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Cuts in matchings of 3-edge-connected cubic graphs

Kolja Knauer* Petru Valicov*

December 19, 2017

Abstract

We discuss relations between several known (some false, some open) conjectures on 3-edge-connected, cubic graphs and how they all fit into the same framework related to cuts in matchings. We then provide a construction of 3-edge-connected digraphs satisfying the property that for every even subgraph E , the graph obtained by contracting the edges of E is not strongly connected. This disproves a recent conjecture of Hochstättler [A flow theory for the dichromatic number. *European Journal of Combinatorics*, 66, 160–167, 2017]. Furthermore, we show that an open conjecture of Neumann-Lara holds for all planar graphs on at most 26 vertices.

1 Introduction

Let us introduce some conjectures on 3-edge-connected graphs in their order of appearance.

Conjecture 1 (Tait 1884). Every 3-edge-connected, cubic, planar graph contains a Hamiltonian cycle.

The first counterexample to Conjecture 1 was given by Tutte [13] and is a graph on 46 vertices. Several smaller counterexamples on 38 vertices were later found by Holton and McKay, who also proved that there is no counterexample with less than 38 vertices [8]. One can observe that all known counterexamples to Tait’s conjecture have odd cycles. Maybe this is essential:

Conjecture 2 (Barnette 1969 [1]). Every 3-edge-connected, cubic, planar, bipartite graph contains a Hamiltonian cycle.

In general Conjecture 2 remains open. It was shown to be true for graphs with at most 66 vertices [7]. There is an announcement (<https://www.fmf.uni-lj.si/~mohar/Problems/P4BarnetteConjecture.html>) claiming that the conjecture was verified for graphs with less than 86 vertices. A few years later a stronger conjecture was proposed:

Conjecture 3 (Tutte 1971 [14]). Every 3-edge-connected, cubic, bipartite graph contains a Hamiltonian cycle.

Conjecture 3 was disproved by Horton [6]. The smallest known counterexample has 50 vertices and was discovered independently by Georges [3] and Kelmans [9]. Moreover, in [9] it is claimed (but no reference given) that Lomonosov and Kelmans proved Conjecture 3 for graphs on at most 30 vertices. We verified this claim by computer.

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Let us now go back to the plane and introduce a seemingly unrelated conjecture in digraphs. The *digirth* of a digraph $D = (V, A)$ is the length of a shortest directed cycle. A digraph is called *oriented graph* if it is of digirth at least 3. A set of vertices $V' \subseteq V$ is acyclic in D if the digraph induced by V' contains no directed cycle. Neumann-Lara stated the following:

Conjecture 4 (Neumann-Lara 1985 [12]). Every planar oriented graph can be vertex-partitioned into two acyclic sets.

Conjecture 4 remains open in general, but was recently proved for graphs with digirth at least 4 [10]. Here we give the first computational evidence for it by showing that it is valid for all planar graphs on at most 26 vertices, see Table 1.

Let us introduce another seemingly unrelated problem. Given a (partially) directed graph $D = (V, A)$, for $E \subseteq A$, let D/E denote the graph obtained from D by contracting the edges of E . An *even* subgraph E of a digraph $D = (V, A)$ is a subset $E \subseteq A$ that is an edge-disjoint union of cycles of D (the cycles are not necessarily directed). Recently, Hochstättler proposed:

Conjecture 5 (Hochstättler 2017 [5]). In every 3-edge-connected digraph $D = (V, A)$ there exists an even subgraph $E \subseteq A$ such that D/E is strongly connected.

In Section 4 we construct a (cubic) counterexample to Conjecture 5 on 122 vertices. On the other hand, we know that Conjecture 5 holds for *cubic* graphs on at most 24 vertices.

Before constructing the counterexample to Conjecture 5 in Section 4 and explaining how we obtained our computational results in Section 5, in the next section we will explain, why all these conjectures arise in the same context and furthermore naturally lead to another new question (Conjecture 6). In Section 3 we give a general method that might also be helpful to search for counterexamples to Conjectures 4 and 6. We conclude the paper in Section 6.

2 Relation between the 5 conjectures and a 6th one

In the present section we will reformulate all of the above conjectures as statements about perfect matchings not containing a (directed) cut in a cubic (planar, bipartite, directed) graph.

First, observe that a cubic graph $G = (V, E)$ contains a Hamiltonian cycle C , if and only if the complement $E \setminus C$ is a perfect matching containing no edge-cut. Therefore, in Conjectures 1, 2, 3 the property *contains a Hamiltonian cycle* can be replaced equivalently by *contains a perfect matching without cut*.

Let us now see why Conjecture 4 is a special case of Conjecture 5 (also explained in [5]). Partitioning the vertex set of an oriented planar graph D in two acyclic sets corresponds to finding an edge-cut whose removal leaves an acyclic directed graph. Since D has digirth at least 3 it can be assumed to be simple. Thus, the planar dual digraph D^* is 3-edge-connected. The cut in D corresponds to an even subgraph in D^* , whereas an acyclic directed planar graph is dual to a strongly connected planar graph. On the other hand, deleting an edge in D corresponds to contracting it in D^* . Thus, Conjecture 4 can be reformulated as: *every 3-edge-connected, planar digraph $D = (V, A)$ contains an even subgraph $E \subseteq A$ such that D/E is strongly connected*, i.e., it is Conjecture 5 restricted to planar digraphs.

We will continue reformulating Conjectures 4 and 5. We show that both are equivalent to their restriction to cubic graphs: One direction is trivial. For the other direction suppose both conjectures

to hold for cubic graphs and let D be a 3-edge-connected directed graph $D = (V, A)$. Replace every vertex v of D with an undirected cycle C_v of length $\deg(v)$ such that the resulting partially directed graph $D' = (V', A')$ is cubic, 3-edge-connected. Let $E' \subseteq A'$ be an even subgraph such that D'/E' is strongly connected. Since D is a contraction of D' and strong connectivity is closed under contraction $D/(E' \cap A)$ is strongly connected. Observe that E' is just a disjoint union of cycles C_1, \dots, C_k of the cubic graph D' . Thus, for any of the cycles of the form C_v in D' either $C_v \in \{C_1, \dots, C_k\}$ or C_v intersects E' in a set of paths. Therefore, after contracting C_v back to a vertex, this vertex has even degree with respect to $E' \cap A$. Thus, $E' \cap A$ is an even subgraph and we are done. Note that his construction preserves planarity and 3-edge-connectivity.

Now, observe that a cubic digraph $D = (V, A)$ contains an even subgraph such that D/E is strongly connected if and only if the complement $A \setminus E$ is an *odd subgraph* (i.e., every vertex has degree 1 or 3 with respect to $A \setminus E$) containing no directed edge-cut.

Let us now see that Conjectures 4 and 5 on cubic graphs are equivalent to their restriction from odd subgraphs to perfect matchings: One direction is trivial, i.e., if there is a perfect matching containing no directed edge cut, then this is also an odd subgraph containing no directed edge cut.

For the other direction, let $D = (V, A)$ be a cubic digraph such that every perfect matching contains a directed cut. We construct a cubic digraph $D' = (V', A')$ such that all odd subgraphs contain a directed cut. Replace each vertex v of D that is neither a sink nor a source as shown in Figure 1. Note that $|V'| = |V| + 6|\{v \in V \mid \delta^+(v) \in \{1, 2\}\}|$. The preserved arcs from D (depicted in bold) are called *suspension arcs*.

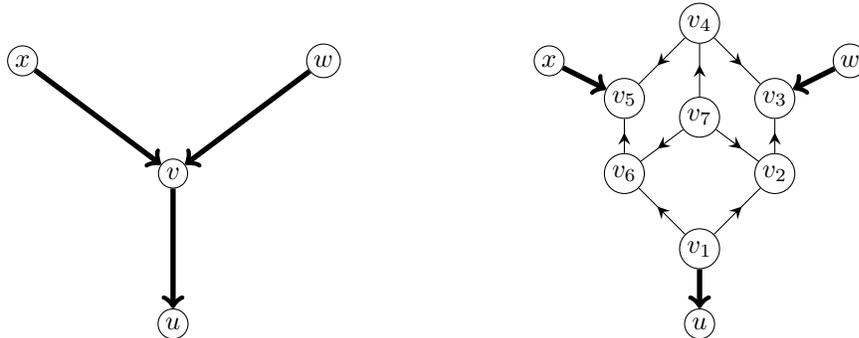


Figure 1: The replacement of R_v of a vertex v of graph D to obtain D' . The bold arcs are the arcs which are preserved (suspension arcs).

Lemma 1. *Let $D = (V, A)$ be a cubic digraph such that every perfect matching contains a directed cut and $D' = (V', A')$ the digraph constructed above. Every odd subgraph of $D' = (V', A')$ contains a directed cut and $|V'| = |V| + 6|\{v \in V \mid \delta^+(v) \in \{1, 2\}\}|$. Furthermore, the construction preserves 3-edge-connectivity, bipartiteness and planarity.*

Proof. Let $O \subseteq A'$ be an odd subgraph of D' . Recall that since D' is cubic, all vertices of O must have degree 1 or 3.

Consider some replacement gadget R_v as in Figure 1. First note that if any of the vertices v_i with i odd has degree three in O , then O contains a directed cut. Second, since R_v has an odd number of vertices v_1, \dots, v_7 and every odd subgraph has an even number of vertices, among the arcs (w, v_3) ,

$(x, v_5), (v_1, u)$ an odd number has to be in O , since otherwise an odd number of vertices remains to be covered in R_v .

Suppose $(w, v_3), (x, v_5), (v_1, u)$ are all in O , then by the previous argument v_1, v_3 and v_5 must be of degree 1 in O . Hence vertices v_2, v_4, v_6, v_7 remain to be covered by arcs of O and thus necessarily v_7 has degree three in O . Thus, O contains a directed cut.

The remaining case is when for every replacement gadget R_v of G , exactly one suspension arc is in O . In this case, by construction of D' , all O induces a perfect matching of D . By assumption, this perfect matching has a directed cut, which also is a directed cut of D' and we are done.

It is easy to see that the construction preserves 3-edge-connectivity, bipartiteness, and planarity. \square

We are now ready to restate Conjectures 4 and 5 in a new way:

Conjecture IV (Neumann-Lara). Every 3-edge-connected, cubic, planar digraph contains a perfect matching without directed cut.

Conjecture V (Hochstättler). Every 3-edge-connected, cubic digraph contains a perfect matching without directed cut.

Note that from the previous discussion a new question follows naturally and, after checking it on some sets of graphs (see Section 5), we propose the following:

Conjecture 6. Every 3-edge-connected, cubic, bipartite digraph contains a perfect matching without directed cut.

See Figure 2 for an illustration of the relations between the conjectures discussed in this section.

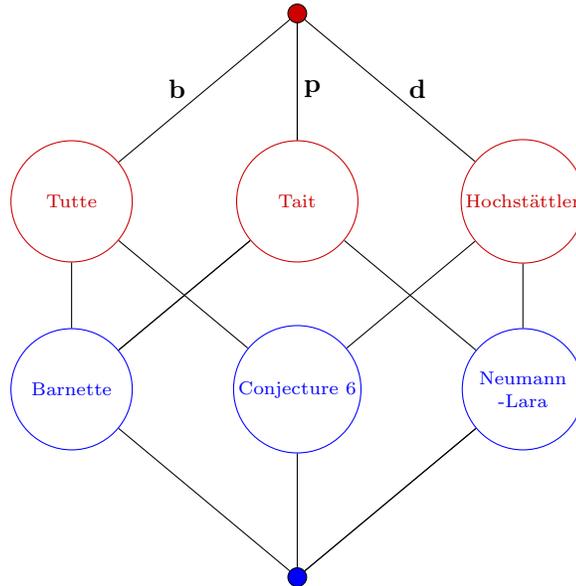
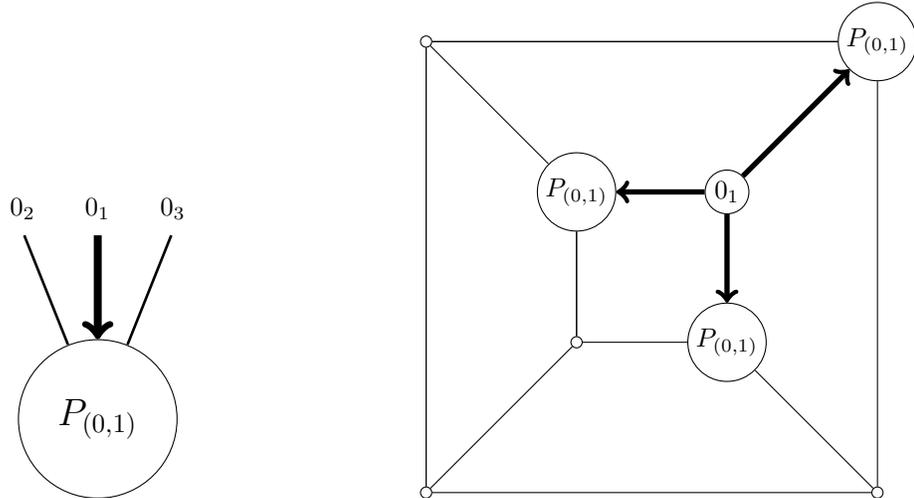


Figure 2: Conjectures of the form "Every 3-edge-connected, cubic, **b**ipartite, **p**lanar, **d**irected graph contains a perfect matching without (directed) cut". The upper four are false, the lower four are open.

Note, that the statement corresponding to the top of Figure 2 is that every cubic, 3-edge-connected graph is Hamiltonian. The first counterexample to this is the Petersen graph. While it is neither bipartite nor planar it can be proved [5] that all of its orientations satisfy Conjecture 5. Actually there it is only shown that any orientation of the Petersen graph has an odd subgraph without directed cut. However, we verified by computer that all orientations even have a perfect matching without directed cut. We will see later that the Petersen graph anyways plays a crucial role in the counterexamples to Conjectures 5 and V.

3 A-arcs

Let $D_{(0,1)}$ be a partially directed, 3-edge-connected, cubic graph with an arc $(0,1)$ such that every perfect matching containing $(0,1)$ has a directed cut. We call such an arc *a-arc*. Observe that the other two arcs incident to 0 can be non-oriented or oriented arbitrarily, this is because a matching can contain at most one edge incident to vertex 0. Call 2 and 3 the other two neighbors of vertex 0. From $D_{(0,1)}$ we build the graph $P_{(0,1)}$ by splitting vertex 0 in order to obtain one arc per vertex. We label these vertices $0_1, 0_2, 0_3$ with the index corresponding to their incident arc. The obtained digraph $P_{(0,1)}$ is depicted in Figure 3a.



(a) The graph $P_{(0,1)}$. The arc $(0,1)$ is the thick one.

(b) The graph P_1 built from the gadgets $P_{(0,1)}$. The three copies of vertex 0_1 are identified in one vertex.

Figure 3: Building a partially directed graph P_1 with all perfect matchings containing a directed cut

Now we build the graph P_1 by replacing three vertices of the cube-graph by copies of $P_{(0,1)}$ and by identifying vertices 0_1 as shown in Figure 3b. The obtained graph is cubic, 3-edge-connected and furthermore the construction preserves planarity and bipartiteness of $D_{(0,1)}$.

Lemma 2. *Every perfect matching of P_1 has a directed cut.*

Proof. For any perfect matching M , exactly one of the arcs incident to vertex 0_1 and to a gadget $P_{(0,1)}$ (call it P') must be in M . Note that by parity either both arcs connecting 0_2 and 0_3 to P' are

in M or none of them. If both are in M , then it is easy to see, that M cannot be a perfect matching of P_1 . Otherwise, M contains one of the perfect matchings M' of P' and, by hypothesis on the gadget $D_{(0,1)}$, M' has a directed cut. This directed cut is also a cut in M . \square

A second way of building a (smaller) cubic, 3-edge-connected digraph from two oppositely oriented copies of $P_{(0,1)}$ yields the graph P_2 , see Figure 4. Note that P_2 is neither bipartite nor planar.

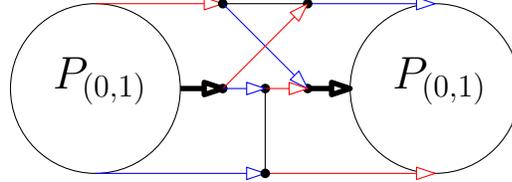


Figure 4: Construction of the graph P_2 . The bold arcs corresponds to $(0,1)$. Red and blue arcs form directed cuts, respectively.

Lemma 3. *Every perfect matching of P_2 has a directed cut.*

Proof. Let M be a perfect matching of P_2 .

First, note that by parity for each copy of $P_{(0,1)}$, either one or all of its arcs incident to vertices $0_1, 0_2, 0_3$ are in M . However, if it was all three, this would yield a directed cut in M .

Second, if M misses both of the two bold arcs, then in order to cover both copies of 0_1 either :

- M contains the two crossing arcs and the vertical edge of P_2 and therefore none of the arcs in either copy of $P_{(0,1)}$ is in M i.e. M cannot be a perfect matching.
 - M covers one of the copies of 0_1 by a horizontal arc and the other by one of the crossing arcs.
- It follows that M contains either the blue directed cut or the red directed cut, so we are done.

We conclude that M contains a bold arc of one of the copies of $P_{(0,1)}$ and one of the perfect matchings M' of the copy and, by hypothesis on the gadget $D_{(0,1)}$, M' has a directed cut. This directed cut is also a cut in M . \square

4 Counterexamples to Conjectures 5 and V

In order to construct counterexamples to Conjecture V we choose the graph $D_{(0,1)}$ to be an orientation of the Petersen graph given in Figure 5. Note that the Petersen graph satisfies Conjecture 5 (see [5]).

Observation 1. *In the partial orientation of the Petersen graph given in Figure 5, the (two) perfect matchings containing edge $(0,1)$ have each a directed cut.*

Thus, the partial orientation of the Petersen graph in Figure 5 together with Lemma 2 or Lemma 3 yields two counterexamples to Conjecture V. The graph coming from Lemma 3 has 24 vertices and is depicted in Figure 6.

Even if the counterexample P_1 from Lemma 2 is larger, i.e., it has 32 vertices, since it has many non-oriented edges, this graph is useful to construct a relatively small counterexample to Conjecture 5 by means of Lemma 1. More precisely, we can fully orient P_1 , such that in every copy of the Petersen

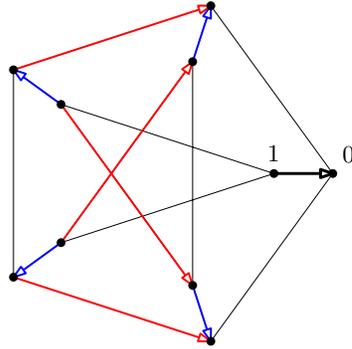


Figure 5: A partial orientation of the Petersen graph with the two matchings containing edge $(0,1)$.

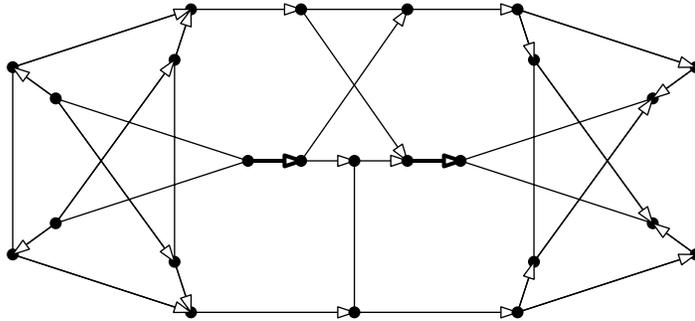


Figure 6: Smallest oriented cubic, 3-edge-connected, partially directed graph where each perfect matching contains a cut (the undirected edges can be oriented arbitrarily)

graph the neighbors of vertices $0_2, 0_3$ are sinks and two more vertices are sources. Finally, the remaining non-oriented edges of the cube can be also oriented so that all of the cube vertices (except the ones where the Petersen graph was inserted), are sinks or sources. Since in Lemma 1 only vertices that are neither sinks nor sources have to be replaced by a gadget of 7 vertices, this results in a cubic counterexample to Conjecture 5 of order $32 + 6 \cdot 5 \cdot 3 = 122$.

5 Computation

In this section we explain how we checked the conjectures on large sets of digraphs computationally. Afterwards we summarize our computational results.

5.1 How to check the conjectures computationally

First, recall that the instances of undirected graphs we are considering are always non-Hamiltonian, that is every perfect matching contains a cut.

In order to verify that for any orientation of a given undirected graph G , there exists a perfect matching without directed cut, our algorithm proceeds by iterating through all perfect matchings of G . For each of them all minimal cuts will be chosen consecutively to be directed and the algorithm

will recurse. This exploration will either lead to a conflict, i.e., at some step of the algorithm the current perfect matching will have only cuts that have already arcs into both directions; or will reduce the set of remaining perfect matchings to consider (since some of their cuts are already directed). In fact what we observed is that many perfect matchings in the graphs have only one single cut and this speeds up the process, since no branching is needed for the choice of which cut to direct.

The program was written in a Python-based language using tools provided by SageMath [15].

5.2 Computational results

Using the Hamiltonicity test implemented in SageMath we verified Conjecture 3 for all 3-edge-connected, cubic, bipartite graphs available on-line (<https://hog.grinvin.org/Cubic>), i.e., all graphs on at most 30 vertices, all of girth at least 6 on at most 34 vertices, and all of girth at least 8 on at most 46 vertices. The list was generated using Minibaum, a computer program written by Brinkmann [2]. In particular, this confirms a claim of Lomonosov and Kelmans that Conjecture 3 holds for all 3-edge-connected cubic bipartite graphs on at most 30 vertices.

Since Conjecture 6 is weaker than Conjecture 3 it holds for all graphs on at most 30 vertices. Furthermore, we verified by computer that no orientation of the known counterexamples to Conjecture 3 (see <http://mathworld.wolfram.com/TutteConjecture.html>) is a counterexample to Conjecture 6.

We verified that all cubic, 3-edge-connected digraphs with at most 22 vertices have a perfect matching without a directed cut. We also checked that the only partially directed graph on 24 vertices admitting an orientation such that every perfect has a directed cut is the one of Figure 6. Therefore, this is the unique smallest counterexample to Conjecture V. Furthermore, we checked that any full orientation of this graph has an odd subgraph without directed cut, i.e., it is not a counterexample to Conjecture 5, which therefore holds for cubic graphs on at most 24 vertices.

Our experimental results show that Conjecture IV holds for planar cubic 3-edge-connected graphs on at most 48 vertices. If moreover the graphs have cyclic connectivity at least 4, we verified it for graphs on 50 vertices too. Therefore, by planar duality, we conclude that Conjecture 4 holds for planar graphs on at most 26 vertices and for 4-vertex-connected planar graphs on at most 27 vertices. Moreover, from columns 4 and 5 of Table 1 we can deduce that Conjecture 4 holds for 4-vertex connected planar graphs with minimum degree 5 on at most 36 vertices.

Recall that the construction of the graph P_1 (Figure 3b) preserves planarity. Thus, we get:

Observation 2. *If Conjecture 4 is true, then no planar, 3-edge-connected, cubic digraph has an a-arc.*

Hence, one way to build a counterexample to Conjecture 4 is to find a planar, 3-edge-connected, cubic digraph having an a-arc. In fact, the underlying undirected graph of such a digraph must have an *a-edge*, i.e., an edge such that each perfect matchings containing it has a cut or equivalently, all Hamiltonian cycles have to use this edge. The list of such planar graphs of order at most 38 was generated by McKay and provided on his webpage : <http://users.cecs.anu.edu.au/~bdm/data/planegraphs.html>. Our computations show that none of them can be oriented such that it has an a-arc.

6 Conclusion

We have disproved Conjecture 5. For its variant Conjecture V we found the unique smallest counterexample on 24 vertices. Is there a counterexample to Conjecture 5 on less than 122 vertices? Similarly

Nr of vertices	No restriction	Cyclically 4-connected	no faces of size 3,4, cyclic connectivity exactly 4	Cyclically 5-connected
≤ 36	0	0	0	0
38	6	0	0	0
40	37	0	0	0
42	277	3	0	0
44	1732	3	1	1
46	11204	19	3	1
48	70614	30	1	0
50	?	126	3	3
52	?	?	6	6
54	?	?	12	2
56	?	?	49	22
58	?	?	126	37
60	?	?	214	31
62	?	?	659	194
64	?	?	1467	298
66	?	?	3247	306
68	?	?	9187	1538
70	?	?	22069	2566
72	?	?	50514	3091
74	?	?	137787	13487
76	?	?	339804	?

Table 1: The counterexamples to Tait’s conjecture on which Conjecture IV was verified.¹

for Conjecture 3 the smallest known counterexample is on 50 vertices and we have reconfirmed that it is true for graphs on at most 30 vertices. What is the size of a smallest counterexample?

As a common weakening of Conjectures V and 3 we proposed Conjecture 6: *Every 3-edge-connected, cubic, bipartite digraph contains a perfect matching without directed cut.* We know that it holds on graphs on at most 30 vertices and for all orientations of the known counterexamples to Conjecture 3.

Note that the weakest statement in Figure 2 (corresponding to the minimum in the diagram) is a natural open case of Conjecture 4: *Every orientation of a planar Eulerian triangulation can be vertex-partitioned into two acyclic sets.*

A well-known parameter of cubic graphs is *cyclic connectivity*, i.e., the size of a smallest *cyclic cut*, where the latter is a minimal edge-cut, such that after its deletion both components have a cycle. Any minimum cyclic cut forms a matching and all edge-cuts that are matchings are cyclic cuts. Clearly, if all cyclic cuts are large, then it is harder to have a cut in every perfect matching. That is why it is natural to consider any of the conjectures on cubic graphs in this paper for high cyclic connectivity.

While Conjecture 1 is false for all possible values of cyclic connectivity (planar graphs have cyclic connectivity at most 5), the highest cyclic connectivity among the known counterexamples to Conjectures 3 and V is 4 and 3, respectively. Indeed, it is even open if there exist general non-Hamiltonian, 3-edge-connected, cubic graphs of high cyclic connectivity. This is conjectured to be true in [4] while Thomassen apparently conjectured the contrary (see [11]). The 28-vertex Coxeter graph has cyclic connectivity 7. We checked that to attain cyclic connectivity 8 and 9 at least 48 and 66 vertices are needed, respectively.

A (probably novel) parameter that is still closer to these problems is the size of a smallest edge-cut contained in a perfect matching. We believe that it deserves further investigation.

¹The list of graphs considered was taken from <http://users.cecs.anu.edu.au/~bdm/data/planegraphs.html>

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