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# Ferromagnetic thin multi-structures

Antonio Gaudiello\* and Rejeb Hadiji†

## Abstract

In this paper, starting from the classical  $3D$  non-convex and nonlocal micromagnetic energy for ferromagnetic materials, we determine, via an asymptotic analysis, the free energy of a multi-structure consisting of a nano-wire in junction with a thin film and of a multi-structure consisting of two joined nano-wires. We assume that the volumes of the two parts composing each multi-structure vanish with same rate. In the first case, we obtain a  $1D$  limit problem on the nano-wire and a  $2D$  limit problem on the thin film, and the two limit problems are uncoupled. In the second case, we obtain two  $1D$  limit problems coupled by a junction condition on the magnetization. In both cases, the limit problem remains non-convex, but now it becomes completely local.

Keywords: micromagnetics, variational problem, thin film, nano-wire, junctions.

2000 AMS subject classifications: 78A25, 49S05, 78M35

## 1 Introduction

In this paper, starting from the classical  $3D$  micromagnetic energy for ferromagnetic materials (see L. D. Landau and E. M. Lifshitz [24] and W. F. Brown [6]), we determine, via an asymptotic analysis, the free energy of a multi-structure consisting of a nano-wire in junction with a thin film and of a multi-structure consisting of two joined nano-wires. These multi-structures appear in nano electronic devices (for instance, see [14] and [27]). For reasons of simplicity and economy, especially by a numerical point of view, one tries to reshape three-dimensional multi-structures, with multi-structures having a smaller size in thin components.

In the sequel,  $x = (x_1, x_2, x_3)$  denotes the generic point of  $\mathbb{R}^3$ . If  $\eta_1, \eta_2, \eta_3 \in \mathbb{R}^3$ , then  $(\eta_1|\eta_2|\eta_3)$  denotes the  $3 \times 3$  real matrix having  $\eta_1^T$  as first column,  $\eta_2^T$  as second column, and  $\eta_3^T$  as third column. In according to this notation, if  $v : A \subset \mathbb{R}^3 \rightarrow \mathbb{R}^3$ , then  $Dv$  denotes the  $3 \times 3$  real matrix  $(D_{x_1}v|D_{x_2}v|D_{x_3}v)$ , where  $D_{x_i}v \in \mathbb{R}^3$ ,  $i=1,2,3$ , stands for the derivative of  $v$  with respect to  $x_i$ .

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Let  $\{h_n\}_{n \in \mathbb{N}} \subset ]0, 1[$  be a vanishing sequence of positive numbers, and let  $\Theta \subset ]0, 1[^2$  be an open connected set with smooth boundary. In this paper, we consider two kinds of thin multistructures in  $\mathbb{R}^3$ . In the first case, for every  $n \in \mathbb{N}$ , we set

$$\Omega_n = (h_n \Theta \times [0, 1]) \cup (\Theta \times ]-h_n^2, 0]),$$

which approximates a wire in junction with a thin film (see Fig. 1), as  $n$  diverges. In the second case, we set

$$\Omega_n = (] - h_n, 0[^2 \times [0, 1]) \times (] - h_n, 1[ \times ] - h_n, 0[^2)$$

which approximates two joined wires (see Fig. 2), as  $n$  diverges. In both cases, the volumes of the two parts of the multi-structure vanish with same rate. The aim of this paper

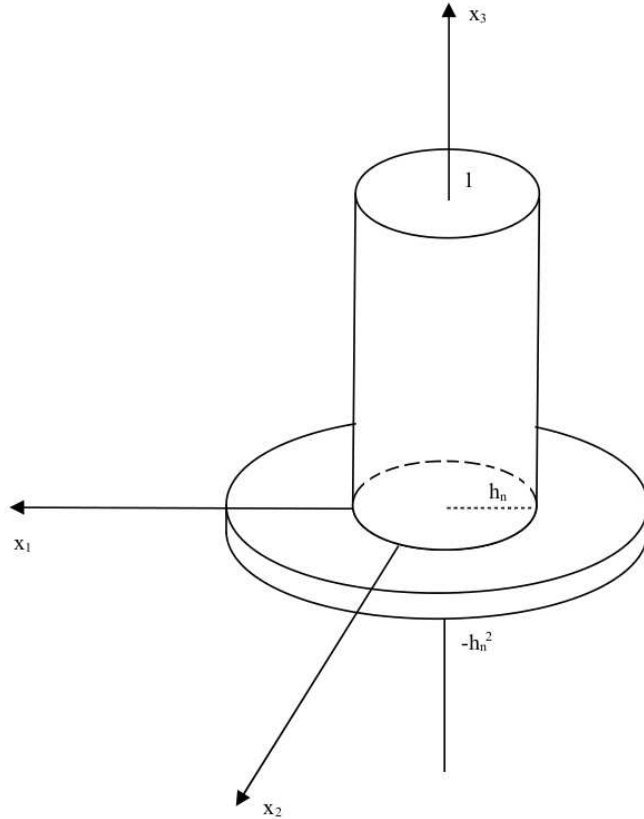


Figure 1:  $\Omega_n$  in the case wire - thin film

is to study the asymptotic behavior, as  $n$  diverges, of the following non-convex, nonlocal variational problem:

$$\left\{ \begin{array}{l} J_n = \min \left\{ \int_{\Omega_n} \left( \lambda |DM|^2 + \varphi(M) + \frac{1}{2} DU_M M - 2F_n M \right) dx : \right. \\ \left. M \in H^1(\Omega_n, S^2) \right\}, \\ \operatorname{div}(-DU_M + M) = 0 \text{ in } \mathbb{R}^3, \end{array} \right. \quad (1.1)$$

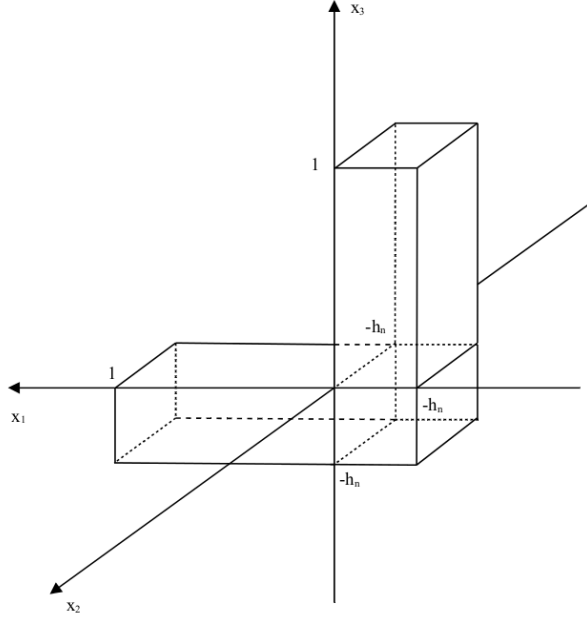


Figure 2:  $\Omega_n$  in the case wire - wire

where  $\lambda$  is a positive constant,  $\varphi : S^2 \rightarrow [0, +\infty[$  is a continuous and even function,  $S^2$  denotes the unit sphere of  $\mathbb{R}^3$ , and  $F_n \in L^2(\Omega_n, \mathbb{R}^3)$ . It is understood that  $M = 0$  in  $\mathbb{R}^3 \setminus \Omega_n$ .

In classical theory of micromagnetics,  $M : \Omega_n \rightarrow \mathbb{R}^3$  denotes the magnetization and the body is always locally magnetized to a saturation magnetization  $|M(x)| = m(T) > 0$  unless the local temperature  $T$  is greater or equal to Curie temperature depending on the body. In the latter case  $m(T) = 0$ , and the material ceases to behave ferromagnetically. In this paper, we suppose constant temperature lower than Curie temperature and, without loss of generality, we assume that  $m = 1$ , that is  $M(x) \in S^2$ . The exchange energy  $\int_{\Omega_n} |DM|^2 dx$  penalizes the spatial variation of  $M$ , driving the body to have large regions of uniform magnetization separated by thin transition layers. The scalar function  $U_M : \mathbb{R}^3 \rightarrow \mathbb{R}$  is the so-called magnetostatic potential. The magnetostatic energy  $\int_{\Omega_n} DU_M M dx = \int_{\mathbb{R}^3} |DU_M|^2 dx$  favors  $\operatorname{div} M = 0$  in  $\Omega_n$  and  $M \cdot \nu = 0$  on  $\partial\Omega_n$ , where  $\nu$  is the exterior unit normal to  $\partial\Omega_n$ . The constant  $\lambda$  is typically on order of 100 nanometers and measures the relative strength of exchange energy with respect to the magnetostatic energy. The anisotropy energy  $\int_{\Omega_n} \varphi(M) dx$  favors magnetization along special crystallographic directions, while the external (Zeeman) energy  $\int_{\Omega_n} F_n M dx$  favors magnetization parallel to an externally applied field.

Reformulating the problem on a fixed domain through appropriate rescalings of the kind proposed by P. G. Ciarlet and P. Destuynder [10], imposing appropriate convergence assumptions on the rescaled exterior fields and using the main ideas of  $\Gamma$ -convergence method introduced by E. De Giorgi [11], we derive the limit problem in both previous cases. Specif-

ically, in the case: wire - thin film, we prove that (see Theorem 3.1)

$$\begin{aligned}
\lim_n \frac{J_n}{h_n^2} &= \min \left\{ |\Theta| \int_0^1 \left( \lambda \left| \frac{d\mu^a}{dx_3} \right|^2 + \varphi(\mu^a) - \frac{2}{|\Theta|} F^a \mu^a \right) dx_3 + \right. \\
&\frac{1}{2} \left( \alpha(\Theta) \int_0^1 |\mu_1^a|^2 dx_3 + \beta(\Theta) \int_0^1 |\mu_2^a|^2 dx_3 + \gamma(\Theta) \int_0^1 \mu_1^a \mu_2^a dx_3 \right) : \\
&\left. \mu^a = (\mu_1^a, \mu_2^a, \mu_3^a) \in H^1(]0, 1[, S^2) \right\} + \\
\min \left\{ \int_{\Theta} \left( \lambda |D\mu^b|^2 + \varphi(\mu^b) + \frac{1}{2} |\mu_3^b|^2 - 2F^b \mu^b \right) dx_1 dx_2 : \right. \\
&\left. \mu^b = (\mu_1^b, \mu_2^b, \mu_3^b) \in H^1(\Theta, S^2) \right\}.
\end{aligned}$$

In the case: wire - wire, we prove that (see Theorem 4.1)

$$\begin{aligned}
\lim_n \frac{J_n}{h_n^2} &= \min \left\{ \int_0^1 \left( \lambda \left| \frac{d\mu^a}{dx_3} \right|^2 + \varphi(\mu^a) - 2F^a \mu^a \right) dx_3 + \right. \\
&\frac{1}{2} \left( \alpha(] - 1, 0[^2) \int_0^1 |\mu_1^a|^2 dx_3 + \beta(] - 1, 0[^2) \int_0^1 |\mu_2^a|^2 dx_3 + \gamma(] - 1, 0[^2) \int_0^1 \mu_1^a \mu_2^a dx_3 \right) + \\
&\int_0^1 \left( \lambda \left| \frac{d\mu^b}{dx_1} \right|^2 + \varphi(\mu^b) - 2G^b \mu^b \right) dx_1 + \\
&\frac{1}{2} \left( \alpha(] - 1, 0[^2) \int_0^1 |\mu_2^b|^2 dx_1 + \beta(] - 1, 0[^2) \int_0^1 |\mu_3^b|^2 dx_1 + \gamma(] - 1, 0[^2) \int_0^1 \mu_2^b \mu_3^b dx_1 \right) : \\
&\left. (\mu^a, \mu^b) = ((\mu_1^a, \mu_2^a, \mu_3^a), (\mu_1^b, \mu_2^b, \mu_3^b)) \in H^1(]0, 1[, S^2) \times H^1(]0, 1[, S^2), \mu^a(0) = \mu^b(0) \right\}.
\end{aligned}$$

Above,  $F^a(x_3)$  is the integral in  $dx_1 dx_2$  of the  $L^2$ -weak limit of the rescaled external field in the vertical domain,  $F^b(x_1, x_2)$  is the integral in  $dx_3$  of the  $L^2$ -weak limit of the rescaled external field in the horizontal domain,  $G^b(x_1)$  is the integral in  $dx_2 dx_3$  of the  $L^2$ -weak limit of the rescaled external field in the horizontal domain. To define coefficients  $\alpha, \beta, \gamma$ , if  $S \subset \mathbb{R}^2$  is a bounded open connected set, we introduce the weak solutions  $p$  and  $q$ , depending on  $S$ ,

of the following problems

$$\left\{ \begin{array}{l} p \in W^1(\mathbb{R}^2), \\ \Delta p = 0 \text{ in } S, \\ \Delta p = 0 \text{ in } \mathbb{R}^2 \setminus S, \\ \left[ \frac{\partial p}{\partial \nu} \right] = \nu e_1 \text{ on } \partial S, \end{array} \right. \quad \left\{ \begin{array}{l} q \in W^1(\mathbb{R}^2), \\ \Delta q = 0 \text{ in } S, \\ \Delta q = 0 \text{ in } \mathbb{R}^2 \setminus S, \\ \left[ \frac{\partial q}{\partial \nu} \right] = \nu e_2 \text{ on } \partial S, \end{array} \right.$$

where  $W^1(\mathbb{R}^2)$  denotes the Beppo-Levi space on  $\mathbb{R}^2$  (see Section 2),  $\nu$  the exterior unit normal to  $\partial S$ ,  $\left[ \frac{\partial \cdot}{\partial \nu} \right]$  the jump of  $\frac{\partial \cdot}{\partial \nu}$  on  $\partial S$ , and  $e_1 = (1, 0)$ ,  $e_2 = (0, 1)$ . Then, we set

$$\alpha(S) = \int_{\mathbb{R}^2} |Dp|^2 dydz, \quad \beta(S) = \int_{\mathbb{R}^2} |Dq|^2 dydz, \quad \gamma(S) = 2 \int_{\mathbb{R}^2} Dp Dq dydz, \quad (1.2)$$

where  $(y, z)$  denote the coordinates in  $\mathbb{R}^2$ . We remark that, if  $S$  is sufficiently smooth, definitions in (1.2) are equivalent to

$$\alpha(S) = \int_{\partial S} p \nu e_1 ds, \quad \beta(S) = \int_{\partial S} q \nu e_2 ds, \quad \gamma(S) = \int_{\partial S} q \nu e_1 ds + \int_{\partial S} p \nu e_2 ds.$$

If  $S = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\}$ , it results that  $\alpha(S) = \beta(S) = \frac{\pi}{2}$  and  $\gamma(S) = 0$  (see Theorem 3.1 in [29]).

In the case: wire - thin film, we obtain a  $1D$  limit problem on the wire and a  $2D$  limit problem on the thin film, and the two limit problems are uncoupled. In particular, if  $\Theta = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\}$ ,  $\varphi = 0$ ,  $F^a = 0$  and  $F^b = 0$ , then the minimum in the wire is attained by  $(0, 0, 1)$  or  $(0, 0, -1)$ , while the minimum in the thin film is attained by every constant  $S^2$ -vector parallel to the thin film.

In the case: wire - wire, we obtain two  $1D$  limit problems coupled by the junction condition on the magnetization  $\mu^a(0) = \mu^b(0)$ .

In both cases, the limit problem remains non-convex, but now it becomes completely local. Strong convergences in  $H^1$ -norm are obtained for the rescaled magnetization.

In Section 2, we recall the definition and some properties of the Beppo Levi space on  $\mathbb{R}^2$ . In Section 3, we study the case wire - thin film. We use two different rescalings: one for the wire and a second one for the thin film. The main difficulty is to identify the limit of the magnetostatic energy. While it is quite classical in the thin film where only the component of the magnetization orthogonal to the film appears in the limit (see [19]), it becomes more complicated in the wire where the following combination of the first two components of the magnetization with coefficients involving solutions of PDE in Beppo Levi space on  $\mathbb{R}^2$  intervene:  $\alpha(\Theta) \int_0^1 |\mu_1^a|^2 dx_3 + \beta(\Theta) \int_0^1 |\mu_2^a|^2 dx_3 + \gamma(\Theta) \int_0^1 \mu_1^a \mu_2^a dx_3$ . These coefficients depend on the geometry of the cross section of the wire. We explicitly remark that, to our knowledge, we are the first to obtain this explicit formula for a wire with a generic cross section. Finally, using the  $\Gamma$ -convergence method with suitable test functions and a density result proved in [16], we identify the limit problem which results uncoupled. In Section 4, we study the case

wire - wire, with wires having rectangular cross section. In this case, the main difficulty is to obtain the junction condition and to perform the limit of the magnetostatic energy. To this aim we have to use different and more sophisticated rescaling and symmetry arguments which, in some sense, take into account the geometry and that the limit problem will be coupled.

Our study can be easily extended to treat multi-structures as in Figure 3, or cruciform multi-structures.

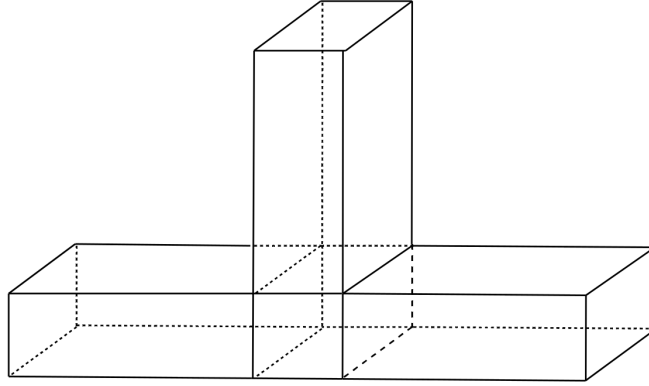


Figure 3:

Several results regarding the study of a single ferromagnetic thin film are present in literature. G. Gioia and R. D. James [19] were the first to prove that the magnetostatic energy behaves, at the limit, like an anisotropic local term which forces the magnetization to be tangent to the thin film. This result was extended by C. Leone and R. Alicandro [1] to the case with non-convex exchange energy, and by M. Baía and E. Zappale [5] to a thin film with nonhomogeneous profile. The case with degenerative coefficients was considered by R. Hadiji and K. Shirakawa [20]. The time-dependent case was treated by H. Ammari, L. Halpern and K. Hamdache [4], and by G. Carbou [7]. F. Alouges, T. Rivière and S. Serfaty [3] and C. Rivière and S. Serfaty [28] considered an infinite cylinder where the magnetization does not depend on the vertical coordinate. In [3] the authors showed that bounded-energy configurations tend to be planar, except in small regions where one can observe vortices. In [28] the magnetization is moreover constrained to be in the horizontal plane, which avoids the vortices. F. Alouges and S. Labbé [2] proposed a model of films with strong convergence of minimizers when the exchange parameter vanishes and with vertically invariant configurations on the cylindrical domain. For reproducing the non uniform states observed experimentally in thin films, very different regimes were considered by A. Desimone, R.V. Kohn, S. Muller and F. Otto [13], and by R.V. Kohn and V.V. Slastikov in [22], where  $\frac{h}{l}$  and  $\frac{\lambda}{l}$  vanish,  $h$  being the film thickness,  $l$  the in-plane diameter and  $\lambda$  the exchange length of the ferromagnetic material. Ferroelectric thin films were studied by A. Gaudiello and K. Hamdache in [18].

Single ferromagnetic nano-wire with circular cross section and finite length was studied by G. Carbou and S. Labbé [8]. In this paper, they also consider a stabilization problem. A similar model of wire with infinite length was studied by G. Carbou, S. Labbé and E. Trélat

[9]. Curved nano-wire was examined by V.V. Slastikov and C. Sonnenberg in [31].

In [17] we considered two joined ferromagnetic thin films and we proved that the limit magnetizations are coupled when the volumes of the two thin films vanish with the same rate.

Multi-structures like in this paper were considered in [16] and [15], where we developed an asymptotic analysis of minimizing maps with values in  $S^2$  for the energy  $\int_{\Omega_n} (|DM|^2 - 2F_n M) dx$ , neglecting the term with the nonlocal magnetostatic energy which characterizes the actual paper.

## 2 Preliminaries

Let

$$W^1(\mathbb{R}^2) = \left\{ \phi \in L^2_{\text{loc}}(\mathbb{R}^2) : D\phi \in (L^2(\mathbb{R}^2))^2 \right\} / \mathbb{R}$$

equipped with the inner product

$$(\phi_1, \phi_2) \in W^1(\mathbb{R}^2) \times W^1(\mathbb{R}^2) \rightarrow \int_{\mathbb{R}^2} D\phi_1 D\phi_2 dydz, \quad (2.1)$$

where  $(y, z)$  denote the coordinates in  $\mathbb{R}^2$ . It is well known that  $W^1(\mathbb{R}^2)$  is a Hilbert space (see [12], Corol. 1.1) and it is separable. Consequently, if  $S \subset \mathbb{R}^2$  is a bounded open set, every one of the following problems

$$\begin{cases} p \in W^1(\mathbb{R}^2), \\ \int_{\mathbb{R}^2} Dp D\phi dydz = \int_S D_y \phi dydz, \quad \forall \phi \in W^1(\mathbb{R}^2), \end{cases} \quad (2.2)$$

$$\begin{cases} q \in W^1(\mathbb{R}^2), \\ \int_{\mathbb{R}^2} Dq D\phi dydz = \int_S D_z \phi dydz, \quad \forall \phi \in W^1(\mathbb{R}^2), \end{cases} \quad (2.3)$$

$$\begin{cases} p_c \in W^1(\mathbb{R}^2), \\ \int_{\mathbb{R}^2} Dp_c D\phi dydz = \int_S c D\phi dydz, \quad \forall \phi \in W^1(\mathbb{R}^2), \end{cases} \quad (2.4)$$

with  $c = (c_1, c_2) \in \mathbb{R}^2$ , admits a unique solution which obviously depends on  $S$ . Then, we set

$$\alpha(S) = \int_{\mathbb{R}^2} |Dp|^2 dydz, \quad \beta(S) = \int_{\mathbb{R}^2} |Dq|^2 dydz, \quad \gamma(S) = 2 \int_{\mathbb{R}^2} Dp Dq dydz. \quad (2.5)$$

In the sequel, we shall use the following evident result.

**Lemma 2.1.** *Let  $p$  and  $q$  be the unique solutions of (2.2) and (2.3), respectively. Then, for every  $c = (c_1, c_2) \in \mathbb{R}^2$ , the unique solution  $p_c$  of (2.4) is given by:*

$$p_c = c_1 p + c_2 q.$$



We recall the Poincaré Lemma (which is well known if the domain is bounded).

**Lemma 2.2.** *Let  $\xi \in (L^2(\mathbb{R}^2))^2$  such that  $\text{rot } \xi = 0$ . Then, there exists a unique  $w \in W^1(\mathbb{R}^2)$  such that  $\xi = Dw$ .*

*Proof.* The fact that  $\text{rot } \xi = 0$  provides the existence of  $T \in \mathcal{D}'(\mathbb{R}^2)$  such that  $\xi = DT$ , and  $T$  is unique up to a constant (see [30], Ch. II, Th. VI, page 59). On the other hand, since  $\xi \in (L^2(\mathbb{R}^2))^2$ , Kryloff Theorem assures that  $T \in L^2_{loc}(\mathbb{R}^2)$  (see [30], Ch. VI, Th. XV, page 181).  $\square$

The following result was suggested by F. Murat [26].

**Proposition 2.3.** *Let  $u \in L^2_{loc}(\mathbb{R}^2)$  be such that  $Du \in (L^2(\mathbb{R}^2))^2$ . Then, there exist a sequence  $\{\varphi_n\}_{n \in \mathbb{N}} \subset C_0^\infty(\mathbb{R}^2)$  such that  $D\varphi_n \rightarrow Du$  strongly in  $(L^2(\mathbb{R}^2))^2$ .*

For sake of completeness, we conclude this section giving another representation of  $W^1(\mathbb{R}^2)$ . There exists a constant  $c > 0$ , and for every  $\phi \in W^1(\mathbb{R}^2)$  there exists  $\bar{\phi} \in \phi$  (we recall that  $\phi$  denotes a class of equivalence) such that (see [23], Th. 6.3)

$$\int_{\mathbb{R}^2} \frac{\bar{\phi}^2}{\left(1 + \log \sqrt{|x|^2 + 1}\right)^2 (|x|^2 + 1)} dx dy \leq c \int_{\mathbb{R}^2} |D\phi|^2 dx dy.$$

Consequently, it results that

$$W^1(\mathbb{R}^2) = \left\{ \phi \in L^2_{loc}(\mathbb{R}^2) : \frac{\phi}{\left(1 + \log \sqrt{|x|^2 + 1}\right) \sqrt{|x|^2 + 1}} \in L^2(\mathbb{R}^2), \quad D\phi \in (L^2(\mathbb{R}^2))^2 \right\} / \mathbb{R}$$

equipped with the inner product in (2.1). About this question see also [25].

### 3 Wire - thin film

This section is devoted to study the asymptotic behavior, as  $n$  diverges, of problem (1.1) in the first case, that is the case wire - thin film.

#### 3.1 The setting of the problem

Let  $\Theta \subset ]0, 1[^2$  be an open connected set with smooth boundary and, for every  $n \in \mathbb{N}$ , let  $\Omega_n^a = h_n \Theta \times [0, 1[$ ,  $\Omega_n^b = \Theta \times ]-h_n^2, 0[$  and  $\Omega_n = \Omega_n^a \cup \Omega_n^b$  (see Fig. 1).

Let  $B = ]-2, 2[^3$ , and set

$$\mathcal{U} = \left\{ U \in L^1_{loc}(\mathbb{R}^3) : U \in L^2(B), \quad DU \in (L^2(\mathbb{R}^3))^3, \quad \int_B U dx = 0 \right\}. \quad (3.1)$$

It is easy to prove that  $\mathcal{U}$  is contained in  $L^2_{loc}(\mathbb{R}^3)$  and it is a Hilbert space with the inner product:  $(U, V) = \int_{\mathbb{R}^3} DUDV dx + \int_B UV dx$ . Moreover, it follows from the Poincaré-Wirtinger

inequality that a norm on  $\mathcal{U}$  equivalent to  $(U, U)^{\frac{1}{2}}$  is given by  $(\int_{\mathbb{R}^3} |DU|^2 dx)^{\frac{1}{2}}$ . Then, Lax-Milgram theorem provides that, for  $M \in L^2(\Omega_n, \mathbb{R}^3)$ , the following equation

$$\begin{cases} U_{M,n} \in \mathcal{U}, \\ \int_{\mathbb{R}^3} DU_{M,n} DU dx = \int_{\Omega_n} MDU dx, \quad \forall U \in \mathcal{U}, \end{cases} \quad (3.2)$$

admits a unique solution and  $U_{M,n}$  is characterized as the unique minimizer of the following problem

$$\min \left\{ \frac{1}{2} \int_{\mathbb{R}^3} |DU - M|^2 dx : U \in \mathcal{U} \right\}, \quad (3.3)$$

where it is understood that  $M = 0$  in  $\mathbb{R}^3 \setminus \Omega_n$ . Moreover,  $U_M$  belongs to  $H^1(\mathbb{R}^3)$  up to an additive constant (see [21]).

Let  $\lambda$  be a positive constant,  $\varphi : S^2 \rightarrow [0, +\infty[$  be a continuous, even function and, for every  $n \in \mathbb{N}$ ,  $F_n \in L^2(\Omega_n, \mathbb{R}^3)$ . The following problem:

$$\min \left\{ \int_{\Omega_n} \left( \lambda |DM|^2 + \varphi(M) + \frac{1}{2} DU_{M,n} M - 2F_n M \right) dx : M \in H^1(\Omega_n, S^2) \right\} \quad (3.4)$$

has at least one solution (see [32]). In general, one can not expect a unique solution, because of the non-convexity of the constraint  $M(x) \in S^2$ . The aim of this section is to study the asymptotic behavior, as  $n$  diverges, of problem (3.4).

## 3.2 The rescaled problem

By setting

$$\begin{cases} \mathbb{R}_a^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0\}, \\ \mathbb{R}_b^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 < 0\}, \end{cases}$$

For every  $n \in \mathbb{N}$ , problem (3.4) will be reformulated on a fixed domain through the following rescaling:

$$(x_1, x_2, x_3) \in \mathbb{R}^3 \rightarrow \begin{cases} (h_n x_1, h_n x_2, x_3), & \text{if } (x_1, x_2, x_3) \in \mathbb{R}_a^3, \\ (x_1, x_2, h_n^2 x_3), & \text{if } (x_1, x_2, x_3) \in \mathbb{R}_b^3. \end{cases}$$

Namely, setting

$$\Omega^a = \Theta \times ]0, 1[, \quad \Omega^b = \Theta \times ]-1, 0[,$$

and

$$B_n^a = \left] -\frac{2}{h_n}, \frac{2}{h_n} \right[ \times ]0, 2[, \quad B_n^b = \left] -2, 2 \right[ \times \left] -\frac{2}{h_n^2}, 0 \right[, \quad \forall n \in \mathbb{N},$$

the space  $\mathcal{U}$  defined in (3.1) is rescaled in the following

$$\begin{aligned} \mathcal{U}_n = \{ & u = (u^a, u^b) \in L^1_{loc}(\overline{\mathbb{R}^3_a}) \times L^1_{loc}(\overline{\mathbb{R}^3_b}) : (u^a|_{B_n^a}, u^b|_{B_n^b}) \in L^2(B_n^a) \times L^2(B_n^b), \\ & (Du^a, Du^b) \in (L^2(\mathbb{R}^3_a))^3 \times (L^2(\mathbb{R}^3_b))^3, \quad \int_{B_n^a} u^a dx + \int_{B_n^b} u^b dx = 0, \\ & u^a(x_1, x_2, 0) = u^b(h_n x_1, h_n x_2, 0), \text{ for } (x_1, x_2) \text{ a.e. in } \mathbb{R}^2 \}. \end{aligned} \quad (3.5)$$

Then, for every  $\underline{m} = (\underline{m}^a, \underline{m}^b) \in L^2(\Omega^a, \mathbb{R}^3) \times L^2(\Omega^b, \mathbb{R}^3)$ , the following equation

$$\left\{ \begin{aligned} & u_{\underline{m},n} = (u_{\underline{m},n}^a, u_{\underline{m},n}^b) \in \mathcal{U}_n, \\ & \int_{\mathbb{R}^3_a} \left( \frac{1}{h_n} D_{x_1} u_{\underline{m},n}^a, \frac{1}{h_n} D_{x_2} u_{\underline{m},n}^a, D_{x_3} u_{\underline{m},n}^a \right) \left( \frac{1}{h_n} D_{x_1} u^a, \frac{1}{h_n} D_{x_2} u^a, D_{x_3} u^a \right) dx + \\ & \int_{\mathbb{R}^3_b} \left( D_{x_1} u_{\underline{m},n}^b, D_{x_2} u_{\underline{m},n}^b, \frac{1}{h_n^2} D_{x_3} u_{\underline{m},n}^b \right) \left( D_{x_1} u^b, D_{x_2} u^b, \frac{1}{h_n^2} D_{x_3} u^b \right) dx = \\ & \int_{\Omega^a} \left( \frac{1}{h_n} D_{x_1} u^a, \frac{1}{h_n} D_{x_2} u^a, D_{x_3} u^a \right) \underline{m}^a dx + \\ & \int_{\Omega^b} \left( D_{x_1} u^b, D_{x_2} u^b, \frac{1}{h_n^2} D_{x_3} u^b \right) \underline{m}^b dx, \quad \forall u = (u^a, u^b) \in \mathcal{U}_n, \end{aligned} \right. \quad (3.6)$$

which rescales equation (3.2), admits a unique solution. Its solution  $u_{\underline{m},n} = (u_{\underline{m},n}^a, u_{\underline{m},n}^b) \in \mathcal{U}_n$  is characterized as the unique minimizer of the following problem rescaling problem (3.3) after the renormalization by  $h_n^2$ :

$$j_{\underline{m},n}(u_{\underline{m},n}) = \min \{ j_{\underline{m},n}(u) : u \in \mathcal{U}_n \}, \quad (3.7)$$

where

$$\begin{aligned} j_{\underline{m},n} : u = (u^a, u^b) \in \mathcal{U}_n \longrightarrow & \frac{1}{2} \int_{\mathbb{R}^3_a} \left| \left( \frac{1}{h_n} D_{x_1} u^a, \frac{1}{h_n} D_{x_2} u^a, D_{x_3} u^a \right) - \underline{m}^a \right|^2 dx + \\ & \frac{1}{2} \int_{\mathbb{R}^3_b} \left| \left( D_{x_1} u^b, D_{x_2} u^b, \frac{1}{h_n^2} D_{x_3} u^b \right) - \underline{m}^b \right|^2 dx, \end{aligned} \quad (3.8)$$

understanding  $\underline{m}^a = 0$  in  $\mathbb{R}^3_a \setminus \Omega^a$  and  $\underline{m}^b = 0$  in  $\mathbb{R}^3_b \setminus \Omega^b$ . We note that  $u_{\underline{m},n} = (u_{\underline{m},n}^a, u_{\underline{m},n}^b)$  belongs to  $H^1(\mathbb{R}^3_a) \times H^1(\mathbb{R}^3_b)$  up to an additive constant.

For every  $n \in \mathbb{N}$ ,  $H^1(\Omega_n, S^2)$ ,  $F_n \in L^2(\Omega_n)$  and the functional involved in problem (3.4) renormalized by  $h_n^2$  are rescaled in

$$\begin{aligned} \mathcal{M}_n = \{ & \underline{m} = (\underline{m}^a, \underline{m}^b) \in H^1(\Omega^a, S^2) \times H^1(\Omega^b, S^2) : \\ & \underline{m}^a(x_1, x_2, 0) = \underline{m}^b(h_n x_1, h_n x_2, 0), \text{ for } (x_1, x_2) \text{ a.e. in } \Theta \}, \end{aligned} \quad (3.9)$$

$$f_n : x \in \Omega^a \cup \Omega^b \rightarrow f_n(x) = \begin{cases} f_n^a(x) = F_n(h_n x_1, h_n x_2, x_3), & \text{for } x \text{ a.e. in } \Omega^a, \\ f_n^b(x) = F_n(x_1, x_2, h_n^2 x_3), & \text{for } x \text{ a.e. in } \Omega^b, \end{cases} \quad (3.10)$$

and

$$\begin{aligned} E_n : \underline{m} = (\underline{m}^a, \underline{m}^b) \in \mathcal{M}_n \longrightarrow \\ \int_{\Omega^a} \left( \lambda \left| \left( \frac{1}{h_n} D_{x_1} \underline{m}^a, \frac{1}{h_n} D_{x_2} \underline{m}^a, D_{x_3} \underline{m}^a \right) \right|^2 + \varphi(\underline{m}^a) - 2f_n^a \underline{m}^a \right) dx + \\ \frac{1}{2} \int_{\Omega^a} \left( \left( \frac{1}{h_n} D_{x_1} u_{\underline{m},n}^a, \frac{1}{h_n} D_{x_2} u_{\underline{m},n}^a, D_{x_3} u_{\underline{m},n}^a \right) \underline{m}^a \right) dx + \\ \int_{\Omega^b} \left( \lambda \left| \left( D_{x_1} \underline{m}^b, D_{x_2} \underline{m}^b, \frac{1}{h_n^2} D_{x_3} \underline{m}^b \right) \right|^2 + \varphi(\underline{m}^b) - 2f_n^b \underline{m}^b \right) dx + \\ \frac{1}{2} \int_{\Omega^b} \left( \left( D_{x_1} u_{\underline{m},n}^b, D_{x_2} u_{\underline{m},n}^b, \frac{1}{h_n^2} D_{x_3} u_{\underline{m},n}^b \right) \underline{m}^b \right) dx, \end{aligned} \quad (3.11)$$

respectively. Then, the function defined by

$$M_n(h_n x_1, h_n x_2, x_3), \text{ for } x \text{ a.e. in } \Omega^a, \quad M_n(x_1, x_2, h_n^2 x_3), \text{ for } x \text{ a.e. in } \Omega^b,$$

with  $M_n$  solution of problem (3.4), is a minimizer of the following problem:

$$\min \{ E_n(\underline{m}) : \underline{m} \in \mathcal{M}_n \}. \quad (3.12)$$

Actually, the goal becomes to study the asymptotic behavior, as  $n$  diverges, of problem (3.12). To this aim, it will be assumed that

$$f_n^a \rightharpoonup f^a \text{ weakly in } L^2(\Omega^a, \mathbb{R}^3), \quad f_n^b \rightharpoonup f^b \text{ weakly in } L^2(\Omega^b, \mathbb{R}^3). \quad (3.13)$$

Note that, setting for every  $n \in \mathbb{N}$

$$\begin{aligned} E_n^{mag} : \underline{m} = (\underline{m}^a, \underline{m}^b) \in L^2(\Omega^a, \mathbb{R}^3) \times L^2(\Omega^b, \mathbb{R}^3) \longrightarrow \\ \frac{1}{2} \int_{\mathbb{R}_a^3} \left| \left( \frac{1}{h_n} D_{x_1} u_{\underline{m},n}^a, \frac{1}{h_n} D_{x_2} u_{\underline{m},n}^a, D_{x_3} u_{\underline{m},n}^a \right) \right|^2 dx + \\ \frac{1}{2} \int_{\mathbb{R}_b^3} \left| \left( D_{x_1} u_{\underline{m},n}^b, D_{x_2} u_{\underline{m},n}^b, \frac{1}{h_n^2} D_{x_3} u_{\underline{m},n}^b \right) \right|^2 dx, \end{aligned} \quad (3.14)$$

by virtue of (3.6), functional  $E_n$  can be rewritten in the following way:

$$\begin{aligned}
E_n(\underline{m}) &= \int_{\Omega^a} \left( \lambda \left| \left( \frac{1}{h_n} D_{x_1} \underline{m}^a \middle| \frac{1}{h_n} D_{x_2} \underline{m}^a \middle| D_{x_3} \underline{m}^a \right) \right|^2 + \varphi(\underline{m}^a) - 2f_n^a \underline{m}^a \right) dx + \\
&\int_{\Omega^b} \left( \lambda \left| \left( D_{x_1} \underline{m}^b \middle| D_{x_2} \underline{m}^b \middle| \frac{1}{h_n^2} D_{x_3} \underline{m}^b \right) \right|^2 + \varphi(\underline{m}^b) - 2f_n^b \underline{m}^b \right) dx + \\
E_n^{mag}(\underline{m}), \quad \forall \underline{m} = (\underline{m}^a, \underline{m}^b) \in \mathcal{M}_n, \quad \forall n \in \mathbb{N}.
\end{aligned} \tag{3.15}$$

### 3.3 The main result

Let

$$\begin{aligned}
\mathcal{M} = \left\{ \mu = (\mu^a, \mu^b) \in H^1(\Omega^a, S^2) \times H^1(\Omega^b, S^2) : \mu^a \text{ is independent of } (x_1, x_2), \right. \\
\left. \mu^b \text{ is independent of } x_3 \right\} \simeq H^1(]0, 1[, S^2) \times H^1(\Theta, S^2),
\end{aligned} \tag{3.16}$$

$$\begin{cases} F^a : x_3 \in ]0, 1[ \longrightarrow \frac{1}{|\Theta|} \int_{\Theta} f^a(x_1, x_2, x_3) dx_1 dx_2, \\ F^b : (x_1, x_2) \in \Theta \longrightarrow \int_{-1}^0 f^b(x_1, x_2, x_3) dx_3, \end{cases} \tag{3.17}$$

and

$$\begin{aligned}
E : \mu = (\mu^a, \mu^b) = ((\mu_1^a, \mu_2^a, \mu_3^a), (\mu_1^b, \mu_2^b, \mu_3^b)) \in \mathcal{M} \longrightarrow \\
|\Theta| \int_0^1 \left( \lambda \left| \frac{d\mu^a}{dx_3} \right|^2 + \varphi(\mu^a) - 2F^a \mu^a \right) dx_3 + \\
\frac{1}{2} \left( \alpha(\Theta) \int_0^1 |\mu_1^a|^2 dx_3 + \beta(\Theta) \int_0^1 |\mu_2^a|^2 dx_3 + \gamma(\Theta) \int_0^1 \mu_1^a \mu_2^a dx_3 \right) + \\
\int_{\Theta} \left( \lambda |D\mu^b|^2 + \varphi(\mu^b) + \frac{1}{2} |\mu_3^b|^2 - 2F^b \mu^b \right) dx_1 dx_2,
\end{aligned} \tag{3.18}$$

where  $\alpha(\Theta)$ ,  $\beta(\Theta)$  and  $\gamma(\Theta)$  are defined by (2.5) with  $S = \Theta$ .

This section is devoted to prove the following main result:

**Theorem 3.1.** *Assume (3.13). For every  $n \in \mathbb{N}$ , let  $\underline{m}_n = (\underline{m}_n^a, \underline{m}_n^b)$  be a solution of (3.12) and  $u_n = (u_n^a, u_n^b)$  be the unique solution of (3.7) corresponding to  $\underline{m}_n$ . Moreover, let  $\mathcal{M}$  and  $E$  be defined by (3.16) and (3.18), respectively. Then, there exist an increasing sequence of positive integer numbers  $\{n_i\}_{i \in \mathbb{N}}$  and  $\widehat{\mu} = (\widehat{\mu}^a, \widehat{\mu}^b) = ((\widehat{\mu}_1^a, \widehat{\mu}_2^a, \widehat{\mu}_3^a), (\widehat{\mu}_1^b, \widehat{\mu}_2^b, \widehat{\mu}_3^b)) \in \mathcal{M}$ , depending on the selected subsequence, such that*

$$\underline{m}_{n_i}^a \rightarrow \widehat{\mu}^a \text{ stongly in } H^1(\Omega^a, S^2), \quad \underline{m}_{n_i}^b \rightarrow \widehat{\mu}^b \text{ strongly in } H^1(\Omega^b, S^2), \tag{3.19}$$

$$\left\{ \begin{array}{l} \frac{1}{h_n} D_{x_1} \underline{m}_n^a \rightarrow 0, \quad \frac{1}{h_n} D_{x_2} \underline{m}_n^a \rightarrow 0 \text{ strongly in } L^2(\Omega^a, \mathbb{R}^3), \\ \frac{1}{h_n^2} D_{x_3} \underline{m}_n^b \rightarrow 0 \text{ strongly in } L^2(\Omega^b, \mathbb{R}^3), \end{array} \right. \quad (3.20)$$

$$\left\{ \begin{array}{l} \frac{1}{h_{n_i}} D_{x_1} u_{n_i}^a \rightarrow \widehat{\xi}_1^a, \quad \frac{1}{h_{n_i}} D_{x_2} u_{n_i}^a \rightarrow \widehat{\xi}_2^a, \quad D_{x_3} u_n^a \rightarrow 0 \quad \text{strongly in } L^2(\mathbb{R}_a^3), \\ D_{x_1} u_n^b \rightarrow 0, \quad D_{x_2} u_n^b \rightarrow 0, \quad \frac{1}{h_{n_i}^2} D_{x_3} u_{n_i}^b \rightarrow \widehat{\mu}_3^b \quad \text{strongly in } L^2(\mathbb{R}_b^3), \end{array} \right. \quad (3.21)$$

as  $n$  and  $i$  diverge, where  $\widehat{\mu}$  is a solution of the following problem:

$$E(\widehat{\mu}) = \min \{E(\mu) : \mu \in \mathcal{M}\}, \quad (3.22)$$

and

$$(\widehat{\xi}_1^a, \widehat{\xi}_2^a)(x_1, x_2, x_3) = \begin{cases} (0, 0), & \text{a.e. in } \mathbb{R}^2 \times ]1, +\infty[, \\ \widehat{\mu}_1^a(x_3) Dp(x_1, x_2) + \widehat{\mu}_2^a(x_3) Dq(x_1, x_2), & \text{a.e. in } \mathbb{R}^2 \times ]0, 1[, \end{cases} \quad (3.23)$$

with  $p$  (resp.  $q$ ) the unique solution of (2.2) (resp. (2.3)). It is understood that  $\widehat{\mu}_3^b = 0$  in  $\mathbb{R}_b^3 \setminus \Omega^b$ . Moreover, the convergence of the energies holds true:

$$\lim_n E_n(\underline{m}_n) = E(\widehat{\mu}). \quad (3.24)$$

### 3.4 A Convergence result for the magnetostatic energy

**Proposition 3.2.** *Let  $\{\underline{m}_n = (\underline{m}_n^a, \underline{m}_n^b)\}_{n \in \mathbb{N}} \subset L^2(\Omega^a, \mathbb{R}^3) \times L^2(\Omega^b, \mathbb{R}^3)$  and  $\mu = (\mu^a, \mu^b) = ((\mu_1^a, \mu_2^a, \mu_3^a), (\mu_1^b, \mu_2^b, \mu_3^b)) \in L^2(\Omega^a, \mathbb{R}^3) \times L^2(\Omega^b, \mathbb{R}^3)$  be such that  $\mu^a$  is independent of  $(x_1, x_2)$ ,  $\mu^b$  is independent of  $x_3$  and*

$$\underline{m}_n^a \rightarrow \mu^a \text{ strongly in } L^2(\Omega^a, \mathbb{R}^3), \quad \underline{m}_n^b \rightarrow \mu^b \text{ strongly in } L^2(\Omega^b, \mathbb{R}^3), \quad (3.25)$$

as  $n$  diverges. Moreover, for every  $n \in \mathbb{N}$ , let  $u_n = (u_n^a, u_n^b)$  be the unique solution of (3.7) corresponding to  $\underline{m}_n$ , and let  $E_n^{\text{mag}}$  be defined by (3.14). Then, it results that

$$\left\{ \begin{array}{l} \frac{1}{h_n} D_{x_1} u_n^a \rightarrow \xi_1^a, \quad \frac{1}{h_n} D_{x_2} u_n^a \rightarrow \xi_2^a, \quad D_{x_3} u_n^a \rightarrow 0 \quad \text{strongly in } L^2(\mathbb{R}_a^3), \\ D_{x_1} u_n^b \rightarrow 0, \quad D_{x_2} u_n^b \rightarrow 0, \quad \frac{1}{h_n^2} D_{x_3} u_n^b \rightarrow \mu_3^b \quad \text{strongly in } L^2(\mathbb{R}_b^3), \end{array} \right. \quad (3.26)$$

as  $n$  diverges, where it is understood that  $\mu_3^b = 0$  in  $\mathbb{R}_b^3 \setminus \Omega^b$ , and

$$(\xi_1^a, \xi_2^a)(x_1, x_2, x_3) = \begin{cases} (0, 0), & \text{a.e. in } \mathbb{R}^2 \times ]1, +\infty[, \\ \mu_1^a(x_3) Dp(x_1, x_2) + \mu_2^a(x_3) Dq(x_1, x_2), & \text{a.e. in } \mathbb{R}^2 \times ]0, 1[, \end{cases} \quad (3.27)$$

with  $p$  (resp.  $q$ ) the unique solution of (2.2) (resp. (2.3)). Furthermore, one has that

$$\begin{aligned} \lim_n E_n^{mag}(\underline{m}_n) &= \frac{1}{2} \left( \int_{\mathbb{R}^2 \times ]0,1[} |\mu_1^a Dp + \mu_2^a Dq|^2 dx + \int_{\Theta} |\mu_3^b|^2 dx_3 \right) = \\ & \frac{1}{2} \left( \alpha(\Theta) \int_0^1 |\mu_1^a|^2 dx_3 + \beta(\Theta) \int_0^1 |\mu_2^a|^2 dx_3 + \gamma(\Theta) \int_0^1 \mu_1^a \mu_2^a dx_3 + \int_{\Theta} |\mu_3^b|^2 dx_1 dx_2 \right), \end{aligned} \quad (3.28)$$

where  $\alpha(\Theta)$ ,  $\beta(\Theta)$  and  $\gamma(\Theta)$  are defined by (2.5) with  $S = \Theta$ .

*Proof.* The proof will be developed in four steps.

By arguing as in the first part of the proof of proposition 5.1 in [17], one can prove that

$$Du_n^a \rightharpoonup 0 \text{ weakly in } (L^2(\mathbb{R}_a^3))^3, \quad Du_n^b \rightharpoonup 0 \text{ weakly in } (L^2(\mathbb{R}_b^3))^3, \quad (3.29)$$

as  $n$  diverges, and that there exist  $\xi^a = (\xi_1^a, \xi_2^a) \in (L^2(\mathbb{R}_a^3))^2$  and  $\xi^b \in L^2(\mathbb{R}_b^3)$  such that, on extraction of a suitable subsequence (not relabeled),

$$\begin{cases} \frac{1}{h_n} D_{x_1} u_n^a \rightharpoonup \xi_1^a \text{ weakly in } L^2(\mathbb{R}_a^3), & \frac{1}{h_n} D_{x_2} u_n^a \rightharpoonup \xi_2^a \text{ weakly in } L^2(\mathbb{R}_a^3), \\ \frac{1}{h_n^2} D_{x_3} u_n^b \rightharpoonup \xi^b \text{ weakly in } L^2(\mathbb{R}_b^3), \end{cases} \quad (3.30)$$

as  $n$  diverges.

The second step is devoted to identify  $\xi^a$ . To this aim, starting from the following evident relation:

$$D_{x_2} \left( \frac{1}{h_n} D_{x_1} u_n^a \right) = D_{x_1} \left( \frac{1}{h_n} D_{x_2} u_n^a \right) \text{ in } \mathcal{D}'(\mathbb{R}_a^3), \quad \forall n \in \mathbb{N},$$

and using the first two limits in (3.30), one obtains that

$$\int_{\mathbb{R}_a^3} \xi_1^a D_{x_2} \varphi dx = \int_{\mathbb{R}_a^3} \xi_2^a D_{x_1} \varphi dx, \quad \forall \varphi \in H_0^1(\mathbb{R}_a^3). \quad (3.31)$$

By taking  $\varphi(x) = \phi(x_1, x_2) \chi(x_3)$  with  $\phi \in H^1(\mathbb{R}^2)$  and  $\chi \in C_0^\infty(]0, +\infty[)$  and recalling that  $H^1(\mathbb{R}^2)$  is separable, it follows from (3.31) that

$$\begin{cases} \text{for } x_3 \text{ a.e. in } ]0, +\infty[, & \int_{\mathbb{R}^2} \xi_1^a(x_1, x_2, x_3) D_{x_2} \phi(x_1, x_2) dx_1 dx_2 = \\ \int_{\mathbb{R}^2} \xi_2^a(x_1, x_2, x_3) D_{x_1} \phi(x_1, x_2) dx_1 dx_2, & \forall \phi \in H^1(\mathbb{R}^2). \end{cases}$$

Consequently, by virtue of the Poincaré Lemma (see Section 2), it results that

$$\begin{cases} \text{for } x_3 \text{ a.e. in } ]0, +\infty[, & \exists! w(\cdot, \cdot, x_3) \in W^1(\mathbb{R}^2) : \\ \xi_1^a(\cdot, \cdot, x_3) = D_{x_1} w(\cdot, \cdot, x_3), & \xi_2^a(\cdot, \cdot, x_3) = D_{x_2} w(\cdot, \cdot, x_3), \quad \text{a.e. in } \mathbb{R}^2. \end{cases} \quad (3.32)$$

Now, in equation (3.6) with  $\underline{m} = \underline{m}_n$  choose  $u^a = \varphi + c_n$  and  $u^b = c_n$ , with  $\varphi \in C_0^\infty(\mathbb{R}_a^3)$  and  $c_n = -(|B_n^a| + |B_n^b|)^{-1} \int_{B_n^a} \varphi dx$  (such that  $(u^a, u^b) \in \mathcal{U}_n$ ). By multiplying this equation by  $h_n$ , one has

$$\left\{ \begin{array}{l} \int_{\mathbb{R}_a^3} \left( \frac{1}{h_n} D_{x_1} u_n^a, \frac{1}{h_n} D_{x_2} u_n^a, D_{x_3} u_n^a \right) (D_{x_1} \varphi, D_{x_2} \varphi, h_n D_{x_3} \varphi) dx = \\ \int_{\Omega^a} (D_{x_1} \varphi, D_{x_2} \varphi, h_n D_{x_3} \varphi) \underline{m}_n^a dx, \quad \forall \varphi \in C_0^\infty(\mathbb{R}_a^3). \end{array} \right. \quad (3.33)$$

Then, passing to the limit, as  $n$  diverges, in (3.33), convergences (3.25), (3.29) and (3.30) give that

$$\int_{\mathbb{R}_a^3} (\xi_1^a, \xi_2^a) (D_{x_1} \varphi, D_{x_2} \varphi) dx = \int_0^1 \left( (\mu_1^a, \mu_2^a) \int_{\Theta} (D_{x_1} \varphi, D_{x_2} \varphi) dx_1 dx_2 \right) dx_3, \quad \forall \varphi \in C_0^\infty(\mathbb{R}_a^3).$$

Consequently, arguing as above, taking into account that  $W^1(\mathbb{R}^2)$  is separable, and using Proposition 2.3 and (3.32), it follows that, for  $x_3$  a.e. in  $]1, +\infty[$ ,  $w(\cdot, \cdot, x_3)$  solves the following problem:

$$\left\{ \begin{array}{l} w(\cdot, \cdot, x_3) \in W^1(\mathbb{R}^2), \\ \int_{\mathbb{R}^2} (D_{x_1} w(x_1, x_2, x_3), D_{x_2} w(x_1, x_2, x_3)) (D_{x_1} \phi(x_1, x_2), D_{x_2} \phi(x_1, x_2)) dx_1 dx_2 = 0, \quad \forall \phi \in W^1(\mathbb{R}^2), \end{array} \right.$$

while, for  $x_3$  a.e. in  $]0, 1[$ ,  $w(\cdot, \cdot, x_3)$  solves the following one:

$$\left\{ \begin{array}{l} w(\cdot, \cdot, x_3) \in W^1(\mathbb{R}^2), \\ \int_{\mathbb{R}^2} (D_{x_1} w(x_1, x_2, x_3), D_{x_2} w(x_1, x_2, x_3)) (D_{x_1} \phi(x_1, x_2), D_{x_2} \phi(x_1, x_2)) dx_1 dx_2 = \\ (\mu_1^a(x_3), \mu_2^a(x_3)) \int_{\Theta} (D_{x_1} \phi(x_1, x_2), D_{x_2} \phi(x_1, x_2)) dx_1 dx_2, \quad \forall \phi \in W^1(\mathbb{R}^2). \end{array} \right. \quad (3.34)$$

Then, by virtue of Lemma 2.1, it results that, for  $x_3$  a.e. in  $]0, +\infty[$ ,

$$w(\cdot, \cdot, x_3) = \begin{cases} 0, & \text{a.e. in } \mathbb{R}^2, \text{ if } x_3 > 1, \\ \mu_1^a(x_3) p(\cdot, \cdot) + \mu_2^a(x_3) q(\cdot, \cdot), & \text{a.e. in } \mathbb{R}^2, \text{ if } x_3 < 1, \end{cases} \quad (3.35)$$

with  $p$  (resp.  $q$ ) the unique solution of (2.2) (resp. (2.3)).

Finally, since Tonelli theorem assures that  $\xi^a$  and  $\mu_1^a Dp_1 + \mu_2^a Dp_2$  belong to  $(L^2(\mathbb{R}_a^3))^2 \subset (L_{\text{loc}}^1(\mathbb{R}_a^3))^2$ , using Fubini theorem with (3.32) and (3.35) one entails that

$$\begin{aligned} \int_{\mathbb{R}_a^3} \xi^a \varphi dx &= \int_0^{+\infty} \left( \int_{\mathbb{R}^2} \xi^a \varphi dx_1 dx_2 \right) dx_3 = \int_0^1 \left( \int_{\mathbb{R}^2} (\mu_1^a Dp + \mu_2^a Dq) \varphi dx_1 dx_2 \right) dx_3 = \\ &\int_{\mathbb{R}^2 \times ]0, 1[} (\mu_1^a Dp + \mu_2^a Dq) \varphi dx, \quad \forall \varphi \in C_0^\infty(\mathbb{R}_a^3), \end{aligned}$$



that is

$$\xi^a(x_1, x_2, x_3) = \begin{cases} (0, 0), & \text{a.e. in } \mathbb{R}^2 \times ]1, +\infty[, \\ \mu_1^a(x_3)Dp(x_1, x_2) + \mu_2^a(x_3)Dq(x_1, x_2), & \text{a.e. in } \mathbb{R}^2 \times ]0, 1[, \end{cases} \quad (3.36)$$

with  $p$  (resp.  $q$ ) the unique solution of (2.2) (resp. (2.3)). Consequently, the first two limits in (3.30) hold true for the whole sequence.

The third step is devoted to identify  $\xi^b$ . To this aim, in equation (3.6) with  $\underline{m} = \underline{m}_n$  choose  $u^a = c_n$  and  $u^b = \varphi + c_n$ , with  $\varphi \in C_0^\infty(\mathbb{R}_b^3)$  and  $c_n = -(|B_n^a| + |B_n^b|)^{-1} \int_{B_n^b} \varphi dx$  (such that  $(u^a, u^b) \in \mathcal{U}_n$ ). By multiplying this equation by  $h_n^2$ , one has

$$\begin{cases} \int_{\mathbb{R}_b^3} \left( D_{x_1} u_n^b, D_{x_2} u_n^b, \frac{1}{h_n^2} D_{x_3} u_n^b \right) (h_n^2 D_{x_1} \varphi, h_n^2 D_{x_2} \varphi, D_{x_3} \varphi) dx = \\ \int_{\Omega^b} (h_n^2 D_{x_1} \varphi, h_n^2 D_{x_2} \varphi, D_{x_3} \varphi) \underline{m}_n^b dx, \quad \forall \varphi \in C_0^\infty(\mathbb{R}_b^3). \end{cases} \quad (3.37)$$

Then, passing to the limit, as  $n$  diverges, in (3.37), convergences (3.25), (3.29) and (3.30) give that

$$\int_{\mathbb{R}_b^3} \xi^b D_{x_3} \varphi dx = \int_{\Omega^b} \mu_3^b D_{x_3} \varphi dx \quad \forall \varphi \in C_0^\infty(\mathbb{R}_b^3),$$

which provides that, for  $(x_1, x_2)$  a.e. in  $\mathbb{R}^2$ , the function  $\xi^b(x_1, x_2, \cdot) - \tilde{\mu}_3^b(x_1, x_2, \cdot)$  is constant in  $] - \infty, 0[$ , where  $\tilde{\mu}_3^b$  denotes the zero extension of  $\mu_3^b$  on  $\mathbb{R}_b^3 \setminus \Omega^b$ . On the other hand, for  $(x_1, x_2)$  a.e. in  $\mathbb{R}^2$ ,  $\xi^b(x_1, x_2, \cdot) - \tilde{\mu}_3^b(x_1, x_2, \cdot) \in L^2(] - \infty, 0[)$ . Then, for  $(x_1, x_2)$  a.e. in  $\mathbb{R}^2$ , it results that

$$\xi^b(x_1, x_2, \cdot) = \tilde{\mu}_3^b(x_1, x_2, \cdot), \quad \text{a.e. in } ] - \infty, 0[,$$

from which, arguing as above, it follows that

$$\xi^b(x_1, x_2, x_3) = \begin{cases} 0, & \text{a.e. in } \mathbb{R}_b^3 \setminus \Omega^b, \\ \mu^b(x_3), & \text{a.e. in } \Omega^b. \end{cases} \quad (3.38)$$

Consequently, also the last limit in (3.30) holds true for the whole sequence.

The last step is devoted to prove that convergences in (3.29) and (3.30) are strong, and to obtain convergence (3.28). By passing to the limit in (3.6) with  $\underline{m} = \underline{m}_n$ ,  $u^a = u_n^a$  and  $u^b = u_n^b$ , and using (3.25), (3.29), (3.30), (3.36), (3.38) and equation (3.34) with test function

$\mu_1^a p + \mu_2^a q$ , one obtains the convergence of the energies:

$$\begin{aligned}
& \lim_n \left[ \int_{\mathbb{R}_a^3} \left| \left( \frac{1}{h_n} D_{x_1} u_n^a, \frac{1}{h_n} D_{x_2} u_n^a, D_{x_3} u_n^a \right) \right|^2 dx + \right. \\
& \left. \int_{\mathbb{R}_b^3} \left| \left( D_{x_1} u_n^b, D_{x_2} u_n^b, \frac{1}{h_n^2} D_{x_3} u_n^b \right) \right|^2 dx \right] = \\
& \lim_n \left[ \int_{\Omega^a} \left( \frac{1}{h_n} D_{x_1} u_n^a, \frac{1}{h_n} D_{x_2} u_n^a, D_{x_3} u_n^a \right) \underline{m}_n^a dx + \right. \\
& \left. \int_{\Omega^b} \left( D_{x_1} u_n^b, D_{x_2} u_n^b, \frac{1}{h_n^2} D_{x_3} u_n^b \right) \underline{m}_n^b dx \right] = \\
& \int_{\Omega^a} (\mu_1^a Dp + \mu_2^a Dq) (\mu_1^a, \mu_2^a) dx + \int_{\Omega^b} |\mu_3^b|^2 dx = \\
& \int_{\mathbb{R}^2 \times ]0,1[} |\mu_1^a Dp + \mu_2^a Dq|^2 dx + \int_{\Omega^b} |\mu_3^b|^2 dx.
\end{aligned} \tag{3.39}$$

By combining (3.29), (3.30), (3.36), (3.38) with (3.39), one deduces limits in (3.26). Limit (3.28) is a consequence of (3.26) and (3.27).  $\square$

### 3.5 Proof of theorem 3.1

*Proof.* By choosing  $\underline{m} = ((0, 1, 0), (0, 1, 0))$  as test function in (3.12), and taking into account (3.13) and that  $|\underline{m}_n| = 1$  a.e. in  $\Omega^a \cup \Omega^b$ , it is easy to see that there exists  $c \in ]0, +\infty[$  such that

$$\begin{aligned}
& \int_{\Omega^a} \left| \left( \frac{1}{h_n} D_{x_1} \underline{m}_n^a, \frac{1}{h_n} D_{x_2} \underline{m}_n^a, D_{x_3} \underline{m}_n^a \right) \right|^2 dx + \int_{\Omega^b} \left| \left( D_{x_1} \underline{m}_n^b, D_{x_2} \underline{m}_n^b, \frac{1}{h_n^2} D_{x_3} \underline{m}_n^b \right) \right|^2 dx \leq \\
& c + E_n^{mag}((0, 1, 0), (0, 1, 0)), \quad \forall n \in \mathbb{N},
\end{aligned}$$

where  $E_n^{mag}$  is defined (3.14). Consequently, since proposition 3.2 provides that the sequence  $\{E_n^{mag}((0, 1, 0), (0, 1, 0))\}_{n \in \mathbb{N}}$  is bounded, one obtains that there exists  $c \in ]0, +\infty[$  such that

$$\begin{cases} \|D_{x_1} \underline{m}_n^a\|_{(L^2(\Omega^a))^3} \leq ch_n, & \|D_{x_2} \underline{m}_n^a\|_{(L^2(\Omega^a))^3} \leq ch_n, & \|D_{x_3} \underline{m}_n^a\|_{(L^2(\Omega^a))^3} \leq c, \\ \|D_{x_1} \underline{m}_n^b\|_{(L^2(\Omega^b))^3} \leq c, & \|D_{x_2} \underline{m}_n^b\|_{(L^2(\Omega^b))^3} \leq c, & \|D_{x_3} \underline{m}_n^b\|_{(L^2(\Omega^b))^3} \leq ch_n^2, \end{cases}$$

for every  $n \in \mathbb{N}$ . Then, taking into account that  $|\underline{m}_n| = 1$  a.e. in  $\Omega^a \cup \Omega^b$ , there exist an increasing sequence of positive integer numbers  $\{n_i\}_{i \in \mathbb{N}}$ ,  $\hat{\mu} = (\hat{\mu}^a, \hat{\mu}^b) \in \mathcal{M}$ ,  $\zeta^a \in (L^2(\Omega^a, \mathbb{R}^3))^2$

and  $\zeta^b \in L^2(\Omega^b, \mathbb{R}^3)$  such that

$$\underline{m}_{n_i}^a \rightharpoonup \widehat{\mu}^a \text{ weakly in } H^1(\Omega^a, \mathbb{R}^3), \quad \underline{m}_{n_i}^b \rightharpoonup \widehat{\mu}^b \text{ weakly in } H^1(\Omega^b, \mathbb{R}^3), \quad (3.40)$$

$$\left\{ \begin{array}{l} \left( \frac{1}{h_{n_i}} D_{x_1} \underline{m}_{n_i}^a, \frac{1}{h_{n_i}} D_{x_2} \underline{m}_{n_i}^a \right) \rightharpoonup \zeta^a \text{ weakly in } (L^2(\Omega^a, \mathbb{R}^3))^2, \\ \frac{1}{h_{n_i}^2} D_{x_3} \underline{m}_{n_i}^b \rightharpoonup \zeta^b \text{ weakly in } L^2(\Omega^b, \mathbb{R}^3), \end{array} \right. \quad (3.41)$$

as  $i$  diverges. Consequently, by virtue of proposition 3.2, limits in (3.21) hold true and it results that

$$\begin{aligned} \lim_i E_{n_i}^{mag}(\underline{m}_{n_i}) &= \frac{1}{2} \left( \alpha(\Theta) \int_0^1 |\widehat{\mu}_1^a|^2 dx_3 + \beta(\Theta) \int_0^1 |\widehat{\mu}_2^a|^2 dx_3 + \right. \\ &\quad \left. \gamma(\Theta) \int_0^1 \widehat{\mu}_1^a \widehat{\mu}_2^a dx_3 + \int_{\Theta} |\widehat{\mu}_3^b|^2 dx_1 dx_2 \right), \end{aligned} \quad (3.42)$$

where  $\alpha(\Theta)$ ,  $\beta(\Theta)$  and  $\gamma(\Theta)$  are defined by (2.5) with  $S = \Theta$ .

Now, the goal is to identify  $\widehat{\mu}$ ,  $\zeta^a$ ,  $\zeta^b$ , to obtain strong convergences in (3.40) and in (3.41), and to prove convergence (3.24). To this aim, for  $(\mu^a, \mu^b) \in \mathcal{M}_{\text{reg}} = \{(\mu^a, \mu^b) \in C^1([0, 1], S^2) \times C^1(\overline{\Theta}, S^2) : \mu^a(0) = \mu^b(0)\}$  let, for every  $n \in \mathbb{N}$ ,  $v_n = (v_n^a, v_n^b) \in \mathcal{M}_n$  be the couple of functions defined in (2.37) of [16] with  $w = \mu^a$  and  $\zeta = \mu^b$ . Then, in [16] it is proved that

$$\begin{aligned} &\lim_n \left[ \int_{\Omega^a} \left( \lambda \left| \left( \frac{1}{h_n} D_{x_1} v_n^a, \frac{1}{h_n} D_{x_2} v_n^a, D_{x_3} v_n^a \right) \right|^2 - 2f_n^a v_n^a \right) dx + \right. \\ &\quad \left. \int_{\Omega^b} \left( \lambda \left| \left( D_{x_1} v_n^b, D_{x_2} v_n^b, \frac{1}{h_n^2} D_{x_3} v_n^b \right) \right|^2 - 2f_n^b v_n^b \right) dx \right] = \\ &\quad |\Theta| \int_0^1 \left( \lambda \left| \frac{d\mu^a}{dx_3} \right|^2 - 2F^a \mu^a \right) dx_3 + \int_{\Theta} \left( \lambda |D\mu^b|^2 - 2F^b \mu^b \right) dx_1 dx_2. \end{aligned} \quad (3.43)$$

Moreover, it is easy to see that

$$v_n^a \rightarrow \mu^a \text{ strongly in } L^2(\Omega^a, \mathbb{R}^3), \quad v_n^b \rightarrow \mu^b \text{ strongly in } L^2(\Omega^b, \mathbb{R}^3), \quad (3.44)$$

as  $n$  diverges. Then, it follows from (3.43), (3.44) and proposition 3.2 that

$$\lim_n E_n(v_n) = E(\mu^a, \mu^b)$$

from which, using l.s.c. arguments, (3.13), (3.40), (3.41) and (3.42), one obtains that

$$\left\{ \begin{array}{l} \lambda \int_{\Omega^a} |\zeta^a|^2 dx + \lambda \int_{\Omega^b} |\zeta^b|^2 dx + E(\widehat{\mu}^a, \widehat{\mu}^b) \leq \liminf_i E_{n_i}(\underline{m}_{n_i}) \leq \\ \limsup_i E_{n_i}(\underline{m}_{n_i}) \leq \lim_i E_{n_i}(v_{n_i}) = E(\mu^a, \mu^b). \end{array} \right. \quad (3.45)$$

Since (3.45) holds true for every  $(\mu^a, \mu^b) \in \mathcal{M}_{\text{reg}}$  and  $\mathcal{M}_{\text{reg}}$  is dense in  $\mathcal{M}$  (see [16]), one has that (3.45) holds also true for every  $(\mu^a, \mu^b) \in \mathcal{M}$ . Consequently,  $\zeta^a = 0$ ,  $\zeta^b = 0$ ,  $(\widehat{\mu}^a, \widehat{\mu}^b)$  solves (3.22) and limit (3.24) holds true. Finally, combining (3.24) with (3.13), (3.40), (3.41) and (3.42) one obtains that convergences in (3.40) and in (3.41) are strong.  $\square$

## 4 Wire - wire

This section is devoted to study the asymptotic behavior, as  $n$  diverges, of problem (1.1) in the second case, that is the case wire - wire. Specifically, for every  $n \in \mathbb{N}$ , let  $\Omega_n^a = ]-h_n, 0[ \times ]0, 1[$ ,  $\Omega_n^{b,l} = ]0, 1[ \times ]-h_n, 0[$  and  $\Omega_n^{b,r} = ]-h_n, 0[$ . Then, we study the asymptotic behavior, as  $n$  diverges, of problem (3.4) with  $\Omega_n = \Omega_n^a \cup \Omega_n^{b,l} \cup \Omega_n^{b,r}$  (see Fig. 2).

### 4.1 The rescaled problem

By setting

$$\begin{cases} \mathbb{R}_a^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0\}, \\ \mathbb{R}_{b,l}^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 < 0, x_1 > 0\}, \\ \mathbb{R}_{b,r}^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 < 0, x_1 < 0\}, \end{cases}$$

for every  $n \in \mathbb{N}$ , problem (3.4) is reformulated on a fixed domain through the following rescaling

$$T_n : (x_1, x_2, x_3) \in \mathbb{R}^3 \rightarrow T_n(x_1, x_2, x_3) = \begin{cases} (h_n x_1, h_n x_2, x_3), & \text{if } (x_1, x_2, x_3) \in \mathbb{R}_a^3, \\ (x_1, h_n x_2, h_n x_3), & \text{if } (x_1, x_2, x_3) \in \mathbb{R}_{b,l}^3, \\ (h_n x_1, h_n x_2, h_n x_3), & \text{if } (x_1, x_2, x_3) \in \mathbb{R}_{b,r}^3. \end{cases} \quad (4.1)$$

Namely, setting

$$\Omega^a = ]-1, 0[ \times ]0, 1[, \quad \Omega^{b,l} = ]0, 1[ \times ]-1, 0[, \quad \Omega^{b,r} = ]-1, 0[$$

and

$$B_n^a = \left] -\frac{2}{h_n}, \frac{2}{h_n} \right[ \times ]0, 2[, \quad B_n^{b,l} = ]0, 2[ \times \left] -\frac{2}{h_n}, 0 \right[, \quad B_n^{b,r} = \left] -\frac{2}{h_n}, 0 \right[, \quad n \in \mathbb{N},$$

the space  $\mathcal{U}$  defined in (3.1) is rescaled in the following

$$\begin{aligned}
\mathcal{U}_n = \{ & u = (u^a, u^{b,l}, u^{b,r}) \in L^1_{loc}(\overline{\mathbb{R}^3_a}) \times L^1_{loc}(\overline{\mathbb{R}^3_{b,l}}) \times L^1_{loc}(\overline{\mathbb{R}^3_{b,r}}) : \\
& (u^a|_{B_n^a}, u^{b,l}|_{B_n^{b,l}}, u^{b,r}|_{B_n^{b,r}}) \in L^2(B_n^a) \times L^2(B_n^{b,l}) \times L^2(B_n^{b,r}), \\
& (Du^a, Du^{b,l}, Du^{b,r}) \in (L^2(\mathbb{R}^3_a))^3 \times (L^2(\mathbb{R}^3_{b,l}))^3 \times (L^2(\mathbb{R}^3_{b,r}))^3, \\
& \int_{B_n^a} u^a dx + \int_{B_n^{b,l}} u^{b,l} dx + h_n \int_{B_n^{b,r}} u^{b,r} dx = 0, \\
& u^a(x_1, x_2, 0) = u^{b,l}(h_n x_1, x_2, 0), \text{ for } (x_1, x_2) \text{ a.e. in } ]0, +\infty[ \times \mathbb{R}, \\
& u^a(x_1, x_2, 0) = u^{b,r}(x_1, x_2, 0), \text{ for } (x_1, x_2) \text{ a.e. in } ]-\infty, 0[ \times \mathbb{R}, \\
& u^{b,l}(0, x_2, x_3) = u^{b,r}(0, x_2, x_3), \text{ for } (x_2, x_3) \text{ a.e. in } \mathbb{R} \times ]-\infty, 0[ \}.
\end{aligned} \tag{4.2}$$

Then, for every  $\underline{m} = (\underline{m}^a, \underline{m}^{b,l}, \underline{m}^{b,r}) \in L^2(\Omega^a, \mathbb{R}^3) \times L^2(\Omega^{b,l}, \mathbb{R}^3) \times L^2(\Omega^{b,r}, \mathbb{R}^3)$ , the following equation

$$\left\{ \begin{aligned}
& u_{\underline{m},n} = (u_{\underline{m},n}^a, u_{\underline{m},n}^{b,l}, u_{\underline{m},n}^{b,r}) \in \mathcal{U}_n, \\
& \int_{\mathbb{R}^3_a} \left( \frac{1}{h_n} D_{x_1} u_{\underline{m},n}^a, \frac{1}{h_n} D_{x_2} u_{\underline{m},n}^a, D_{x_3} u_{\underline{m},n}^a \right) \left( \frac{1}{h_n} D_{x_1} u^a, \frac{1}{h_n} D_{x_2} u^a, D_{x_3} u^a \right) dx + \\
& \int_{\mathbb{R}^3_{b,l}} \left( D_{x_1} u_{\underline{m},n}^{b,l}, \frac{1}{h_n} D_{x_2} u_{\underline{m},n}^{b,l}, \frac{1}{h_n} D_{x_3} u_{\underline{m},n}^{b,l} \right) \left( D_{x_1} u^{b,l}, \frac{1}{h_n} D_{x_2} u^{b,l}, \frac{1}{h_n} D_{x_3} u^{b,l} \right) dx + \\
& \frac{1}{h_n} \int_{\mathbb{R}^3_{b,r}} (D_{x_1} u_{\underline{m},n}^{b,r}, D_{x_2} u_{\underline{m},n}^{b,r}, D_{x_3} u_{\underline{m},n}^{b,r}) (D_{x_1} u^{b,r}, D_{x_2} u^{b,r}, D_{x_3} u^{b,r}) dx = \\
& \int_{\Omega^a} \left( \frac{1}{h_n} D_{x_1} u^a, \frac{1}{h_n} D_{x_2} u^a, D_{x_3} u^a \right) \underline{m}^a dx + \\
& \int_{\Omega^{b,l}} \left( D_{x_1} u^{b,l}, \frac{1}{h_n} D_{x_2} u^{b,l}, \frac{1}{h_n} D_{x_3} u^{b,l} \right) \underline{m}^{b,l} dx + \\
& \int_{\Omega^{b,r}} (D_{x_1} u^{b,r}, D_{x_2} u^{b,r}, D_{x_3} u^{b,r}) \underline{m}^{b,r} dx, \quad \forall u = (u^a, u^{b,l}, u^{b,r}) \in \mathcal{U}_n,
\end{aligned} \right. \tag{4.3}$$

which rescales equation (3.2), admits a unique solution. We note that  $u_{\underline{m},n} = (u_{\underline{m},n}^a, u_{\underline{m},n}^{b,l}, u_{\underline{m},n}^{b,r})$  belongs to  $H^1(\mathbb{R}^3_a) \times H^1(\mathbb{R}^3_{b,l}) \times H^1(\mathbb{R}^3_{b,r})$  up to an additive constant.

For every  $n \in \mathbb{N}$ ,  $H^1(\Omega_n, S^2)$ ,  $F_n \in L^2(\Omega_n)$  and the functional involved in problem (3.4)

with  $\Omega_n = \Omega_n^a \cup \Omega_n^{b,l} \cup \Omega_n^{b,r}$  and renormalized by  $h_n^2$  are rescaled in

$$\begin{aligned} \mathcal{M}_n = \left\{ \underline{m} = (\underline{m}^a, \underline{m}^{b,l}, \underline{m}^{b,r}) \in H^1(\Omega^a, S^2) \times H^1(\Omega^{b,l}, S^2) \times H^1(\Omega^{b,r}, S^2) : \right. \\ \underline{m}^a(x_1, x_2, 0) = \underline{m}^{b,r}(x_1, x_2, 0), \text{ for } (x_1, x_2) \text{ a.e. in } ]-1, 0]^2, \\ \underline{m}^{b,l}(0, x_2, x_3) = \underline{m}^{b,r}(0, x_2, x_3), \text{ for } (x_2, x_3) \text{ a.e. in } ]-1, 0]^2 \left. \right\}, \\ f_n : x \in \Omega^a \cup \Omega^{b,l} \cup \Omega^{b,r} \longrightarrow \end{aligned} \quad (4.4)$$

$$f_n(x) = \begin{cases} f_n^a(x) = F_n(h_n x_1, h_n x_2, x_3), & \text{for } x \text{ a.e. in } \Omega^a, \\ f_n^{b,l}(x) = F_n(x_1, h_n x_2, h_n x_3), & \text{for } x \text{ a.e. in } \Omega^{b,l}, \\ f_n^{b,r}(x) = F_n(h_n x_1, h_n x_2, h_n x_3), & \text{for } x \text{ a.e. in } \Omega^{b,r}, \end{cases} \quad (4.5)$$

and

$$\begin{aligned} E_n : \underline{m} = (\underline{m}^a, \underline{m}^{b,l}, \underline{m}^{b,r}) \in \mathcal{M}_n \longrightarrow \\ \int_{\Omega^a} \left( \lambda \left| \left( \frac{1}{h_n} D_{x_1} \underline{m}^a \middle| \frac{1}{h_n} D_{x_2} \underline{m}^a \middle| D_{x_3} \underline{m}^a \right) \right|^2 + \varphi(\underline{m}^a) - 2f_n^a \underline{m}^a \right) dx + \\ \frac{1}{2} \int_{\Omega^a} \left( \left( \frac{1}{h_n} D_{x_1} u_{\underline{m},n}^a, \frac{1}{h_n} D_{x_2} u_{\underline{m},n}^a, D_{x_3} u_{\underline{m},n}^a \right) \underline{m}^a \right) dx + \\ \int_{\Omega^{b,l}} \left( \lambda \left| \left( D_{x_1} \underline{m}^{b,l} \middle| \frac{1}{h_n} D_{x_2} \underline{m}^{b,l} \middle| \frac{1}{h_n} D_{x_3} \underline{m}^{b,l} \right) \right|^2 + \varphi(\underline{m}^{b,l}) - 2f_n^{b,l} \underline{m}^{b,l} \right) dx + \\ \frac{1}{2} \int_{\Omega^{b,l}} \left( \left( D_{x_1} u_{\underline{m},n}^{b,l}, \frac{1}{h_n} D_{x_2} u_{\underline{m},n}^{b,l}, \frac{1}{h_n} D_{x_3} u_{\underline{m},n}^{b,l} \right) \underline{m}^{b,l} \right) dx + \\ h_n \int_{\Omega^{b,r}} \left( \lambda \left| \left( \frac{1}{h_n} D_{x_1} \underline{m}^{b,r} \middle| \frac{1}{h_n} D_{x_2} \underline{m}^{b,r} \middle| \frac{1}{h_n} D_{x_3} \underline{m}^{b,r} \right) \right|^2 + \varphi(\underline{m}^{b,r}) - 2f_n^{b,r} \underline{m}^{b,r} \right) dx + \\ \frac{1}{2} \int_{\Omega^{b,r}} \left( (D_{x_1} u_{\underline{m},n}^{b,r}, D_{x_2} u_{\underline{m},n}^{b,r}, D_{x_3} u_{\underline{m},n}^{b,r}) \underline{m}^{b,r} \right) dx, \end{aligned} \quad (4.6)$$

respectively. Then, the function defined by

$$\begin{cases} M_n(h_n x_1, h_n x_2, x_3), & \text{for } x \text{ a.e. in } \Omega^a, \\ M_n(x_1, h_n x_2, h_n x_3), & \text{for } x \text{ a.e. in } \Omega^{b,l}, \\ M_n(h_n x_1, h_n x_2, h_n x_3), & \text{for } x \text{ a.e. in } \Omega^{b,r}, \end{cases}$$

with  $M_n$  solution of problem (3.4) with  $\Omega_n = \Omega_n^a \cup \Omega_n^{b,l} \cup \Omega_n^{b,r}$ , is a minimizer of the following problem:

$$\min \{E_n(\underline{m}) : \underline{m} \in \mathcal{M}_n\}. \quad (4.7)$$

Actually, the goal of this section becomes to study the asymptotic behavior, as  $n$  diverges, of problem (4.7). To this aim, it will be assumed that

$$\begin{cases} f_n^a \rightharpoonup f^a \text{ weakly in } L^2(\Omega^a, \mathbb{R}^3), \\ f_n^{b,l} \rightharpoonup f^{b,l} \text{ weakly in } L^2(\Omega^{b,l}, \mathbb{R}^3), \\ f_n^{b,r} \rightharpoonup f^{b,r} \text{ weakly in } L^2(\Omega^{b,r}, \mathbb{R}^3). \end{cases} \quad (4.8)$$

## 4.2 The main result

Let

$$\mathcal{M} = \left\{ \mu = (\mu^a, \mu^{b,l}) \in H^1(\Omega^a, S^2) \times H^1(\Omega^{b,l}, S^2) : \mu^a \text{ is independent of } (x_1, x_2), \right. \\ \left. \mu^b \text{ is independent of } (x_2, x_3), \quad \mu^a(0) = \mu^{b,l}(0) \right\} \simeq \quad (4.9)$$

$$\left\{ \mu = (\mu^a, \mu^{b,l}) \in H^1(]0, 1[, S^2) \times H^1(]0, 1[, S^2) : \mu^a(0) = \mu^{b,l}(0) \right\}, \\ \begin{cases} F^a : x_3 \in ]0, 1[ \longrightarrow \int_{-1}^0 \int_{-1}^0 f^a(x_1, x_2, x_3) dx_1 dx_2, \\ F^{b,l} : x_1 \in ]0, 1[ \longrightarrow \int_{-1}^0 \int_{-1}^0 f^{b,l}(x_1, x_2, x_3) dx_2 dx_3, \end{cases} \quad (4.10)$$

and

$$E : \mu = (\mu^a, \mu^{b,l}) = ((\mu_1^a, \mu_2^a, \mu_3^a), (\mu_1^{b,l}, \mu_2^{b,l}, \mu_3^{b,l})) \in \mathcal{M} \longrightarrow \\ \int_0^1 \left( \lambda \left| \frac{d\mu^a}{dx_3} \right|^2 + \varphi(\mu^a) - 2F^a \mu^a \right) dx_3 + \\ \frac{1}{2} \left( \alpha(] - 1, 0[^2) \int_0^1 |\mu_1^a|^2 dx_3 + \beta(] - 1, 0[^2) \int_0^1 |\mu_2^a|^2 dx_3 + \gamma(] - 1, 0[^2) \int_0^1 \mu_1^a \mu_2^a dx_3 \right) + \\ \int_0^1 \left( \lambda \left| \frac{d\mu^{b,l}}{dx_1} \right|^2 + \varphi(\mu^{b,l}) - 2F^{b,l} \mu^{b,l} \right) dx_1 + \\ \frac{1}{2} \left( \alpha(] - 1, 0[^2) \int_0^1 |\mu_2^{b,l}|^2 dx_1 + \beta(] - 1, 0[^2) \int_0^1 |\mu_3^{b,l}|^2 dx_1 + \gamma(] - 1, 0[^2) \int_0^1 \mu_2^{b,l} \mu_3^{b,l} dx_1 \right) \quad (4.11)$$

where  $\alpha(\cdot - 1, 0[{}^2]$ ,  $\beta(\cdot - 1, 0[{}^2]$  and  $\gamma(\cdot - 1, 0[{}^2]$  are defined by (2.5) with  $S = \cdot - 1, 0[{}^2$ .

This section is devoted to prove the following main result

**Theorem 4.1.** *Assume (4.8). For every  $n \in \mathbb{N}$ , let  $\underline{m}_n = (\underline{m}_n^a, \underline{m}_n^{b,l}, \underline{m}_n^{b,r})$  be a solution of problem (4.7) and  $u_n = (u_n^a, u_n^{b,l}, u_n^{b,r})$  be the unique solution of (4.3) corresponding to  $\underline{m}_n$ . Moreover, let  $\mathcal{M}$  and  $E$  be defined by (4.9) and (4.11), respectively. Then, there exist an increasing sequence of positive integer numbers  $\{n_i\}_{i \in \mathbb{N}}$  and  $\hat{\mu} = (\hat{\mu}^a, \hat{\mu}^{b,l}) = ((\hat{\mu}_1^a, \hat{\mu}_2^a, \hat{\mu}_3^a), (\hat{\mu}_1^{b,l}, \hat{\mu}_2^{b,l}, \hat{\mu}_3^{b,l})) \in \mathcal{M}$ , depending on the selected subsequence, such that*

$$\begin{cases} \underline{m}_{n_i}^a \rightarrow \hat{\mu}^a \text{ strongly in } H^1(\Omega^a, S^2), \\ \underline{m}_{n_i}^{b,l} \rightarrow \hat{\mu}^{b,l} \text{ strongly in } H^1(\Omega^{b,l}, S^2), \\ \underline{m}_{n_i}^{b,r} \rightarrow \hat{\mu}^a(0) = \hat{\mu}^{b,l}(0) \text{ strongly in } H^1(\Omega^{b,r}, S^2), \end{cases} \quad (4.12)$$

$$\begin{cases} \frac{1}{h_n} D_{x_1} \underline{m}_n^a \rightarrow 0, \quad \frac{1}{h_n} D_{x_2} \underline{m}_n^a \rightarrow 0 \text{ strongly in } L^2(\Omega^a, \mathbb{R}^3), \\ \frac{1}{h_n} D_{x_2} \underline{m}_n^{b,l} \rightarrow 0, \quad \frac{1}{h_n} D_{x_3} \underline{m}_n^{b,l} \rightarrow 0 \text{ strongly in } L^2(\Omega^{b,l}, \mathbb{R}^3), \\ \frac{1}{\sqrt{h_n}} D \underline{m}_n^{b,r} \rightarrow 0 \text{ strongly in } (L^2(\Omega^{b,r}, \mathbb{R}^3))^3, \end{cases} \quad (4.13)$$

$$\begin{cases} \frac{1}{h_{n_i}} D_{x_1} u_{n_i}^a \rightharpoonup \xi_1^a, \quad \frac{1}{h_{n_i}} D_{x_2} u_{n_i}^a \rightharpoonup \xi_2^a, \quad D_{x_3} u_{n_i}^a \rightharpoonup 0 \quad \text{weakly in } L^2(\mathbb{R}_a^3), \\ D_{x_1} u_{n_i}^{b,l} \rightarrow 0, \quad \frac{1}{h_{n_i}} D_{x_2} u_{n_i}^{b,l} \rightharpoonup \xi_2^{b,l}, \quad \frac{1}{h_{n_i}} D_{x_3} u_{n_i}^{b,l} \rightharpoonup \xi_3^{b,l} \quad \text{weakly in } L^2(\mathbb{R}_{b,l}^3), \\ D u_{n_i}^{b,r} \rightarrow 0 \quad \text{strongly in } (L^2(\mathbb{R}_{b,r}^3))^3, \end{cases} \quad (4.14)$$

as  $n$  and  $i$  diverge, where  $\hat{\mu}$  is a solution of the following problem:

$$E(\hat{\mu}) = \min \{E(\mu) : \mu \in \mathcal{M}\}, \quad (4.15)$$

and

$$(\xi_1^a, \xi_2^a)(x_1, x_2, x_3) = \begin{cases} (0, 0), \text{ a.e. in } \mathbb{R}^2 \times ]1, +\infty[, \\ \mu_1^a(x_3) Dp(x_1, x_2) + \mu_3^a(x_3) Dq(x_1, x_2), \text{ a.e. in } \mathbb{R}^2 \times ]0, 1[, \end{cases} \quad (4.16)$$

$$(\xi_2^{b,l}, \xi_3^{b,l})(x_1, x_2, x_3) = \begin{cases} (0, 0), \text{ a.e. in } ]1, +\infty[ \times \mathbb{R} \times ]-\infty, 0[, \\ \mu_2^{b,l}(x_1) Dp(x_2, x_3) + \mu_3^{b,l}(x_1) Dq(x_2, x_3), \text{ a.e. in } ]0, 1[ \times \mathbb{R} \times ]-\infty, 0[, \end{cases} \quad (4.17)$$

with  $p$  (resp.  $q$ ) the unique solution of (2.2) (resp. (2.3)). Moreover, the convergence of the energies holds true, i.e.

$$\lim_n E_n(\underline{m}_n) = E(\hat{\mu}). \quad (4.18)$$



### 4.3 A convergence result for the magnetostatic energy

**Proposition 4.2.** *Let  $\{\underline{m}_n = (\underline{m}_n^a, \underline{m}_n^{b,l}, \underline{m}_n^{b,r})\}_{n \in \mathbb{N}} \subset L^2(\Omega^a, S^2) \times L^2(\Omega^{b,l}, S^2) \times L^2(\Omega^{b,r}, S^2)$ , and let  $\mu^a = (\mu_1^a, \mu_2^a, \mu_3^a) \in L^2(\Omega^a, S^2)$  be independent of  $(x_1, x_2)$  and  $\mu^{b,l} = (\mu_1^{b,l}, \mu_2^{b,l}, \mu_3^{b,l}) \in L^2(\Omega^{b,l}, S^2)$  be independent of  $(x_2, x_3)$  such that*

$$\begin{cases} \underline{m}_n^a \rightarrow \mu^a \text{ strongly in } L^2(\Omega^a, \mathbb{R}^3), \\ \underline{m}_n^{b,l} \rightarrow \mu^{b,l} \text{ strongly in } L^2(\Omega^{b,l}, \mathbb{R}^3), \end{cases} \quad (4.19)$$

as  $n$  diverges. Moreover, for every  $n \in \mathbb{N}$  let  $u_n = (u_n^a, u_n^{b,l}, u_n^{b,r})$  be the unique solution of (4.3) corresponding to  $\underline{m}_n$ . Then, it results that

$$\begin{cases} \frac{1}{h_n} D_{x_1} u_n^a \rightharpoonup \xi_1^a, & \frac{1}{h_n} D_{x_2} u_n^a \rightharpoonup \xi_2^a, & D_{x_3} u_n^a \rightharpoonup 0 & \text{weakly in } L^2(\mathbb{R}_a^3), \\ D_{x_1} u_n^{b,l} \rightharpoonup 0, & \frac{1}{h_n} D_{x_2} u_n^{b,l} \rightharpoonup \xi_2^{b,l}, & \frac{1}{h_n} D_{x_3} u_n^{b,l} \rightharpoonup \xi_3^{b,l} & \text{weakly in } L^2(\mathbb{R}_{b,l}^3), \\ D u_n^{b,r} \rightarrow 0 & \text{strongly in } (L^2(\mathbb{R}_{b,r}^3))^3, \end{cases} \quad (4.20)$$

as  $n$  diverges, where

$$\begin{aligned} (\xi_1^a, \xi_2^a)(x_1, x_2, x_3) &= \begin{cases} (0, 0), & \text{a.e. in } \mathbb{R}^2 \times ]1, +\infty[, \\ \mu_1^a(x_3) D p(x_1, x_2) + \mu_3^a(x_3) D q(x_1, x_2), & \text{a.e. in } \mathbb{R}^2 \times ]0, 1[, \end{cases} \\ (\xi_2^{b,l}, \xi_3^{b,l})(x_1, x_2, x_3) &= \begin{cases} (0, 0), & \text{a.e. in } ]1, +\infty[ \times \mathbb{R} \times ]-\infty, 0[, \\ \mu_2^{b,l}(x_1) D p(x_2, x_3) + \mu_3^{b,l}(x_1) D q(x_2, x_3), & \text{a.e. in } ]0, 1[ \times \mathbb{R} \times ]-\infty, 0[, \end{cases} \end{aligned} \quad (4.21)$$

with  $p$  (resp.  $q$ ) the unique solution of (2.2) (resp. (2.3)). Furthermore, one has that

$$\begin{aligned} & \lim_n \left[ \int_{\Omega^a} \left( \frac{1}{h_n} D_{x_1} u_n^a, \frac{1}{h_n} D_{x_2} u_n^a, D_{x_3} u_n^a \right) \underline{m}_n^a dx + \right. \\ & \int_{\Omega^{b,l}} \left( D_{x_1} u_n^{b,l}, \frac{1}{h_n} D_{x_2} u_n^{b,l}, \frac{1}{h_n} D_{x_3} u_n^{b,l} \right) \underline{m}_n^{b,l} dx + \\ & \left. \int_{\Omega^{b,r}} (D_{x_1} u_n^{b,r}, D_{x_2} u_n^{b,r}, D_{x_3} u_n^{b,r}) \underline{m}_n^{b,r} dx \right] = \end{aligned} \quad (4.23)$$

$$\alpha(] - 1, 0[^2) \int_0^1 |\mu_1^a|^2 dx_3 + \beta(] - 1, 0[^2) \int_0^1 |\mu_2^a|^2 dx_3 + \gamma(] - 1, 0[^2) \int_0^1 \mu_1^a \mu_2^a dx_3 +$$

$$\alpha(] - 1, 0[^2) \int_0^1 |\mu_2^{b,l}|^2 dx_1 + \beta(] - 1, 0[^2) \int_0^1 |\mu_3^{b,l}|^2 dx_1 + \gamma(] - 1, 0[^2) \int_0^1 \mu_2^{b,l} \mu_3^{b,l} dx_1,$$

where  $\alpha(] - 1, 0[^2)$ ,  $\beta(] - 1, 0[^2)$  and  $\gamma(] - 1, 0[^2)$  are defined by (2.5) with  $S = ] - 1, 0[^2$ .

*Proof.* By choosing  $u = u_n$  as test function in (4.3) and taking into account that  $\{(\underline{m}_n^a, \underline{m}_n^{b,l}, \underline{m}_n^{b,r})\}_{n \in \mathbb{N}} \subset L^2(\Omega^a, S^2) \times L^2(\Omega^{b,l}, S^2) \times L^2(\Omega^{b,r}, S^2)$ , there exists  $c \in ]0, +\infty[$  such that

$$\left\{ \begin{array}{l} \left\| \left( \frac{1}{h_n} D_{x_1} u_n^a, \frac{1}{h_n} D_{x_2} u_n^a, D_{x_3} u_n^a \right) \right\|_{(L^2(\mathbb{R}_a^3))^9} \leq c, \\ \left\| \left( D_{x_1} u_n^{b,l}, \frac{1}{h_n} D_{x_2} u_n^{b,l}, \frac{1}{h_n} D_{x_3} u_n^{b,l} \right) \right\|_{(L^2(\mathbb{R}_{b,l}^3))^9} \leq c, \\ \frac{1}{\sqrt{h_n}} \left\| (D_{x_1} u_n^{b,r}, D_{x_2} u_n^{b,r}, D_{x_3} u_n^{b,r}) \right\|_{(L^2(\mathbb{R}_{b,r}^3))^9} \leq c, \end{array} \right. \quad (4.24)$$

for every  $n \in \mathbb{N}$ .

The last estimate in (4.24) gives the last limit in (4.20).

By arguing as in the first part of the proof of proposition 5.1 in [17], from the first two estimates in (4.24) one derives the third and the fourth limit in (4.20).

By arguing as in the first two steps of the proof of proposition 3.2, from the first limit in (4.19) and the first estimate in (4.24) one obtains the first two limits in (4.20) with  $(\xi_1^a, \xi_2^a)$  defined in (4.21). Finally, using the first limit in (4.19), the first three limits in (4.20) and also the last one, taking into account that  $\{(\underline{m}_n^{b,r})\}_{n \in \mathbb{N}} \subset L^2(\Omega^{b,r}, S^2)$ , and using equation (3.34) with test function  $\mu_1^a p + \mu_2^a q$ , one obtains that

$$\left\{ \begin{array}{l} \lim_n \int_{\Omega^a} \left( \frac{1}{h_n} D_{x_1} u_n^a, \frac{1}{h_n} D_{x_2} u_n^a, D_{x_3} u_n^a \right) \underline{m}_n^a dx = \\ \alpha(] - 1, 0[^2) \int_0^1 |\mu_1^a|^2 dx_3 + \beta(] - 1, 0[^2) \int_0^1 |\mu_2^a|^2 dx_3 + \gamma(] - 1, 0[^2) \int_0^1 \mu_1^a \mu_2^a dx_3, \\ \lim_n \int_{\Omega^{b,r}} (D_{x_1} u_n^{b,r}, D_{x_2} u_n^{b,r}, D_{x_3} u_n^{b,r}) \underline{m}_n^{b,r} dx = 0. \end{array} \right. \quad (4.25)$$

To prove the fifth and the sixth limit in (4.20), we introduce other rescalings. Specifically, by setting

$$\left\{ \begin{array}{l} \mathbb{R}_{a,r}^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0, x_1 < 0\}, \\ \mathbb{R}_l^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 > 0\}, \end{array} \right.$$

for every  $n \in \mathbb{N}$ , problem (3.2) will be reformulated on a fixed domain through the following rescaling:

$$\mathcal{T}_n : (x_1, x_2, x_3) \in \mathbb{R}^3 \rightarrow \mathcal{T}_n(x_1, x_2, x_3) = \left\{ \begin{array}{l} (h_n x_1, h_n x_2, x_3), \text{ if } (x_1, x_2, x_3) \in \mathbb{R}_{a,r}^3, \\ (x_1, h_n x_2, h_n x_3), \text{ if } (x_1, x_2, x_3) \in \mathbb{R}_l^3, \\ (h_n x_1, h_n x_2, h_n x_3), \text{ if } (x_1, x_2, x_3) \in \mathbb{R}_{b,r}^3, \end{array} \right. \quad (4.26)$$

(note that  $T_n|_{\mathbb{R}_{b,r}^3} = \mathcal{T}_n|_{\mathbb{R}_{b,r}^3}$ , and  $T_n(\Omega^a) = \mathcal{T}_n(\Omega^a) = \Omega_n^a$ ,  $T_n(\Omega^b) = \mathcal{T}_n(\Omega^b) = \Omega_n^{b,l}$ ). Namely, setting

$$B_n^{a,r} = \left] -\frac{2}{h_n}, 0 \right[ \times ]0, 2[, \quad B_n^l = ]0, 2[ \times \left] -\frac{2}{h_n}, \frac{2}{h_n} \right[ , \quad B_n^{b,r} = \left] -\frac{2}{h_n}, 0 \right[ , \quad n \in \mathbb{N},$$

space  $\mathcal{U}$  defined in (3.1) is rescaled in the following

$$\begin{aligned} \mathcal{V}_n = \{ & v = (v^{a,r}, v^l, v^{b,r}) \in L^1_{loc}(\overline{\mathbb{R}_{a,r}^3}) \times L^1_{loc}(\overline{\mathbb{R}_l^3}) \times L^1_{loc}(\overline{\mathbb{R}_{b,r}^3}) : \\ & (v^{a,r}|_{B_n^{a,r}}, v^l|_{B_n^l}, v^{b,r}|_{B_n^{b,r}}) \in L^2(B_n^{a,r}) \times L^2(B_n^l) \times L^2(B_n^{b,r}), \\ & (Dv^{a,r}, Dv^l, Dv^{b,r}) \in (L^2(\mathbb{R}_{a,r}^3))^3 \times (L^2(\mathbb{R}_l^3))^3 \times (L^2(\mathbb{R}_{b,r}^3))^3, \\ & \int_{B_n^{a,r}} v^{a,r} dx + \int_{B_n^l} v^l dx + h_n \int_{B_n^{b,r}} v^{b,r} dx = 0, \\ & v^l(0, x_2, x_3) = v^{a,r}(0, x_2, h_n x_3), \text{ for } (x_2, x_3) \text{ a.e. in } \mathbb{R} \times ]0, +\infty[, \\ & v^l(0, x_2, x_3) = v^{b,r}(0, x_2, x_3), \text{ for } (x_2, x_3) \text{ a.e. in } \mathbb{R} \times ]-\infty, 0[, \\ & v^{a,r}(x_1, x_2, 0) = v^{b,r}(x_1, x_2, 0), \text{ for } (x_1, x_2) \text{ a.e. in } ]-\infty, 0[ \times \mathbb{R} \}. \end{aligned} \tag{4.27}$$

Then, for every  $\underline{m} = (\underline{m}^a, \underline{m}^{b,l}, \underline{m}^{b,r}) \in L^2(\Omega^a, \mathbb{R}^3) \times L^2(\Omega^{b,l}, \mathbb{R}^3) \times L^2(\Omega^{b,r}, \mathbb{R}^3)$ , the following equation:

$$\left\{ \begin{aligned} & v_{\underline{m},n} = (v_{\underline{m},n}^{a,r}, v_{\underline{m},n}^l, v_{\underline{m},n}^{b,r}) \in \mathcal{V}_n, \\ & \int_{\mathbb{R}_{a,r}^3} \left( \frac{1}{h_n} D_{x_1} v_{\underline{m},n}^{a,r}, \frac{1}{h_n} D_{x_2} v_{\underline{m},n}^{a,r}, D_{x_3} v_{\underline{m},n}^{a,r} \right) \left( \frac{1}{h_n} D_{x_1} v^{a,r}, \frac{1}{h_n} D_{x_2} v^{a,r}, D_{x_3} v^{a,r} \right) dx + \\ & \int_{\mathbb{R}_l^3} \left( D_{x_1} v_{\underline{m},n}^l, \frac{1}{h_n} D_{x_2} v_{\underline{m},n}^l, \frac{1}{h_n} D_{x_3} v_{\underline{m},n}^l \right) \left( D_{x_1} v^l, \frac{1}{h_n} D_{x_2} v^l, \frac{1}{h_n} D_{x_3} v^l \right) dx + \\ & \frac{1}{h_n} \int_{\mathbb{R}_{b,r}^3} (D_{x_1} v_{\underline{m},n}^{b,r}, D_{x_2} v_{\underline{m},n}^{b,r}, D_{x_3} v_{\underline{m},n}^{b,r}) (D_{x_1} v^{b,r}, D_{x_2} v^{b,r}, D_{x_3} v^{b,r}) dx = \\ & \int_{\Omega^a} \left( \frac{1}{h_n} D_{x_1} v^{a,r}, \frac{1}{h_n} D_{x_2} v^{a,r}, D_{x_3} v^{a,r} \right) \underline{m}^a dx + \\ & \int_{\Omega^{b,l}} \left( D_{x_1} v^l, \frac{1}{h_n} D_{x_2} v^l, \frac{1}{h_n} D_{x_3} v^l \right) \underline{m}^{b,l} dx + \\ & \int_{\Omega^{b,r}} (D_{x_1} v^{b,r}, D_{x_2} v^{b,r}, D_{x_3} v^{b,r}) \underline{m}^{b,r} dx, \quad \forall v = (v^{a,r}, v^l, v^{b,r}) \in \mathcal{V}_n, \end{aligned} \right. \tag{4.28}$$

which rescales equation (3.2) by rescaling (4.26), admits a unique solution.

For every  $n \in \mathbb{N}$ , let  $v_n = (v_n^{a,r}, v_n^l, v_n^{b,r})$  be the unique solution of (4.28) corresponding to  $\underline{m}_n$ . Arguing as in the first part of this proof, for a symmetric argument, one can easily prove that

$$D_{x_1} v_n^l \rightharpoonup 0, \quad \frac{1}{h_n} D_{x_2} v_n^l \rightharpoonup \xi_2^l, \quad \frac{1}{h_n} D_{x_3} v_n^l \rightharpoonup \xi_3^l \text{ weakly in } L^2(\mathbb{R}_l^3), \quad (4.29)$$

as  $n$  diverges, where

$$(\xi_2^l, \xi_3^l)(x_1, x_2, x_3) = \begin{cases} (0, 0), & \text{a.e. in } ]1, +\infty[ \times \mathbb{R}^2, \\ \mu_2^b(x_1) Dp(x_2, x_3) + \mu_3^b(x_1) Dq(x_2, x_3), & \text{a.e. in } ]0, 1[ \times \mathbb{R}^2, \end{cases}$$

with  $p$  (resp.  $q$ ) the unique solution of (2.2) (resp. (2.3)). Furthermore, one has that

$$\begin{aligned} \lim_n \int_{\Omega^b} \left( D_{x_1} v_n^l, \frac{1}{h_n} D_{x_2} v_n^l, \frac{1}{h_n} D_{x_3} v_n^l \right) \underline{m}_n^b dx = \\ \alpha(] - 1, 0[^2) \int_0^1 |\mu_2^b|^2 dx_1 + \beta(] - 1, 0[^2) \int_0^1 |\mu_3^b|^2 dx_1 + \gamma(] - 1, 0[^2) \int_0^1 \mu_2^b \mu_3^b dx_1, \end{aligned} \quad (4.30)$$

where  $\alpha(] - 1, 0[^2)$ ,  $\beta(] - 1, 0[^2)$  and  $\gamma(] - 1, 0[^2)$  are defined by (2.5) with  $S = ] - 1, 0[^2$ .

Now, to conclude it is enough to note that

$$\begin{aligned} \mathcal{T}_n^{-1}(T_n(x)) = x, \quad \forall x \in \mathbb{R}^3 \setminus \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 \geq 0, x_3 \geq 0\}, \quad \forall n \in \mathbb{N}, \\ v_n(\mathcal{T}_n^{-1}(T_n(x))) = u_n(x), \quad \forall x \in \mathbb{R}^3, \quad \forall n \in \mathbb{N}. \end{aligned}$$

Consequently, it results that

$$v_n(x) = u_n(x), \quad \forall x \in \mathbb{R}^3 \setminus \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 \geq 0, x_3 \geq 0\}, \quad \forall n \in \mathbb{N}. \quad (4.31)$$

Then, combining (4.29) and (4.30) with (4.31), one obtains the fifth and the sixth limit in (4.20) and

$$\begin{aligned} \lim_n \int_{\Omega^b} \left( D_{x_1} u_n^l, \frac{1}{h_n} D_{x_2} u_n^l, \frac{1}{h_n} D_{x_3} u_n^l \right) \underline{m}_n^b dx = \\ \alpha(] - 1, 0[^2) \int_0^1 |\mu_2^b|^2 dx_1 + \beta(] - 1, 0[^2) \int_0^1 |\mu_3^b|^2 dx_1 + \gamma(] - 1, 0[^2) \int_0^1 \mu_2^b \mu_3^b dx_1. \end{aligned} \quad (4.32)$$

Finally, combining (4.25) with (4.32), also limit (4.23) holds true.  $\square$

#### 4.4 Proof of theorem 4.1

*Proof.* By choosing  $\underline{m} = ((0, 1, 0), (0, 1, 0), (0, 1, 0))$  as test function in (4.7), taking into account (4.8) and that  $|\underline{m}_n| = 1$  a.e. in  $\Omega^a \cup \Omega^{b,l} \cup \Omega^{b,r}$ , using proposition 4.2 and arguing

as in the proof of theorem 3.1, it is easy to prove the existence of  $c \in ]0, +\infty[$  such that

$$\left\{ \begin{array}{l} \|D_{x_1} \underline{m}_n^a\|_{(L^2(\Omega^a))^3} \leq ch_n, \quad \|D_{x_2} \underline{m}_n^a\|_{(L^2(\Omega^a))^3} \leq ch_n, \quad \|D_{x_3} \underline{m}_n^a\|_{(L^2(\Omega^a))^3} \leq c, \\ \|D_{x_1} \underline{m}_n^{b,l}\|_{(L^2(\Omega^{b,l}))^3} \leq c, \quad \|D_{x_2} \underline{m}_n^{b,l}\|_{(L^2(\Omega^{b,l}))^3} \leq h_n c, \quad \|D_{x_3} \underline{m}_n^{b,l}\|_{(L^2(\Omega^{b,l}))^3} \leq ch_n, \\ \|D \underline{m}_n^{b,r}\|_{(L^2(\Omega^{b,l}))^9} \leq c\sqrt{h_n}, \end{array} \right.$$

for every  $n \in \mathbb{N}$ . Then, taking into account again that  $|\underline{m}_n| = 1$  a.e. in  $\Omega^a \cup \Omega^{b,l} \cup \Omega^{b,r}$ , there exist an increasing sequence of positive integer numbers  $\{n_i\}_{i \in \mathbb{N}}$ ,  $\widehat{\mu}^a \in H^1(\Omega^a, S^2)$  independent of  $(x_1, x_2)$ ,  $\widehat{\mu}^{b,l} \in H^1(\Omega^{b,l}, S^2)$  independent of  $(x_2, x_3)$  and  $c \in S^2$ ,  $\zeta^a \in (L^2(\Omega^a, \mathbb{R}^3))^2$ ,  $\zeta^{b,l} \in (L^2(\Omega^{b,l}, \mathbb{R}^3))^2$ ,  $\zeta^{b,r} \in (L^2(\Omega^{b,r}, \mathbb{R}^3))^3$  such that

$$\left\{ \begin{array}{l} \underline{m}_{n_i}^a \rightharpoonup \widehat{\mu}^a \text{ weakly in } H^1(\Omega^a, \mathbb{R}^3), \\ \underline{m}_{n_i}^{b,l} \rightharpoonup \widehat{\mu}^{b,l} \text{ weakly in } H^1(\Omega^{b,l}, \mathbb{R}^3), \\ \underline{m}_{n_i}^{b,r} \rightharpoonup c \text{ weakly in } H^1(\Omega^{b,r}, \mathbb{R}^3), \end{array} \right. \quad (4.33)$$

$$\left\{ \begin{array}{l} \left( \frac{1}{h_{n_i}} D_{x_1} \underline{m}_{n_i}^a, \frac{1}{h_{n_i}} D_{x_2} \underline{m}_{n_i}^a \right) \rightharpoonup \zeta^a \text{ weakly in } (L^2(\Omega^a, \mathbb{R}^3))^2, \\ \left( \frac{1}{h_{n_i}} D_{x_2} \underline{m}_{n_i}^{b,l}, \frac{1}{h_{n_i}} D_{x_3} \underline{m}_{n_i}^{b,l} \right) \rightharpoonup \zeta^{b,l} \text{ weakly in } (L^2(\Omega^{b,l}, \mathbb{R}^3))^2, \\ \frac{1}{\sqrt{h_{n_i}}} D \underline{m}_{n_i}^{b,r} \rightharpoonup \zeta^{b,r} \text{ weakly in } (L^2(\Omega^{b,r}, \mathbb{R}^3))^3, \end{array} \right. \quad (4.34)$$

as  $i$  diverges. Consequently, since one has that

$$\left\{ \begin{array}{l} \underline{m}_n^a(x_1, x_2, 0) = \underline{m}_n^{b,r}(x_1, x_2, 0), \text{ for } (x_1, x_2) \text{ a.e. in } ]-1, 0]^2, \\ \underline{m}_n^{b,l}(0, x_2, x_3) = \underline{m}_n^{b,r}(0, x_2, x_3), \text{ for } (x_2, x_3) \text{ a.e. in } ]-1, 0]^2, \end{array} \right.$$

for every  $n \in \mathbb{N}$ , it follows that  $\widehat{\mu}^a(0) = c = \widehat{\mu}^{b,l}(0)$ , that is  $\widehat{\mu} = (\widehat{\mu}^a, \widehat{\mu}^{b,l}) \in \mathcal{M}$ . Moreover, by virtue of proposition 4.2, limits in (4.14) hold true and it results that

$$\begin{aligned} & \lim_i \left[ \int_{\Omega^a} \left( \frac{1}{h_{n_i}} D_{x_1} u_{n_i}^a, \frac{1}{h_{n_i}} D_{x_2} u_{n_i}^a, D_{x_3} u_{n_i}^a \right) \underline{m}_{n_i}^a dx + \right. \\ & \int_{\Omega^{b,l}} \left( D_{x_1} u_{n_i}^{b,l}, \frac{1}{h_{n_i}} D_{x_2} u_{n_i}^{b,l}, \frac{1}{h_{n_i}} D_{x_3} u_{n_i}^{b,l} \right) \underline{m}_{n_i}^{b,l} dx + \\ & \left. \int_{\Omega^{b,r}} \left( D_{x_1} u_{n_i}^{b,r}, D_{x_2} u_{n_i}^{b,r}, D_{x_3} u_{n_i}^{b,r} \right) \underline{m}_{n_i}^{b,r} dx \right] = \end{aligned} \quad (4.35)$$

$$\begin{aligned} & \alpha(\cdot - 1, 0[{}^2] \int_0^1 |\mu_1^a|^2 dx_3 + \beta(\cdot - 1, 0[{}^2] \int_0^1 |\mu_2^a|^2 dx_3 + \gamma(\cdot - 1, 0[{}^2] \int_0^1 \mu_1^a \mu_2^a dx_3 + \\ & \alpha(\cdot - 1, 0[{}^2] \int_0^1 |\mu_2^{b,l}|^2 dx_1 + \beta(\cdot - 1, 0[{}^2] \int_0^1 |\mu_3^{b,l}|^2 dx_1 + \gamma(\cdot - 1, 0[{}^2] \int_0^1 \mu_2^{b,l} \mu_3^{b,l} dx_1, \end{aligned}$$

where  $\alpha(\cdot - 1, 0[{}^2]$ ,  $\beta(\cdot - 1, 0[{}^2]$  and  $\gamma(\cdot - 1, 0[{}^2]$  are defined by (2.5) with  $S = \cdot - 1, 0[{}^2$ .

Now, the goal is to identify  $\widehat{\mu}$ ,  $\zeta^a$ ,  $\zeta^{b,l}$ ,  $\zeta^{b,r}$ , to obtain strong convergences in (4.33) and in (4.34), and to prove limit in (4.18). To this aim, for  $(\widehat{\mu}^a, \widehat{\mu}^{b,l}) \in \mathcal{M}$ , let us set

$$v = \begin{cases} \widehat{\mu}^a, & \text{in } \Omega^a, \\ \widehat{\mu}^{b,l}, & \text{in } \Omega^{b,l}, \\ \widehat{\mu}^a(0) = \widehat{\mu}^{b,l}(0), & \text{in } \Omega^{b,r}. \end{cases}$$

Obviously,  $v \in \mathcal{M}_n$ , for every  $n \in N$ . Then, by virtue of l.s.c. arguments, (4.8), (4.33), (4.34) and (4.35) and proposition 4.2, it results that

$$\lambda \int_{\Omega^a} |\zeta^a|^2 dx + \lambda \int_{\Omega^{b,l}} |\zeta^{b,l}|^2 dx + \lambda \int_{\Omega^{b,r}} |\zeta^{b,r}|^2 dx + E(\widehat{\mu}^a, \widehat{\mu}^b) \leq \liminf_i E_{n_i}(\underline{m}_{n_i}) \leq \quad (4.36)$$

$$\limsup_i E_{n_i}(\underline{m}_{n_i}) \leq \lim_i E_{n_i}(v) = E(\widehat{\mu}^a, \widehat{\mu}^b), \quad \forall (\widehat{\mu}^a, \widehat{\mu}^{b,l}) \in \mathcal{M}.$$

Consequently,  $\zeta^a = 0$ ,  $\zeta^{b,l} = 0$ ,  $\zeta^{b,r} = 0$ ,  $(\widehat{\mu}^a, \widehat{\mu}^b)$  solves (4.15) and limit (4.18) holds true. Finally, combining (4.18) with (4.8), (4.33), (4.34) and (4.35) one obtains that limits in (4.33) and in (4.34) are strong.  $\square$

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