Step 1, we know $[f(\cdot, x_2)]_\beta \lesssim [u(\cdot, x_2)]_\alpha \lesssim [u]_\alpha$ for every $x_2 \in [0, 1)$. Since the function $u(\cdot, x_2) - u(\cdot, y_2)$ is Lipschitz (by Lemma 12), using the same argument as in (62) and Step 1, we have for every $x_1, x_2, y_2 \in (0, 1)$: $|f(x_1, x_2) - f(x_1, y_2)| \lesssim [f(\cdot, x_2) - f(\cdot, y_2)]_{(\alpha-\beta)/\alpha} \lesssim [u(\cdot, x_2) - u(\cdot, y_2)]_{(\alpha-\beta)/\alpha} \lesssim [u]_\alpha |x_2 - y_2|^{2\beta/3}$ because $\min\{|x_1 - y_1|, |x_2 - y_2|^{2\alpha/3}\} \leq |x_1 - y_1|^{(\alpha-\beta)/\alpha} |x_2 - y_2|^{2\beta/3}$. The conclusion is now straightforward. \square

5 Proof of the main results: Theorems 2 and 9

The twin Theorems 2 and 9 are an immediate consequence of the following purely deterministic result, which relies on a fixed point argument based on the Schauder theory of Lemma 5 and the regular product result of Lemma 6.

Proposition 11. *For given $0 < \epsilon < \frac{1}{8}$, there exists a (possibly large) constant $C > 0$ with the following property: Suppose we are given a function v of vanishing average*

⁹We use $\min\{a, b\} \leq a^\epsilon b^{1-\epsilon}$ for $\epsilon \in (0, 1)$ and $a, b \geq 0$.

and a distribution F , both periodic, and small in the sense of

$$[F]_{-\frac{3}{4}-\epsilon} + [v]_{\frac{3}{4}-\epsilon} \leq \frac{1}{C}. \quad (63)$$

Then there exists a unique periodic function w of vanishing average in x_1 with

$$[w]_{\frac{5}{4}-2\epsilon} \leq \frac{1}{C}$$

and that satisfies (in a distributional sense)

$$\begin{aligned} & (-\partial_1^2 - |\partial_1|^{-1}\partial_2^2)w \\ & + P\left(F + v\partial_2 R w + w\partial_2 R v + w\partial_2 R w + \partial_2 \frac{1}{2}R(w+v)^2 - (w+v)\partial_1 \frac{1}{2}R(w+v)^2\right) = 0. \end{aligned}$$

Moreover, we have the a priori estimate

$$[w]_{\frac{5}{4}-2\epsilon} \lesssim [F]_{-\frac{3}{4}-\epsilon} + [v]_{\frac{3}{4}-\epsilon}^2. \quad (64)$$

Finally, if \tilde{w} denotes the solution for another data pair (\tilde{v}, \tilde{F}) , then we have

$$[w - \tilde{w}]_{\frac{5}{4}-2\epsilon} \lesssim [F - \tilde{F}]_{-\frac{3}{4}-\epsilon} + [v - \tilde{v}]_{\frac{3}{4}-\epsilon}. \quad (65)$$

We postpone for the moment the proof of Proposition 11, and we use it in order to prove our main results:

Proof of Theorems 2 and 9. Fix $0 < \epsilon < \frac{1}{8}$. From the stochastic Lemmas 4 and 8, and using the Schauder theory of Lemma 5 in order to pass from ξ to v , we know that the random variable

$$\sup_{\ell \leq 1} \left(([v_\ell]_{\frac{3}{4}-\epsilon} + [F^\ell]_{-\frac{3}{4}-\epsilon}^{\frac{1}{2}}) + \ell^{-\frac{\epsilon}{2}} ([v_\ell - v]_{\frac{3}{4}-\epsilon} + [F^\ell - F]_{-\frac{3}{4}-\epsilon}^{\frac{1}{2}}) \right)$$

has bounded moments of all order p . Hence there exists a random variable $\sigma_0 \geq 0$ that is almost surely positive such that on the one hand, we have for all $\ell \leq 1$

$$\sigma_0 [v_\ell]_{\frac{3}{4}-\epsilon} + \sigma_0^2 [F^\ell]_{-\frac{3}{4}-\epsilon} \leq \frac{1}{C}, \quad (66)$$

$$\sigma_0 [v_\ell - v]_{\frac{3}{4}-\epsilon} + \sigma_0^2 [F^\ell - F]_{-\frac{3}{4}-\epsilon} \leq \ell^{\frac{\epsilon}{2}}, \quad (67)$$

where C denotes the constant in Proposition 11, and on the other hand, $\frac{1}{\sigma_0}$ has bounded moments of all order.

In view of (66), for fixed $0 \leq \sigma \leq \sigma_0$ and $0 < \ell \leq 1$ we may apply Proposition 11 with $(\sigma^2 F^\ell, \sigma v_\ell)$ playing the role of (F, v) . It yields a unique periodic function w^ℓ of

vanishing average in x_1 with $[w^\ell]_{\frac{5}{4}-2\epsilon} \leq \frac{1}{C}$ and such that

$$\begin{aligned} & (-\partial_1^2 - |\partial_1|^{-1}\partial_2^2)w^\ell \\ & + P\left(\sigma^2 F^\ell + \sigma v_\ell \partial_2 R w^\ell + \sigma w^\ell \partial_2 R v_\ell + w^\ell \partial_2 R w^\ell\right. \\ & \left. + \partial_2 \frac{1}{2} R(w^\ell + \sigma v_\ell)^2 - (w^\ell + \sigma v_\ell) \partial_1 \frac{1}{2} R(w^\ell + \sigma v_\ell)^2\right) = 0. \end{aligned}$$

Since the convolution parameter $\ell > 0$ is present, we have by definition $F^\ell = v_\ell \partial_2 R v_\ell$ so that in terms of $u^\ell := \sigma v_\ell + w^\ell$ and by definition of v through $(-\partial_1^2 - |\partial_1|^{-1}\partial_2^2)v = P\xi$, the above equation turns into the desired Euler-Lagrange equation

$$\begin{aligned} & (-\partial_1^2 - |\partial_1|^{-1}\partial_2^2)u^\ell \\ & + P\left(u^\ell \partial_2 R u^\ell + \partial_2 \frac{1}{2} R(u^\ell)^2 - u^\ell \partial_1 \frac{1}{2} R(u^\ell)^2 - \sigma \xi_\ell\right) = 0. \end{aligned}$$

The a priori estimate (64) on w^ℓ turns into the desired $[u^\ell - \sigma v_\ell]_{\frac{5}{4}-2\epsilon} \lesssim [\sigma^2 F^\ell]_{-\frac{3}{4}-\epsilon} + [\sigma v_\ell]_{\frac{3}{4}-\epsilon}^2 \lesssim \left(\frac{\sigma}{\sigma_0}\right)^2$, where we used (66) in the last estimate.

We now turn to the first convergence statement as $\ell \downarrow 0$ in (18), which assumes the form $\lim_{\ell \downarrow 0} [w^\ell - w]_{\frac{5}{4}-2\epsilon} = 0$, where w is the solution provided by Proposition 11 for $(\sigma^2 F, \sigma v)$ playing the role of (F, v) there. In particular by (64) and (66) for $\ell = 0$, we obtain (20). It follows from the convergence (67) of the data, which we need for $\sigma \leq \sigma_0$ but only in the qualitative form of

$$\lim_{\ell \downarrow 0} ([\sigma v_\ell - \sigma v]_{\frac{3}{4}-\epsilon} + [\sigma^2 F^\ell - \sigma^2 F]_{-\frac{3}{4}-\epsilon}) = 0,$$

and the continuity property (65) of the fixed point w^ℓ in the data $(\sigma^2 F^\ell, \sigma v_\ell)$. The second convergence statement in (18) is contained in (67) via Lemma 5. \square

Proof of Proposition 11. We will apply Banach's contraction mapping theorem on the application

$$\begin{aligned} (F, v, w) \mapsto \Phi(F, v, w) := & -(-\partial_1^2 - |\partial_1|^{-1}\partial_2^2)^{-1} P\left(F + v \partial_2 R w + w \partial_2 R v + w \partial_2 R w\right. \\ & \left. + \partial_2 \frac{1}{2} R(w + v)^2 - (w + v) \partial_1 \frac{1}{2} R(w + v)^2\right); \end{aligned}$$

we are interested in its fixed points $w = \Phi(F, v, w)$. We make the standing assumptions

$$[F]_{-\frac{3}{4}-\epsilon}, [v]_{\frac{3}{4}-\epsilon}, [w]_{\frac{5}{4}-2\epsilon} \leq 1 \tag{68}$$

with v and w of vanishing average. Existence and uniqueness under the smallness condition (63) will follow from the following boundedness and Lipschitz continuity of Φ in the w -variable

$$[\Phi(F, v, w)]_{\frac{5}{4}-2\epsilon} \lesssim [F]_{-\frac{3}{4}-\epsilon} + ([v]_{\frac{3}{4}-\epsilon} + [w]_{\frac{5}{4}-2\epsilon})^2, \quad (69)$$

$$[\Phi(F, v, w) - \Phi(F, v, w')]_{\frac{5}{4}-2\epsilon} \lesssim ([v]_{\frac{3}{4}-\epsilon} + [w]_{\frac{5}{4}-2\epsilon} + [w']_{\frac{5}{4}-2\epsilon})[w - w']_{\frac{5}{4}-2\epsilon}. \quad (70)$$

Indeed, the first property ensures that under the smallness condition (63) on (F, v) , $w \mapsto \Phi(F, v, w)$ is a self-map on a sufficiently small ball w. r. t. $[w]_{\frac{5}{4}-2\epsilon}$. The second property ensures that on such a sufficiently small ball, and perhaps strengthening the smallness condition on v , this map $w \mapsto \Phi(F, v, w)$ is a contraction w. r. t. $[w]_{\frac{5}{4}-2\epsilon}$. Since the space of all periodic w 's with vanishing average in x_1 is complete when endowed with $[w]_{\frac{5}{4}-2\epsilon}$, we are done. For the a priori estimate (64) on the fixed point w we write $w = \Phi(F, v, 0) + (\Phi(F, v, w) - \Phi(F, v, 0))$ and note that by (69) the first RHS term is estimated by the desired $[F]_{-\frac{3}{4}-\epsilon} + [v]_{\frac{3}{4}-\epsilon}^2$, whereas the second RHS term can be absorbed into the l. h. s. by contractivity. For the continuity (65) of the fixed point $w = w(F, v)$ in (F, v) , it is sufficient to establish the following Lipschitz continuity of Φ in the (F, v) -variables

$$[\Phi(F, v, w) - \Phi(F', v', w)]_{\frac{5}{4}-2\epsilon} \lesssim [F - F']_{-\frac{3}{4}-\epsilon} + [v - v']_{\frac{3}{4}-\epsilon}. \quad (71)$$

Indeed, (65) follows from writing $w - w' = \Phi(F, v, w) - \Phi(F', v', w') = \Phi(F, v, w) - \Phi(F', v', w) + \Phi(F', v', w) - \Phi(F', v', w')$ and applying (71) on the first RHS term and appealing to contractivity to absorb the second RHS term.

We now turn to the proof (69), (70), and (71) — always under the assumption (68). By the Schauder theory of Lemma 5, it is sufficient to consider

$$\begin{aligned} \Psi(v, w) := & v\partial_2 R w + w\partial_2 R v + w\partial_2 R w \\ & + \partial_2 \frac{1}{2} R (w + v)^2 - (w + v)\partial_1 \frac{1}{2} R (w + v)^2 \end{aligned} \quad (72)$$

so that $\Phi(F, v, w) = \mathcal{L}^{-1} P F + \mathcal{L}^{-1} P \Psi(v, w)$ and to establish

$$\begin{aligned} [\Psi(v, w)]_{-\frac{3}{4}-2\epsilon} & \lesssim ([v]_{\frac{3}{4}-\epsilon} + [w]_{\frac{5}{4}-2\epsilon})^2, \\ [\Psi(v, w) - \Psi(v, w')]_{-\frac{3}{4}-2\epsilon} & \lesssim ([v]_{\frac{3}{4}-\epsilon} + [w]_{\frac{5}{4}-2\epsilon} + [w']_{\frac{5}{4}-2\epsilon})[w - w']_{\frac{5}{4}-2\epsilon}, \\ [\Psi(v, w) - \Psi(v', w)]_{-\frac{3}{4}-2\epsilon} & \lesssim [v - v']_{\frac{3}{4}-\epsilon}. \end{aligned}$$

It is convenient to separate Ψ into a quadratic and a cubic part so that it is sufficient (also using the ordering of negative exponent $[\cdot]_{-\frac{3}{4}-2\epsilon} \lesssim [\cdot]_{-\frac{1}{4}-3\epsilon} \lesssim [\cdot]_{-\frac{1}{4}-2\epsilon}$ and positive exponent $[\cdot]_{\frac{3}{4}-2\epsilon} \lesssim [\cdot]_{\frac{3}{4}-\epsilon} \lesssim [\cdot]_{\frac{5}{4}-2\epsilon}$ Hölder norms, see Remark 2), to show the multi-

linear estimates for v_1, v_2, v_3 of vanishing average:

$$[v_1 \partial_2 R v_2]_{-\frac{1}{4}-3\epsilon} \lesssim [v_1]_{\frac{3}{4}-\epsilon} [v_2]_{\frac{5}{4}-2\epsilon}, \quad (73)$$

$$[v_1 \partial_2 R v_2]_{-\frac{3}{4}-2\epsilon} \lesssim [v_1]_{\frac{5}{4}-2\epsilon} [v_2]_{\frac{3}{4}-\epsilon}, \quad (74)$$

$$[\partial_2 R(v_1 v_2)]_{-\frac{3}{4}-2\epsilon} \lesssim [v_1]_{\frac{3}{4}-\epsilon} [v_2]_{\frac{3}{4}-\epsilon}, \quad (75)$$

$$[v_1 \partial_1 R(v_2 v_3)]_{-\frac{1}{4}-2\epsilon} \lesssim [v_1]_{\frac{3}{4}-\epsilon} [v_2]_{\frac{3}{4}-\epsilon} [v_3]_{\frac{3}{4}-\epsilon}. \quad (76)$$

This is easy for (75): For periodic functions with vanishing average we have the algebra property $[v_1 v_2]_{\frac{3}{4}-\epsilon} \lesssim [v_1]_{\frac{3}{4}-\epsilon} [v_2]_{\frac{3}{4}-\epsilon}$, cf. Lemma 12 in Appendix; by Lemma 7, we may get rid of the Hilbert transform R at the prize of an ϵ : $[R(v_1 v_2)]_{\frac{3}{4}-2\epsilon} \lesssim [v_1 v_2]_{\frac{3}{4}-\epsilon}$; finally, by the definition 3 of negative exponent Hölder norms we have $[\partial_2 R(v_1 v_2)]_{-\frac{3}{4}-2\epsilon} \leq [R(v_1 v_2)]_{\frac{3}{4}-2\epsilon}$.

For (76) we use the same strategy: we note that by the algebra property $[v_2 v_3]_{\frac{3}{4}-\epsilon} \lesssim [v_2]_{\frac{3}{4}-\epsilon} [v_3]_{\frac{3}{4}-\epsilon}$, that by the boundedness of the Hilbert transform $[R(v_2 v_3)]_{\frac{3}{4}-2\epsilon} \lesssim [v_2 v_3]_{\frac{3}{4}-\epsilon}$, and that by definition of the negative exponent Hölder norms $[\partial_1 R(v_2 v_3)]_{-\frac{1}{4}-2\epsilon} \lesssim [R(v_2 v_3)]_{\frac{3}{4}-2\epsilon}$. As a new element, we need to appeal to Lemma 6 to obtain $[v_1 \partial_1 R(v_2 v_3)]_{-\frac{1}{4}-2\epsilon} \lesssim [v_1]_{\frac{3}{4}-\epsilon} [\partial_1 R(v_2 v_3)]_{-\frac{1}{4}-2\epsilon}$ (recall that v_1 is of vanishing average), which requires $(\frac{3}{4} - 2\epsilon) + (-\frac{1}{4} - 2\epsilon) = \frac{1}{2} - 4\epsilon > 0$.

Estimates (73) and (74) use the same ingredients

$$\begin{aligned} [v_1 \partial_2 R v_2]_{-\frac{1}{4}-3\epsilon} &\lesssim [v_1]_{\frac{3}{4}-\epsilon} [\partial_2 R v_2]_{-\frac{1}{4}-3\epsilon} \quad \text{for } (\frac{3}{4} - \epsilon) + (-\frac{1}{4} - 3\epsilon) > 0 \\ &\lesssim [v_1]_{\frac{3}{4}-\epsilon} [R v_2]_{\frac{5}{4}-3\epsilon} \lesssim [v_1]_{\frac{3}{4}-\epsilon} [v_2]_{\frac{5}{4}-2\epsilon}, \\ [v_1 \partial_2 R v_2]_{-\frac{3}{4}-2\epsilon} &\lesssim [v_1]_{\frac{5}{4}-2\epsilon} [\partial_2 R v_2]_{-\frac{3}{4}-2\epsilon} \quad \text{for } (\frac{5}{4} - 2\epsilon) + (-\frac{3}{4} - 2\epsilon) > 0 \\ &\lesssim [v_1]_{\frac{5}{4}-2\epsilon} [R v_2]_{\frac{3}{4}-2\epsilon} \lesssim [v_1]_{\frac{5}{4}-\epsilon} [v_2]_{\frac{3}{4}-\epsilon}. \end{aligned}$$

□

6 Estimates of the stochastic terms

6.1 Estimate of the white noise. Proof of Lemma 4

Proof of Lemma 4. To simplify the notation, we will denote $\xi := P\xi$. We divide the proof in several steps:

Step 1. In this step, we consider the Fourier coefficients $\xi_\ell(k) := \int_{[0,1]^2} e^{-ik \cdot x} \xi_\ell(x) dx$, $k \in (2\pi\mathbb{Z})^2$, of $\xi_\ell = \xi * \phi_\ell$ and its logarithmic derivative $\ell \frac{\partial}{\partial \ell} \xi_\ell(k)$ in the convolution

scale ℓ . For $k \neq 0$, we claim the following stochastic (second moment) bounds

$$\langle |\xi_\ell(k)|^2 \rangle \lesssim 1, \quad \langle |\ell \frac{\partial}{\partial \ell} \xi_\ell(k)|^2 \rangle \lesssim \min\{1, \ell^2 d^2(k, 0)\}, \quad \forall k \in (2\pi\mathbb{Z})^2.$$

Recall that \lesssim means that the (generic) constant only depends on ϕ in this context.

Within our identification $\xi := P\xi$, $\xi_\ell(k)$ vanishes for $k_1 = 0$, so that we may restrict to $k \neq 0$. Recall

$$\xi_\ell(k) := \phi(\ell k_1, \ell^{\frac{3}{2}} k_2) \xi(k),$$

and where $\phi(k)$ denotes the Fourier *transform* of the Schwartz mask ϕ of the convolution kernel $\phi_\ell(x_1, x_2) = \ell^{-\frac{5}{2}} \phi(\frac{x_1}{\ell}, \frac{x_2}{\ell^{\frac{3}{2}}})$, $x \in \mathbb{R}^2$ so that in Fourier space, $\phi_\ell(k) = \phi(\ell k_1, \ell^{\frac{3}{2}} k_2)$, $k \in \mathbb{R}^2$. We are also interested in the sensitivities $\ell \frac{\partial}{\partial \ell}$ with respect to the convolution length ℓ . It is convenient to consider the derivative in this logarithmic form $\ell \frac{\partial}{\partial \ell} = \frac{\partial}{\partial \ln \ell}$ since it preserves the structure of the convolution:

$$\ell \frac{\partial}{\partial \ell} [\phi(\ell k_1, \ell^{\frac{3}{2}} k_2)] = \ell k_1 \frac{\partial \phi}{\partial k_1}(\ell k_1, \ell^{\frac{3}{2}} k_2) + \frac{3}{2} \ell^{\frac{3}{2}} k_2 \frac{\partial \phi}{\partial k_2}(\ell k_1, \ell^{\frac{3}{2}} k_2) =: \delta \phi_\ell(k) \quad (77)$$

is the rescaled Fourier transform of another Schwartz function $\delta \phi$ given through $\delta \phi := (k_1 \frac{\partial}{\partial k_1} + \frac{3}{2} k_2 \frac{\partial}{\partial k_2}) \phi$, which in real space assumes the form $\delta \phi = -\partial_1(x_1 \phi) - \frac{3}{2} \partial_2(x_2 \phi) = -(\frac{5}{2} + x_1 \partial_1 + \frac{3}{2} x_2 \partial_2) \phi$. Therefore, this prompts the definition of

$$\delta \xi_\ell(k) := \ell \frac{\partial}{\partial \ell} \xi_\ell(k) = \delta \phi(\ell k_1, \ell^{\frac{3}{2}} k_2) \xi(k).$$

By the relation between convolution and Fourier series (as explained at the beginning of Section 4) we have $\xi_\ell(k) = \phi_\ell(k) \xi(k)$ where $\xi(k)$ is the Fourier coefficient of ξ . By the characterizing property of white noise we have

$$\langle |\xi(k)|^2 \rangle = \int_{[0,1]^2} dx \int_{[0,1]^2} dx' e^{ik \cdot (x' - x)} \langle \xi(x) \bar{\xi}(x') \rangle = \int_{[0,1]^2} dx = 1.$$

Hence, we deduce the desired estimates:

$$\langle |\xi_\ell(k)|^2 \rangle = |\phi_\ell|^2(k) \lesssim 1 \quad \langle |\ell \frac{\partial}{\partial \ell} \xi_\ell(k)|^2 \rangle = |\delta \phi_\ell|^2(k) \lesssim \min\{1, \ell^2 d^2(k, 0)\}, \quad k \neq 0,$$

where we used that $|\delta \phi(k)| \leq \min\{1, d(k, 0)\}$ for all $k \in \mathbb{R}^2$ (because by definition, $\delta \phi$ vanishes for $k = 0$).

Step 2. We claim that for all $\ell \leq 1$, $x \in \mathbb{R}^2$, and $T > 0$ we have the estimate

$$\langle (\xi_{\ell, T}(x))^2 \rangle^{\frac{1}{2}} \lesssim (T^{\frac{1}{3}})^{-\frac{5}{4}}, \quad (78)$$

$$\langle (\ell \frac{\partial}{\partial \ell} \xi_{\ell, T}(x))^2 \rangle^{\frac{1}{2}} \lesssim \min\{(T^{\frac{1}{3}})^{-\frac{5}{4}}, \ell (T^{\frac{1}{3}})^{-\frac{9}{4}}\}, \quad (79)$$

where $\xi_{\ell,T} := \xi * \phi_\ell * \psi_T$ and ψ_T is given by (21). Indeed, since the distribution ξ and its translation $\xi(\cdot + h)$ by some translation vector h have the same distribution under $\langle \cdot \rangle$, this shift-invariance carries over to ξ_ℓ and $\xi_{\ell,T}$. This implies that $\langle (\xi_{\ell,T}(x))^2 \rangle$ does not depend on x . Hence for (78) it is enough to establish the space-integrated version

$$\left\langle \int_{[0,1]^2} \xi_{\ell,T}^2 dx \right\rangle \lesssim (T^{\frac{1}{3}})^{-\frac{5}{2}}.$$

In conjunction with the periodicity of $\xi_{\ell,T} = \psi_T * \xi_\ell$, this allows us to appeal to Plancherel and the relation between convolution and Fourier series (see the beginning of Section 4), into which we insert the Fourier characterization (21) of ψ_T :

$$\left\langle \int_{[0,1]^2} \xi_{\ell,T}^2 dx \right\rangle = \sum_{k \in (2\pi\mathbb{Z})^2} \exp(-2T(|k_1|^3 + k_2^2)) \langle |\xi_\ell(k)|^2 \rangle \lesssim \sum_{k \in (2\pi\mathbb{Z})^2, k \neq 0} \exp(-T(|k_1|^3 + k_2^2))$$

where we used Step 1 (and that $\xi_\ell(k=0) = \xi(k=0) = 0$ thanks to the presence of P in the definition of $\xi := P\xi$). It is thus sufficient to show:

$$\sum_{k \in (2\pi\mathbb{Z})^2 \setminus \{0\}} \exp(-Td^3(k, 0)) \lesssim (T^{\frac{1}{3}})^{-\frac{5}{2}}, \quad \forall T > 0. \quad (80)$$

To prove this kind of estimate, we will systematically use the following algorithm: by the obvious ‘‘volume scaling’’

$$\#\{k \in (2\pi\mathbb{Z})^2 \mid \ell d(k, 0) \leq 1\} \lesssim \ell^{-\frac{5}{2}}, \quad \ell \leq 1, \quad (81)$$

we do a decomposition into dyadic annuli (based on the distance d), so that (81) implies that the above integral may be estimated as their Euclidean counterpart in dimension $\frac{5}{2}$:

$$\begin{aligned} \sum_{k \in (2\pi\mathbb{Z})^2 \setminus \{0\}} \exp(-Td^3(k, 0)) &\leq \sum_{n \in \mathbb{Z}} \sum_{T^{\frac{1}{3}}d(k,0) \in (2^{n-1}, 2^n], k \in (2\pi\mathbb{Z})^2} \exp(-2^{3(n-1)}) \\ &\leq \sum_{n \in \mathbb{Z}} \exp(-2^{3(n-1)}) \#\{k \neq 0 \mid T^{\frac{1}{3}}d(k, 0) \leq 2^n\} \\ &\lesssim (T^{-\frac{1}{3}})^{\frac{5}{2}} \sum_{n \in \mathbb{Z}} 2^{\frac{5n}{2}} \exp(-2^{3(n-1)}) \lesssim (T^{-\frac{1}{3}})^{\frac{5}{2}}. \end{aligned}$$

Turning to (79), which differs from (78) through the presence of $\ell \frac{\partial}{\partial \ell}$, we see by analogous arguments that we need to establish (due to Step 1):

$$\sum_{k \in (2\pi\mathbb{Z})^2 \setminus \{0\}} \exp(-Td^3(k, 0)) \min\{1, \ell^2 d^2(k, 0)\} \lesssim \min\{(T^{\frac{1}{3}})^{-\frac{5}{2}}, \ell^2 (T^{\frac{1}{3}})^{-\frac{9}{2}}\}, \quad \forall \ell \leq 1, T > 0,$$

which splits into (80) and

$$\sum_{k \neq 0} \exp(-Td^3(k, 0)) d^2(k, 0) = T^{-\frac{2}{3}} \sum_{k \neq 0} \exp(-Td^3(k, 0)) (T^{\frac{1}{3}}d(k, 0))^2 \lesssim (T^{\frac{1}{3}})^{-\frac{9}{2}} \quad (82)$$

which follows by the same argument as above.

Step 3. We claim that for all $\ell \leq 1$, $1 \leq p < \infty$, and $T > 0$ we have the estimate

$$\langle \|\xi_{\ell,T}\|^p \rangle^{\frac{1}{p}} \lesssim (T^{\frac{1}{3}})^{-\frac{5}{4}-\frac{1}{p}\frac{5}{2}}, \quad (83)$$

$$\langle \|\ell \frac{\partial}{\partial \ell} \xi_{\ell,T}\|^p \rangle^{\frac{1}{p}} \lesssim \min\{(T^{\frac{1}{3}})^{-\frac{5}{4}-\frac{1}{p}\frac{5}{2}}, \ell (T^{\frac{1}{3}})^{-\frac{9}{4}-\frac{1}{p}\frac{5}{2}}\}. \quad (84)$$

Note that both the left-hand side (LHS) becomes larger (Jensen's) and the RHS smaller (at least for $T \leq 1$) as p increases; however, the constant hidden in \lesssim now depends on p and blows up as $p \uparrow \infty$. Estimates (83) and (84) follow from Step 2 in two stages. Here comes the first, stochastic stage: The two random variables considered in Step 2, i.e., $f \in \{\xi_{\ell,T}(x), \ell \frac{\partial}{\partial \ell} \xi_{\ell,T}(x)\}$, are linear expressions in the Gaussian field ξ and thus centered Gaussian random variables. As such, they satisfy an inverse Jensen's inequality $\langle |f|^p \rangle^{\frac{1}{p}} \lesssim \langle f^2 \rangle^{\frac{1}{2}}$ for all $1 \leq p < \infty$. Hence we obtain from Step 2, after integration in x

$$\langle \int_{[0,1]^2} |\xi_{\ell,T}|^p \rangle^{\frac{1}{p}} \lesssim (T^{\frac{1}{3}})^{-\frac{5}{4}}, \quad \langle \int_{[0,1]^2} |\ell \frac{\partial}{\partial \ell} \xi_{\ell,T}|^p \rangle^{\frac{1}{p}} \lesssim \min\{(T^{\frac{1}{3}})^{-\frac{5}{4}}, \ell (T^{\frac{1}{3}})^{-\frac{9}{4}}\}.$$

We now turn to the second, deterministic stage. It follows from the fact that for all the fields $f \in \{\xi_{\ell}, \ell \frac{\partial}{\partial \ell} \xi_{\ell}\}$ we have by the semi-group property (23) in form of $f_T = \psi_{\frac{T}{2}} * f_{\frac{T}{2}}$ so that for $p \in (1, \infty)$:

$$\begin{aligned} \|f_T\| &\leq \left(\int_{\mathbb{R}^2} |\psi_{\frac{T}{2}}|^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}} \left(\int_{[0,1]^2} |f_{\frac{T}{2}}|^p \right)^{\frac{1}{p}} \\ &\stackrel{(22)}{=} ((T^{\frac{1}{3}})^{\frac{5}{2}})^{-\frac{1}{p}} \left(\int_{\mathbb{R}^2} |\psi|^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}} \left(\int_{[0,1]^2} |f_{\frac{T}{2}}|^p \right)^{\frac{1}{p}}, \end{aligned}$$

where the ‘‘volume’’ factor $(T^{\frac{1}{3}})^{\frac{5}{2}}$ arises from the rescaling of dx . The combination of $\psi \in L^1$ (see (26)) with $\|\psi\| \lesssim \int |\psi(k)| dk < \infty$ yields $(\int_{\mathbb{R}^2} |\psi|^{\frac{p}{p-1}})^{\frac{p-1}{p}} \lesssim 1$, so that the above turns into the ‘‘inverse estimate’’

$$\|f_T\| \lesssim (T^{\frac{1}{3}})^{-\frac{5}{2}\frac{1}{p}} \left(\int_{[0,1]^2} |f_{\frac{T}{2}}|^p \right)^{\frac{1}{p}}. \quad (85)$$

This proves (83)-(84) for $p > 1$. For $p = 1$, the conclusion follows by Jensen's inequality.

Step 4. We claim that for all $\varepsilon \in (0, 1)$, $1 \leq p < \infty$ and $\ell_0 \leq 1$ we have for every $T \leq 1$:

$$\langle \sup_{\ell \leq 1} \|\xi_{\ell,T}\|^p \rangle^{\frac{1}{p}} \lesssim (T^{\frac{1}{3}})^{-\frac{5}{4}-2\varepsilon}, \quad (86)$$

$$\langle \sup_{\ell, \ell' \leq \ell_0} \|\xi_{\ell,T} - \xi_{\ell',T}\|^p \rangle^{\frac{1}{p}} \lesssim \ell_0^\varepsilon (T^{\frac{1}{3}})^{-\frac{5}{4}-2\varepsilon}, \quad (87)$$

where now, the constant also depends on $\varepsilon > 0$. Clearly, by the triangle inequality w.r.t. to $\langle |\cdot|^p \rangle^{\frac{1}{p}}$, (87) follows from

$$\langle \left(\int_0^{\ell_0} \left\| \frac{\partial}{\partial \ell} \xi_{\ell, T} \right\| d\ell \right)^p \rangle^{\frac{1}{p}} \lesssim \ell_0^\varepsilon (T^{\frac{1}{3}})^{-\frac{5}{4}-2\varepsilon}, \quad (88)$$

whereas (86) then is a consequence of this for $\ell_0 = 1$ and (83) for $\ell = 1$ provided that p is so large that $\frac{1}{p} \frac{5}{2} \leq 2\varepsilon$; then, Jensen's inequality will also lead to (86) for smaller p . Estimate (88) in turn follows from (84) and the triangle inequality w.r.t. to $\langle |\cdot|^p \rangle^{\frac{1}{p}}$ in form of

$$\langle \left(\int_0^{\ell_0} \left\| \frac{\partial}{\partial \ell} \xi_{\ell, T} \right\| d\ell \right)^p \rangle^{\frac{1}{p}} \leq \int_0^{\ell_0} \langle \left\| \frac{\partial}{\partial \ell} \xi_{\ell, T} \right\|^p \rangle^{\frac{1}{p}} d\ell = \int_0^{\ell_0} \langle \left\| \ell \frac{\partial}{\partial \ell} \xi_{\ell, T} \right\|^p \rangle^{\frac{1}{p}} \frac{d\ell}{\ell}$$

and the fact that the RHS of (84) can be majorated as follows

$$\min\left\{ (T^{\frac{1}{3}})^{-\frac{5}{4}-\frac{1}{p}\frac{5}{2}}, \ell (T^{\frac{1}{3}})^{-\frac{9}{4}-\frac{1}{p}\frac{5}{2}} \right\} \leq \ell^\varepsilon (T^{\frac{1}{3}})^{-\frac{5}{4}-2\varepsilon}$$

because $\min\{a, b\} \leq a^\varepsilon b^{1-\varepsilon}$ for $a, b > 0$ and $\varepsilon \in (0, 1)$ and for $p \geq 1$ so large that $\frac{1}{p} \frac{5}{2} \leq \varepsilon$. For smaller p , the estimate (88) follows by Jensen's inequality.

Step 5. We claim that for every $0 < 2\varepsilon < \frac{1}{4}$ and any $p \geq 1$:

$$\langle \sup_{\ell \leq 1} [\xi_\ell]_{-\frac{5}{4}-2\varepsilon}^p \rangle^{\frac{1}{p}} \lesssim 1, \quad \langle \sup_{\ell, \ell' \leq \ell_0} [\xi_\ell - \xi_{\ell'}]_{-\frac{5}{4}-2\varepsilon}^p \rangle^{\frac{1}{p}} \lesssim \ell_0^\varepsilon, \quad \ell_0 \leq 1.$$

First, by Jensen's inequality in $\langle \cdot \rangle$, we note that Step 4 may be reformulated as follows: For every $0 < 2\varepsilon < \frac{1}{4}$, $p \geq 1$, $\ell_0 \leq 1$ and any $T \in (0, 1]$:

$$\langle \sup_{\ell \leq 1} \|\xi_{\ell, T}\|^p \rangle^{\frac{1}{p}} \lesssim (T^{\frac{1}{3}})^{-\frac{5}{4}-2\varepsilon}, \quad \langle \sup_{\ell, \ell' \leq \ell_0} \|\xi_{\ell, T} - \xi_{\ell', T}\|^p \rangle^{\frac{1}{p}} \lesssim \ell_0^\varepsilon (T^{\frac{1}{3}})^{-\frac{5}{4}-2\varepsilon}. \quad (89)$$

Let $T_n := 2^n$, $n \in \{0, -1, -2, \dots\}$. Note that for $T \in (0, 1]$ we choose $T_n < T \leq T_{n+1}$ and we have

$$|\xi_{\ell, T}(x)| = |(\xi_{\ell, T_n})_{T-T_n}(x)| \leq \|\xi_{\ell, T_n}\| \int_{\mathbb{R}^2} |\psi_{T-T_n}(y)| dy \lesssim \|\xi_{\ell, T_n}\|$$

as well as $|\xi_{\ell, T}(x) - \xi_{\ell', T}(x)| \lesssim \|\xi_{\ell, T_n} - \xi_{\ell', T_n}\|$. Then we have for some $0 < \varepsilon' < \varepsilon$:

$$\begin{aligned} \left\langle \sup_{\ell \leq 1} \sup_{T \in (0, 1]} \left((T^{\frac{1}{3}})^{\frac{5}{4}+2\varepsilon} \|\xi_{\ell, T}\| \right)^p \right\rangle &\lesssim \left\langle \sup_{\ell \leq 1} \sup_{n \in \mathbb{Z}, n \leq 0} \left((T_n^{\frac{1}{3}})^{\frac{5}{4}+2\varepsilon} \|\xi_{\ell, T_n}\| \right)^p \right\rangle \\ &\leq \left\langle \sup_{\ell \leq 1} \sum_{n \in \mathbb{Z}, n \leq 0} \left((T_n^{\frac{1}{3}})^{\frac{5}{4}+2\varepsilon} \|\xi_{\ell, T_n}\| \right)^p \right\rangle \\ &\leq \sum_{n \in \mathbb{Z}, n \leq 0} \left((T_n^{\frac{1}{3}})^{p(\frac{5}{4}+2\varepsilon)} \underbrace{\left\langle \sup_{\ell \leq 1} \|\xi_{\ell, T_n}\|^p \right\rangle}_{\stackrel{(89)}{\lesssim} (T_n^{\frac{1}{3}})^{-(\frac{5}{4}+2\varepsilon')p}} \right) \lesssim 1. \end{aligned}$$

By the same argument, relying on (87), we also obtain:

$$\left\langle \sup_{\ell, \ell' \leq \ell_0} \sup_{T \in (0,1]} \left((T^{\frac{1}{3}})^{\frac{5}{4}+2\varepsilon} \|\xi_{\ell,T} - \xi_{\ell',T}\| \right)^p \right\rangle^{\frac{1}{p}} \lesssim \ell_0^\varepsilon.$$

By Lemma 10 to convert the convolution-based norm into a negative exponent Hölder norm we obtain the conclusion of Step 5.

As $\{\xi_\ell\}$ is a Cauchy “sequence” in the Banach space $C^{-\frac{5}{4}-2\varepsilon}$ we deduce that the limit ξ of ξ_ℓ as $\ell \rightarrow 0$ satisfies $\left\langle \sup_{\ell \leq \ell_0} [\xi - \xi_\ell]_{-\frac{5}{4}-2\varepsilon}^p \right\rangle^{1/p} \lesssim \ell_0^\varepsilon$ for every $p \in [1, \infty)$. \square

6.2 Estimate of off-line term. Proof of Lemma 8

Proof of Lemma 8. We consider the Fourier coefficients $F^\ell(k) := \int_{[0,1)^2} e^{-ik \cdot x} F^\ell(x) dx$, $k \in (2\pi\mathbb{Z})^2$, of F^ℓ and its logarithmic derivative $\ell \frac{\partial}{\partial \ell} F^\ell(k)$ in the convolution scale ℓ . For $k \in (2\pi\mathbb{Z})^2 \setminus \{0\}$, we claim the following stochastic (second moment) bounds:

$$\langle |F^\ell(k)|^2 \rangle \lesssim d^{-1}(k, 0), \quad \langle |\ell \frac{\partial}{\partial \ell} F^\ell(k)|^2 \rangle \lesssim \min\{d^{-1}(k, 0), \ell^2 d(k, 0)\}, \quad \forall \ell \leq 1. \quad (90)$$

Here, \lesssim means up to a (generic) constant that only depends on ϕ .

Here comes the argument: Because of the projection P , $F^\ell(k)$ vanishes for $k_1 = 0$, so that we may restrict to $k_1 \neq 0$. We appeal to the formula for the product

$$F^\ell(k) = \sum_{k'+k''=k} v_\ell(k') (\partial_2 R v_\ell)(k'') = - \sum_{k'+k''=k} v_\ell(k') (\operatorname{sgn} k_1'') k_2'' v_\ell(k''),$$

where we used that the Fourier multiplier of R is $i \operatorname{sgn} k_1$ and that of ∂_2 is ik_2 . By definition of v via $Pv = v$ and $(-\partial_1^2 - |\partial_1|^{-1} \partial_2^2)v = P\xi$ we have on the Fourier level

$$v_\ell(k) = G_\ell(k) \xi(k), \quad k \in (2\pi\mathbb{Z})^2,$$

with the abbreviations ¹⁰

$$G(k) := \frac{|k_1|}{|k_1|^3 + k_2^2} \quad \text{and} \quad G_\ell(k) := \phi(\ell k_1, \ell^{\frac{3}{2}} k_2) G(k); \quad (91)$$

recall that $\phi(k)$ denotes the Fourier *transform* of the symmetric Schwartz mask ϕ of the convolution kernel $\phi_\ell(x_1, x_2) = \ell^{-\frac{5}{2}} \phi(\frac{x_1}{\ell}, \frac{x_2}{\ell^{\frac{3}{2}}})$. As $\phi(x)$ is symmetric, the Fourier transform $\phi(k)$ is real valued. With the abbreviations

$$\tilde{G}(k) := -(\operatorname{sgn} k_1) k_2 G(k) \quad \text{and} \quad \tilde{G}_\ell(k) := \phi(\ell k_1, \ell^{\frac{3}{2}} k_2) \tilde{G}(k) \quad (92)$$

¹⁰Do not confound G with the “heat kernel” used in the proof of Lemma 7.

we thus obtain the formula

$$F^\ell(k) = \sum_{k'+k''=k} G_\ell(k') \tilde{G}_\ell(k'') \xi(k') \xi(k''). \quad (93)$$

Recall the definition of $\delta\phi_\ell$ introduced in (77) as well as the sensitivities $\ell \frac{\partial}{\partial \ell}$ with respect to the convolution length ℓ . In line with the second items in (91) and (92), this prompts the definition of

$$\delta G_\ell(k) := \delta\phi(\ell k_1, \ell^{\frac{3}{2}} k_2) G(k), \quad \delta \tilde{G}_\ell(k) := \delta\phi(\ell k_1, \ell^{\frac{3}{2}} k_2) \tilde{G}(k). \quad (94)$$

Hence from (93), we find by Leibniz' rule

$$\ell \frac{\partial}{\partial \ell} F^\ell(k) = \sum_{k'+k''=k} (\delta G_\ell(k') \tilde{G}_\ell(k'') + G_\ell(k') \delta \tilde{G}_\ell(k'')) \xi(k') \xi(k''). \quad (95)$$

Step 1. The first step for (90) is to prove the following identities for the white noise: for $k', k'', l', l'' \neq 0$,

$$\begin{aligned} & \langle \xi(k') \xi(k'') \overline{\xi(l')} \overline{\xi(l'')} \rangle \\ &= \left\{ \begin{array}{ll} \langle |\xi(k')|^2 |\xi(k'')|^2 \rangle & \text{for } \{k', k''\} = \{l', l''\} \\ \langle |\xi(k')|^2 |\xi(l')|^2 \rangle & \text{for } k' + k'' = l' + l'' = 0 \\ 0 & \text{else} \end{array} \right\}. \end{aligned} \quad (96)$$

Indeed, it follows easily from the characterization of white noise ξ that the real-valued random variables in the family $\{\mathcal{R}e\xi(k), \mathcal{I}m\xi(k)\}_{k \neq 0}$ are centered, of variance $\frac{1}{2}$ and of vanishing covariances. Since these variables are also jointly Gaussian, they are in fact independent (and identically distributed) besides the linear constraint $\overline{\xi(k)} = \xi(-k)$. For the sake of completeness, let us give an argument for (96) in form of

$$\begin{aligned} & \langle \xi(k^1) \xi(k^2) \xi(k^3) \xi(k^4) \rangle \\ &= \left\{ \begin{array}{ll} \langle |\xi(k^1)|^2 |\xi(k^2)|^2 \rangle & \text{for } \{-k^1, -k^2\} = \{k^3, k^4\} \\ 0 & \text{if } \{k^1, k^2, k^3, k^4\} \text{ is not composed} \\ & \text{of two pairs that sum to zero} \end{array} \right\}. \end{aligned} \quad (97)$$

Note that the RHS of (97) does not cover all cases explicitly; the missing cases are implicitly covered by permutation symmetry of the lhs. Here comes the argument for (97): Since for $k^1, k^2 \neq 0$, $\xi(k^1) = \overline{\xi(-k^1)}$ and $\xi(k^2) = \overline{\xi(-k^2)}$ are independent unless $|k^1| = |k^2|$, the expression on the lhs of (97) vanishes unless (k^1, k^2, k^3, k^4) is composed by two pairs of indices that agree up to the sign. By permutation we may w.l.o.g. assume that $\{|k_1|, |k_2|\} = \{|k_3|, |k_4|\}$. We have to distinguish 4 cases: 1) $\{-k^1, -k^2\} = \{k^3, k^4\}$, in which case the expression turns into the desired $\langle |\xi(k^1)|^2 |\xi(k^2)|^2 \rangle$. 2) $\{-k^1, k^2\} = \{k^3, k^4\}$, in which case the expression turns into $\langle |\xi(k^1)|^2 \xi^2(k^2) \rangle$. This expression vanishes, since shifting ξ to $\xi(\cdot + h)$ by a shift vector

h with $h \cdot k^2 = -\frac{\pi}{2}$ does not change the white-noise distribution but changes the Fourier coefficient $\xi(k^2)$ by a factor of i , while $|\xi(k^1)|^2$ is preserved. 3) $\{k^1, -k^2\} = \{k^3, k^4\}$, in which case we obtain 0 for the same reason. 4) $\{-k^1, -k^2\} = \{k^3, k^4\}$, in which case we obtain $\langle \xi^2(k^1) \xi^2(k^2) \rangle$; here, we have to distinguish the three sub cases: 4a) $|k^1| \neq |k^2|$ in which case $\xi^2(k^1)$ and $\xi^2(k^2)$ are independent so that the expression assumes the form $\langle \xi^2(k^1) \rangle \langle \xi^2(k^2) \rangle$ with both factors vanishing (by the argument under 2)). 4b) $-k^1 = k^2$, in which $\{k^1, k^2, k^3, k^4\}$ is composed of two pairs that sum to zero and thus does not fall under the second case in (97). 4c) $k^1 = k^2$ in which case the expression turns into $\langle \xi^4(k^1) \rangle$ which can be seen to vanish by shifting ξ by a vector h with $h \cdot k^1 = -\frac{\pi}{4}$.

Step 2. We prove

$$\begin{aligned} \langle |F^\ell(k)|^2 \rangle &\lesssim \sum_{\substack{k'+k''=k \\ k', k'' \neq 0}} d^{-4}(k', 0) d^{-1}(k'', 0), \\ \langle |\ell \frac{\partial}{\partial \ell} F^\ell(k)|^2 \rangle &\lesssim \sum_{\substack{k'+k''=k \\ k', k'' \neq 0}} \min^2\{1, \ell(d(k', 0) + d(k'', 0))\} d^{-4}(k', 0) d^{-1}(k'', 0). \end{aligned}$$

For that, we make use of (96); because of $k_1 \neq 0$, the middle case of (96) does not occur when applying it to the square of (93):

$$\begin{aligned} \langle |F^\ell(k)|^2 \rangle &= \sum_{k'+k''=k} ((G_\ell(k') \tilde{G}_\ell(k''))^2 + (G_\ell(k') \tilde{G}_\ell(k''))(G_\ell(k'') \tilde{G}_\ell(k'))) \\ &\quad \times \langle |\xi(k')|^2 |\xi(k'')|^2 \rangle. \end{aligned}$$

Because of the same structure, starting from (95) we get

$$\begin{aligned} \langle |\ell \frac{\partial}{\partial \ell} F^\ell(k)|^2 \rangle &= \sum_{k'+k''=k} \left((\delta G_\ell(k') \tilde{G}_\ell(k'') + G_\ell(k') \delta \tilde{G}_\ell(k''))^2 \right. \\ &\quad \left. + (\delta G_\ell(k') \tilde{G}_\ell(k'') + G_\ell(k') \delta \tilde{G}_\ell(k'')) \times (\delta G_\ell(k'') \tilde{G}_\ell(k') + G_\ell(k'') \delta \tilde{G}_\ell(k')) \right) \\ &\quad \times \langle |\xi(k')|^2 |\xi(k'')|^2 \rangle. \end{aligned}$$

With help of Young's inequality (in conjunction with the symmetry of $\langle |\xi(k')|^2 |\xi(k'')|^2 \rangle$ in $k' \leftrightarrow k''$) we may simplify to

$$\langle |F^\ell(k)|^2 \rangle \leq 2 \sum_{k'+k''=k} G_\ell^2(k') \tilde{G}_\ell^2(k'') \langle |\xi(k')|^2 |\xi(k'')|^2 \rangle,$$

and, applying this argument twice,

$$\begin{aligned} &\langle |\ell \frac{\partial}{\partial \ell} F^\ell(k)|^2 \rangle \\ &\leq 4 \sum_{k'+k''=k} (\delta G_\ell^2(k') \tilde{G}_\ell^2(k'') + G_\ell^2(k') \delta \tilde{G}_\ell^2(k'')) \langle |\xi(k')|^2 |\xi(k'')|^2 \rangle. \end{aligned}$$

We appeal to the Cauchy-Schwarz inequality in form of $\langle |\xi(k')|^2 |\xi(k'')|^2 \rangle \leq (\langle |\xi(k')|^4 \rangle \langle |\xi(k'')|^4 \rangle)^{\frac{1}{2}}$, the identical distribution of $\{\xi(k)\}_{k \neq 0}$ in form of $\langle |\xi(k')|^4 \rangle = \langle |\xi(k'')|^4 \rangle$, the independence and identical distribution of $\mathcal{Re}\xi(k)$ and $\mathcal{Im}\xi(k)$ (for $k \neq 0$) in form of $\langle |\xi(k)|^4 \rangle = 2(\langle (\mathcal{Re}\xi(k))^4 \rangle + \langle (\mathcal{Re}\xi(k))^2 \rangle^2)$, and the (standard) Gaussianity of $\mathcal{Re}\xi(k)$ in form of

$$\langle (\mathcal{Re}\xi(k))^4 \rangle = \left(\frac{1}{2}\right)^4 \underbrace{\langle (2\mathcal{Re}\xi(k))^4 \rangle}_{\text{of variance 1}} = \left(\frac{1}{2}\right)^4 3 \langle (2\mathcal{Re}\xi(k))^2 \rangle = \frac{3}{16}$$

to conclude $\langle |\xi(k')|^2 |\xi(k'')|^2 \rangle \lesssim 1$, which we insert:

$$\langle |F^\ell(k)|^2 \rangle \lesssim \sum_{\substack{k'+k''=k \\ k', k'' \neq 0}} G_\ell^2(k') \tilde{G}_\ell^2(k''), \quad (98)$$

$$\langle |\ell \frac{\partial}{\partial \ell} F^\ell(k)|^2 \rangle \lesssim \sum_{\substack{k'+k''=k \\ k', k'' \neq 0}} (\delta G_\ell^2(k') \tilde{G}_\ell^2(k'') + G_\ell^2(k') \delta \tilde{G}_\ell^2(k'')). \quad (99)$$

By the following (build-in) relation between the symbol $G(k) = \frac{|k_1|}{|k_1|^3 + k_2^2}$ of $\mathcal{L}^{-1}P$ and the intrinsic metric (which we obtain with help of the Young inequality), namely

$$G(k) \lesssim d^{-2}(k, 0) \quad \text{and thus} \quad |\tilde{G}(k)| \lesssim d^{-\frac{1}{2}}(k, 0), \quad (100)$$

we obtain from the definitions (91) & (92)

$$|G_\ell(k)| \lesssim d^{-2}(k, 0) \quad \text{and thus} \quad |\tilde{G}_\ell(k)| \lesssim d^{-\frac{1}{2}}(k, 0),$$

where we used that for our Schwartz kernel $|\phi(k)| \lesssim 1$. By definition, $\delta\phi$ vanishes for $k = 0$ so that here, we even have $|\delta\phi(k)| \lesssim \min\{1, d(k, 0)\}$. Using this in the definitions (94) we obtain

$$|\delta G_\ell(k)| \lesssim \min\{1, \ell d(k, 0)\} d^{-2}(k, 0) \quad \text{and} \quad |\delta \tilde{G}_\ell(k)| \lesssim \min\{1, \ell d(k, 0)\} d^{-\frac{1}{2}}(k, 0).$$

Inserting this into (98) and (99) we obtain the conclusion of Step 2.

Step 3. In order to conclude with the claim (90), in view of Step 2, it remains to show

$$\sum_{\substack{k'+k''=k \\ k', k'' \neq 0}} d^{-4}(k', 0) d^{-1}(k'', 0) \lesssim d^{-1}(k, 0) \quad (101)$$

and for $\ell \leq 1$ and $k \neq 0$,

$$\sum_{\substack{k'+k''=k \\ k', k'' \neq 0}} \min^2\{1, \ell(d(k', 0) + d(k'', 0))\} d^{-4}(k', 0) d^{-1}(k'', 0) \lesssim \min\{d^{-1}(k, 0), \ell^2 d(k, 0)\}. \quad (102)$$

We will focus on the more subtle (102). Obviously, (102) splits into the three statements

$$\sum_{\substack{k'+k''=k \\ k',k'' \neq 0}} \frac{1}{d^4(k',0)d(k'',0)} \lesssim \frac{1}{d(k,0)}, \quad (103)$$

$$\sum_{\substack{k'+k''=k \\ k',k'' \neq 0}} \frac{1}{d^2(k',0)d(k'',0)} \lesssim d(k,0), \quad (104)$$

$$\sum_{\substack{k'+k''=k \\ k',k'' \neq 0}} \frac{d(k'',0)}{d^4(k',0)} \lesssim d(k,0). \quad (105)$$

As we shall see, all these statements rely on

$$\sum_{k' \neq 0} \frac{1}{d^4(k',0)} \lesssim \sum_{k' \neq 0} \frac{1}{d^3(k',0)} \lesssim 1, \quad (106)$$

which is an immediate consequence of (81), expressing that the effective dimension of the k -space is $\frac{5}{2} < 3$, and the triangle inequality in form of

$$|d(k'',0) - d(k',0)| \leq d(k,0) \leq d(k',0) + d(k'',0), \quad (107)$$

as a consequence of $k' + k'' = k$. Indeed, by (107) in form of $d(k'',0) \leq d(k',0) + d(k,0)$ and (106), the LHS of (105) is estimated by $1 + d(k,0) \lesssim d(k,0)$, where we used $k \neq 0$ in the last step. Turning to (104), we split the domain of summation into $\{d(k',0) \leq d(k'',0)\}$ and $\{d(k',0) > d(k'',0)\}$. By (106) (with k' replaced by k'' for the second contribution), both contributions are estimated by $1 \lesssim d(k,0)$. Finally addressing (103) (which coincides with (101)), we split the domain of summation into the same two sets. On the first domain $\{d(k',0) \leq d(k'',0)\}$, by the second inequality in (107) we must have $d(k'',0) \geq \frac{1}{2}d(k,0)$, so that by (106), the corresponding contribution is estimated as desired. On the second domain $\{d(k',0) > d(k'',0)\}$, we use that for the same reason $d^4(k',0)d(k'',0) \geq \frac{1}{2}d(k,0)d^4(k'',0)$, so that by (106) (with k'' playing the role of k') also this contribution is estimated as desired.

Step 4. We claim that for all $\ell \leq 1$, $x \in \mathbb{R}^2$, and $T > 0$ we have the estimate

$$\langle (F_T^\ell(x))^2 \rangle^{\frac{1}{2}} \lesssim (T^{\frac{1}{3}})^{-\frac{3}{4}}, \quad (108)$$

$$\langle (\ell \frac{\partial}{\partial \ell} F_T^\ell(x))^2 \rangle^{\frac{1}{2}} \lesssim \min\{(T^{\frac{1}{3}})^{-\frac{3}{4}}, \ell(T^{\frac{1}{3}})^{-\frac{7}{4}}\}. \quad (109)$$

We focus on (109) and use the same type of arguments as in Step 2 of the proof of Lemma 4. Since the distribution ξ and its translation $\xi(\cdot + h)$ by some translation vector h have the same distribution under $\langle \cdot \rangle$, and since v arises as the solution of a (linear) constant-coefficient (pseudo-) differential operator with RHS ξ , also v and

$v(\cdot + h)$ have the same distribution as fields. This shift-invariance carries over to v_ℓ , $\partial_2 v_\ell$, $\partial_2 Rv_\ell$ and thus to $v_\ell \partial_2 Rv_\ell$, F^ℓ and F_T^ℓ . This implies that $\langle (\ell \frac{\partial}{\partial \ell} F_T^\ell(x))^2 \rangle$ does not depend on x so that it is enough to show

$$\langle \int_{[0,1]^2} (\ell \frac{\partial}{\partial \ell} F_T^\ell)^2 dx \rangle \lesssim \min\{(T^{\frac{1}{3}})^{-\frac{3}{2}}, \ell^2 (T^{\frac{1}{3}})^{-\frac{7}{2}}\}.$$

By Plancherel, this assumes the form of

$$\sum_{k \neq 0} \langle |\ell \frac{\partial}{\partial \ell} F_T^\ell(k)|^2 \rangle \lesssim \min\{(T^{\frac{1}{3}})^{-\frac{3}{2}}, \ell^2 (T^{\frac{1}{3}})^{-\frac{7}{2}}\}.$$

Because of the identity $\ell \frac{\partial}{\partial \ell} F_T^\ell(k) = \psi_T(k) \ell \frac{\partial}{\partial \ell} F^\ell(k)$, the inequality $0 \leq \psi_T(k) \leq \exp(-Td^3(k, 0))$, and the previous steps in form of (90) this reduces to

$$\sum_{k \neq 0} \exp(-2Td^3(k, 0)) \min\{d^{-1}(k, 0), \ell^2 d(k, 0)\} \lesssim \min\{(T^{\frac{1}{3}})^{-\frac{3}{2}}, \ell^2 (T^{\frac{1}{3}})^{-\frac{7}{2}}\}.$$

The latter obviously splits into

$$\begin{aligned} \sum_{k \neq 0} \exp(-2Td^3(k, 0)) d^{-1}(k, 0) &\lesssim (T^{\frac{1}{3}})^{-\frac{3}{2}}, \\ \sum_{k \neq 0} \exp(-2Td^3(k, 0)) d(k, 0) &\lesssim (T^{\frac{1}{3}})^{-\frac{7}{2}}, \end{aligned}$$

which both follows from the effective dimension $\frac{5}{2}$ of k -space, cf. (81).

Step 5. We claim that for all $\ell \leq 1$, $1 \leq p < \infty$, and $T > 0$ we have the estimate

$$\langle \|F_T^\ell\|^p \rangle^{\frac{1}{p}} \lesssim (T^{\frac{1}{3}})^{-\frac{3}{4} - \frac{1}{p} \frac{5}{2}}, \quad (110)$$

$$\langle \|\ell \frac{\partial}{\partial \ell} F_T^\ell\|^p \rangle^{\frac{1}{p}} \lesssim \min\{(T^{\frac{1}{3}})^{-\frac{3}{4} - \frac{1}{p} \frac{5}{2}}, \ell (T^{\frac{1}{3}})^{-\frac{7}{4} - \frac{1}{p} \frac{5}{2}}\}. \quad (111)$$

The argument proceeds as in Step 3 of the proof of Lemma 4 with the one notable difference that we now can no longer simply appeal to Gaussianity to get the inverse Hölder estimate

$$\langle |F_T^\ell(x)|^p \rangle^{\frac{1}{p}} \lesssim \langle (F_T^\ell(x))^2 \rangle^{\frac{1}{2}},$$

and the analogous statement for $\ell \frac{\partial}{\partial \ell} F_T^\ell$. However, such an estimate remains true because $F_T^\ell(x)$ is a quadratic functional of ξ , and as such an element of what is called the Second Wiener Chaos. In this situation, the estimate is known as Nelson's estimate, see [25, Proposition 3.3] for a proof.

Step 6. We claim that for all $\varepsilon \in (0, 1)$, $1 \leq p < \infty$ and $\ell_0 \leq 1$ we have for every $T \leq 1$:

$$\langle \sup_{\ell \leq 1} \|F_T^\ell\|^p \rangle^{\frac{1}{p}} \lesssim (T^{\frac{1}{3}})^{-\frac{3}{4}-2\varepsilon}, \quad (112)$$

$$\langle \sup_{\ell, \ell' \leq \ell_0} \|F_T^\ell - F_T^{\ell'}\|^p \rangle^{\frac{1}{p}} \lesssim \ell_0^\varepsilon (T^{\frac{1}{3}})^{-\frac{3}{4}-2\varepsilon}, \quad (113)$$

where now, the constant also depends on ε . This follows by the same argument as in Step 4 in the proof of Lemma 4.

Step 7. By the same argument as in Step 5 in the proof of Lemma 4, we have for any $0 < 2\varepsilon < \frac{1}{4}$ and any $p \in [1, \infty)$:

$$\langle \sup_{\ell \leq 1} [F^\ell]_{-\frac{3}{4}-2\varepsilon}^p \rangle^{\frac{1}{p}} \lesssim 1, \quad \langle \sup_{\ell, \ell' \leq \ell_0} [F^\ell - F^{\ell'}]_{-\frac{3}{4}-2\varepsilon}^p \rangle^{\frac{1}{p}} \lesssim \ell_0^\varepsilon,$$

where the constant inside \lesssim depends on p and ε .

Step 8. We now give the argument that the limit F of F^ℓ in the Banach space defined through the norm $\langle [\cdot]_{-\frac{3}{4}-2\varepsilon}^p \rangle^{\frac{1}{p}}$ is independent of the (normalized) symmetric Schwartz kernel ϕ that entered the definition of F^ℓ via convolution. Take another (symmetric) Schwartz kernel $\tilde{\phi}$ and (with a benign misuse of language) denote by $\tilde{v}_\ell := \tilde{\phi}_\ell * v$ the corresponding convolutions with the rescaled kernel $\tilde{\phi}_\ell$ on x_1 -scale ℓ , cf. (11). An inspection of the above proof, in particular the claim (90), shows that the second estimate of this lemma holds with v replaced by \tilde{v}_ℓ , since the relevant Fourier multipliers $G(k)$, cf. (91), and $\tilde{G}(k)$, cf. (92), that lead from ξ to v and $\partial_2 Rv$, respectively, are dominated, cf. (100), in an identical way. Hence also $F^{\ell\tilde{\ell}} := P(v_{\ell\tilde{\ell}} \partial_2 Rv_{\ell\tilde{\ell}})$ is a Cauchy sequence in ℓ and thus has a limit, all w.r.t. to the Banach-space norm $\langle [\cdot]_{-\frac{3}{4}-2\varepsilon}^p \rangle^{\frac{1}{p}}$, and the limit is uniform w.r.t. to $\tilde{\ell}$ in view of the above mentioned uniform-in- ℓ estimates on the Fourier multipliers. On the other hand, thanks to $\tilde{\ell} > 0$, this limit is classical and given by $F^{\tilde{\ell}} := P(v_{\tilde{\ell}} \partial_2 Rv_{\tilde{\ell}})$. By symmetry, the same double-indexed object $F^{\ell\tilde{\ell}}$ has a limit in $\tilde{\ell}$ (always w.r.t. $\langle [\cdot]_{-\frac{3}{4}-2\varepsilon}^p \rangle^{\frac{1}{p}}$), this limit is uniform in ℓ , and for $\ell > 0$ is given by F^ℓ . Thus by the triangle inequality w.r.t. $\langle [\cdot]_{-\frac{3}{4}-2\varepsilon}^p \rangle^{\frac{1}{p}}$, $F^{\tilde{\ell}}$ and F^ℓ get closer and closer for ℓ and $\tilde{\ell}$ tending to zero. This implies that their respective limits in $\langle [\cdot]_{-\frac{3}{4}-2\varepsilon}^p \rangle^{\frac{1}{p}}$ coincide, as desired. □

7 Appendix

7.1 The linearized energy

For $\sigma > 0$, we note that the linearized energy functional of (9) on $[0, 1]^2$:

$$E_{\text{lin}}(u) = \int_{[0,1]^2} \left(|\partial_1 u|^2 + (\partial_2 u) |\partial_1|^{-1} (\partial_2 u) - 2\sigma \xi u \right) dx$$

only admits critical points that have (negative) infinite energy (with positive probability). Indeed, if u is a solution of the Euler-Lagrange equation $\mathcal{L}u = P\xi$, then we can explicitly solve it in Fourier space, obtaining

$$u(k) = \sigma(k_1^2 + |k_1|^{-1}k_2^2)^{-1}\xi(k) \text{ for } k_1 \neq 0, \quad u(0, k_2) = 0 \text{ for } k_2 \in 2\pi\mathbb{Z}.$$

Since $\sum_{k_1 \neq 0} |k|^{-2}$ diverges logarithmically,

$$\langle E_{\text{lin}}(u) \rangle = -\sigma^2 \sum_{k \neq 0} \frac{\langle |\xi(k)|^2 \rangle}{|k_1|^2 + |k_1|^{-1}k_2^2} \sim - \sum_{k_1 \neq 0} \frac{1}{|k_1|^2 + |k_1|^{-1}k_2^2} \leq - \sum_{k_1 \neq 0} \frac{1}{|k|^2} = -\infty.$$

7.2 The anisotropic Hölder space C^α for $\alpha \in (0, \frac{3}{2})$

For the reader convenience, we give the following (standard) result for our anisotropic Hölder spaces C^α , $\alpha > 0$:

Lemma 12. *Let $u : [0, 1]^2 \rightarrow \mathbb{R}$ be a 1-periodic function. If $\alpha \in (0, 1]$, then $\|u\| \lesssim [u]_\alpha$ provided that u is of vanishing average.¹¹ If $\alpha \in (1, \frac{3}{2})$, then $\|\partial_1 u\| \lesssim [\partial_1 u]_{\alpha-1} \lesssim [u]_\alpha$ for every periodic function u . Moreover, if $\frac{3}{2} > \alpha \geq \beta > 0$, then $[u]_\beta \lesssim [u]_\alpha$, i.e., $C^\alpha \subset C^\beta$. Also, for two periodic functions $u \in C^\alpha$ and $f \in C^\beta$ on $[0, 1]^2$ with $\alpha \geq \beta > 0$, then $uf \in C^\beta$ with*

$$[uf]_\beta \lesssim \left([u]_\alpha + \left| \int_{[0,1]^2} u dx \right| \right) \left([f]_\beta + \left| \int_{[0,1]^2} f dx \right| \right).$$

Proof. Step 1. If $\alpha \in (0, 1]$, then $\|u\| \lesssim [u]_\alpha$ for every periodic function u of vanishing average. Indeed, we have $u(x) - u(y) \leq [u]_\alpha d^\alpha(x, y) \lesssim [u]_\alpha$ for every $x, y \in [0, 1]^2$. Integrating in $y \in [0, 1]^2$, as u has zero average, we obtain $u(x) \lesssim [u]_\alpha$ for all $x \in [0, 1]^2$. Similarly, we have $-u(x) \lesssim [u]_\alpha$ on $[0, 1]^2$, so that the first claim follows.

Step 2. If $\alpha \in (1, \frac{3}{2})$, then $[\partial_1 u]_{\alpha-1} \lesssim [u]_\alpha$ for every periodic function u . Indeed, let $u \in C^\alpha$ for $\alpha \in (1, \frac{3}{2})$, i.e., $|u(y) - u(x) - \partial_1 u(x)(y_1 - x_1)| \leq [u]_\alpha d^\alpha(x, y)$ for every $x, y \in [0, 1]^2$. Interchanging x and y , by summation, we deduce that $|[\partial_1 u(x) - \partial_1 u(y)](y_1 -$

¹¹In particular, the assumption is satisfied if u is of vanishing average in x_1 .

$x_1)$ $\leq 2[u]_\alpha d^\alpha(x, y)$. In particular, for $x_2 = y_2$, we deduce that $|\partial_1 u(x_1, x_2) - \partial_1 u(y_1, x_2)| \leq 2[u]_\alpha |x_1 - y_1|^{\alpha-1}$ for every $x_1, y_1, x_2 \in [0, 1]^2$. We conclude that for every $x_1, x_2, y_2 \in [0, 1]^2$:

$$\begin{aligned} |\partial_1 u(x_1, x_2) - \partial_1 u(x_1, y_2)| &\leq |\partial_1 u(x) - \partial_1 u(y)| + |\partial_1 u(x_1, y_2) - \partial_1 u(y_1, y_2)| \\ &\leq 2[u]_\alpha \left(\frac{d^\alpha(x, y)}{|x_1 - y_1|} + |x_1 - y_1|^{\alpha-1} \right) \lesssim [u]_\alpha |x_2 - y_2|^{2(\alpha-1)/3} \end{aligned}$$

for y_1 chosen such that $|x_1 - y_1| = |x_2 - y_2|^{2\alpha/3}$. As $\partial_1 u$ is of vanishing average, Step 1 implies that $\|\partial_1 u\| \lesssim [u]_{\alpha-1}$.

Step 3. We prove that $[u]_\beta \lesssim [u]_\alpha$ for every periodic function u if $0 < \beta \leq \alpha < \frac{3}{2}$. Indeed, this is straightforward by Definition 1 if $\alpha \leq 1$, respectively by Definition 3 if $\beta > 1$. It remains to treat the case $\beta \leq 1 < \alpha$. By Step 2, we already know that $|u(x_1, x_2) - u(x_1, y_2)| \leq [u]_\alpha |x_2 - y_2|^{2\alpha/3}$ and $|\partial_1 u(x_1, x_2) - \partial_1 u(y_1, x_2)| \leq 2[u]_\alpha |x_1 - y_1|^{\alpha-1}$ for every $x_1, y_1, x_2, y_2 \in [0, 1]^2$. As u is periodic and $\partial_1 u$ is of vanishing average, we deduce that $\|\partial_1 u\| \lesssim [u]_\alpha$. Therefore

$$\begin{aligned} |u(x) - u(y)| &\leq |u(x_1, x_2) - u(x_1, y_2)| + |u(x_1, y_2) - u(y_1, y_2)| \\ &\leq [u]_\alpha |x_2 - y_2|^{2\alpha/3} + \|\partial_1 u\| |x_1 - y_1| \lesssim [u]_\alpha d^\beta(x, y). \end{aligned}$$

Step 4. We prove that $uf \in C^\beta$ for two periodic functions $u \in C^\alpha$ and $f \in C^\beta$ in $[0, 1]^2$ with $\alpha \geq \beta > 0$. If $\beta \leq 1$, then by Step 3 we know that $u \in C^\alpha \subset C^\beta$ and we conclude that $uf \in C^\beta$ with the desired inequality for the product uf because $\|u\| \lesssim [u]_\alpha + |\int_{[0,1]^2} u dx|$ and $\|f\| \lesssim [f]_\beta + |\int_{[0,1]^2} f dx|$. If $\beta > 1$, then we have for $x = (x_1, x_2), y = (y_1, y_2) \in [0, 1]^2$:

$$\begin{aligned} &|(uf)(x) - (uf)(y) - \partial_1(uf)(y_1 - x_1)| \\ &\leq |u(x)[f]_\beta d^\beta(x, y) + |f(y)[u]_\alpha d^\alpha(x, y) + |\partial_1 u(x)||f(x) - f(y)||y_1 - x_1| \\ &\lesssim \|u\|[f]_\beta d^\beta(x, y) + \|f\|[u]_\alpha d^\beta(x, y) + \|\partial_1 u\|([f]_\beta d^\beta(x, y) + \|\partial_1 f\||y_1 - x_1|)|y_1 - x_1| \\ &\lesssim ([u]_\alpha + |\int_{[0,1]^2} u dx|)([f]_\beta + |\int_{[0,1]^2} f dx|)d^\beta(x, y) \end{aligned}$$

as $\|\partial_1 u\| \lesssim [u]_\alpha$ and $\|\partial_1 f\| \lesssim [f]_\beta$. □

As a consequence we prove the following compactness result:

Lemma 13. *Let $(f_n)_n \subset \mathcal{D}'([0, 1]^2)$ be a sequence of periodic distributions satisfying the uniform bound $\limsup_{n \rightarrow \infty} [f_n]_\beta < \infty$ for some $\beta \in (-\frac{3}{2}, \frac{3}{2}) \setminus \{0\}$. If $\beta \in (0, \frac{3}{2})$, we assume in addition, that $\{\|f_n\|\}_n$ is uniformly bounded. Then there exists $f \in C^\beta$ in $[0, 1]^2$ such that along a subsequence we have $f_n \rightharpoonup f$ in $\mathcal{D}'([0, 1]^2)$ for $n \rightarrow \infty$ and $[f]_\beta \leq \liminf_{n \rightarrow \infty} [f_n]_\beta$.*

Proof of Lemma 13. We can always assume that $[f_n]_\beta \rightarrow \liminf_n [f_n]_\beta =: C_0$ (by extracting eventually a subsequence).

If $\beta \in (0, 1]$, then by Definition 1 we know that $\{f_n\}$ is equicontinuous and uniformly bounded; thus the Ascoli theorem implies the conclusion. If $\beta \in (1, \frac{3}{2})$, by Lemma 12, we know that $\{[f_n]_1\}$ is uniformly bounded as well as $\|\partial_1 f_n\| \lesssim [\partial_1 f_n]_{\beta-1} \lesssim C_0$; applying again the Ascoli theorem to $\{f_n\}$ and $\{\partial_1 f_n\}$, we deduce the conclusion.

We prove the case $\beta \in (-1, 0)$ (the case $\beta \in (-3/2, 1]$ is similar). By Definition 3, there exist $g_n \in C^{1+\beta}([0, 1]^2, d)$ and $h_n \in C^{3/2+\beta}([0, 1]^2, d)$ (of vanishing average) such that $f_n = \int_{[0,1]^2} f_n dx + \partial_1 g_n + \partial_2 h_n \in \mathcal{D}'([0, 1]^2)$ and

$$|\int_{[0,1]^2} f_n dx| + [g_n]_{1+\beta} + [h_n]_{3/2+\beta} \rightarrow C_0 \quad \text{as } n \rightarrow 0$$

and $\{\|g_n\|\}_n$ and $\{\|h_n\|\}_n$ are uniformly bounded. By the previous cases, we know that $\int_{[0,1]^2} f_n dx \rightarrow A$, $g_n \rightarrow g$ and $h_n \rightarrow h$ for $n \rightarrow \infty$ along a subsequence where $g \in C^{1+\beta}$ and $h \in C^{3/2+\beta}$ in $[0, 1]^2$ are of vanishing average and $[g]_{\beta+1} \leq \liminf_{n \rightarrow \infty} [g_n]_{\beta+1}$ as well as $[h]_{\beta+3/2} \leq \liminf_{n \rightarrow \infty} [h_n]_{\beta+3/2}$. In particular, defining $f := A + \partial_1 g + \partial_2 h$, Definition 3 implies $f \in C^\beta$ in $[0, 1]^2$ (of average A) with $[f]_\beta \leq C_0$ and $f_n \rightharpoonup f$ distributionally because for every $\zeta \in C^\infty([0, 1]^2)$:

$$\begin{aligned} \int_{[0,1]^2} (f_n - A)\zeta dx &= - \int_{[0,1]^2} (g_n \partial_1 \zeta + h_n \partial_2 \zeta) dx \\ &\xrightarrow{n \rightarrow \infty} - \int_{[0,1]^2} (g \partial_1 \zeta + h \partial_2 \zeta) dx = \int_{[0,1]^2} (f - A)\zeta dx. \end{aligned}$$

□

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