HAL
open science

RATE OF CONVERGENCE TO ALPHA STABLE LAW USING ZOLOTAREV DISTANCE : TECHNICAL REPORT

Solym Manou-Abi

▶ **To cite this version:**

Solym Manou-Abi. RATE OF CONVERGENCE TO ALPHA STABLE LAW USING ZOLOTAREV DISTANCE : TECHNICAL REPORT. [Research Report] CUFR de Mayotte et IMAG Montpellier, France. 2017. hal-01672804

HAL Id: hal-01672804

https://hal.science/hal-01672804v1

Submitted on 27 Dec 2017

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

RATE OF CONVERGENCE TO ALPHA STABLE LAW USING ZOLOTAREV DISTANCE : TECHNICAL REPORT

Solym Mawaki MANOU-ABI [1,2]

[1] CUFR de Mayotte
Département Sciences Technologies, Dembeni, France
solym.manou-abi@univ-mayotte.fr

[2] IMAG, Univ Montpellier
CNRS, Montpellier, France
solym-mawaki.manou-abi@umontpellier.fr

Abstract. This paper considers the question of the rate of convergence to α-stable laws, using arguments based on the Zolotarev distance to prove bounds. We provide a rate of convergence to α-stable random variable where $1 < \alpha < 2$, in the generalized CLT, that is, for the partial sums of independent identically distributed random variables which are not assumed to be square integrable. This work is a technical report based on the Zolotarev paper in [1].

AMS Subject Classification : 60F05; 60G52; 60E07.
Key words and phrases: CLT, Probability metric, Rate of convergence, Stable law.

1. Introduction

When studying limit theorems in probability theory it is important to try to assess the rates at which these converges. It is known that taking into account probability metrics in the space of random variables, allow to approximate partial sums for sequences of random variables. It is also known that stable laws are the only limits in the study of the limit law for such partial sums properly normalized. However, the lack of explicit formulas for their density functions except some stable distributions (Gaussian, Cauchy, Lévy) significantly limits their use in practice. A random variable ϑ has a stable distribution if there exists some coefficients $C_n > 0$ and $D_n \in \mathbb{R}$ such that:

$$\vartheta_1 + \ldots + \vartheta_n \stackrel{d}{=} C_n \vartheta + D_n \quad \forall n \geq 1,$$

where $\vartheta_1, \ldots, \vartheta_n$ are i.i.d copies of ϑ. In particular such random variables are infinitely divisible. It is shown that $C_n = n^{1/\alpha}$ with $\alpha \in (0, 2]$ (see for instance [2], Section VI.1). The coefficient α is the index of stability. We shall say that ϑ is an α-stable (real-valued) random variable. The case $\alpha = 2$ and $D_n = 0$ corresponds to a Gaussian random variable. The Lévy-Khintchine representation in the case $\alpha \in (0, 2)$ reads as follows: $\mathbb{E}[e^{iu\vartheta}] = e^{\psi(u)}$ where ψ is the characteristic exponent

given by

$$\psi(u) = ibu + \int_{-\infty}^{+\infty} (e^{iux} - 1 - iux1_{|x|\leq 1}) \frac{1}{|x|^{1+\alpha}} (c_1 1_{x<0} + c_2 1_{x>0}) dx,$$

with $b \in \mathbb{R}$, $c_1, c_2 \geq 0$ and $c_1 + c_2 > 0$. We say in this case that ϑ is an α-stable random variable with characteristics (b, c_1, c_2). For strictly α-stable random variable we have $b = 0$ and the case $b = 0$, $c_1 = c_2$ corresponds to a symmetric α-stable random variable.

In the context of the Central Limit Theorem (CLT) it is well known that, for some given sequence of iid random variables $(V_n)_{n\geq 1}$, the partial sum properly normalized convergence to an α-stable random variable, $\alpha \in (0, 2]$, if the random variable V_1 is in a domain of attraction of the α-stable random variable. We say that the random variable V_1 is in the domain of attraction of an α-stable with tails parameters c_1 and c_2 if

$$\mathbb{P}[V_1 > x] = \frac{c_1 + h(x)}{x^\alpha}, \quad \text{and} \quad \mathbb{P}[V_1 \leq -x] = \frac{c_2 + h(x)}{x^\alpha}, \quad x > 0, \tag{1}$$

where h is a function such that $\lim_{x\to\infty} h(x) = 0$. The case of domain of attraction of a symmetric α-stable random variable corresponds to $c_1 = c_2$. The reader can refer to [3] for the domain of attraction conditions.

We shall also say that the random variable X is in a strong domain of attraction of the α-stable if in (1), there exists $\gamma_\alpha > 0$ such that $h(x) = \mathcal{O}(\frac{1}{|x|^{\gamma_\alpha}})$ at $+\infty$ meaning that $\exists K, x_0 \in \mathbb{R}_+$ and $|h(x)| \leq K|x|^{-\gamma_\alpha}$ for all $x \geq x_0$:

$$\mathbb{P}[V_1 > x] - \frac{c_1}{x^\alpha} = \mathcal{O}(\frac{1}{x^{\alpha+\gamma_\alpha}}), \quad \mathbb{P}[V_1 \leq -x] - \frac{c_2}{|x|^\alpha} = \mathcal{O}(\frac{1}{|x|^{\alpha+\gamma_\alpha}}), \quad x > 0\,.$$

In the sequel we call γ_α the attraction index.

In this paper, we obtain a rate of convergence to an α-stable random variable, $\alpha \in (1, 2)$, for partial sums of iid random variables $(V_n)_{n\geq 1}$ (the generalized CLT) between characteristic functions.

To this end, we show in Theorem 3.1 a bound of order $n^{\frac{\alpha-r}{\alpha}}$ in the Zolotarev ideal probability metric ζ_r for $r \in]\alpha, 2]$, provided the random variable V_1 is in a strong domain of attraction of the α-stable random variable with an attraction index $\gamma_\alpha > r - \alpha$. We then applied the result to derive the rate of convergence between characteristic functions, see Theorem 3.2.

The layout of this short paper is as follows. We start in Section 2, by the notion of ideal probability metrics and introduce the Zolotarev ideal probability metric of order $r > 0$. We give in Section 3 a rate of convergence to an α-stable random variable, $\alpha \in (1, 2)$, for iid random variables between characteristic functions in Theorem 3.2.

2. Preliminaries

Denote by Σ the space of real random variables. In this section we introduce the notion of ideal probability distance and as an example we introduce the Zolotarev distance.

On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ consider two real random variables X and Y, denote by P_X and P_Y their probability laws respectively.

Definition 2.1. *A map $d(.,.)$ defined in the space $\Sigma \times \Sigma \to [0,\infty]$ is said to be a probability distance in Σ if for all random variables X, Y and Z the following statements hold:*

(1)
$$\mathbb{P}(X = Y) = 1 \Rightarrow d(X,Y) = 0,$$

(2)
$$d(X,Y) = d(Y,X),$$

(3)
$$d(X,Y) \leq d(X,Z) + d(Z,Y).$$

If the values of $d(X,Y)$ are determined by the marginal distributions P_X and P_Y then one says that the distance d is simple.
An example of a simple distance is:

- the total variation distance defined by:
$$d_{VT}(X,Y) = \sup_{||f||_\infty \leq 1} |\mathbb{E}[f(X)] - \mathbb{E}[f(Y)]|,$$
where $||f||_\infty = \sup_{x\in\mathbb{R}} |f(x)|$.

Now let us define an ideal probability distance.

Definition 2.2. *A simple distance d in Σ is called an ideal probability distance of order $r \geq 0$, if the following statements hold:*

(4)
$$d(X+Z, Y+Z) \leq d(X,Y), \tag{2}$$
for Z independent of X and Y. (Regularity)

(5)
$$d(cX, cY) = |c|^r d(X,Y) \tag{3}$$
for any $c \in \mathbb{R}^*$. (Homogeneity of order r)

We now give as example the Zolotarev distance.

Definition 2.3. *Let $r > 0$ with the representation $r = m + \beta$ where $\beta \in]0,1]$ and $m \in \mathbb{N}$. We define the following simple metric :*
$$\zeta_r(X,Y) = \sup_{f\in\Lambda_r} |\mathbb{E}[f(X)] - \mathbb{E}[f(Y)]|, \tag{4}$$
where Λ_r is the set of bounded functions $f : \mathbb{R} \to \mathbb{R}$ which are m-times continuously differentiable and such that
$$|f^{(m)}(x) - f^{(m)}(y)| \leq |x-y|^\beta, \quad \beta + m = r, \tag{5}$$
where $f^{(m)}$ is the derivate function of order m.

The metric ζ_r is called the Zolotarev probability distance and it is an ideal probability metric of order r. We refer the reader to Theorem 1.4.2 in [4] for the proof. Note that by a simple application of the Taylor formula with integral remainder, one can show that $\zeta_r(X,Y) < \infty$ provided
$$\mathbb{E}[|X|^r],\ \mathbb{E}[|Y|^r] < \infty \tag{6}$$
and
$$\mathbb{E}[X^k] = \mathbb{E}[Y^k] \quad 0 \leq k \leq m. \tag{7}$$

Note also that the condition in (7) is necessarily but the moment condition in (6) can be relaxed using domain of attraction conditions.
Some interesting cases are $r = 1$ and $r = 2$.

(a) The Zolotarev metric of order 1 is defined by:

$$\zeta_1(X, Y) = \sup_{f \in \Lambda_1} |\mathbb{E}[f(X)] - \mathbb{E}[f(Y)]|,$$

where Λ_1 is the set of bounded continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $|f(x) - f(y)| \leq |x - y|$.
By the famous Kantorovich-Rubinstein duality, it rewrites as :

$$\mathbb{W}(X, Y) = \inf_{(P_X, P_Y)} \mathbb{E}[|X - Y|],$$

where the infimum runs over all coupling of the marginal distributions P_X and P_Y.

(b) The Zolotarev metric of order 2 is defined by

$$\zeta_2(X, Y) = \sup_{f \in \Lambda_2} |\mathbb{E}[f(X)] - \mathbb{E}[f(Y)]|,$$

where Λ_2 is the set of bounded functions $f : \mathbb{R} \to \mathbb{R}$ which are 1-times continuously differentiable and such that $|f'(x) - f'(y)| \leq |x - y|$.

3. Main results

Lemma 3.1.

$$|x - y| \max(|x|^{r-1}, |y|^{r-1}) \leq 2 \left| x|x|^{r-1} - y|y|^{r-1} \right|, \quad \forall x, y \in \mathbb{R}, \quad r \geq 1.$$

Proof. If we apply the following triangle inequality

$$||a| - |b|| \leq |a - b|,$$

for $a = x|x|^{r-1}$ and $b = x|x|^{r-1}$, then we have:

$$||x|^r - |y|^r| \leq \left| x|x|^{r-1} - y|y|^{r-1} \right|,$$

so that

$$\left| x|x|^{r-1} - y|y|^{r-1} \right| + ||x|^r - |y|^r| \leq 2 \left| x|x|^{r-1} - y|y|^{r-1} \right|. \tag{8}$$

Now assume that $|x| \geq |y|$. We have

$$\begin{aligned}
|x - y||x|^{r-1} &= |x|x|^{r-1} - y|x|^{r-1}| \\
&= |x|x|^{r-1} - y|y|^{r-1} + y|y|^{r-1} - y|x|^{r-1}| \\
&\leq \left| x|x|^{r-1} - y|y|^{r-1} \right| + |y|(|x|^{r-1} - |y|^{r-1}) \\
&= \left| x|x|^{r-1} - y|y|^{r-1} \right| + |y||x|^{r-1} - |y|^r \\
&\leq \left| x|x|^{r-1} - y|y|^{r-1} \right| + |x|^r - |y|^r.
\end{aligned}$$

This leads to the following fact:

$$|x - y| \max(|x|^{r-1}, |y|^{r-1}) \leq \left| x|x|^{r-1} - y|y|^{r-1} \right| + ||x|^r - |y|^r|. \tag{9}$$

Bringing together the inequalities (8) and (9), we thus obtain the desired inequality. □

Theorem 3.1. *Given a sequence of integrable iid random variables $(V_n)_{n\geq 1}$, set*

$$\tilde{S}_n := n^{-1/\alpha}\sum_{k=1}^{n}(V_k - \mathbb{E}[V_k]).$$

Assume that V_1 is in the strong domain of attraction of a symmetric α-stable random variable ϑ with $\alpha \in (1,2)$ and an attraction index $\gamma_\alpha > r-\alpha$ where $r \in (\alpha, 2]$. Then there exists a constant $C > 0$ such that :

$$\zeta_r(\tilde{S}_n, \vartheta) \leq C\, n^{\frac{\alpha - r}{\alpha}}.$$

One can deduce the rate of convergence between characteristic functions.

Theorem 3.2. *Given a sequence of integrable iid random variables $(V_n)_{n\geq 1}$, set*

$$\tilde{S}_n := n^{-1/\alpha}\sum_{k=1}^{n}(V_k - \mathbb{E}[V_k]).$$

Assume that V_1 is in the strong domain of attraction of a symmetric α-stable random variable ϑ with $\alpha \in (1,2)$ and an attraction index $\gamma_\alpha > 2-\alpha$. Then there exists a constant $C(t) > 0$ such that :

$$\chi_t(\tilde{S}_n, \vartheta) = |\mathbb{E}e^{it\tilde{S}_n} - \mathbb{E}e^{it\vartheta}| \leq C(t)\, n^{\frac{\alpha-2}{\alpha}}.$$

Proof. Set

$$\chi_t(\tilde{S}_n, \vartheta) = \left|\mathbb{E}[e^{it\tilde{S}_n}] - \mathbb{E}[e^{it\vartheta}]\right|, \quad t \in \mathbb{R},$$

and observe that, using the definition of the ideal probability distance ζ_2, we have

$$\chi_t(\tilde{S}_n, \vartheta) \leq t^2 \zeta_2(\tilde{S}_n, \vartheta),$$

since the function $f_t(x) = e^{itx}$, $t \in \mathbb{R}$, $x \in \mathbb{R}$ is bounded and the derivative f_t' is a t^2-Lipschitz function.
We thus obtain the result by applying Theorem 3.1 with $r = 2$.

□

Now let us start the proof of Theorem 3.1

Proof of Theorem 3.1. *Without loss of generality, we assume that the sequence $(V_n)_{n\geq 1}$ is centered. Consider a sequence $(\vartheta_n)_{n\geq 1}$ of i.i.d copies of ϑ. We have the following identity:*

$$\vartheta \overset{\mathcal{L}}{=} \frac{\vartheta_1 + \ldots + \vartheta_n}{n^{1/\alpha}}, \quad \forall n \geq 1.$$

We choose the sequence $(\vartheta_n)_{n\geq 1}$ to be independent of the sequence $(V_n)_{n\geq 1}$. Since ζ_r is a simple distance, we have

$$\zeta_r(n^{-1/\alpha}\sum_{k=1}^{n} V_k, \vartheta) = \zeta_r\left(n^{-1/\alpha}\sum_{k=1}^{n} V_k, n^{-1/\alpha}\sum_{k=1}^{n}\vartheta_k\right).$$

For $n = 2$, since ζ_r is an ideal probability metric of order r we have:

$$\begin{aligned}
\zeta_r\left(2^{-1/\alpha}(V_1+V_2), 2^{-1/\alpha}(\vartheta_1+\vartheta_2)\right) &\leq \zeta_r\left(2^{-1/\alpha}(V_1+V_2), 2^{-1/\alpha}(\vartheta_2+V_1)\right) \\
&\quad + \zeta_r\left(2^{-1/\alpha}(\vartheta_2+V_1), 2^{-1/\alpha}(\vartheta_1+\vartheta_2)\right) \\
&\leq 2^{-\frac{r}{\alpha}}\left(\zeta_r(V_1,\vartheta_1) + \zeta_r(V_2,\vartheta_2)\right) \\
&= 2^{1-\frac{r}{\alpha}}\,\zeta_r(V_1,\vartheta_1) = 2^{1-\frac{r}{\alpha}}\,\zeta_r(V_1,\vartheta),
\end{aligned}$$

where we use in the first inequality the regularity property and in the second inequality the homogeneity of order r of the probability metric ζ_r; the last equality holds since ζ_r is a simple distance.
Thus, by induction on n, we have

$$\zeta_r\left(n^{-1/\alpha}\sum_{k=1}^{n}V_k, n^{-1/\alpha}\sum_{k=1}^{n}\vartheta_k\right) \leq n^{-\frac{r}{\alpha}}\sum_{k=1}^{n}\zeta_r(V_k,\vartheta_k),$$

so that

$$\begin{aligned}\zeta_r\left(n^{-1/\alpha}\sum_{k=1}^{n}V_k, n^{-1/\alpha}\sum_{k=1}^{n}\vartheta_k\right) &\leq n^{1-\frac{r}{\alpha}}\zeta_r(V_1,\vartheta_1)\\ &= n^{1-\frac{r}{\alpha}}\zeta_r(V_1,\vartheta)\\ &= C\,n^{1-\frac{r}{\alpha}},\end{aligned}$$

provided $C := \zeta_r(V_1,\vartheta)$ is finite.
Now the remainder of the proof is devoted to show that C is finite.

Consider two real random variables X, Y and define

$$\kappa_r(X,Y) = \sup_{f\in\Gamma}|\mathbb{E}[f(X)] - \mathbb{E}[f(Y)]|,$$

where Γ is the set of bounded functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$|f(x)-f(y)| \leq \left|x|x|^{r-1} - y|y|^{r-1}\right|.$$

By [5], *κ_r rewrites as*

$$\kappa_r(X,Y) = \inf_{(P_X,P_Y)}\left(\mathbb{E}\left[\left|X|X|^{r-1} - Y|Y|^{r-1}\right|\right]\right),$$

where the infimum runs over all coupling of the marginal distributions P_X and P_Y.

Let V_1 and ϑ be the optimal coupling for $\kappa_r(V_1,\vartheta)$ and $f:\mathbb{R}\to\mathbb{R}$, bounded and 1-time continuously differentiable such that:

$$|f'(x)-f'(y)| \leq |x-y|^{r-1}.$$

Letting $Z = V_1 - \vartheta$, we have

$$f(V_1) - f(\vartheta) = f(\vartheta+Z) - f(\vartheta).$$

By the mean value theorem there exists $\lambda \in]0,1]$ such that:

$$f(\vartheta+Z) - f(\vartheta) = f'(\vartheta+\lambda Z)Z.$$

Since $\mathbb{E}[V_1] = \mathbb{E}[\vartheta] = 0$ we have,

$$\mathbb{E}[f(\vartheta+Z) - f(\vartheta)] = \mathbb{E}[f'(\vartheta+\lambda Z)Z] - \mathbb{E}[f'(0)Z]$$

and

$$\begin{aligned}|\mathbb{E}[f(\vartheta+Z)] - \mathbb{E}[f(\vartheta)]| &\leq \mathbb{E}\left[|f'(\vartheta+\lambda Z)Z - f'(0)Z|\right]\\ &\leq \mathbb{E}\left[|\vartheta+\lambda Z|^{r-1}|Z|\right]\\ &= \mathbb{E}\left[|(1-\lambda)\vartheta + \lambda V_1|^{r-1}|V_1-\vartheta|\right].\end{aligned}$$

Since

$$(1-\lambda)|x| + \lambda|y| \leq \max(|x|,|y|), \quad \lambda\in]0,1],$$

we have,

$$|\mathbb{E}[f(\vartheta+Z)]-\mathbb{E}[f(\vartheta)]| \leq \mathbb{E}\left[|V_1-\vartheta|\max(|V_1|^{r-1},|\vartheta|^{r-1})\right].$$

Now taking into account Lemma 3.1 we obtain

$$|\mathbb{E}[f(\vartheta+Z)]-\mathbb{E}[f(\vartheta)]| \leq 2\,\mathbb{E}\left[\left|V_1|V_1|^{r-1}-\vartheta|\vartheta|^{r-1}\right|\right],$$

and thus

$$\begin{aligned}\zeta_r(V_1,\vartheta) &\leq 2\,\mathbb{E}\left[\left|V_1|V_1|^{r-1}-\vartheta|\vartheta|^{r-1}\right|\right]\\ &= 2\,\kappa_r(V_1,\vartheta).\end{aligned}$$

By [4]*, the following representation hold:*

$$\kappa_r(V_1,\vartheta) \;=\; r\int_{\mathbb{R}}|u|^{r-1}|F_{V_1}(u)-F_\vartheta(u)|du,$$

and thus

$$\begin{aligned}\kappa_r(V_1,\vartheta) &= r\int_0^{+\infty}|u|^{r-1}\left|\mathbb{P}[V_1>u]-\mathbb{P}[\vartheta>u]\right|du\\ &+ r\int_0^{+\infty}|u|^{r-1}\left|\mathbb{P}[V_1<-u]-\mathbb{P}[\vartheta<-u]\right|du.\end{aligned}$$

Now recall that for any α-stable random variable Θ with local characteristic (c_1,c_2), we have the following expansion, cf. [6]*, when $u\to+\infty$:*

$$\mathbb{P}[\Theta>u] \;=\; \frac{c_1}{u^\alpha}+\frac{c_2}{u^{2\alpha}}+\mathcal{O}(\frac{1}{u^{3\alpha}}),\quad \mathbb{P}[\Theta<-u] \;=\; \frac{c_2}{u^\alpha}+\frac{c_1}{u^{2\alpha}}+\mathcal{O}(\frac{1}{u^{3\alpha}}).$$

This means that every stable random variable is in its own strong domain of attraction :

$$|\mathbb{P}[V_1>u]-\mathbb{P}[\vartheta>u]| \;\leq\; \left|\mathbb{P}[V_1>u]-c\,u^{-\alpha}\right|+\mathcal{O}(u^{-2\alpha}),$$

and

$$|\mathbb{P}[V_1<-u]-\mathbb{P}[\vartheta<-u]| \;\leq\; \left|\mathbb{P}[V_1<-u]-c\,u^{-\alpha}\right|+\mathcal{O}(u^{-2\alpha}),$$

where c is the local characteristic of the symmetric α-stable random variable ϑ with $\alpha\in(1,2)$.

Therefore we get

$$\kappa_r(V_1,\vartheta)<\infty \quad\Longleftrightarrow\quad \int_1^\infty |u|^{r-1-\alpha-\gamma_\alpha}\,du<\infty \quad\Longleftrightarrow\quad \gamma_\alpha>r-\alpha,$$

which corresponds to our assumptions. Since

$$\zeta_r(V_1,\vartheta) \;\leq\; 2\,\kappa_r(V_1,\vartheta),$$

we then have :

$$\zeta_r(\tilde{S}_n,\vartheta)\leq C\,n^{\frac{\alpha-r}{\alpha}}.$$

References

[1] V.M. Zolotarev. *Approximation of distributions of sums of independent random variables with values in infinite-dimensional spaces. Theory Probab. Appl. 21 (4) 1976, 721-737.*

[2] W. Feller. *An introduction to probability theory and its applications. John Wiley and Sons, 2008.*

[3] B.V. Gnedenko and A.N. Kolmogorov. *Limit distributions for sums of independent random variables. Addison-Wesley Reading, Massachusetts, 1968*

[4] V.M. Zolotarev. *Probability metrics. Theory of Probability and its Applications. 28 (2), 1983, 278-302.*

[5] S. T. Rachev and L. Rüschendorf. *Approximation of sums by compound Poisson distributions with respect to stop-loss distances. Advances in applied probability, 350-374, 1990*

[6] J. Mijnheer. *On the rate of convergence to a stable limit law II. Lithuanian Mathematical Journal, 26(3), 255-259, 1986.*