A product form and a sub-additive theorem for the general stochastic matching model

Pascal Moyal, Ana Bušić, Jean Mairesse

A product form and a sub-additive theorem for the general stochastic matching model

Pascal Moyal, Ana Bušić and Jean Mairesse*

December 25, 2017

Abstract

We consider a stochastic matching model with a general matching graph, as introduced in [14]. We show that the natural necessary condition of stability of the system exhibited therein is also sufficient whenever the matching policy is First Come, First Matched (FCFM). For doing so, we exhibit a stationary distribution under a remarkable product form, by using an original dynamic reversibility inspired by that of [2] for the bipartite matching model. Second, we observe that most common matching policies (including FCFM, priorities and random) satisfy a remarkable sub-additive property, which we exploit to derive in many cases, a coupling result to the steady state, using a constructive backwards scheme *à la* Loynes. We then use these results to explicitly construct perfect bi-infinite matchings.

1 Introduction

Consider a general stochastic matching model (GM), as introduced in [14]: items of various classes enter a system one by one, to be matched by couples. Two items are compatible if and only if their classes are adjacent in a compatibility graph G that is fixed beforehand. The classes of the entering items are drawn following a prescribed probability measure on $\mathcal{V}$, the set of nodes of G. This model is a variant of the Bipartite Matching model (BM) introduced in [9], see also [1]. In the BM, the compatibility graph is bipartite (say $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$). Along the various natural applications of this model, the nodes of $\mathcal{V}_1$ and $\mathcal{V}_2$ represent respectively classes of customers and servers, kidneys and patients, blood givers and blood receivers, houses and applicants, and so on. The items are matched by couples of $\mathcal{V}_1 \times \mathcal{V}_2$, and also arrive by couples of $\mathcal{V}_1 \times \mathcal{V}_2$. The classes of the elements of the entering couples are random, and it is assumed in the aforementioned references that the class of the entering element of $\mathcal{V}_1$ is always independent of that of the entering element of $\mathcal{V}_2$.

An important generalization of the BM is the so-called Extended Bipartite Matching model (EBM) introduced in [4], where this independent assumption is relaxed. Possible entering couples are element of a bipartite arrival graph on the bipartition $\mathcal{V}_1 \cup \mathcal{V}_2$. Importantly, one can observe that the GM is in fact a particular case of EBM, taking the bipartite double cover of G as compatibility graph, and duplicating arrivals with a copy of an item of the same class.

The main question raised in [14] is the shape of the stability region of the model, that is, the set of probability measures on $\mathcal{V}$ rendering the corresponding system stable. Partly relying on the aforementioned connection between GM and EBM, and the results of [4], [14] show that the stability region is always included in a designated set, namely the set of measures satisfying the natural necessary condition (8) below. The form of the stability region is then heavily dependent on the structural properties of the compatibility graph, and on the matching policy, i.e. the rule of choice of a match for an

*P.M. is with LMAC, Université de Technologie de Compiegne, A.B. is with Inria Paris and DI ENS, École Normale Supérieure, CNRS, PSL Research University, and J.M. is with CNRS and Université Pierre et Marie Curie.

entering item whenever several possible matches are possible. A matching policy is then said to have a maximal stability region for G if the system is stable for any measure satisfying (8). It is shown in [14] that a bipartite G makes the stability region empty, that a designated class of graphs (the so-called non-bipartite separable ones - precisely defined below) make the stability region maximal for all matching policies, and that the policy 'Match the Longest' always has a maximal stability region for a non-bipartite G. Applying fluid (in)stability arguments to a continuous-time version of the GM, [15] show that, aside for a very particular class of graphs, whenever G is not separable there always exists a policy of the strict priority type that does not have a maximal stability region, and that the 'Uniform' random policy (natural in the case where no information is available to the entering items on the state of the system) never has a maximal stability region, thereby providing a partial converse of the result in [14]. A related model is studied in [11], which draws a comparison of matching policies in the case where the matching structure is a particular hypergraph, that is, items may be matched by groups of more than two.

In the first part of this work, we are concerned with the stability region of the GM under the 'First Come, First Matched' (FCFM) policy, consisting in always performing the match of the entering item with the oldest compatible item in the system. Compared to the aforementioned stability studies, this matching policy raises technical problems, mainly due to the infinite dimension of the state space of its natural Markov representation. In the history of study of the BM, the corresponding 'First Come, First Served' (FCFS) policy was the first one considered in the seminal papers [9, 1], where the existence of a stationary matching under a natural resource pooling condition analog to (8) is shown. Further, [2] recently show that the stationary state can be obtained in a remarkable product form, which is obtained using an original dynamic reversibility argument. However, these results cannot be directly applied to the present context, for the GM is not a particular case of a BM, but of an EBM, for which the latter reversibility argument is unlikely to hold in general. Moreover the maximality of the stability region of FCFS for the EBM is conjectured, but left as on open problem in [4]. We show hereafter that we can in fact construct a reversibility scheme that is closely related to the one proposed in [2] for the BM, to show that the stability region of FCFM is indeed maximal for the GM, and that the stationary state of the Markov representation also satisfies a product form.

It is well known since the pioneering works of Loynes [13] and then Borovkov [5], that backwards schemes and specifically strong backwards coupling convergence, can lead to an explicit construction of the stationary state of the system under consideration within its stability region. One can then use pathwise representations to compare systems in steady state, via the stochastic ordering of a given performance metric (see Chapter 4 of [3] on such comparison results for queues). Moreover, we know since the seminal work of Propp and Wilson [16] that coupling from the past algorithms (which essentially use backwards coupling convergence) provide a powerful tool for simulating the steady state of the system. In the second part of this work, we aim at achieving such results for the general matching model: under various conditions, we construct a stationary version of the system under general stationary ergodic assumptions, via a stochastic recursive representation of the system on the canonical space of its bi-infinite input. For this, we first observe that most usual matching policies (including FCFM, the optimal 'Match the Longest' policy, and - possibly randomized - priorities) satisfy a remarkable sub-additive property, which allows to construct the appropriate backwards scheme to achieve this explicit construction. These results lead to stability conditions for various models, under stationary ergodic assumptions that subsume the markovian (i.e., iid) settings. Second, in some cases (including iid), we construct a unique (up to the natural parity of the model, in a sense that will be specified below) stationary perfect matching, by strong backwards coupling.

The paper is organized as follows. In Section 2 we introduce and formalize our model. In Section 3 we develop our reversibility scheme for the FCFM system, leading to our main result of the first part, Theorem 1 in subsection 3.4, which establishes the existence of a stationary probability under a product form for the natural Markov representation of the system, under the natural condition (8). Our coupling result is then presented in Section 4, including the algebraic study of sub-additive policies in Section 4.2, the construction of renovating events *à la* Borovkov and Foss in Section 4.4, and the explicit constructions of perfect bi-infinite matchings for sub-additive policies, in Section 4.6.

2 The model

2.1 General notation

Denote by $\mathbb{R}$ the real line, by $\mathbb{N}$ the set of non-negative integers and by $\mathbb{N}_+$, the subset of positive integers. For any two integers m and n, denote by $[\![m,n]\!] = [m,n] \cap \mathbb{N}$. For any finite set A, let S_A be the group of permutations of A, and for all permutation $s \in S_A$ and any $a \in A$, let $s[a]$ be the image of a by s. Let A^* be the set of finite words over the alphabet A. Denote by $\emptyset$, the empty word of A^*. For any word $w \in A^*$ and any subset B of A, we let $|w|_B$ be the number of occurrences of elements of B in w. For any letter $a \in A$, we denote $|w|_a := |w|_{\{a\}}$ the number of occurrences of the letter a in w, and we let $|w| = \sum_{a \in A} |w|_a$ be the *length* of w. For a word $w \in A^*$ of length $|w| = q$, we write $w = w_1 w_2 ... w_q$, i.e. w_i is the i-th letter of the word w. In the proofs below, we understand the word $w_1...w_k$ as $\emptyset$ whenever $k = 0$. Also, for any $w \in A^*$ and any $i \in [\![1, |w|]\!]$, we denote by $w_{[i]}$, the word of length $|w| - 1$ obtained from w by deleting its i-th letter. We let $[w] := (|w|_a)_{a \in A} \in \mathbb{N}^A$ be the *commutative image* of w. Finally, a *right sub-word* of the word $w = w_1...w_k$ is a word $w_j...w_k$ obtained by deleting the first $j-1$ letters of w, for $j \in [\![1,k]\!]$.

For any $p \in \mathbb{N}_+$, a vector x in the set A^p is denoted $x = (x(1), ..., x(p))$. For any $i \in [\![1,p]\!]$, we denote by $\mathbf{e}_i$ the i-th vector of the canonical basis of $\mathbb{R}^p$, i.e. $\mathbf{e}_i(j) = \delta_{ij}$ for any $j \in [\![1,p]\!]$. The ℓ_1 norm of $\mathbb{R}^p$ is denoted $\| \, . \, \|$.

Consider a simple graph $G = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ denotes the set of nodes, and $\mathcal{E} \subset \mathcal{E} \times \mathcal{E}$ is the set of edges. We use the notation $u{-}v$ for $(u,v) \in \mathcal{E}$ and $u \not- v$ for $(u,v) \notin \mathcal{E}$. For $U \subset \mathcal{V}$, we define $U^c = \mathcal{V} \setminus U$ and

$$\mathcal{E}(U) = \{v \in \mathcal{V} \,:\, \exists u \in U,\ u - v\}\,.$$

An *independent set* of G is a non-empty subset $\mathcal{I} \subset \mathcal{V}$ which does not include any pair of neighbors, *i.e.* $\left(\forall i \neq j \in \mathcal{I},\ i \not- j\right)$. Let $\mathbb{I}(G)$ be the set of independent sets of G. An independent set $\mathcal{I}$ is said to be *maximal* if $\mathcal{I} \cup \{j\} \notin \mathbb{I}(G)$ for any $j \notin \mathcal{I}$.

2.2 Formal definition of the model

We consider a *general stochastic matching model*, as was defined in [14]: items enter one by one a system, and each of them belongs to a determinate class. The set of classes is denoted by $\mathcal{V}$, and identified with $[\![1, |\mathcal{V}|]\!]$. We fix a connected simple graph $G = (\mathcal{V}, \mathcal{E})$ having set of nodes $\mathcal{V}$, termed *compatibility graph*. Upon arrival, any incoming item of class, say, $i \in \mathcal{V}$ is either matched with an item present in the buffer, of a class j such that $i{-}j$, if any, or if no such item is available, it is stored in the buffer to wait for its match. Whenever several possible matches are possible for an incoming item i, a *matching policy* ϕ decides what is the match of i without ambiguity. Each matched pair departs the system right away.

We assume that the successive classes of entering items, and possibly their choices of match, are random. We fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which all random variables (r.v.'s, for short) are defined, and view, throughout, the input as a bi-infinite sequence $(V_n, \Sigma_n)_{n \in \mathbb{Z}}$ that is defined as follows: first, for any $n \in \mathbb{Z}$ we let $V_n \in \mathcal{V}$ denote the class of the n-th incoming item. Second, we introduce the set

$$\mathcal{S} = S_{\mathcal{E}(1)} \times ... \times S_{\mathcal{E}(|\mathcal{V}|)},$$

in other words for any $\sigma = (\sigma(1), ..., \sigma(|\mathcal{V}|)) \in \mathcal{S}$ and $i \in \mathcal{V}$, $\sigma(i)$ is a permutation of the classes of items that are compatible with i (which are identified with their indexes in $[\![1, |\mathcal{V}|]\!]$). Any array of permutations $\sigma \in \mathcal{S}$ is called *list of preferences*. For any $n \in \mathbb{Z}$, we let Σ_n denote the list of preferences at time n, i.e. if $\Sigma_n = \sigma$ and $V_n = v$, then the permutation $\sigma(v)$ represents the order of preference of the entering v-item at n, among the classes of its possible matches.

Along the various results in this work, we will consider the following statistical assumptions on the input sequence $(V_n, \Sigma_n)_{n\in\mathbb{Z}}$,

(H1) The sequence $((V_n, \Sigma_n))_{n\in\mathbb{Z}}$ is stationary and ergodic, drawn at all n from a distribution having first marginal μ on $\mathcal{V}$ and second marginal ν_ϕ on $\mathcal{S}$.

(H1') The sequence $((V_{2n}, \Sigma_{2n}, V_{2n+1}, \Sigma_{2n+1}))_{n\in\mathbb{Z}}$ is stationary and ergodic, drawn at all n from a distribution of first marginal μ^0 on $\mathcal{V}$, third marginal μ^1 on $\mathcal{V}$, and second and fourth marginals ν_ϕ on $\mathcal{S}$. We denote $\mu := \frac{\mu^0+\mu^1}{2}$.

(H1") The sequence $((V_{2n-1}, \Sigma_{2n-1}, V_{2n}, \Sigma_{2n}))_{n\in\mathbb{Z}}$ is stationary and ergodic, drawn at all n from a distribution of first marginal μ^0 on $\mathcal{V}$, third marginal μ^1 on $\mathcal{V}$, and second and fourth marginals ν_ϕ on $\mathcal{S}$. We denote $\mu := \frac{\mu^0+\mu^1}{2}$.

(IID) The sequence $((V_n, \Sigma_n))_{n\in\mathbb{Z}}$ is iid from the distribution $\mu \otimes \nu_\phi$ on $\mathcal{V} \times \mathcal{S}$.

Under either one of the above conditions, we assume that μ has full support $\mathcal{V}$ (we write $\mu \in \mathcal{M}(\mathcal{V})$).

Then, the matching policy ϕ will be formalized by an operator mapping the system state onto the next one, given the class of the entering item and the list of preferences at this time. The matching policies we consider are presented in detail in Section 2.4.

Altogether, the matching graph G, the matching policy ϕ and the measure μ (under assumptions (H1) and (IID)) or μ^0 and μ^1 (under assumptions (H1') and (H1")) fully specify the model, which we denote for short general matching (GM) model associated with (G, μ, ϕ) under assumptions (H1) and (IID), or $(G, (\mu^0, \mu^1), \phi)$ under assumptions (H1') and (H1").

2.3 State spaces

Fix the compatibility graph $G = (\mathcal{V}, \mathcal{E})$ until the end of this section.

Fix an integer $n_0 \geq 1$, and two realizations $v_1, ...v_{n_0}$ of $V_1, ..., V_{n_0}$ and $\sigma_1, ..., \sigma_{n_0}$ of $\Sigma_1, ..., \Sigma_{n_0}$. Define the two words $v \in \mathcal{V}^*$ and $\sigma \in \mathcal{S}^*$ respectively by $z := v_1v_2...v_{n_0}$ and $\varsigma := \sigma_1\sigma_2...\sigma_{n_0}$. Then, for any matching policy ϕ, there exists a unique *matching* of the word z associated to σ, that is, a graph having set of nodes $\{v_1, ..., v_{n_0}\}$ and whose edges represent the matches performed in the system until time n_0, if the successive arrivals are given by z and the lists of preferences by σ. This matching is denoted by $M_\phi(z, \varsigma)$. The state of the system is then defined as the word $Q_\phi(z, \varsigma) \in \mathcal{V}^*$, whose letters are the classes of the unmatched items at n_0, i.e. the isolated vertices in the matching $M_\phi(z, \varsigma)$, in their order of arrivals. The word $Q_\phi(z, \varsigma)$ is called *queue detail* at time n_0. Observe that any admissible queue detail belongs to the set

$$\mathbb{W} = \Big\{ w \in \mathcal{V}^* \ : \ \forall (i,j) \in \mathcal{E}, \ |w|_i|w|_j = 0 \Big\}. \tag{1}$$

As will be seen below, depending on the service discipline ϕ we can also restrict the available information on the state of the system at time n_0, to a vector only keeping track of the number of items of the various classes remaining unmatched at n_0, that is, of the number of occurrences of the various letters of the alphabet $\mathcal{V}$ in the word $Q_\phi(z, \varsigma)$. This restricted state thus equals the commutative image of $Q_\phi(z, \varsigma)$, and is called *class detail* of the system. It takes values in the set

$$\mathbb{X} = \Big\{ x \in \mathbb{N}^{|\mathcal{V}|} \ : \ x(i)y(j) = 0 \text{ for any } (i,j) \in \mathcal{E} \Big\} = \Big\{ [w] \, ; \, w \in \mathbb{W} \Big\}. \tag{2}$$

2.4 Matching policies

We now present and define formally, the set of matching policies which we consider.

Definition 1. *A matching policy ϕ is said* admissible *if the choice of match of an incoming item depends* solely *on the queue detail and the list of preferences drawn upon arrival.*

In other words, if a matching policy ϕ is admissible there exists a mapping $\odot_\phi : \mathbb{W} \times (\mathcal{V} \times \mathcal{S}) \to \mathbb{W}$ such that, denoting by w the queue detail at a given time, and by w' the queue detail if the input is augmented by the arrival of a couple $(v, \sigma) \in \mathcal{V} \times \mathcal{S}$, then w' and w are connected by the relation

$$w' = w \odot_\phi (v, \sigma). \tag{3}$$

2.4.1 FCFM and LFCM

The first two policies we introduce are First Come, First Matched and Last Come, First matched. For both policies, the order of preference of each class is irrelevant, and so the following construction is independent of the preference σ.

First Come, First Matched. In First Come, First Matched (FCFM) the map $\odot_{\text{FCFM}}$ is given for all $w \in \mathbb{W}$ and all couples (v, σ), by

$$w \odot_{\text{FCFS}} (v, \sigma) = \begin{cases} wv & \text{if } |w|_{\mathcal{E}(v)} = 0; \\ w_{[\Phi(w,v)]} & \text{else, where } \Phi(w, v) = \arg\min\{|w_k| : k \in \mathcal{E}(v)\}. \end{cases}$$

Last Come, First Matched. For the last come, first matched (LCFM) matching policy, the updating map $\odot_{\text{LCFM}}$ is analog to $\odot_{\text{FCFM}}$, for $\Phi(w, v) = \arg\max\{|w_k| : k \in \mathcal{E}(v)\}$.

2.4.2 Matching policies that only depend on the class detail

A matching policy ϕ is said to be *class-admissible* if it can be implemented by knowing only the class detail of the system. Let us define for any $v \in \mathcal{V}$ and $x \in \mathbb{X}$,

$$\mathcal{P}(x, v) = \Big\{ j \in \mathcal{E}(v) \,:\, x(j) > 0 \Big\},$$

the set of classes of available compatible items with the entering class v-item, if the class detail of the system is given by x. Then, a class-admissible policy ϕ is fully characterized by the probability distribution ν_ϕ on $\mathcal{S}$, together with a mapping p_ϕ such that $p_\phi(x, v, \sigma)$ denotes the class of the match chosen by the entering v-item under ϕ for a list of preferences σ, in a system of class detail x such that $\mathcal{P}(x, v)$ is non-empty. Then the arrival of v and the draw of σ from ν_ϕ corresponds to the following action on the class detail,

$$x \circledcirc_\phi (v, \sigma) = \begin{cases} x + \mathbf{e}_v & \text{if } \mathcal{P}(x, v) = \emptyset, \\ x - \mathbf{e}_{p_\phi(x,v,\sigma)} & \text{else.} \end{cases} \tag{4}$$

Remark 1. *As is easily seen, to any class-admissible policy ϕ corresponds an admissible policy, if one makes precise the rule of choice of match for the incoming items* within *the class that is chosen by ϕ, in the case where more than one item of that class is present in the system. In this paper, we always make the assumption that within classes, the item chosen is always the* oldest *in line, i.e. we always apply a FCFM policy* within classes. *Under this convention, any class-admissible policy ϕ is admissible, that is, the mapping $\circledcirc_\phi$ from $\mathbb{X} \times (\mathcal{V} \times \mathcal{S})$ to $\mathbb{X}$ can be detailed into a map $\odot_\phi$ from $\mathbb{W} \times (\mathcal{V} \times \mathcal{S})$ to $\mathbb{W}$, as in (3), that is such that for any queue detail w and any (v, σ),*

$$[w \odot_\phi (v, \sigma)] = [w] \circledcirc_\phi (v, \sigma).$$

Random policies. In a random policy, the only information that is needed to determine the choice of matches for the incoming items, is whether their various compatible classes have an empty queue or not. Specifically, the order of preference of each incoming item is drawn upon arrival following the prescribed probability distribution. Then the considered item investigates its compatible classes in that order, until it finds one having a non-empty buffer, if any. The item is then matched with an item of the latter class. In other words, a list of preferences $\sigma = (\sigma(1), ..., \sigma(|\mathcal{V}|))$ is drawn from ν_ϕ on $\mathcal{S}$, and we set

$$p_\phi(x, v, \sigma) = \sigma(v)[k], \text{ where } k = \min\Big\{i \in \mathcal{E}(v)]\!] \,:\, \sigma(v)[i] \in \mathcal{P}(x, v)\Big\}. \tag{5}$$

Priority policies. A strict priority policy is such that, for any $v \in \mathcal{V}$, the order of preference of v in $\mathcal{E}(v)$ is deterministic. This is thus a particular case of random policy in which a list of preference $\sigma^* \in \Sigma$ is fixed beforehand, in other words $\nu_\phi = \delta_{\sigma^*}$ and (5) holds for $\sigma := \sigma^*$.

Uniform. The uniform policy U is another particular case of random policy, such that ν_ϕ is the uniform distribution on $\mathcal{S}$. Consequently, for any $i \in \mathcal{V}$ and any j such that $j{-}i$, $\sigma(i)[j]$ is drawn uniformly in $\mathcal{E}(i)$.

Match the Longest. In 'Match the Longest' (ML), the newly arrived item chooses an item of the compatible class that has the longest line. Ties are broken by a uniform draw between classes having queues of the same length. Formally, set for all x and v such that $\mathcal{P}(x, v) \neq \emptyset$,

$$L(x, v) = \max\{x(j) \,:\, j \in \mathcal{E}(v)\} \qquad \text{and} \qquad \mathcal{L}(x, v) = \{i \in \mathcal{E}(v) \,:\, x(i) = L(x, v)\} \subset \mathcal{P}(x, v).$$

Then, set ν_ϕ as the uniform distribution on $\mathcal{S}$. If the resulting sample is σ, we have

$$p_{\text{ML}}(x, v, \sigma) = \sigma(v)[k], \text{ where } k = \min\Big\{i \in \mathcal{E}(v) \,:\, \sigma(v)[i] \in \mathcal{L}(y, c)\Big\}.$$

Match the Shortest. The 'Match the Shortest' policy is defined similarly to 'Match the Longest', except that the shortest queue is chosen instead of the longest. It is denoted MS.

2.5 Markov representation

Fix a (possibly random) word $w \in \mathbb{W}$ and a word $\varsigma \in \mathcal{S}^*$ having the same length as w. Denote for all $n \geq 0$ by $W_n^{[w]}$ the buffer content at time n (i.e. just before the arrival of item n) if the buffer content at time 0 was set to w, in other words

$$W_n^{[w]} = Q_\phi\left(wV_0...V_n\,,\,\varsigma\Sigma_0...\Sigma_n\right).$$

It follows from (3) that the buffer-content sequence is stochastic recursive, since we have that

$$\begin{cases} W_0^{[w]} &= w; \\ W_{n+1}^{[w]} &= W_n^{[w]} \odot_\phi (V_n, \Sigma_n),\ n \in \mathbb{N}, \end{cases} \qquad \text{a.s..}$$

Second, it follows from (4) that for any matching policy ϕ that only depends on the class detail of the system (e.g. $\phi = $ RANDOM, ML or MS), for any initial conditions as above, the $\mathbb{X}$-valued sequence $(X_n)_{n\in\mathbb{N}}$ of class-details also is stochastic recursive: for any initial condition $x \in \mathbb{X}$,

$$\begin{cases} X_0^{[x]} &= x; \\ X_{n+1}^{[x]} &= X_n^{[x]} \circledcirc_\phi (V_n, \Sigma_n),\ n \in \mathbb{N}, \end{cases} \qquad \text{a.s..} \tag{6}$$

3 A product form for the FCFM model

Throughout this section, suppose that the input of the system satisfies assumption (IID). Then, for any connected graph G and any admissible policy ϕ, the queue detail $(W_n)_{n\in\mathbb{N}}$ is a $\mathbb{W}$-valued $\mathcal{F}_n$-Markov chains, where $(\mathcal{F}_n)_{n\in\mathbb{N}}$ is the natural filtration of the sequence $((V_n,\Sigma_n))_{n\in\mathbb{N}}$. This sequence will be termed *natural chain* of the system. In line with [14], for any admissible matching policy ϕ we define the *stability region* associated to G and ϕ as the set of measures

$$\textsc{stab}(G,\phi) := \left\{\mu\in\mathcal{M}\left(\mathcal{V}\right)\,:\,\left(W_n\right)_{n\in\mathbb{N}} \text{ is positive recurrent}\right\}. \tag{7}$$

Consider also the set

$$\textsc{Ncond}(G) : \{\mu\in\mathcal{M}\left(\mathcal{V}\right)\,:\,\text{for any } \mathcal{I}\in\mathbb{I}(G), \mu\left(\mathcal{I}\right)<\mu\left(\mathcal{E}\left(\mathcal{I}\right)\right)\} \tag{8}$$

which, from Theorem 1 in [14], is non-empty if and only if G is non-bipartite. From Proposition 2 in [*ibid.*], we know that $\textsc{stab}(G,\phi)\subset\textsc{Ncond}(G)$ for any admissible ϕ. The policy ϕ is said to be *maximal* if these two sets coincide. Theorem 2 in [*ibid.*] establishes the maximality of ML for any non-bipartite graph, however priority policies and the uniform policy are in general not maximal (respectively, Theorem 3 and Proposition 7 in [15]).

This section is devoted to proving the maximality of First Come, First Matched, by constructing explicitly the stationary distribution of the natural chain on $\mathbb{W}$. Interestingly enough, this probability distribution has a remarkable product form, detailed in (9).

Theorem 1. *Let $G=(\mathcal{V},\mathcal{E})$ be a non-bipartite graph. Then the sets $\textsc{stab}(G,\textsc{fcfm})$ and $\textsc{Ncond}(G)$ coincide, in other words the general stochastic matching model $(G,\mu,\textsc{fcfm})$ is stable if and only if μ satisfies condition (8). In that case, the following is the only stationary probability of the natural chain $(W_n)_{n\in\mathbb{N}}$:*

$$\Pi_W\left(w\right)=\alpha\prod_{\ell=1}^{q}\frac{\mu(w_\ell)}{\mu\Big(\mathcal{E}\left(\{w_1,...,w_\ell\}\right)\Big)},\ \textit{ for any } w=w_1...w_q\in\mathcal{V}^*, \tag{9}$$

where α is the normalizing constant of the measure Π_B defined by (11) below.

The remainder of this Section is devoted to the proof of Theorem 1, which, as will be demonstrated below, is based on a subtle reversibility scheme that is related to the proof of reversibility for the BM model in [2]. Observe however that the GM model is *not* a particular case of BM model, so the proof below presents many specificities with respect to [2].

3.1 Other notation

Before proceeding, we first need to introduce an additional piece of notation. For $w=w_1w_2\dots w_q\in\mathcal{V}^*$, we denote by $\overleftarrow{w}$ the reversed version of w, i.e.

$$\overleftarrow{w}=w_qw_{q-1}...w_2w_1.$$

Let $\overline{\mathcal{V}}$ be an *independent copy* of the set $\mathcal{V}$, i.e., $\overline{\mathcal{V}}$ is a set of cardinality $|\mathcal{V}|$ and we define the bijection

$$\begin{cases}\mathcal{V} & \longrightarrow \overline{\mathcal{V}};\\ a & \longmapsto \overline{a}.\end{cases}$$

For any $\overline{a}\in\overline{\mathcal{V}}$, let us also denote $\overline{\overline{a}}=a$. Then, we say that $\overline{a}$ is the *counterpart* of a and vice-versa.

Let $\mathbf{V}:=\mathcal{V}\cup\overline{\mathcal{V}}$. For any word $\mathbf{w}\in\mathbf{V}^*$, denote by $\mathcal{V}(\mathbf{w})$ (respectively, $\overline{\mathcal{V}}(\mathbf{w})$) the set of letters of $\mathcal{V}$ (resp., $\overline{\mathcal{V}}$) that are present in $\mathbf{w}$, in other words

$$\mathcal{V}(\mathbf{w})=\left\{a\in\mathcal{V}\,:\,|\mathbf{w}|_a>0\right\};\qquad \overline{\mathcal{V}}(\mathbf{w})=\left\{\overline{a}\in\overline{\mathcal{V}}\,:\,|\mathbf{w}|_{\overline{a}}>0\right\}.$$

For any $\mathbf{w} \in \mathbf{V}^*$, the *restriction* of $\mathbf{w}$ to $\mathcal{V}$ (respectively, to $\overline{\mathcal{V}}$) is the word $\mathbf{w}|_{\mathcal{V}} \in \mathcal{V}^*$ (resp., $\mathbf{w}|_{\overline{\mathcal{V}}} \in \overline{\mathcal{V}}^*$) of size $|\mathbf{w}|_{\mathcal{V}}$ (resp. of size $|\mathbf{w}|_{\overline{\mathcal{V}}}$), obtained by keeping only the letters belonging to $\mathcal{V}$ (resp. to $\overline{\mathcal{V}}$) in $\mathbf{w}$, in the same order. The *dual* $\overline{\mathbf{w}}$ of the word $\mathbf{w} = \mathbf{w}_1\mathbf{w}_2...\mathbf{w}_q \in \mathbf{V}^*$ is the word obtained by exchanging the letter of $\mathbf{w}$ belonging to $\mathcal{V}$ with their counterpart in $\overline{\mathcal{V}}$, and vice-versa. In other words,

$$\overline{\mathbf{w}} = \overline{\mathbf{w}_1}\,\overline{\mathbf{w}_2}\,...\,\overline{\mathbf{w}_q}.$$

Example 1. *Take for instance* $\mathbf{w} = a\,b\,\overline{a}\,c\,\overline{b}\,\overline{c}\,\overline{b}\,d\,a$. *Then we obtain*

$$\begin{aligned} \mathcal{V}(\mathbf{w}) &= \{a,b,c,d\}\,, \quad \overline{\mathcal{V}}(\mathbf{w}) = \left\{\overline{a},\overline{b},\overline{c}\right\};\\ \mathbf{w}|_{\mathcal{V}} &= a\,b\,c\,d\,a, \quad \mathbf{w}|_{\overline{\mathcal{V}}} = \overline{a}\,\overline{b}\,\overline{c}\,\overline{b};\\ \overline{\mathbf{w}} &= \overline{a}\,\overline{b}\,a\,\overline{c}\,b\,c\,b\,\overline{d}\,\overline{a}, \quad \overleftarrow{\mathbf{w}} = a\,d\,\overline{b}\,\overline{c}\,\overline{b}\,c\,\overline{a}\,b\,a. \end{aligned}$$

3.2 Auxiliary Markov representations

We now introduce two auxiliary Markov representations of the system: the $\mathbf{V}^*$-valued Backwards and Forwards detailed chains, similar in construction to the backwards and forwards 'pair by pair detailed FCFS matching processes', introduced in subsection 5.1 of [2].

Backwards detailed chain. We define the $\mathbf{V}^*$-valued backwards detailed process $(B_n)_{n\in\mathbb{N}}$ as follows: $B_0 = \emptyset$ and for any $n \geq 1$,

- if $W_n = \emptyset$ (i.e. all the items arrived up to time n are matched at time n), then we set $B_n = \emptyset$ as well;
- if not, we let $i(n) \leq n$ be the index of the oldest item in line. Then, the word B_n is of length $n - i(n) + 1$, and for any $\ell \in [\![1, n - i(n) + 1]\!]$, we set

$$B_n(\ell) = \begin{cases} V_{i(n)+\ell-1} & \text{if } V_{i(n)+\ell-1} \text{ has not been matched up to time } n;\\ \overline{V_k} & \text{if } V_{i(n)+\ell-1} \text{ is matched at of before time } n, \text{ with item } V_k \text{ (where } k \leq n). \end{cases}$$

In other words, the word B_n gathers the class indexes of all unmatched items entered up to n, and the copies of the class indexes of the items matched after the arrival of the oldest unmatched item at n, at the place of the class index of the item they have been matched to. Observe that we necessarily have that $B_n(1) = V_{i(n)} \in \mathcal{V}$. Moreover, the word B_n necessarily contains all the letters of W_n. More precisely, we have

$$B_n = W_n|_{\mathcal{V}}, \;\; n \geq 0. \tag{10}$$

It is easily seen that $(B_n)_{n\in\mathbb{N}}$ also is a $\mathcal{F}_n$-Markov chain: for any $n \geq 0$, the value of B_{n+1} can be deduced from that of B_n and the class V_{n+1} of the item entered at time $n+1$.

Forward detailed chain. The $\mathbf{V}^*$-valued forward detailed process $(F_n)_{n\in\mathbb{N}}$ is defined as follows: $F_0 = \emptyset$ and for any $n \geq 1$,

- if $W_n = \emptyset$, then we also set $F_n = \emptyset$;
- if not, let $j(n) > n$ be the largest index of an item that is matched with an item entered up to n ($j(n)$ necessarily exists, otherwise all items entered up to n would be have been matched by time n, and we would have $W_n = \emptyset$). Then, the word F_n is of size $j(n) - n$ and for any $\ell \in [\![1, j(n) - n]\!]$, we set

$$F_n(\ell) = \begin{cases} V_{n+\ell} & \text{if } V_{n+\ell} \text{ is not matched with an item arrived up to } n;\\ \overline{V_k} & \text{if } V_{n+\ell} \text{ is matched with item } V_k, \text{ where } k \leq n. \end{cases}$$

In other words, the word F_n contains the copies of all the class indexes of the items entered up to time n and matched after n, together with the class indexes of all unmatched items entered before the last item matched with an item entered up to n. Observe that $F_n\left(j(n)-n\right) \in \overline{\mathcal{V}}$ since by definition, the item $V_{j(n)}$ is matched with some V_k for $k \leq n$, and therefore $F_n\left(j(n)-n\right) = \overline{V}_k$. It is also clear that $\left(F_n\right)_{n\in\mathbb{N}}$ is a $\mathcal{F}_n$-Markov chain, as for any $n \geq 0$, the value of F_{n+1} depends solely on F_n and the class index $V_{n+j(n)+1}$ of the item entered at time $n+j(n)+1$.

Example 2. Consider the matching graph of Figure 1, addressed in Section 5 of [14] (this is the smallest graph that is neither bipartite nor separable). An arrival scenario together with successive values of the natural chain, the backwards and the forwards chain are represented in Figure 2.

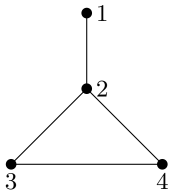

Figure 1: Matching graph of Example 2.

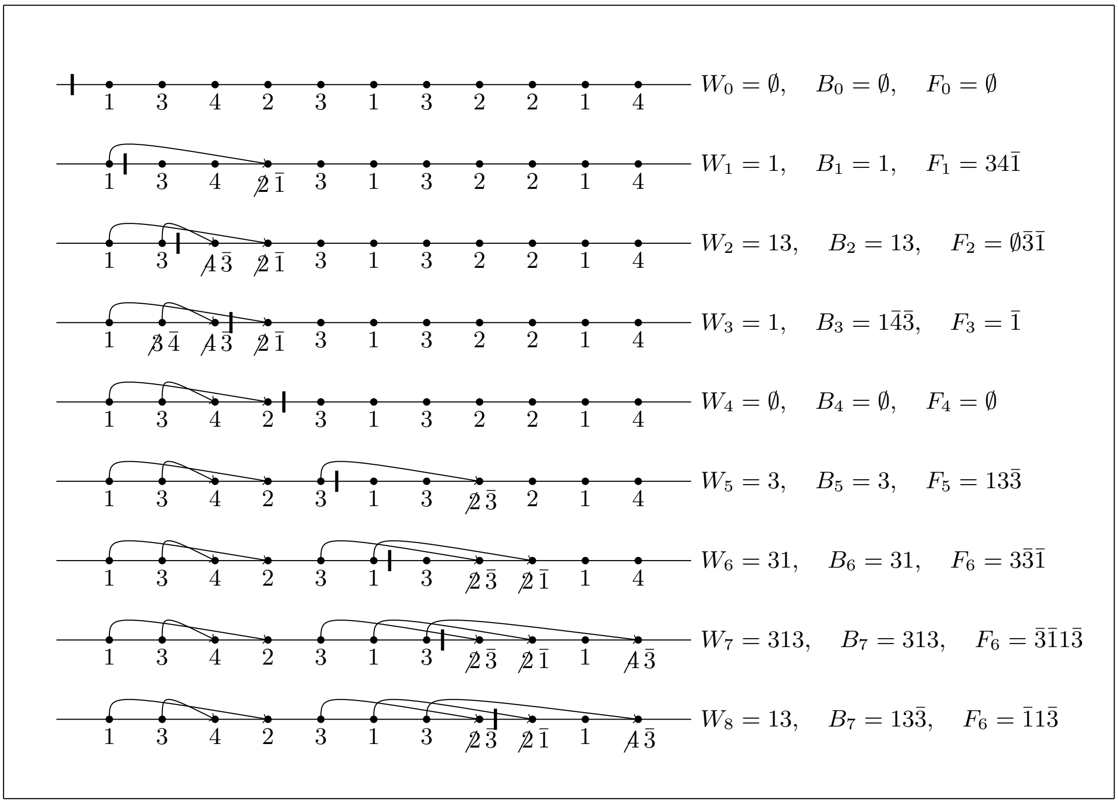

Figure 2: An arrival scenario on the matching graph of Figure 1, and the trajectories of the three Markov chains.

3.3 Reversibility

For both chains $(B_n)_{n\in\mathbb{N}}$ and $(F_n)_{n\in\mathbb{N}}$, a state $\mathbf{w} \in \mathbf{V}^*$ is said admissible if it can be reached by the chain under consideration under FCFM. We denote

$$\text{ADM}_B := \left\{ \mathbf{w} \in \mathbf{V}^* \;:\; \mathbf{w} \text{ is admissible for } (B_n)_{n\in\mathbb{N}} \right\};$$
$$\text{ADM}_F := \left\{ \mathbf{w} \in \mathbf{V}^* \;:\; \mathbf{w} \text{ is admissible for } (F_n)_{n\in\mathbb{N}} \right\}.$$

We have the following result,

Proposition 1. *Suppose that condition (8) holds. Then the Backwards detailed Markov chain $(B_n)_{n\in\mathbb{N}}$ and the Forwards detailed Markov chain $(F_n)_{n\in\mathbb{N}}$ both admit the following unique stationary distribution (defined up to a constant):*

$$\Pi_B(\mathbf{w}) = \prod_{i=1}^{p} \mu(i)^{\|\mathbf{w}\|_i + \|\overline{\mathbf{w}}\|_i}, \tag{11}$$

for any admissible state $\mathbf{w} \in \mathbf{V}^*$ of the respective chain.

Observe that by the very definition (11), the measure of a word $\mathbf{w}$ does not change whenever any of its letters a is exchanged with $\overline{a}$. In particular, we have $\Pi_B(\overleftarrow{\mathbf{w}}) = \Pi_B(\mathbf{w})$ for any $\mathbf{w}$. Before showing Proposition 1 we need to introduce a couple of technical results. Let us first observe that

Lemma 1. *Let $\mathbf{w} = \mathbf{w}_1 ... \mathbf{w}_q \in \mathbf{V}^*$. Then $\mathbf{w} \in \text{ADM}_B$ if, and only if the following two conditions hold,*

$$\forall k, \ell \in [\![1, q]\!] \text{ such that } \mathbf{w}_k \in \mathcal{V} \text{ and } \mathbf{w}_\ell \in \overline{\mathcal{V}},\; \mathbf{w}_k \not\!\!-\, \mathbf{w}_j; \tag{12}$$
$$\forall k \in [\![1, q]\!], \forall j \in [\![k+1, q]\!] \text{ such that } \mathbf{w}_k \in \mathcal{V},\, \mathbf{w}_j \in \overline{\mathcal{V}},\; \mathbf{w}_k \not\!\!-\, \overline{\mathbf{w}_j}. \tag{13}$$

Proof of Lemma 1. The necessity of (12) is obvious: would an element $\mathbf{w}$ of ADM_B contain two compatible letters in $\mathcal{V}$, the two corresponding items would have been matched. Let us prove the necessity of (13). Fix two such indexes k and j in $[\![1.q]\!]$. This means that $V_{i(n)+j-1}$ is matched with an item V_ℓ of class $\overline{\mathbf{w}_j}$. Suppose that $\mathbf{w}_k - \overline{\mathbf{w}_j}$. Then we have the following alternative,

- if $\ell < i(n) + k - 1$, then V_ℓ is present in the system when the item $V_{i(n)+k-1}$ of class $\mathbf{w}_k$ enters. As $\overline{\mathbf{w}_j} - \mathbf{w}_k$, these two items would have been matched;
- if $\ell \in [\![i(n) + k - 1, i(n) + j - 2]\!]$, then V_ℓ finds $V_{i(n)+k-1}$ available in the system, and thus the two items would have been matched;
- if $\ell \in [\![i(n) + j, q]\!]$, then V_ℓ finds both $V_{i(n)+k-1}$ and $V_{i(n)+j-1}$ available in the system, and as the policy is FCFM, choses the oldest one: again, V_ℓ and $V_{i(n)+k-1}$ are matched.

Consequently, $\mathbf{w}_k - \overline{\mathbf{w}_j}$ would imply in all cases that V_ℓ is matched with $V_{i(n)+k-1}$, an absurdity since $V_{i(n)+k-1}$ of class $\mathbf{w}_k$ is still unmatched at n. This completes the proof of necessity.

Regarding sufficiency, fix a state $\mathbf{w}$ satisfying both (12) and (13):

$$\mathbf{w} = b_1\, \overline{a_{11}}\, \overline{a_{12}} \ldots \overline{a_{1k_1}}\, b_2\, \overline{a_{21}} \ldots \overline{a_{2k_2}}\, b_3 \ldots b_q\, \overline{a_{q1}} \ldots \overline{a_{qk_q}},$$

where $q \geq 1$, $k_q \in \mathbb{N}$ for all q, $b_\ell \in \mathcal{V}$ for all ℓ and $a_{\ell j} \in \mathcal{V}$ for all ℓ, j. In particular, from (12) we have that $b_i \not\!\!-\, b_j$ for any $i \neq j$ whereas from (13), $a_{\ell,j} \not\!\!-\, b_i$ for any j and any $i \leq \ell$. Let us show that the chain $(B_n)_{n\in\mathbb{N}}$ can reach the state $\mathbf{w}$. For this we construct inductively an arrival vector V leading to $\mathbf{w}$ from the state $\emptyset$. At first, set

$$V := (V_{n-q+1}, V_{n-q+2}, ..., V_n) = (b_1, ..., b_q).$$

Then, we investigate all elements $\overline{a_{\ell j}}$ from left to right, as follows. We start from a_{11}:

(1) if there is no element $\overline{a_{\ell' j'}}$ to the right of $\overline{a_{11}}$ such that $a_{11} {-} a_{\ell' j'}$, then set $V_{n-q-1} = a_{11}$, $V_{n-q} = b_1$, $V_{n-q+1} = c \in \mathcal{E}(a_{11})$, in a way that the vertex of class a_{11} is matched with that of class b, and we retrieve the letter $\overline{a_{11}}$ to the right of b_1 because the item of class c is matched with a_{11};

(2) if there exists an element $\overline{a_{\ell' j'}}$ to the right of $\overline{a_{11}}$ such that $a_{11} {-} a_{\ell' j'}$, we investigate all terms $\overline{a_{\ell'' j''}}$ to the left of $\overline{a_{\ell' j'}}$. If one of them, $\overline{a_{\ell'' j''}}$ is such that $a_{\ell'' j''} {-} a_{\ell' j'}$, then $a_{\ell' j'}$ could be matched in FCFM with another item arrived before a_{11}. Then we do as in case (1): $V_{n-q-1} = a_{11}$, $V_{n-q} = b_1$, $V_{n-q+1} = c \in \mathcal{E}(a_{11})$;

(3) if there exists an element $\overline{a_{\ell' j'}}$ to the right of $\overline{a_{11}}$ such that $a_{11} {-} a_{\ell' j'}$, and no term $\overline{a_{\ell'' j''}}$ to the left of $\overline{a_{\ell', j'}}$ is such that $a_{\ell'' j''} {-} a_{\ell' j'}$, we interpose a term $a_{\ell' j'}$ in V between b_1 and b_2 (or at the extreme right of V if $q = 1$), and a term $a_{\ell j}$ between $b_{\ell'}$ and $b_{\ell'+1}$ (or at the extreme right of V if $q = \ell'$), in a way that the two corresponding items are matched.

Then, by induction we investigate in the same way all the terms $\overline{a_{\ell j}}$ not yet considered:

(1) if there is no element $\overline{a_{\ell' j'}}$ to the right of $\overline{a_{\ell j}}$ such that $a_{\ell j} {-} a_{\ell' j'}$, then in V we interpose a letter $a_{\ell j}$ just to the left of b_1, and a letter $c \in \mathcal{E}(a_{\ell j})$ to the left of $b_{\ell+1}$ if $\ell < q$ (or at the extreme right of V if $\ell = q$); the items of classes c and $a_{\ell, j}$ are matched and a term $\overline{a_{\ell j}}$ appears at the right place in the detailed state of the system;

(2) we do as in case (1) if there exists an element $\overline{a_{\ell' j'}}$ to the right of $\overline{a_{\ell j}}$ such that $a_{\ell j} {-} a_{\ell' j'}$, but one of the non yet investigated terms $\overline{a_{\ell' j'}}$ to the left of $\overline{a_{\ell' j'}}$ is such that $a_{\ell'' j''} {-} a_{\ell' j'}$.

(3) if there exists an element $\overline{a_{\ell' j'}}$ to the right of $\overline{a_{\ell j}}$ such that $a_{\ell j} {-} a_{\ell' j'}$, and no term $\overline{a_{\ell'' j''}}$ to the left of $\overline{a_{\ell' j'}}$ is such that $a_{\ell'' j''} {-} a_{\ell' j'}$, then in V we interpose a term $a_{\ell' j'}$ just to the left of $b_{\ell+1}$ (or at the extreme right of V if $\ell = q$), and a term $a_{\ell j}$ to the immediate left of $b_{\ell'+1}$ (or at the extreme right of V if $\ell' = q$), in a way that the two corresponding items are matched.

We continue this construction on and on, until all the letters $\overline{a_{\ell j}} \in \mathbf{w}|_{\overline{\mathcal{V}}}$ are investigated and the corresponding items are matched. The final arrival vector V that we obtain is of size q', where

$$q' \in \left[\!\!\left[q + \sum_{\ell=1}^{q} k_\ell, q + 2\sum_{\ell=1}^{q} k_\ell \right]\!\!\right].$$

Indeed, the number of items added to V is at least equal to the number of letters of $\mathbf{w}|_{\overline{\mathcal{V}}}$, and at most equal to twice the latter number (which is the case if all the corresponding items entered the system before the item of class b_1 and are matched after the arrival time of the latter). Finally, for any $n \geq q + 2\sum_{\ell=1}^{q} k_\ell$, if $B_{n-q'} = \emptyset$, an arrival scenario V for the q' following time epochs yields to a state $B_n = \mathbf{w}$. This concludes the proof. ■

As a consequence,

Lemma 2. *The two subsets* ADM_B *and* ADM_F *are isomorphic. More precisely, the following is a one-to-one relation,*

$$\begin{cases} \text{ADM}_B & \longleftrightarrow \text{ADM}_F \\ \mathbf{w} & \longleftrightarrow \overleftarrow{\mathbf{w}}. \end{cases}$$

Proof. From Lemma 1, it is sufficient to prove that a state $\mathbf{w}$ belongs to ADM_F if and only if $\overleftarrow{\mathbf{w}}$ satisfies both (12) and (13). The proof of this statement is similar to that of Lemma 1. ■

We can now state the following strong connexion between the dynamics of $(B_n)_{n\in\mathbb{N}}$ and $(F_n)_{n\in\mathbb{N}}$,

Proposition 2. *Let Π_B be the measure on $\mathbf{V}^*$ defined by (11). Then for any two admissible states $\mathbf{w}, \mathbf{w}' \in \mathbf{V}^*$ for $(B_n)_{n\in\mathbb{N}}$, the states $\overleftarrow{\overline{\mathbf{w}}}$ and $\overleftarrow{\mathbf{w}'}$ are admissible for $(F_n)_{n\in\mathbb{N}}$ and we have that*

$$\Pi_B(\mathbf{w})\mathbb{P}\left[B_{n+1} = \mathbf{w}' | B_n = \mathbf{w}\right] = \Pi_B\left(\overleftarrow{\mathbf{w}'}\right)\mathbb{P}\left[F_{n+1} = \overleftarrow{\overline{\mathbf{w}}} | F_n = \overleftarrow{\mathbf{w}'}\right]. \tag{14}$$

Proof of Proposition 2. Fix $\mathbf{w} \in \text{ADM}_B$ (so that $\overleftarrow{\overline{\mathbf{w}}} \in \text{ADM}_B$ from Lemma 2). We address the 5 possible cases for the transition of $(B_n)_{n\in\mathbb{N}}$. In cases (1)-(4) hereafter, we assume that $\mathbf{w} \neq \emptyset$ and set $\mathbf{w} = \mathbf{w}_1...\mathbf{w}_q \in \mathbf{V}^*$. Remember that $\mathbf{w}_1 \in \mathcal{V}$.

(1) Let us first address the case where $\mathbf{w}' = \mathbf{w}a$, for some $a \in \mathcal{V}$. Plainly, such a state is admissible if and only if $a \in \mathcal{E}\left(\mathcal{V}(w)\right)^c$. The backwards chain moves from $\mathbf{w}$ to $\mathbf{w}a$ at $n+1$ whenever $V_{n+1} = a$, so we have that

$$\mathbb{P}\left[B_{n+1} = \mathbf{w}a | B_n = \mathbf{w}\right] = \mu(a).$$

On the other hand, $F_n = \overleftarrow{\mathbf{w}a} = \overline{a}\,\overline{\mathbf{w}_q}\,\overline{\mathbf{w}_{q-1}}...\overline{\mathbf{w}_2}\,\overline{\mathbf{w}_1}$ entails that the item entering at $n+1$ is matched with an item of class a entered before or at time n. Therefore we necessarily have that $F_{n+1} = \overline{\mathbf{w}_q}\,\overline{\mathbf{w}_{q-1}}...\overline{\mathbf{w}_2}\,\overline{\mathbf{w}_1} = \overleftarrow{\overline{\mathbf{w}}}$, in other words

$$\begin{aligned}\Pi_B\left(\overleftarrow{\mathbf{w}a}\right)\mathbb{P}\left[F_{n+1} = \overleftarrow{\overline{\mathbf{w}}} | F_n = \overleftarrow{\mathbf{w}a}\right] &= \Pi_B\left(\overleftarrow{\mathbf{w}a}\right)\\ &= \Pi_B\left(\overleftarrow{\overline{\mathbf{w}}}\right)\mu(a)\\ &= \Pi_B(\mathbf{w})\mu(a) = \Pi_B(\mathbf{w})\mathbb{P}\left[B_{n+1} = \mathbf{w}a | B_n = \mathbf{w}\right].\end{aligned}$$

(2) Suppose now that $\mathbf{w}' = \mathbf{w}_1...\mathbf{w}_{k-1}\,\overline{a}\,\mathbf{w}_{k+1}...\mathbf{w}_q\overline{\mathbf{w}_k}$. This means that $\mathbf{w}_k \in \mathcal{V}$ and that the item V_{n+1} is of class a, where $a \in \mathcal{E}\left(\mathbf{w}_k\right) \cap \mathcal{E}\left(\mathcal{V}\left(\mathbf{w}_1....\mathbf{w}_{k-1}\right)\right)^c$, so that in FCFM, V_{n+1} is matched with the item $V_{i(n)+k-1}$ of class $\mathbf{w}_k$. Suppose that

$$F_n = \overleftarrow{\mathbf{w}'} = \mathbf{w}_k\overline{\mathbf{w}_q}\,\overline{\mathbf{w}_{q-1}}\,...\overline{\mathbf{w}_{k+1}}\,a\,\overline{\mathbf{w}_{k-1}}\,...\,\overline{\mathbf{w}_1}.$$

Then from Lemma 1, $\mathbf{w}_k$ is not adjacent to any of the elements of $\mathcal{V}\left(\overline{\mathbf{w}_q}\,\overline{\mathbf{w}_{q-1}}\,...\,\overline{\mathbf{w}_{k+1}}\right)$. But $\mathbf{w}_k - a$, so the item V_{n+1} of class $\mathbf{w}_k$ is matched with the item V_{n+k+2} of class a, and we have with probability 1,

$$F_{n+1} = \overline{\mathbf{w}_q}\,\overline{\mathbf{w}_{q-1}}\,...\overline{\mathbf{w}_{k+1}}\,\overline{\mathbf{w}_k}\,\overline{\mathbf{w}_{k-1}}\,...\,\overline{\mathbf{w}_1} = \overleftarrow{\overline{w}}.$$

Therefore, in this case,

$$\begin{aligned}\Pi_B\left(\overleftarrow{\mathbf{w}'}\right)\mathbb{P}\left[F_{n+1} = \overleftarrow{\overline{\mathbf{w}}} | F_n = \overleftarrow{\mathbf{w}'}\right] &= \Pi_B\left(\overleftarrow{\mathbf{w}'}\right)\\ &= \Pi_B\left(\overleftarrow{\overline{\mathbf{w}}}\right)\mu(a)\\ &= \Pi_B(\mathbf{w})\mu(a) = \Pi_B(\mathbf{w})\mathbb{P}\left[B_{n+1} = \mathbf{w}' | B_n = \mathbf{w}\right].\end{aligned}$$

(3) Now, suppose that $\mathbf{w}' = \mathbf{w}_k\mathbf{w}_{k+1}...\mathbf{w}_q\overline{\mathbf{w}_1}$ for some $k \in [\![1, q]\!]$. This means that the class of item V_{n+1} belongs to $\mathcal{E}\left(\mathbf{w}_1\right)$, so V_{n+1} is matched with the oldest item in line $V_{i(n)}$. Then $\mathbf{w}_2, \mathbf{w}_3, ..., \mathbf{w}_{k-1}$ all belong to $\overline{\mathcal{V}}$, and so $\mathbf{w}_k \in \mathcal{V}$ and is the class of the oldest item in line after $V_{i(n)}$, now becoming the new oldest one. Suppose that

$$F_n = \overleftarrow{\mathbf{w}'} = \mathbf{w}_1\overline{\mathbf{w}_q}\,\overline{\mathbf{w}_{q-1}}\,...\overline{\mathbf{w}_{k+1}}\,\overline{\mathbf{w}_k}.$$

Applying again Lemma 1, we obtain that $\mathbf{w}_1$ is not adjacent to any of the elements of the set $\mathcal{V}\left(\overline{\mathbf{w}_q}\,\overline{\mathbf{w}_{q-1}}\,...\,\overline{\mathbf{w}_k}\right)$, so the state $\overleftarrow{\mathbf{w}'}$ is admissible. All the same, again in view of Lemma 1, $\mathbf{w}_1$ is not adjacent to any of the elements $\overline{\mathbf{w}_2}, \overline{\mathbf{w}_3}, ..., \overline{\mathbf{w}_{k-1}}$, which are all of $\mathcal{V}$. So we obtain

$$F_{n+1} = \overline{\mathbf{w}_q}\,\overline{\mathbf{w}_{q-1}}\,...\overline{\mathbf{w}_{k+1}}\,\overline{\mathbf{w}_k}\,\overline{\mathbf{w}_{k-1}}\,...\overline{\mathbf{w}_2}\,\overline{\mathbf{w}_1} = \overleftarrow{\overline{\mathbf{w}}}$$

if the incoming items V_{n+2},...,V_{n+k-1} are of respective classes $\mathbf{w}_2, ..., \mathbf{w}_{k-1}$ and V_{n+k-2} is of a class belonging to $\mathcal{E}(\mathbf{w}_1)$. This occurs with probability

$$\mu\Big(\overline{\mathbf{w}_2}\Big)\mu\Big(\overline{\mathbf{w}_3}\Big)...\mu\Big(\overline{\mathbf{w}_{k-1}}\Big)\mu\Big(\mathcal{E}(\mathbf{w}_1)\Big).$$

Gathering all the above we obtain that

$$\begin{aligned}\Pi_B\left(\overleftarrow{\overline{\mathbf{w}'}}\right)\mathbb{P}\left[F_{n+1}=\overleftarrow{\overline{\mathbf{w}}}|F_n=\overleftarrow{\overline{\mathbf{w}'}}\right] &= \Pi_B\left(\overleftarrow{\overline{\mathbf{w}'}}\right)\mu\Big(\overline{\mathbf{w}_2}\Big)\mu\Big(\overline{\mathbf{w}_3}\Big)...\mu\Big(\overline{\mathbf{w}_{k-1}}\Big)\mu\Big(\mathcal{E}(\mathbf{w}_1)\Big)\\ &= \Pi_B\left(\overleftarrow{\overline{\mathbf{w}}}\right)\mu\Big(\mathcal{E}(\mathbf{w}_1)\Big)\\ &= \Pi_B(\mathbf{w})\mu\Big(\mathcal{E}(\mathbf{w}_1)\Big)= \Pi_B(\mathbf{w})\mathbb{P}\left[B_{n+1}=\mathbf{w}'|B_n=\mathbf{w}\right].\end{aligned}$$

(4) Suppose now that $\mathbf{w}'=\emptyset$, which is possible only if $\mathbf{w}_1\in\mathcal{V}$, the incoming item V_{n+1} belongs to $\mathcal{E}(\mathbf{w}_1)$, and if $q\ge 2$, $\mathbf{w}_2,...,\mathbf{w}_q\in\overline{\mathcal{V}}$, which implies again from Lemma 1 that $\mathbf{w}_1 \not- \overline{\mathbf{w}_j}$ for any $j\in[\![2,q]\!]$. Thus, $F_n=\emptyset$ leads to the state $F_{n+1}=\overleftarrow{\overline{\mathbf{w}}}=\overline{\mathbf{w}_q}\,\overline{\mathbf{w}_{q-1}}\,....\,\overline{\mathbf{w}_2}\,\overline{\mathbf{w}_1}$ if and only if V_{n+1} is of class $\mathbf{w}_1$, and then V_{n+2} is of class $\overline{\mathbf{w}_q}$, V_{n+3} is of class $\overline{\mathbf{w}_{q-1}}$, and so on ..., V_{n+q} is of class $\overline{\mathbf{w}_2}$ and V_{n+q+1} is of a class belonging to $\mathcal{E}(\mathbf{w}_1)$. This event occurs with probability $\mu(1)\mu(\overline{\mathbf{w}_q})....\mu(\overline{\mathbf{w}_2})\,\mu(\mathcal{E}(\mathbf{w}_1))$. So we obtain

$$\begin{aligned}\Pi_B(\emptyset)\,\mathbb{P}\left[F_{n+1}=\overleftarrow{\overline{\mathbf{w}}}|F_n=\emptyset\right] &= \mu(\mathbf{w}_1)\,\mu\Big(\overline{\mathbf{w}_2}\Big)\mu\Big(\overline{\mathbf{w}_3}\Big)...\mu\Big(\overline{\mathbf{w}_q}\Big)\mu\Big(\mathcal{E}(\mathbf{w}_1)\Big)\\ &= \Pi_B(\mathbf{w})\mu\Big(\mathcal{E}(\mathbf{w}_1)\Big)= \Pi_B(\mathbf{w})\mathbb{P}\left[B_{n+1}=\mathbf{w}'|B_n=\mathbf{w}\right].\end{aligned}$$

(5) The only case that remains to be treated is when $\mathbf{w}=\emptyset$. Then for any $a\in\mathcal{V}$, we obtain $B_{n+1}=a$ provided that V_{n+1} is of class a, which occurs with probability $\mu(a)$. Then, $F_n=\bar{a}$ means that V_{n+1} is matched with an item of class a that was entered before n. Then we necessarily have that $F_{n+1}=\emptyset$, and thus

$$\Pi_B(\bar{a})\,\mathbb{P}\left[F_{n+1}=\emptyset|F_n=\bar{a}\right]=\Pi_B(a)=\mu(a)=\Pi_B(\emptyset)\mathbb{P}\left[B_{n+1}=a|B_n=\emptyset\right].$$

This completes the proof. ■

We can now turn to the proof of Proposition 1.

Proof of Proposition 1. We first show that the measure Π_B defined by (11) is finite on ADM_B under condition (8). From Lemma 1, we know that for any word $\mathbf{w}$ in ADM_B, the letters of $\mathcal{V}$ present in $\mathbf{w}$ form an independent set of $\mathcal{V}$, that is $\mathcal{V}(\mathbf{w})\in\mathbb{I}(G)$, and intermediate letters of $\bar{\mathcal{V}}$ whose counterparts in $\mathcal{V}$ are not adjacent of any prior letter of $\mathbf{w}$ in $\mathcal{V}$. Therefore we have that

$$\begin{aligned}&\Pi_B(\text{ADM}_B)=\Pi_B(\emptyset)\\ &+\sum_{\mathcal{I}\in\mathbb{I}(G)}\sum_{q\in\mathbb{N}_+}\sum_{\substack{(b_1,...,b_q)\in\mathcal{I}^q,\\ (k_1,k_2,...,k_q)\in\mathbb{N}^q}}\sum_{\substack{(a_{i1},..,a_{ik_i},a_{21},..,a_{q1},..,a_{qk_q})\in\mathcal{V}^{\sum_{l=1}^q k_l}:\\ a_{ij}\in\mathcal{E}(\{b_1,...,b_i\})^c\text{for all } i\in[\![1,q]\!],j\in[\![1,k_i]\!]}}\Pi_B\left(b_1\overline{a_{11}}\,\overline{a_{12}}...\overline{a_{1k_1}}b_2....b_q\overline{a_{q1}}...\overline{a_{qk_q}}\right)\\ &\quad=1+\sum_{\mathcal{I}\in\mathbb{I}(G)}\sum_{q\in\mathbb{N}_+}\sum_{(b_1,...,b_q)\in\mathcal{I}^q}\prod_{i=1}^q\left(\mu(b_i)\left(1+\sum_{k\in\mathbb{N}_+}\sum_{(a_{i1},...,a_{ik})\in(\mathcal{E}(\{b_1,...,b_i\})^c)^k}\prod_{\ell=1}^k\mu(a_{i\ell})\right)\right)\\ &\quad\le 1+\sum_{\mathcal{I}\in\mathbb{I}(G)}\sum_{q\in\mathbb{N}_+}\left(1+\sum_{k\in\mathbb{N}*}\sum_{(a_1,...,a_k)\in(\mathcal{E}(\mathcal{I})^c)^k}\prod_{\ell=1}^k\mu(a_\ell)\right)^q\left(\sum_{(b_1,...,b_q)\in\mathcal{I}^q}\prod_{i=1}^q\mu(b_i)\right)\\ &\quad\le\sum_{\mathcal{I}\in\mathbb{I}(G)}\sum_{q\in\mathbb{N}}\left(\sum_{k\in\mathbb{N}}\mu\left(\mathcal{E}(\mathcal{I})^c\right)^k\right)^q\mu(\mathcal{I})^q\\ &\quad=\sum_{\mathcal{I}\in\mathbb{I}(G)}\sum_{q\in\mathbb{N}}\left(\frac{\mu(\mathcal{I})}{\mu\left(\mathcal{E}(\mathcal{I})\right)}\right)^q,\end{aligned}$$

which is clearly finite under (8).

Let α be the normalizing constant of Π_B. Then it suffices to apply Kelly's Lemma ([12], Section 1.7): define for any two admissible states $\mathbf{w}, \mathbf{w}' \in \text{ADM}_B$,

$$P_{\mathbf{w}',\mathbf{w}} = \frac{\mathbb{P}\left[B_{n+1} = \mathbf{w}' \mid B_n = \mathbf{w}\right] \alpha \Pi_B(\mathbf{w})}{\alpha \Pi_B(\mathbf{w}')}. \tag{15}$$

Then, Π_B is the only stationary distribution of $(B_n)_{n\in\mathbb{N}}$ if P defines a transition operator on $\mathbf{V}^*$. But this is a simple consequence of Proposition 2: for any $\mathbf{w}' \in \mathbf{V}^*$, we have that

$$\begin{aligned}
\sum_{w\in\mathbf{V}^*} P_{\mathbf{w}',\mathbf{w}} &= \sum_{\mathbf{w}\in\mathbf{V}^*} \frac{\mathbb{P}\left[B_{n+1} = \mathbf{w}' \mid B_n = \mathbf{w}\right] \alpha \Pi_B(\mathbf{w})}{\alpha \Pi_B(\mathbf{w}')}\\
&= \sum_{\mathbf{w}\in\mathbf{V}^*} \frac{\Pi_B\left(\overleftarrow{\overline{\mathbf{w}'}}\right) \mathbb{P}\left[F_{n+1} = \overleftarrow{\overline{\mathbf{w}}} | F_n = \overleftarrow{\overline{\mathbf{w}'}}\right]}{\Pi_B(\mathbf{w}')}\\
&= \sum_{\mathbf{w}\in\mathbf{V}^*} \mathbb{P}\left[F_{n+1} = \overleftarrow{\overline{\mathbf{w}}} | F_n = \overleftarrow{\overline{\mathbf{w}'}}\right]\\
&= 1,
\end{aligned}$$

where we use the fact that $(F_n)_{n\in\mathbb{N}}$ is a Markov chain and Lemma 2. This concludes the proof for $(B_n)_{n\in\mathbb{N}}$. We can now reverse the argument by exchanging the roles of B_n, B_{n+1} and F_n, F_{n+1} in the definition (15). This entails that $(F_n)_{n\in\mathbb{N}}$ is the reversed Markov chain of $(B_n)_{n\in\mathbb{N}}$, on a sample space where arrivals are reversed in time and exchanged with their match. In particular $(F_n)_{n\in\mathbb{N}}$ also has the same stationary probability Π_B. ■

3.4 Proof of Theorem 1

We are now in position to prove the main result of this section. As Π_B is the only stationary distribution of $(B_n)_{n\in\mathbb{N}}$, from (10) it is clearly sufficient to check that

$$\Pi_W(w) = \sum_{\mathbf{w}\in\text{ADM}_B:\,\mathbf{w}|_{\mathcal{V}}=w} \Pi_B(\mathbf{w}) \quad \text{for any } w \in \mathcal{V}^*.$$

Let $w = w_1...w_q \in \mathcal{V}^*$. Then, from Lemma 1, any $\mathbf{w} \in \text{ADM}_B$ such that $\mathbf{w}|_{\mathcal{V}} = w$ is of the form

$$\mathbf{w} = w_1\overline{a_{11}}\,\overline{a_{12}}\,...\,\overline{a_{1k_1}}\,w_2\,\overline{a_{21}}...\overline{a_{2k_2}}\,w_3\,...\,w_q\overline{a_{q1}}\,...\,\overline{a_{qk_q}},$$

where any of the elements $a_{\ell j}$ is such that $a_{\ell j} \not- w_i$ for any $i \leq \ell$. We therefore obtain that

$$\begin{aligned}
\sum_{\mathbf{w}\in\text{ADM}_B:\,\mathbf{w}|_{\mathcal{V}}=w} \Pi_B(\mathbf{w}) &= \alpha \prod_{\ell=1}^{q} \left(\mu(w_\ell)\left(1 + \sum_{k\in\mathbb{N}_+}\ \sum_{(a_{\ell 1},...,a_{\ell k})\in(\mathcal{E}(\{w_1,...,w_\ell\})^c)^k}\ \prod_{j=1}^{k} \mu\left(a_{\ell j}\right)\right)\right)\\
&= \alpha \prod_{\ell=1}^{q} \left(\mu(w_\ell) \sum_{k\in\mathbb{N}} \left(\mu\left(\mathcal{E}\left(\{w_1,...,w_\ell\}\right)^c\right)\right)^k\right)\\
&= \alpha \prod_{\ell=1}^{q} \frac{\mu(w_\ell)}{\mu\left(\mathcal{E}\left(\{w_1,...,w_\ell\}\right)\right)} = \Pi_W(w),
\end{aligned}$$

which concludes the proof of Theorem 1.

4 Coupling in the General Matching model

In this Section we construct explicitly for a wide range of models, a stationary version of the buffer-content process $(W_n)_{n\in\mathbb{N}}$. This will be done beyond the iid case, by the strong backwards coupling method of Borovkov and Foss. By doing so, we show in which cases a unique stationary buffer content exists and thereby, a unique stationary complete matching, in a sense that will be specified below. The argument will be more easily developed in an ergodic-theoretical framework that we introduce in Section 4.1. Then, we show in Section 4.2 that many matching policies (including ML and FCFM) satisfy a remarkable property of sub-additivity, which will be the essential tool of our proofs, together with the existence of (strong) erasing words in non-bipartite graphs, introduced in Section 4.3. Our coupling results will then be given in Section 4.4 under assumption (H1') or (H1"), and in Section 4.5 under assumption (IID).

4.1 General settings

The general matching model is intrinsically periodic: arrivals are simple but departure are pairwise, so the size of the system has the parity of the original size at all even times - in particular a system started empty can possibly be empty only every other time, and two systems cannot couple unless their initial sizes have almost surely the same parity. To circumvent this difficulty, at first we track the system only at even times. Equivalently, we change the time scale and see the arrivals by *pairs* of items (as in the original bipartite matching model [9, 4, 2]) which play different roles: the first one investigates first all possible matchings in the buffer upon its arrival according to ϕ, before possibly considering the second one if no match is available, whereas the second one applies ϕ to all available items, including the first one. By doing so, it is immediate to observe that we obtain exactly a GM model as presented thus far, only do we track it at even times.

Throughout this sub-section, suppose that assumption (H1') holds. To formalize the above observation, we let $(U_n)_{n\in\mathbb{N}}$ be the buffer content sequence at even times (we will use the term "*even* buffer content"), that is, $U_n = W_{2n}$, $n \in \mathbb{N}$. We will primarily construct a (possibly unique) stationary version of the sequence $(U_n)_{n\in\mathbb{N}}$ of even buffer content, by coupling. For this, we work on the canonical space $\Omega^0 := (\mathcal{V}\times\mathcal{S}\times\mathcal{V}\times\mathcal{S})^{\mathbb{Z}}$ of the bi-infinite sequence $((V_{2_n},\Sigma_{2n},V_{2n+1},\Sigma_{2n+1}))_{n\in\mathbb{Z}}$, on which we define the bijective shift operator θ by $\theta\left((\omega_n)_{n\in\mathbb{Z}}\right) = (\omega_{n+1})_{n\in\mathbb{Z}}$ for all $(\omega_n)_{n\in\mathbb{Z}} \in \Omega$. We denote by θ^{-1} the reciprocal operator of θ, and by θ^n and θ^{-n} the n-th iterated of θ and θ^{-1}, respectively, for all $n \in \mathbb{N}$. We equip Ω^0 with a sigma-field $\mathscr{F}^0$ and with the image probability measure $\mathbb{P}^0$ of the sequence $((V_{2_n},\Sigma_{2n},V_{2n+1},\Sigma_{2n+1}))_{n\in\mathbb{Z}}$ on Ω^0. Observe that under (H1'), $\mathbb{P}^0$ is compatible with the shift, i.e. for any $\mathscr{A} \in \mathscr{F}$, $\mathbb{P}^0\left[\mathscr{A}\right] = \mathbb{P}^0\left[\theta^{-1}\mathscr{A}\right]$ and any θ-invariant event $\mathscr{B}$ (*i.e.* such that $\mathscr{B} = \theta^{-1}\mathscr{B}$) is either $\mathbb{P}^0$-negligible or almost sure. Altogether, the quadruple $\mathscr{Q}^0 := \left(\Omega^0,\mathscr{F}^0,\mathbb{P}^0,\theta\right)$ is thus stationary ergodic, and will be referred to as *Palm space* of the input at even times. For more details about this framework, we refer the reader to the monographs [8], [3] (Sections 2.1 and 2.5) and [17] (Chapter 7).

Let the r.v. $\left(V^0,\Sigma^0,V^1,\Sigma^1\right)$ be the projection of sample paths over their 0-coordinate. Thus $\left(V^0,\Sigma^0,V^1,\Sigma^1\right)$ can be interpreted as the input brought to the system at time 0, i.e. at 0 an item of class V^0 and then an item of class V^1 enter the systems, having respective lists of preference Σ^0 and Σ^1 over $\mathcal{V}$, and the order of arrival between the two is kept track of (V^0 and *then* V^1). Then for any $n \in \mathbb{Z}$, the r.v. $\left(V^0\circ\theta^n,\, \Sigma^0\circ\theta^n,\, V^1\circ\theta^n,\, \Sigma^1\circ\theta^n\right)$ corresponds to the input brought to the system at time n. Define the subset

$$\mathbb{W}_2 = \{w \in \mathbb{W} \,:\, |w| \text{ is even }\}.$$

For any $\mathbb{W}_2$-valued r.v. Y, we define on Ω^0 the sequence $\left(U_n^{[Y]}\right)_{n\in\mathbb{N}}$ as the even buffer content sequence of the model initiated at value Y, i.e.

$$\begin{cases} U_0^{[Y]} &= Y;\\ U_{n+1}^{[Y]} &= \left(U_n^{[Y]} \odot_\phi \left(V^0\circ\theta^n,\Sigma^0\circ\theta^n\right)\right)\odot_\phi \left(V^1\circ\theta^n,\Sigma^1\circ\theta^n\right),\, n\in\mathbb{N}, \end{cases} \qquad \mathbb{P}^0-\text{ a.s.} \tag{16}$$

A stationary version of (16) is thus a recursion satisfying (16) and compatible with the shift, i.e. a sequence $(U \circ \theta^n)_{n\in\mathbb{Z}}$, where the $\mathbb{W}_2$-valued r.v. U satisfies the equation

$$U \circ \theta = \left(U \odot_\phi (V^0, \Sigma^0)\right) \odot_\phi (V^1, \Sigma^1),\ \mathbb{P}^0 - \text{ a.s.}, \tag{17}$$

see Section 2.1 of [3] for details. To any stationary buffer content $(U \circ \theta^n)_{n\in\mathbb{Z}}$ corresponds a unique stationary probability for the sequence $(W_{2n})_{n\in\mathbb{N}}$ on the original probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Moreover, provided that $\mathbb{P}^0[U = \emptyset] > 0$, the bi-infinite sequence $(U \circ \theta^n)_{n\in\mathbb{Z}}$ corresponds on $\mathscr{Q}^0$ to a unique stationary matching by ϕ (we write a *ϕ-matching*), that is obtained by using the (bi-infinite) family of construction points $\{n \in \mathbb{Z} : U \circ \theta^n = \emptyset\}$, and matching the incoming items by ϕ, within each finite block between construction points.

In other words, obtaining constructively a stationary buffer-content at even times and thereby, a stationary ϕ-matching on $\mathbb{Z}$, amounts to solving on $\mathscr{Q}^0$ the almost-sure equation (17). This will be done by constructing the associated backwards scheme, as in [13]: for a $\mathbb{W}_2$-valued r.v. Y and any fixed $n \geq 0$, the r.v. $U_n^{[Y]} \circ \theta^{-n}$ represents the even buffer content at time 0, whenever initiated at value Y, n time epochs in the past. Loynes' theorem shows the existence of a solution to (17), as the $\mathbb{P}^0$-almost sure limit of the non-decreasing sequence $\left(U_n^{[\emptyset]} \circ \theta^{-n}\right)_{n\in\mathbb{N}}$, whenever the random map driving the recursion $(U_n)_{n\in\mathbb{N}}$ is almost surely non-decreasing in the state variable. As the present model does not exhibit any particular monotonic structure, such a result is *a pariori* out of reach. We thus resort to Borovkov's and Foss theory of Renovation, see [6, 7].

Following [5], we say that the buffer content sequence $\left(U_n^{[Y]}\right)_{n\in\mathbb{N}}$ converges with *strong backwards coupling* to the stationary buffer content sequence $(U \circ \theta^n)_{n\in\mathbb{N}}$ if, $\mathbb{P}^0$-almost surely there exists $N^* \geq 0$ such that for all $n \geq N^*$, $U_n^{[Y]} = U$. Note that strong backwards coupling implies the (forward) coupling between $\left(U_n^{[Y]}\right)_{n\in\mathbb{N}}$ and $(U \circ \theta^n)_{n\in\mathbb{N}}$, i.e. there exists a.s. an integer $N \geq 0$ such that $U_n^{[Y]} = U \circ \theta^n$ for all $n \geq N$. In particular the distribution of $U_n^{[Y]}$ converges in total variation to that of U, see e.g. Section 2.4 of [3].

4.2 Sub-additivity

We show hereafter that most of the models we have introduced above satisfy a sub-additivity property that will prove crucial in the main results of this section.

Definition 2 (Sub-additivity)**.** *An admissible matching policy ϕ is said to be* sub-additive *if, for all $z', z'' \in \mathcal{V}^*$, for all $\varsigma', \varsigma'' \in \mathcal{S}^*$ whose letters are drawn by ν_ϕ and such that $|\varsigma'| = |z'|$ and $|\varsigma''| = |z''|$, we have that*

$$|Q_\phi(z'z'', \varsigma'\varsigma'')| \leq |Q_\phi(z', \varsigma')| + |Q_\phi(z'', \varsigma'')|.$$

Proposition 3. *The matching policies* FCFS*,* LCFS*, Random (including Priorities and* U*) and* ML *are sub-additive.*

Before turning to the proof of Proposition 3 in the remainder of this section, let us show by a counter-example that, on the other hand, the policy 'Match the Shortest' is not sub-additive:

Example 3 (MS is not sub-additive)**.** *Take as a matching graph, the graph of Figure 1, and the arrival scenario depicted in Figure 3.*

In the first case above we let $z' = 11$ and $z'' = 133224$. Then, for any ς' and ς'' we get $Q_{\text{MS}}(z', \varsigma') = 11$ and $Q_{\text{MS}}(z'', \varsigma'') = \emptyset$, whereas performing $Q_{\text{MS}}(z'z'', \varsigma'\varsigma'') = 1114$. So

$$4 = |Q_{\text{MS}}(z'z'', \varsigma'\varsigma'')| = 4 > 2 = |Q_{\text{MS}}(z', \varsigma')| + |Q_{\text{MS}}(z', \varsigma')| = 2.$$

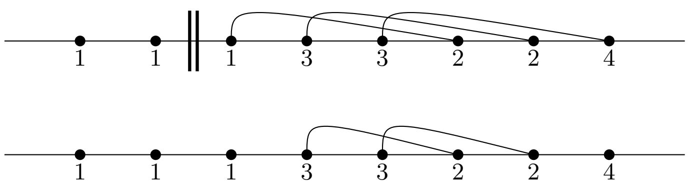

Figure 3: 'Match the Shortest' is not sub-additive.

4.2.1 Non-expansiveness

In the framework of stochastic recursions, the *non-expansiveness* property with respect to the ℓ_1-norm, as introduced by Crandall and Tartar [10], amounts to the 1-Lipschitz property of the driving map of the recursion. Similarly,

Definition 3 (Non-expansiveness). *A class-admissible policy ϕ is said* non-expansive *if for any x and x' in $\mathbb{X}$, any $v \in \mathcal{V}$ and any $\sigma \in \mathcal{S}$ that can be drawn by ν_ϕ,*

$$\|x' \circledcirc_\phi (v,\sigma) - x \circledcirc_\phi (v,\sigma)\| \leq \|x' - x\|. \tag{18}$$

Proposition 4. *Any random matching policy (in particular, priority and* U*) is non-expansive.*

Proof. The result has been proven for priority and U in [15]: this is precisely the inductive argument, respectively in the proofs of Lemma 4 and Lemma 7 therein. As is easily seen, the same argument can be generalized to any random policy ϕ, once the list of preference that is drawn from ν_ϕ is common to both systems. Indeed, the following consistency property holds: for any states x and x', any incoming item v and any list of preferences σ drawn from ν_ϕ,

$$\left[\left\{p_\phi(x,v,\sigma), p_\phi(x',v,\sigma)\right\} \subset \mathcal{P}(x,v) \cap \mathcal{P}(x',v)\right] \implies \left[p_\phi(x,v,\sigma) = p_\phi(x',v,\sigma)\right], \tag{19}$$

in other words, the choice of match of v cannot be different in the two systems, if both options were available in both systems. The result follows for any random policy. ■

Proposition 5. ML *is non-expansive.*

Proof. The proof is similar to that for random policies, except for the consistency property (19), which does not hold in this case. Specifically, an entering item can be matched with items of two different classes in the two systems, whereas the queues of these two classes are non-empty in both systems. Let us consider that case: specifically, a v-item enters the system, and for a common draw σ according to the (uniform) distribution ν_{ML}, we obtain $p_{\text{ML}}(x,v,\sigma) = k$ and $p_{\text{ML}}(x',v,\sigma) = k'$ for $\{k,k'\} \subset \mathcal{P}(x,v) \cap \mathcal{P}(x',v)$ and $k \neq k'$. Thus we have

$$\|x' \circledcirc_{\text{ML}} (v,\sigma) - x \circledcirc_{\text{ML}} (v,\sigma)\| = \sum_{i \neq k,k'} |x(i) - x'(i)| + R, \tag{20}$$

where

$$R = |(x(k)-1) - x'(k)| + |x(k') - (x'(k')-1)|\,.$$

We are in the following alternatives,

1. if $x(k) > x'(k)$ and $x'(k') > x(k')$, then

$$R = (x(k) - 1 - x'(k)) + (x'(k') - 1 - x(k')) = |x(k) - x'(k)| + |x(k') - x'(k')| - 2.$$

2. if $x(k) \le x'(k)$ and $x'(k') > x(k')$, then

$$R = (x'(k) - x(k) + 1) + (x(k') - 1 - x'(k')) = |x(k) - x'(k)| + |x(k') - x'(k')|\,.$$

3. if $x(k) > x'(k)$ and $x'(k') \le x(k')$, we also have

$$R = (x(k) - 1 - x'(k)) + (x(k') - x'(k') + 1) = |x(k) - x'(k)| + |x(k') - x'(k')|\,.$$

Observe that the case $x(k) \le x'(k)$ and $x'(k') \le x(k')$ cannot occur. Indeed, by the definition of ML we have that

$$x(k') \le x(k) \text{ and } x'(k) \le x'(k'),$$

which would imply in turn that

$$x(k) = x(k') = x'(k) = x'(k').$$

This is impossible since, in that case, under the common list of preferences σ both systems would have chosen the same match for the new v-item.

As a conclusion, in view of (20), in all possible cases we obtain that

$$\|x' \circledcirc_{\text{ML}} (v,\sigma) - x \circledcirc_{\text{ML}} (v,\sigma)\| \le \sum_{i \ne k,k'} |x(i) - x'(i)| + |x(k) - x'(k)| + |x(k') - x'(k')| = \|x' - x\|,$$

which concludes the proof. ■

4.2.2 Proof of Proposition 3 for Non-expansive policies

As we prove below, the non-expansiveness for the ℓ_1-norm, which is satisfied by all random policies (Proposition 4) and ML (Proposition 5), entails simply the sub-additivity of the corresponding model.

Fix a non-expansive matching policy ϕ. Keeping the notations of Definition 2, let us define the two arrays $(x_i)_{i=1,\dots,|v''|}$ and $(x'_i)_{i=1,\dots,|v''|}$ to be the class details of the system at arrival times, starting respectively from an empty system and from a system of buffer content w', and having a common input $(v''_i, \sigma''_i)_{i=1,\dots,|v''|}$, where $(\sigma''_i)_{i=1,\dots,|u''|}$ are drawn from ν_ϕ on $\mathcal{S}$. In other words, we set

$$\begin{cases} x_0 &= \mathbf{0};\\ x'_0 &= [w'] \end{cases}$$

and by induction,

$$\begin{cases} x_{n+1} &= x_n \circledcirc_\phi \left(v''_{n+1}, \sigma''_{n+1}\right),\ n \in \{0,\dots,|v''|-1\}\,;\\ x'_{n+1} &= x'_n \circledcirc_\phi \left(v''_{n+1}, \sigma''_{n+1}\right),\ n \in \{0,\dots,|v''|-1\}\,. \end{cases}$$

Applying (18) at all n, we obtain by induction that for all $n \in \{0,\dots,|v''|\}$,

$$\|x'_n - x_n\| \le \|x'_0 - x_0\| = |w'|. \tag{21}$$

Now observe that by construction, $x_{|v''|} = [w'']$ which, together with (21), implies that

$$|w| = \left\|x'_{|v''|}\right\| \le \left\|x'_{|v''|} - x_{|v''|}\right\| + \left\|x_{|v''|}\right\| \le |w'| + |w''|,$$

hence the sub-additivity of ϕ.

4.2.3 Proof of Proposition 3 for FCFM and LCFM

For the disciplines FCFM and LCFM, we cannot exploit a non-expansiveness property similar to (18). Indeed, it is straightforward that a given common arrival can very well increase the distance between the commutative images of the queue details of two systems:

Example 4. *Consider the graph of Figure 1. Then, regardless of σ we have for instance that*

$$\begin{aligned}&\|[133 \odot_{\text{FCFM}} (2,\sigma)] - [311 \odot_{\text{FCFM}} (2,\sigma)]\|\\ &\quad = \|[33] - [11]\| = \|(0,0,2,0) - (2,0,0,0)\| = 4 > 2 = \|(1,0,2,0) - (2,0,1,0)\| = \|[133] - [311]\|\end{aligned}$$

whereas for LCFM*,*

$$\|[331 \odot_{\text{LCFM}} (2,\sigma)] - [113 \odot_{\text{LCFM}} (2,\sigma)]\| = \|[33] - [11]\| = 4 > 2 = \|[331] - [11]\|\,,$$

see Figure 4.

Figure 4: FCFM (left) and LCFM (right) are not non-expansive.

.

As we cannot apply the arguments of Section 4.2.2 we resort to a direct proof for both FCFM (for which our argument is related to the proof of Lemma 4 in [2]), and LCFM. We keep the notation of Definition 2, where we drop for short the dependence on ς in the notations $M_{\text{FCFM}}(.)$ and $M_{\text{LCFM}}(.)$, as the various FCFM and LCFM matchings do not depend on any list of preferences.

FCFM. Start with the policy FCFM. We proceed in two steps,

Step I: Let $|z'| = 1$, and assume that $M_{\text{FCFM}}(z'')$ has K unmatched items. We need to show that $M_{\text{FCFM}}(z'z'')$ has at most $K+1$ unmatched items. There are three possible cases:

(a) The item u'_1 is unmatched in $M_{\text{FCFM}}(z'z'') = M_{\text{FCFM}}(z'_1z'')$. Then, by the definition of FCFM $z'_1 \not- z''_j$ for any letter z''_j of z''. Again from the definition of FCFM, the presence in line of this incompatible item z'_1 does not influence the choice of match of any subsequent item of the word z''. Thus the matched pairs in $M_{\text{FCFM}}(z'z'')$ are exactly the ones in $M_{\text{FCFM}}(z'')$, so there are $K+1$ unmatched items in $M_{\text{FCFM}}(z'z'')$.

(b) The item z'_1 gets matched in $M_{\text{FCFM}}(z'z'')$ with an unmatched item z''_{j_1} of $M_{\text{FCFM}}(z')$. Then, any unmatched item in $M_{\text{FCFM}}(z'')$ remains unmatched in $M_{\text{FCFM}}(z'z'')$. On another hand, for any matched item z''_i in $M_{\text{FCFM}}(z'')$ (let z''_j be its match), either $z''_i \not- z''_{j_1}$, and thus choses its match in $M_{\text{FCFM}}(v)$ regardless of whether z''_{j_1} is matched or not, and thus choses again z''_j, or $z''_i - z''_{j_1}$ and thus from the FCFM property, we have $j < j_1$ and in turn z''_j remains matched with z''_j in $M_{\text{FCFM}}(z'z'')$. Therefore the matching induced by the letters of z'' in $M_{\text{FCFM}}(z'z'')$ remains precisely $M_{\text{FCFM}}(z'')$, so $M_{\text{FCFM}}(z'z'')$ has $K-1$ unmatched items.

(c) The item z'_1 gets matched with an item z''_{j_1} that was matched in $M_{\text{FCFM}}(z'')$ to some item z''_{i_1}. The FCFM matching of z'_1 with z''_{j_1} breaks the old match (z''_{i_1}, z''_{j_1}), so we now need to search a new match for z''_{i_1}. Either there is no FCFM match for z''_{i_1} and we stop, or we find a match z''_{j_2}. The new pair (z''_{i_1}, z''_{j_2}) potentially broke an old pair (z''_{i_2}, z''_{j_2}). We continue on and on, until either z''_{i_k} cannot find a new match or z''_{j_k} was not previously matched, and consequently, with K unmatched items in the first case and $K-1$ in the second. Observe that due to the FCFM property, we have $i_\ell \leq i_{\ell+1}$ and $j_\ell \leq j_{\ell+1}$ for all $\ell \leq k$.

Step II: Consider now an arbitrary finite word z'. Observe, that if $(z'_i, z'_j) \in M_{\text{FCFM}}(z')$, then $(z'_i, z'_j) \in M_{\text{FCFM}}(z'z'')$, as is the case for any admissible policy. Thus, denoting $w' = Q_{\text{FCFM}}(z')$, we have $Q_{\text{FCFM}}(z'z'') = Q_{\text{FCFM}}(w'z'')$. Denote $w' = w'_1 \dots w'_p$. We will consider one by one the items in w', starting from the right to the left. If we denote for all $1 \le i \le p$, $M^i_{\text{FCFM}} = M_{\text{FCFM}}(w'_{p-i+1} \dots w'_p z'')$ and K_i, the number of unmatched items in M^i_{FCFM}, Step I entails by an immediate induction that for all $1 \le i \le p$, $K_i \le i + |Q_\phi(z'')|$. Hence we finally have

$$|Q_\phi(z'z'')| = K_p \le p + |Q_\phi(z'')| = |Q_\phi(z')| + |Q_\phi(z'')|,$$

which concludes the proof for FCFM.

LCFM. We now turn to LCFM, for which we apply the same procedure as above,

Step I: Set $|z'| = 1$ and assume that $M_{\text{FCFM}}(z'')$ has K unmatched items. The three different cases are the same as above,

(a) If z'_1 is unmatched in $M_{\text{LCFM}}(z'z'')$, then z'_1 is incompatible with z''_1, otherwise the two items would have been matched. In turn, if follows from the definition of LCFM that the presence in line of z'_1 does not influence the choice of match of any item z''_j that is matched in $M_{\text{LCFM}}(z'')$, even though $z'_1 - z''_j$. So $M_{\text{LCFM}}(z'z'')$ has exactly $K + 1$ unmatched customers items.

(b) Whenever z'_1 is matched in $M_{\text{LCFM}}(z'z'')$ with an item z''_{j_1} that was unmatched in $M_{\text{LCFM}}(z'')$, any matched item z''_i in $M_{\text{LCFM}}(z'')$ that is compatible with z''_{j_1} has found in z'' a more recent compatible match z''_j. The matching of z''_i with z''_j still occurs in $M_{\text{LCFM}}(z'z'')$. Thus, as above the matching induced in $M_{\text{LCFM}}(z'z'')$ by the nodes of z'' is not affected by the match (z'_1, z''_{j_1}), so there are are $K - 1$ unmatched items in $M_{\text{LCFM}}(z'z'')$.

(c) Suppose now that z'_1 is matched with a server z''_{j_1} that is matched in $M_{\text{LCFM}}(z'')$. We proceed as for FCFM, by constructing the new corresponding matchings $\left(z'_1, z''_{j_1}\right)$, $\left(z''_{i_1}, z''_{j_2}\right)$, $\left(z''_{i_2}, z''_{j_3}\right)$, and so on, until we reach the same conclusion as for FCFM (with the only difference that in LCFM the indexes $i_1, i_2, \dots$ and $j_1, j_2, \dots$ are not necessarily ordered increasingly).

Therefore, at Step I we reach the same conclusions as for FCFM.

Step II: The construction for FCFM remains valid for any admissible policy, and in particular for LCFM.

4.3 Erasing words

The concepts of *erasing words* and *strong* erasing words will also be useful in the construction below.

Definition 4. *Let $G = (\mathcal{V}, \mathcal{E})$ be a connected graph, and ϕ be an admissible matching policy. Let $u \in \mathbb{W}_2$. We say that the word $z \in \mathcal{V}^*$ is an* erasing word *of u for (G, ϕ) if $|z|$ is even and for any two words ς' and ς possibly drawn by ν_ϕ on $\mathcal{S}^*$ and having respectively the same size as z and u, we have that*

$$Q_\phi(z, \varsigma') = \emptyset \qquad \text{and} \qquad Q_\phi(uz, \varsigma\varsigma') = \emptyset. \tag{22}$$

In other words, an erasing word of u has the twofold property of being perfectly matchable by ϕ alone, and together with u. The following proposition guarantees the existence of erasing words for any stabilizable graph and any sub-additive policy.

Proposition 6. *Let G be a non-bipartite graph and ϕ be a sub-additive matching policy. Then any word $u \in \mathbb{W}_2$ admits an erasing word for (G, ϕ).*

Proof. As will be clear below, the arguments of this proof do not depend on the drawn lists of preferences, as long as they are fixed upon arrival. For notational convenience, we thus skip this parameter from all notations (i.e. we write for instance $Q_\phi(u)$ instead of $Q_\phi(u,\varsigma)$, and so on). We first show that any admissible word of size 2 admits an erasing word y; so let us consider a word ij where $i \not - j$.

As G is connected, i and j are connected at distance, say, $p \geq 2$, i.e. there exists a minimal path $i - i_1 - ... - i_{p-1} - j$ connecting i to j. If p is odd, then just set $y = i_1 i_2 ... i_{p-1}$. Clearly, $Q_\phi(w) = \emptyset$ and as the path is minimal, in $M_\phi(ijz)$ i_1 is matched with i, i_3 is matched with i_2, and so on, until i_{p-1} is matched with j. So $Q_\phi(ijy) = \emptyset$, and (22) follows.

We now assume that p is even. Set $y^1 = i_1 i_2 ... i_{p-1} i_{p-1}$. Then, in $M_\phi(ijy^1)$ i_1 is matched with i, i_3 with i_2, and so on, until both j and i_{p-2} are matched with an i_{p-1} item. So $Q_\phi(ijy^1) = \emptyset$, however $Q_\phi(y^1) = i_{p-1} i_{p-1}$. But as G is non-bipartite, it contains an odd cycle. Thus (see e.g. the proof of Lemma 3 in [15]) there necessarily exists an *induced* odd cycle in G, say of length $2r+1$, $r \geq 1$. As G is connected, there exists a path connecting i_{p-1} to any element of the latter cycle. Take the shortest one (which may intersect with the path between i to j, or coincide with a part of it), and denote it $i_{p-1} - j_1 - j_2 - ... - j_q - k_1$, where k_1 is the first element of the latter path belonging to the odd cycle, and by $k_1 - k_2 - ... - k_{2r+1} - k_1$, the elements of the cycle. See an example in Figure 5.

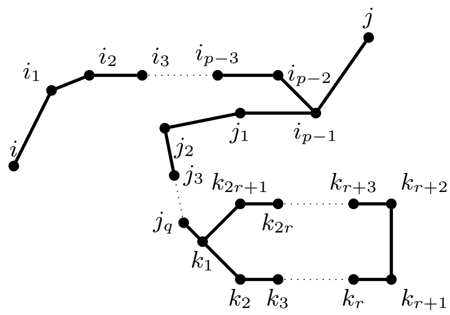

Figure 5: The path from i to j and then to an odd cycle

Then set
$$y^2 = j_1 j_1 j_2 j_2 ... j_q j_q k_1 k_1 k_2 k_3 ... k_{2r} k_{2r+1}.$$
We are in the following alternative:

- if q is even, then in $M_\phi(y^1 y^2)$ the two nodes i_{p-1} are matched with the two noded j_1, the two j_2 with the two j_3, and so on, until the two j_q are matched with the two k_1, and then, as the cycle is induced, k_2 is matched with k_3, k_4 with k_5 and so on, until k_{2p} is matched with k_{2p+1}. On the other hand, in $M_\phi(y^2)$, the two j_1 are matched with the two j_2, the two j_3 with the two j_4, and so on, until the two j_{q-1} are matched with the two j_q. Then, a k_1 is matched with k_2, k_3 with k_4 and so on, until k_{2p-1} is matched with k_{2p} and k_{2p+1} is matched with the remaining k_1.
- if q is odd, then the edges of $M_\phi(y^1 y^2)$ are as in the first case, until the two nodes j_{q-1} are matched with the two nodes j_q. But then, whatever ϕ is, one of the two nodes k_1 is matched with k_2, k_3 with k_4, and so on, until k_{2p-1} is matched with k_{2p}, and k_{2p+1} is matched with the remaining k_1. Also, in $M_\phi(y^2)$, the two j_1 are matched with the two j_2, the two j_3 with the two j_4, and so on, until the two j_{q-2} are matched with the two j_{q-1}. Then, the two j_q are matched with the two k_1, k_2 is matched with k_3, and so on, until k_{2p} is matched with k_{2p+1}.

In both cases, we obtain that both $Q_\phi(y^1 y^2) = \emptyset$ and that $Q_\phi(y^2) = \emptyset$. In particular, as $Q_\phi(ijy^1) = \emptyset$ we have $Q_\phi(ijy^1 y^2) = \emptyset$. Therefore $y = y^1 y^2$ is an erasing word for ij. See an example in Figure 6.

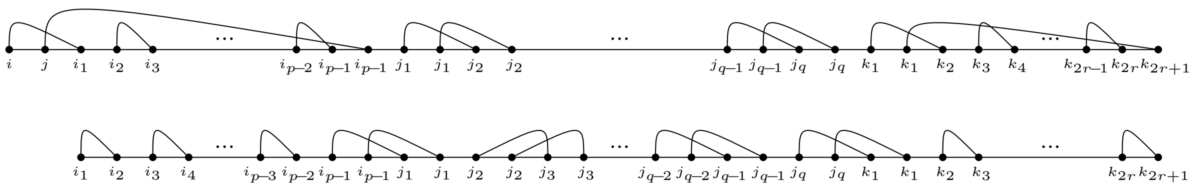

Figure 6: The two perfect matchings $M_{\text{FCFM}}(ijy^1y^2)$ and $M_{\text{FCFM}}(y^1y^2)$, for an even p and an odd q.

We now consider any word $u \in \mathbb{W}_2$, say $u = u_1u_2...u_{2r_1}$. First, as we just proved, there exists an erasing word, say z^1, for the two-letter word $u_{2r_1-1}u_{2r_1}$. In particular, we have that $Q_\phi\left(u_{2r_1-1}u_{2r_1}z^1\right) = \emptyset$. Thus, the sub-additivity of ϕ entails that

$$\left|Q_\phi\left(uz^1\right)\right| \le \left|Q_\phi\left(u_1u_2...u_{2r_1-2}\right)\right| + \left|Q_\phi\left(u_{2r_1-1}u_{2r_1}z^1\right)\right| = \left|Q_\phi\left(u\right)\right| - 2,$$

in other words the input of z^1 strictly decreases the size of the buffer content u, that is, if we let $u^2 = Q_\phi\left(uz^1\right)$, then u^2 is of even length $2r_2$, where $r_2 < r_1$. We then apply the same argument as above for u^2 instead of u: there exists an erasing word z^2 for the two-letter word $u^2_{2r_2-1}u^2_{2r_2}$ gathering the last two letters of u^2, so as above,

$$\left|Q_\phi\left(uz^1z^2\right)\right| = \left|Q_\phi\left(u^2z^2\right)\right| \le \left|Q_\phi\left(u^2\right)\right| - 2.$$

We can continue this construction by induction, until we reach an index ℓ such that

$$Q_\phi\left(uz^1z^2...z^\ell\right) = \emptyset. \tag{23}$$

Observe that, as $z^1, ...z^\ell$ are all erasing words, we have that $Q_\phi(z^1) = Q_\phi(z^2) = ... = Q_\phi(z^\ell) = \emptyset$. Thus $Q_\phi(z^1z^2...z^\ell) = \emptyset$, which shows, together with (23), that $z = z^1z^2...z^\ell$ is an erasing word for u. ■

Clearly, uniqueness of the erasing does not hold true. In particular, if z^1 and z^2 are both erasing words of the same word u for (G, ϕ), then z^1z^2 also is. Hence the following definition,

Definition 5. *Let $u \in \mathbb{W}_2$. An erasing word z of u for (G, ϕ) is said to be* reduced*, if z cannot be written as $z = z^1z^2$, where z^1 and z^2 are both non-empty erasing words of u. A reduced erasing word z of u is said to be* minimal*, if it is of minimal length among all reduced erasing words of u.*

Definition 6. *A word $z \in \mathcal{V}^*$ of even length $2p$ is said to be a* strong erasing word *for the graph $G = (\mathcal{V}, \mathcal{E})$ and the matching policy ϕ if*

1. *z is completely matchable by ϕ together with any two-letter word, i.e. for any $i, j \in \mathcal{V}$ such that $i \not- j$, and any two words ς' and ς of $\mathcal{S}^*$ whose letters can be possibly drawn by ν_ϕ, and of respective length 2 and $2p$, we have that $Q_\phi\left(ijz, \varsigma\varsigma'\right) = \emptyset$;*
2. *any right sub-word of z of even length is completely matchable by ϕ, i.e. for any $\ell \in [\![0, p-1]\!]$ and any ς' of length $2p$ and whose letters can possibly be drawn by ν_ϕ, $Q_\phi\left(z_{2\ell+1}...z_{2p}, \varsigma'_{2\ell+1}...\varsigma'_{2p}\right) = \emptyset$.*

Plainly, a strong erasing word for (G, ϕ) is a an erasing word for any two-letter word ij with $i \not- j$. Also observe that condition 2 above is typically met whenever the letters of z form a cycle of G - this fact will be exploited below.

Lemma 3. *Let ϕ be a sub-additive matching policy and z be a strong erasing word for $G = (\mathcal{V}, \mathcal{E})$ and ϕ. Then for any $\varsigma' \in \mathcal{S}^*$ of length $|z|$, any word $u \in \mathbb{W}_2$ and any $\varsigma \in \mathcal{S}^*$ of length $|u|$, we have that $|Q_\phi(uz, \varsigma\varsigma')| \le |Q_\phi(u, \varsigma)| - 2$.*

Proof. From the sub-additivity of ϕ, if $|u| = 2r$,

$$\begin{aligned}|Q_\phi(uz, \varsigma\varsigma')| &\leq |Q_\phi\left(u_1...u_{2r-2}, \varsigma_1...\varsigma_{2r-2}\right)| + |Q_\phi\left(u_{2r-1}u_{2r}z, \varsigma_{2r-1}\varsigma_{2r}\varsigma'\right)|\\ &= |Q_\phi\left(u_1...u_{2r-2}, \varsigma_1...\varsigma_{2r-2}\right)| = |Q_\phi(u, \varsigma)| - 2,\end{aligned}$$

using the fact that $u_1...u_{2r-2} \in \mathbb{W}_2$. ■

To address the question of existence of strong erasing words for a given pair (G, ϕ), we need the following Lemma,

Lemma 4. *Any connected non-bipartite graph $G = (\mathcal{V}, \mathcal{E})$ can be spanned by an odd cycle, i.e. there exists a cycle of odd length in which all the nodes of $\mathcal{V}$ appear at least once.*

Proof. As G is non-bipartite, G contains an odd cycle $\mathcal{C} := c_1{-}c_2{-}\ ...\ c_{2q+1}$. Let $p \in \mathbb{N}$ be the number of nodes of $\mathcal{V}$ which do not appear in the latter cycle, and denote by $i_1, ..., i_p$, these nodes. By connectedness, there exists for any $j \in [\![1, p]\!]$, a minimal path $\mathcal{P}_j$ of length, say, ℓ_j, from k_1 to i_j. Then, we can connect k_1 to itself by following, first, the cycle $\mathcal{C}$, and then all the paths $\mathcal{P}_j$ from k_1 to i_j and then the reversed path of $\mathcal{P}_j$ from i_j to k_1, successively for all $j \in [\![1, p]\!]$. The resulting path is a cycle connecting to k_1 to itself and spanning the whole set $\mathcal{V}$, and its length is $2q + 1 + \sum_{j=1}^{p} 2\ell_j$, an odd number. ■

Let us recall (see [14] for details), that a connected graph $G = (\mathcal{V}, \mathcal{E})$ is said to be *separable of order* p, $p \geq 2$, if there exists a partition of $\mathcal{V}$ into maximal independent sets $\mathcal{I}_1, \ldots, \mathcal{I}_p$, such that

$$\forall i \neq j, \forall u \in \mathcal{I}_i, \forall v \in \mathcal{I}_j,\ u{-}v\ .$$

In particular, a separable graph G is non-bipartite if and only if its order is at least 3.

Proposition 7. *The following conditions are sufficient for the existence of a strong erasing word for (G, ϕ):*

(i) G is non-bipartite separable and ϕ is any admissible policy;

(ii) G is non-bipartite and $\phi = \textsc{lcfm}$.

Proof. (i) Suppose that G is separable of order $p \geq 3$, and let $\mathcal{I}_1, ..., \mathcal{I}_p$ be the corresponding maximal independent sets. Let z be a word of length $2p$ which contains exactly two letters of each maximal independent set, but whose last two letters do not represent the same independent set. Then it is immediate that z, and any even right sub-word of z is completely matchable by any ϕ. Second, if we let i and $j \in \mathcal{V}$ such that $i \not{-} j$, which is true if and only if i and j belong to the same maximal independent set, say $\mathcal{I}_\ell$. Then it is immediate that $Q_\phi(ijz) = \emptyset$ for any ϕ, since any incoming item of a class in any other independent set than $\mathcal{I}_\ell$ can be matched on the fly with any element of $\mathcal{I}_\ell$.

(ii) As an immediate consequence of Lemma 4, there exists a cycle $\mathscr{C} = c_1{-}c_2{-}...{-}c_{2q+1}$ such that

$$\mathcal{E}\left(\{c_1, c_2, ..., c_{2q+1}\}\right) = \mathcal{V}, \tag{24}$$

which is true in particular for the odd cycle that spans $\mathcal{V}$. We let z be the word consisting of all the nodes of $\mathscr{C}$ visited 4 times in that order, i.e.

$$z = c_1c_2...c_{2q+1}c_1...c_{2q+1}c_1...c_{2q+1}c_1...c_{2q+1}.$$

We drop again the lists of permutations from all notation. First observe that, as $\mathscr{C}$ is a cycle we clearly get $Q_{\textsc{lcfm}}(z) = \emptyset$, as for any admissible policy. Second, as $\mathscr{C}$ is a cycle it is also clear that any right sub-word of z of even size is completely matchable by any admissible policy. Now fix i and j in $\mathcal{V}$ such that $i \not- j$. We need to show that $Q_{\text{LCFM}}(ijz) = \emptyset$. For this let us define the following sets for $k \in \{i, j\}$,

$$\mathcal{H}(k) = \{\text{even indexes } 2\ell \text{ in } [\![1, 2q+1]\!] \,:\, c_{2\ell} - k\}\,;$$
$$\mathcal{O}(k) = \{\text{odd indexes } 2\ell+1 \text{ in } [\![1, 2q+1]\!] \,:\, c_{2\ell+1} - k\}\,.$$

We are in the following alternative:

Case 1: $\mathcal{O}(i) \cup \mathcal{O}(j) \neq \emptyset$, i.e. i or j (or both) neighbor a node of odd index in $\mathscr{C}$. Let $2p+1 = \min \mathcal{O}(i) \cup \mathcal{O}(j)$. First observe that, by the definition of LCFM all items of even indexes in $[\![1, 2p]\!]$ are matched with the immediate preceding item of odd index, so the entering c_{2p+1} item finds only i and j in the system, and is matched with j if $c_{2p+1} - j$, or with i if $j \not- c_{2p+1}$ and $i - c_{2p+1}$. Let us assume that we are in the first case, the second one can be treated analogously. So we have $Q_{\text{LCFM}}(ijc_1...c_{2p+1}) = i$. Let us now define

$$\tilde{\mathcal{H}}(i) = \{\text{even indexes } 2\ell \text{ in } [\![2p+2, 2q]\!] \,:\, c_{2\ell} - i\}\,.$$

We have three sub-cases:

Sub-case 1a: $\tilde{\mathcal{H}}(i) \neq \emptyset$. Set $2r = \min \tilde{\mathcal{H}}(i)$. Then the i item is matched with c_{2r}. Indeed, in LCFM all items of odd indexes in $[\![2p+2, 2r]\!]$ are matched with the immediate preceding item, even if they are compatible with i. Then, after the i item is matched with the c_{2r} item, all items of odd indexes in $[\![2r+1, 2q-1]\!]$ (if the latter is non-empty) in the first exploration of $\mathscr{C}$ are matched with the immediate following item, until the first c_{2q+1} item is matched with the second c_1 item. After that, in the second exploration of $\mathscr{C}$ all items of even nodes are matched with the following item of odd index, until the second c_{2q} item is matched with the second $2q+1$ item, so we get a perfect matching of ij with the first two explorations of $\mathscr{C}$. Then the last two visits of $\mathscr{C}$ are perfectly matched on the fly, since $\mathscr{C}$ is a cycle. So $Q_{\text{LCFM}}(ijz) = \emptyset$.

Sub-case 1b: $\tilde{\mathcal{H}}(i) = \emptyset$ and $\mathcal{O}(i) \neq \emptyset$. Due to the LCFM policy, in the first exploration of $\mathscr{C}$ all odd items are matched with the immediate preceding item of even index, until c_{2q+1}, in a way that $Q_{\text{LCFM}}(ijc_1...c_{2q+1}) = i$. Let $2s+1 = \min \mathcal{O}(i)$. Then the remaining i item is matched with the second c_{2s+1}, since in LCFM, all items of even indexes less than $2s+1$ that are compatible with i, are matched with the preceding item of odd index. After that, if $s < q$ then all remaining items of even indexes in the second exploration of $\mathscr{C}$ are matched with the immediate following item, until the second c_{2q} item is matched with the second c_{2q+1}. Thus $Q_{\phi}(ijz) = \emptyset$, and we conclude as in 1a.

Sub-case 1c: $\tilde{\mathcal{H}}(i) = \emptyset$ and $\mathcal{O}(i) = \emptyset$. From (24) there necessarily exists an even index (take the smallest one) $2u \in [\![2, 2p]\!]$ such that $i - c_{2u}$. Then, as in 1b we have $Q_{\text{LCFM}}(ijc_1...c_{2q+1}) = i$. Then, in the second exploration of $\mathscr{C}$, in LCFM all items of even indexes are matched with the preceding item of odd index, until the second c_{2q+1} remains unmatched, i.e. $Q_{\text{LCFM}}(ijc_1...c_{2q+1}c_1...c_{2q+1}) = ic_{2q+1}$. Then the remaining c_{2q+1} item is matched with the third c_1, and in the third visit of $\mathscr{C}$, all items of even indexes are matched with the following item of odd index, until c_{2u} is matched with i. To finish the third exploration, if $u < q$ then all items of odd index in $[\![2u+1, 2q-1]\!]$ are matched with the following item of even index, until the third c_{2q+1} remains alone unmatched, i.e. $Q_{\text{LCFM}}(ijc_1...c_{2q+1}c_1...c_{2q+1}c_1...c_{2q+1}) = c_{2q+1}$. At this point, the forth c_1 is matched with the third c_{2q+1}, and then in the fourth exploration of $\mathscr{C}$ all items of even index are matched with the following item of odd index, until the last c_{2q} is matched with the last c_{2q+1} item. We end up again with $Q_{\text{LCFM}}(ijz) = \emptyset$.

Case 2: $\mathcal{O}(i) \cup \mathcal{O}(j) = \emptyset$. In that case i and j both have only neighbors of even indexes in $\mathscr{C}$, in particular from (24) $\mathcal{H}(i)$ and $\mathcal{H}(j)$ are both non-empty. Let $2p = \min \mathcal{H}(i)$ and $2p' = \min \mathcal{H}(j)$. Again from the definition of LCFM, in the first exploration of $\mathscr{C}$, all items of even indexes are matched with the preceding item of odd index, until c_{2q+1} remains unmatched, so $Q_{\text{LCFM}}(ijc_1...c_{2q+1}) = ijc_{2q+1}$. Then the first c_{2q+1} item is matched with the second

c_1, and if $2 < \min(2p, 2p')$, in the second exploration of $\mathscr{C}$ all items of even index in $[\![2, \min(2p, 2p') - 2]\!]$ are matched with the following item of odd index. We have again, two sub-cases:

Sub-case 2a: $p' \leq p$, so in LCFM the j-item is matched with the second c_{2p}. In the second exploration of $\mathscr{C}$, after the c_{2p} item has been matched with the j item, if $p < q$ all items of odd indexes in $[\![2p+1, 2q-1]\!]$ are matched on the fly with the immediate following item of even index, until only the second c_{2q+1} item remains unmatched, so $Q_{\text{LCFM}}\left(ijc_1...c_{2q+1}c_1...c_{2q+1}\right) = ic_{2q+1}$. Then the second c_{2q+1} is matched with the third c_1. In the third exploration, if $p > 2$, all items of even indexes in $[\![2, 2p-2]\!]$ are matched with the following item, until the c_{2p} item is matched with i. We then conclude as in 1c, and end up again with $Q_{\text{LCFM}}(ijz) = \emptyset$.

Sub-case 2b: $p < p'$, so the i-item is matched with the second c_{2p}. Then the j item remains to be matched, and we conclude exactly as in 2a, by matching the j item with the third $c_{2p'}$ (instead of i with the third c_{2p}). This concludes the proof.

■

Remark 2. Observe that the conclusion of (ii) of Proposition 7 clearly hold true for any policy ϕ that emulates LCFM on the input ijz, for $z = c_1...c_{2q+1}c_1....c_{2q+1}c_1...c_{2q+1}c_1...c_{2q+1}$ (keeping the notation of the above proof), and for any $i \neq j \in \mathcal{V}$. This is true in particular if ϕ is a priority policy such that for any index $j \geq 2$, c_j prioritizes c_{j-1} over any other node, and c_1 prioritizes c_{2q+1} over any other node.

4.4 Renovating events

Define the following family of events for any $\mathbb{W}_2$-valued r.v. Y,

$$\mathscr{A}_n(Y) = \left\{U_n^{[Y]} = \emptyset\right\} = \left\{Q_\phi(YV^0V^1V^0 \circ \theta\, V^1 \circ \theta \ldots V^0 \circ \theta^{n-1}V^1 \circ \theta^{n-1}) = \emptyset\right\}, \quad n \geq 0.$$

Let us first observe that

Proposition 8. *Let $G = (\mathcal{V}, \mathcal{E})$ be a matching graph and ϕ be an admissible matching policy. Suppose that assumption (H1') holds, and let Y be a $\mathbb{W}_2$-valued random variable. If the following condition holds:*

$$\lim_{n\to\infty} \mathbb{P}^0 \left[\bigcap_{k=0}^{\infty} \bigcup_{l=0}^{n} \mathscr{A}_l(Y) \cap \theta^k \mathscr{A}_{l+k}(Y)\right] = 1, \tag{25}$$

then $\left(U_n^{[Y]}\right)_{n\in\mathbb{N}}$ converges with strong backwards coupling to a stationary buffer content sequence $\left(U \circ \theta^n\right)_{n\in\mathbb{N}}$.

Proof. Clearly, $\left(\mathscr{A}_n(Y)\right)_{n\in\mathbb{N}}$ is a sequence of renovating events of length 1 for $\left(U_n^{[Y]}\right)_{n\in\mathbb{N}}$ (see [6, 7]). The result then follows from Theorem 2.5.3 of [3]. ■

The above result applies to any initial word of even size, however it does not guarantee the uniqueness of a solution to (17). As we now demonstrate, this stronger property holds at least under the additional assumption

(H2) There exists a strong erasing word for (G, ϕ).

Let us define the following sets of random variables:

$$\mathscr{Y}_2^r = \left\{\mathbb{W}_2 - \text{ valued r.v. } Y\colon\ |Y| \leq 2r \text{ a.s.}\right\}, r \in \mathbb{N}_+,$$

and let

$$\mathscr{Y}_2^\infty := \bigcup_{r=1}^{+\infty} \mathscr{Y}_2^r.$$

Lemma 5. *Let $G=(\mathcal{V},\mathcal{E})$, ϕ be a sub-additive policy, and suppose that (H2) holds. Fix a positive integer r, and define the events*

$$\mathscr{B}^r(z^1,...,z^r) = \left\{V^0V^1\,V^0\circ\theta\,V^1\circ\theta\,....\,V^0\circ\theta^{m-1}V^1\circ\theta^{m-1} = z^1z^2...z^r\right\}; \tag{26}$$

$$\mathscr{C}_n^r(z^1,...,z^r) = \mathscr{A}_n(\emptyset)\cap\theta^{-n}\mathscr{B}^r(z^1,...,z^r), \tag{27}$$

where the z^i's are (possibly identical) strong erasing words for (G,ϕ) and $m=\sum_{i=1}^r|z^i|/2$. Then for any any r.v. $Y\in\mathscr{Y}_2^r$ and $n\geq 1$, up to a negligible event we have that $\mathscr{C}_n^r(z)\subset\mathscr{A}_{n+m}(Y)$.

Proof. Fix $n\geq 1$, $r\geq 1$ and $Y\in\mathscr{Y}_2^r$. All the arguments in this proof hold for any fixed sequence of lists of preferences, so we drop again that parameter of all notations for short. Throughout this proof, let us also fix a sample in $\mathscr{C}_n^r(z)$. First, as $U_n^{[\emptyset]}=\emptyset$, the matching of the arrivals up to n is complete, i.e. we have that $Q_\phi\left(V^0\,V^1\,V^0\circ\theta\;V^1\circ\theta\,...\,V^0\circ\theta^{n-1}\,V^1\circ\theta^{n-1}\right)=\emptyset$. Thus from the sub-additivity of ϕ we have

$$\begin{aligned}\left|U_{2n}^{[Y]}\right| &= \left|Q_\phi(Y\,V^0\,V^1\,V^0\circ\theta\;V^1\circ\theta\,...\,V^0\circ\theta^{n-1}\,V^1\circ\theta^{n-1})\right|\\ &\leq |Q_\phi(Y)| + \left|Q_\phi(V^0\,V^1\,V^0\circ\theta\;V^1\circ\theta\,...\,V^0\circ\theta^{n-1}\,V^1\circ\theta^{n-1})\right| = |Y|\leq 2r.\end{aligned} \tag{28}$$

Now, as z is a strong erasing word for (G,ϕ), from Lemma 3 for any $l\in[\![1,r]\!]$ we have that

$$\left|U^{[Y]}_{n+\sum_{i=1}^{l}|z^i|/2}\right| = \left|Q_\phi\left(U^{[Y]}_{n+\sum_{i=1}^{l-1}|z^i|/2}z^l\right)\right| \leq \left|Q_\phi\left(U^{[Y]}_{n+\sum_{i=1}^{l-1}|z^i|/2}\right)\right|-2 = \left|U^{[Y]}_{n+\sum_{i=1}^{l-1}|z^i|/2}\right|-2,$$

where we understand $\sum_{i=1}^{0}.$ as 0. This together with (28) and the fact that $\left|U_n^{[Y]}\right|$ is even, entails that for some index $n'\in[\![n+1,n+m]\!]$ (take the smallest one), we have $U_{n'}^{[Y]}=\emptyset$. Let $k\in[\![0,r]\!]$ be the largest integer such that $n'\geq n+\sum_{i=1}^k|z^i|/2$, that is, such that k full strong erasing words $z^1,z^2,...z^k$ (or none if $k=0$) have entered the system until time n'. Let also, if $k<r$, $j\in[\![0,z^{k+1}]\!]$ be the (even) number of letters of z^{k+1} that have entered the system up to time $n'-1$ included. In other words the input of letters between times n and $n'-1$ reads

$$V^0\circ\theta^n\,V^1\circ\theta^n\,...\,V^0\circ\theta^{n'-1}\,V^1\circ\theta^{n'-1} = \begin{cases} z^1z^2...z^k\,z_1^{k+1}z_2^{k+1}...z_j^{k+1}, & \text{if } k<r;\\ z^1z^2...z^r, & \text{else.}\end{cases}$$

If $k<r$, as for any right sub-word of z^{k+1} of even size, $z_{j+1}^{k+1}z_{j+2}^{k+1}...z_{|z^{k+1}|}^{k+1}$ is completely matchable by ϕ. Thus, as the following strong erasing words (if $k<r-1$) are also perfectly matchable, we obtain that

$$U_{n+rm}^{[Y]} = Q_\phi\left(z_{j+1}^{k+1}z_{j+2}^{k+1}...z_{|z^{k+1}|}^{k+1}z^{k+2}...z^r\right)=\emptyset,$$

which completes the proof. ■

We thus have the following

Proposition 9. *Let $G=(\mathcal{V},\mathcal{E})$, ϕ be a sub-additive policy, and suppose that (H1') holds together with (H2). Let $r\in\mathbb{N}_+$. Suppose that there exists a family of (possibly identical) r strong erasing words $z^1,...z^r$ such that*

$$\lim_{n\to\infty}\mathbb{P}^0\left[\bigcap_{k=0}^{\infty}\bigcup_{l=0}^{n}\mathscr{A}_l(\emptyset)\cap\theta^k\mathscr{A}_{l+k}(\emptyset)\cap\theta^{-l}\mathscr{B}^r(z^1,...,z^r)\right]=1, \tag{29}$$

for $\mathscr{B}^r(z^1,...,z^r)$ defined by (26). Then, there exists a solution U^r to (17) in $\mathscr{Y}_2^\infty$, to which all sequences $\left(U_n^{[Y]}\right)_{n\in\mathbb{N}}$, for $Y\in\mathscr{Y}_2^r$, converge with strong backwards coupling.

Proof. Lemma 5 shows that the events $\left(\mathscr{C}_n^r(z^1,...,z^r)\right)_{n\in\mathbb{N}}$ defined by (27) form a a sequence of renovating events of length $m=\sum_{i=1}^r |z^i|/2$ for the recursion $\left(U_n^{[Y]}\right)_{n\in\mathbb{N}}$, for any $Y\in\mathscr{Y}_2^r$. Indeed, for any n, on $\mathscr{C}_n^r(z^1,...,z^r)$ the value of $\left(U_{n+m}^{[Y]}\right)_{n\in\mathbb{N}}$ equals the empty set and does not depend on the input up to n. Observe that, by θ-invariance, (29) is equivalent to the sufficient condition in Theorem 2.5.3 of [3] applied to $\left(\mathscr{C}_n^r(z^1,...,z^r)\right)_{n\in\mathbb{N}}$. It thus follows again from that Theorem, that all such sequences $\left(U_n^{[Y]}\right)_{n\in\mathbb{N}}$ converge with strong backwards coupling to a solution U^r to (17). ■

Consequently,

Theorem 2. *If the conditions of Proposition 9 are satisfied for any* $r\in\mathbb{N}_+$*, then the solution* U *to (17) is unique in* $\mathscr{Y}_2^\infty$*, and all sequences* $\left(U_n^{[Y]}\right)_{n\in\mathbb{N}}$*,* $Y\in\mathscr{Y}_2^\infty$*, converge with strong backwards coupling to* U*.*

Proof. For any $r\geq 1$ we can apply Proposition 9, and then the same argument to $r+1$, yielding that all sequences $\left(U_n^{[Y]}\right)_{n\in\mathbb{N}}$ for $Y\in\mathscr{Y}_2^{r+1}$, also converge with strong backwards coupling to a solution U^{r+1}. As this is true in particular for any $Y\in\mathscr{Y}_2^r$, and by uniqueness of the backwards coupling limit, U^r and U^{r+1} coincide $\mathbb{P}^0$-almost surely. We conclude by and immediate induction on r that all solutions U^r, $r\geq 1$, coincide almost surely, and let U^* be this common limit. Uniqueness of the solution of (17) can be shown using Remark 2.5.3 in [3]: any two solutions U^* and U^{**} in $\mathscr{Y}_2^\infty$, belong to some set $\mathscr{Y}_2^r$, $r\geq 0$. Thus for some strong erasing words $z^1,...,z^r$ for (G,ϕ), $\left(\mathscr{C}_n^r(z^1,...,z^r)\right)_{n\in\mathbb{N}}$ forms a sequence of renovating events for both sequences $(U^*\circ\theta^n)_{n\in\mathbb{N}}$ and $(U^{**}\circ\theta^n)_{n\in\mathbb{N}}$ which, as they converge with strong backwards coupling to the same limit and are both stationary, necessarily coincide almost surely. ■

The renovation conditions (25) and (29) have the following intuitive interpretation: in (25), with overwhelming probability, any recursion started at Y at some point in the past, couples at value $\emptyset$ with the recursion started at Y at time 0, before a future horizon that goes large. In condition (29), for an initially empty system the first m arrivals after this coupling time form a sequence of r strong erasing word for (G,ϕ). As will be shown in Section 4.5, these conditions take a much simpler form whenever the input is i.i.d.. On another hand, in Example 5 we show a simple case where Theorem 2 applies, for a separable graph G and a non-independent input.

We conclude this section by observing that these renovation conditions cannot be satisfied unless the measure μ introduced in assumption (H1') is an element of NCOND(G) (defined by (8)).

Proposition 10. *Under assumption (H1'), conditions (25) and (29) entail that* $\mu\in$ NCOND(G)*.*

Proof. If for some independent set $\mathcal{I}$ of G,

$$\mu(\mathcal{I})=\frac{\mu^0(\mathcal{I})+\mu^1(\mathcal{I})}{2}>\frac{\mu^0(\mathcal{E}(\mathcal{I}))+\mu^1(\mathcal{E}(\mathcal{I}))}{2},$$

Birkhoff's Theorem entails the total number of arrivals of elements of $\mathcal{I}$ (from elements of $\left(V^0\circ\theta^n\right)_{n\in\mathbb{Z}}$ and $\left(V^1\circ\theta^n\right)_{n\in\mathbb{Z}}$ almost surely exceeds the total number of arrivals of elements of the neighboring classes of $\mathcal{I}$ by a quantity that is of order n in the long run . All the same, if we replace the inequality above by an equality, the Markov chain is at most null recurrent, as in the proof of Theorem 2 in [14]. So $\left(U_n^{[Y]}\right)_{n\in\mathbb{N}}$ cannot visit zero infinitely often with probability one. Hence (25) for $Y=\emptyset$ a.s., and thereby (29), cannot hold. ■

4.5 Independent Case

In this section, we reformulate the renovation condition (25), in the particular case where the input is iid and thereby, where the recursion $(U_n)_{n\in\mathbb{N}}$ is a Markov chain, under a natural stability condition which we now specify. Denote for any $\mathbb{W}_2$-valued r.v. Y and any $j \in \mathbb{N}^*$, by $\tau_j(Y)$ the j-th visit time to $\emptyset$ (or return time if $Y \equiv \emptyset$) for the process $\left(U_n^{[Y]}\right)_{n\in\mathbb{N}}$, that is

$$\tau_1(Y) := \inf\{n > 0, U_n^{[Y]} = \emptyset\}, \quad \tau_j(Y) := \inf\{n > \tau_{j-1}(Y), U_n^{[Y]} = \emptyset\},\ j \geq 2.$$

We define the following stability condition depending on the initial condition Y,

(H3) The stopping time $\tau_1(Y)$ is integrable.

Under assumption (IID), the Markov chain $(U_n)_{n\in\mathbb{N}}$ is clearly irreducible on $\mathbb{W}_2$. So (H3) holds true whenever the chain is positive recurrent. Therefore applying Theorem 1 for FCFM, and Theorem 2 of [14] we obtain the following list of sufficient conditions for (H3):

Proposition 11. *Condition (H3) holds true for any $\mathbb{W}_2$-valued initial condition Y, whenever G is non-bipartite, (IID) holds, $\mu \in$ NCOND(G), and in either one of the following cases:*

1. $\phi =$ FCFM*;*
2. $\phi =$ ML*;*
3. *ϕ is any admissible policy and G is separable.*

We have the following result,

Proposition 12. *If (IID) holds and $Y \in \mathscr{Y}_2^\infty$, then (H3) entails (25).*

Proof. Fix throughout $\varepsilon > 0$, and $r \in \mathbb{N}_+$ such that $Y \in \mathscr{Y}_2^r$. First observe that, as a consequence of (H3) the random variable

$$\kappa = \sup\left\{k \in \mathbb{N} \,:\, \tau_1(Y) \circ \theta^{-k} > k\right\}$$

that is, the largest horizon in the past from which the first visit to $\emptyset$ takes place after time 0, is a.s. finite. In particular there exists a positive integer K_ε such that

$$\mathbb{P}^0\left[\kappa > K_\varepsilon\right] < \frac{\varepsilon}{5}. \tag{30}$$

Again in view of (H3), there exists an integer $T_\varepsilon > 0$ such that

$$\mathbb{P}^0\left[\tau_1(Y) > T_\varepsilon\right] < \frac{\varepsilon}{5}, \tag{31}$$

and let us denote

$$H_\varepsilon := 2K_\varepsilon + 2r + T_\varepsilon.$$

We know from Proposition 6 that any word admits at least one minimal erasing word. Also, there are finitely many words in $\mathbb{W}_2$ of size less than H_ε, and thus finitely many minimal erasing words of those words. So the following integer is well defined, and depends only on H_ε,

$$\ell_\varepsilon = \frac{1}{2} \max_{u\in\mathbb{W}_2 : |w| \leq H_\varepsilon} \min_{\substack{z\in\mathcal{V}^* : \\ z \text{ minimal erasing word of } u}} |z|. \tag{32}$$

We now define the sequence $(\tilde{\tau}_i)_{i\in\mathbb{N}_+}$ (where we drop the dependence on Y for notational convenience), as the following subsequence of $(\tau_i(Y))_{i\in\mathbb{N}_+}$:

$$\tilde{\tau}_1 := \tau_1(Y), \quad \tilde{\tau}_i := \inf\{n > \tilde{\tau}_{i-1} + \ell_\varepsilon,\, U_n^{[Y]} = \emptyset\},\ i \geq 2.$$

Also define the following family of events: for all $k \in \mathbb{N}$ and $i \in \mathbb{N}_+$,

$$\mathcal{D}_i^k(Y) = \bigcup_{m=1}^{\ell_\varepsilon} \left\{ V^0 \circ \theta^{\tilde{\tau}_i+k}\, V^1 \circ \theta^{\tilde{\tau}_i+k} \dots V^0 \circ \theta^{\tilde{\tau}_i+k+m-1}\, V^1 \circ \theta^{\tilde{\tau}_i+k+m-1} \text{ is an erasing word of } U^{[Y]}_{\tilde{\tau}_i+k} \right\},$$

and for any $k, n \in \mathbb{N}$,

$$\mathcal{D}^{k,n}(Y) = \bigcup_{\substack{i\in\mathbb{N}_+:\\ \tilde{\tau}_i+\ell_\varepsilon\leq 2n}} \mathcal{D}_i^k(Y), \quad k \in \mathbb{N},\, n \in \mathbb{N}_+. \tag{33}$$

For any $k \in \mathbb{N}$ and $i \in \mathbb{N}_+$, on $\theta^k \mathcal{D}_i^k(Y)$ we first have that for some (unique, and even) integer $m \leq \ell_\varepsilon$, $U^{[Y]}_{\tilde{\tau}_i+k+m} \circ \theta^{-k} = \emptyset$, and second, that $U^{[Y]}_{\tilde{\tau}_i+m} = \emptyset$, since

$$\begin{aligned}
&\left|U^{[Y]}_{\tilde{\tau}_i+m}\right| \\
&= \left|Q_\phi\left(Y\, V^0\, V^1 \dots V^0 \circ \theta^{\tilde{\tau}_i-1}\, V^1 \circ \theta^{\tilde{\tau}_i-1} V^0 \circ \theta^{\tilde{\tau}_i}\, V^1 \circ \theta^{\tilde{\tau}_i} \dots V^0 \circ \theta^{\tilde{\tau}_i+m-1}\, V^1 \circ \theta^{\tilde{\tau}_i+m-1}\right)\right| \\
&\leq \left|Q_\phi\left(Y\, V^0\, V^1 \dots V^0 \circ \theta^{\tilde{\tau}_i-1}\, V^1 \circ \theta^{\tilde{\tau}_i-1}\right)\right| + \left|Q_\phi\left(V^0 \circ \theta^{\tilde{\tau}_i}\, V^1 \circ \theta^{\tilde{\tau}_i} \dots V^0 \circ \theta^{\tilde{\tau}_i+m-1}\, V^1 \circ \theta^{\tilde{\tau}_i+m-1}\right)\right| = 0,
\end{aligned}$$

where the two terms in the third line above are zero from the very definitions of $\tilde{\tau}_i$ and an erasing word. Consequently, we have that

$$\theta^k \mathcal{D}^{-k,n}(Y) \subseteq \bigcup_{l=0}^{n} \mathscr{A}_l(Y) \cap \theta^k \mathscr{A}_{l+k}(Y), \quad k, n \in \mathbb{N}. \tag{34}$$

Second, fix $n \in \mathbb{N}$ and a sample $\omega \in \{\kappa \leq K_\varepsilon\} \cap \bigcap_{k'=0}^{K_\varepsilon} \theta^{k'} \mathcal{D}^{k',n}(\emptyset)$ and an integer $k \geq K+1$. By the definition of κ, $U_0\left(\theta^{-k}\omega\right) = Y(\theta^{-k}\omega)$ entails that $U_{k-k'}\left(\theta^{-k}\omega\right) = \emptyset$ for some $k' \leq K_\varepsilon$; in other words $U_n^{[\emptyset]}\left(\theta^{-k'}\omega\right)$ equals $U^{[Y]}_{n+k-k'}\left(\theta^{-k}\omega\right)$ for any $n \geq 0$. But as $\theta^{-k'}\omega \in \mathcal{D}^{k',n}(\emptyset)$ by assumption, we obtain that $\theta^{-k}\omega \in \mathcal{D}^{k,n}(Y)$. Consequently we have that

$$\{\kappa \leq K_\varepsilon\} \cap \bigcap_{k=0}^{K_\varepsilon} \theta^{k'} \mathcal{D}^{k',n}(\emptyset) \subseteq \{\kappa \leq K_\varepsilon\} \cap \bigcap_{k=K+1}^{\infty} \theta^k \mathcal{D}^{k,n}(Y)$$

and thereby

$$\{\kappa \leq K_\varepsilon\} \cap \bigcap_{k=0}^{K_\varepsilon} \theta^k \left(\mathcal{D}^{k,n}(Y) \cap \mathcal{D}^{k,n}(\emptyset)\right) \subseteq \{\kappa \leq K_\varepsilon\} \cap \bigcap_{k=0}^{\infty} \theta^k \mathcal{D}^{k,n}(Y).$$

This, together with (34), yields to

$$\{\kappa \leq K_\varepsilon\} \cap \bigcap_{k=0}^{K_\varepsilon} \theta^k \left(\mathcal{D}^{k,n}(Y) \cap \mathcal{D}^{k,n}(\emptyset)\right) \subseteq \{\kappa \leq K_\varepsilon\} \cap \bigcap_{k=0}^{\infty} \bigcup_{l=0}^{n} \mathscr{A}_l(Y) \cap \theta^k \mathscr{A}_{l+k}(Y), \qquad n \in \mathbb{N}. \tag{35}$$

Now recall (32). In words, ℓ_ε is (half of) the minimal length of word that can accommodate at least one erasing word of any admissible word of even size bounded by H_ε. Therefore, in view of the iid assumptions the following is a well defined element of $]0,1[$:

$$\beta_\varepsilon = \min_{u\in\mathbb{W}_2\,:\,|u|\leq H_\varepsilon} \mathbb{P}^0 \left[\bigcup_{m=1}^{\ell_\varepsilon} \left\{ V^0\, V^1\, V^0 \circ \theta\, V^1 \circ \theta \dots V^0 \circ \theta^{m-1}\, V^0 \circ \theta^{m-1} \text{ is a minimal erasing word of } u \right\} \right]. \tag{36}$$

Let

$$M_\varepsilon = \left\lceil \frac{\text{Log}\varepsilon - \text{Log}5 - \text{Log}(K_\varepsilon + 1)}{\text{Log}(1-\beta_\varepsilon)} \right\rceil,$$

that is, the least integer that is such that

$$(1-\beta_\varepsilon)^{M_\varepsilon} < \frac{\varepsilon}{5(K_\varepsilon+1)}. \tag{37}$$

Again from (H3) and (IID), there exists a positive integer N_ε such that

$$\mathbb{P}^0\left[\tilde{\tau}_{M_\varepsilon} + \ell_\varepsilon > N_\varepsilon\right] < \frac{\varepsilon}{5}. \tag{38}$$

All in all, we obtain that for all $n > N_\varepsilon$,

$$\begin{aligned}
&\mathbb{P}^0\left[\overline{\bigcap_{k=0}^{\infty}\bigcup_{l=0}^{n} \mathscr{A}_l(Y)\cap\theta^k\mathscr{A}_{l+k}(Y)}\right]\\
&\le \mathbb{P}^0\left[\overline{\bigcap_{k=0}^{\infty}\bigcup_{l=0}^{n} \mathscr{A}_l(Y)\cap\theta^k\mathscr{A}_{l+k}(Y)} \cap \{\tilde{\tau}_{M_\varepsilon}+\ell_\varepsilon \le N_\varepsilon\}\cap\{\kappa\le K_\varepsilon\}\cap\{\tau_1(Y)\le T_\varepsilon\}\right]\\
&\qquad\qquad + \mathbb{P}^0\left[\tilde{\tau}_{M_\varepsilon}+\ell_\varepsilon > N_\varepsilon\right] + \mathbb{P}^0\left[\kappa > K_\varepsilon\right] + \mathbb{P}^0\left[\tau_1(Y) > T_\varepsilon\right]\\
&\le \mathbb{P}^0\left[\overline{\bigcap_{k=0}^{K_\varepsilon}\theta^k\left(\mathcal{D}^{k,n}(Y)\cap\mathcal{D}^{k,n}(\emptyset)\right)} \cap\{\tilde{\tau}_{M_\varepsilon}+\ell_\varepsilon\le N_\varepsilon\}\cap\{\tau_1(Y)\le T_\varepsilon\}\right] + \frac{3\varepsilon}{5}\\
&\le \sum_{k=0}^{K_\varepsilon}\mathbb{P}^0\left[\left(\bigcap_{i=1}^{M_\varepsilon}\overline{\theta^k\mathcal{D}_i^k(Y)}\right)\cap\{\tau_1(Y)\le T_\varepsilon\}\right] + \sum_{k=0}^{K_\varepsilon}\mathbb{P}^0\left[\left(\bigcap_{i=1}^{M_\varepsilon}\overline{\theta^k\mathcal{D}_i^k(\emptyset)}\right)\cap\{\tau_1(Y)\le T_\varepsilon\}\right] + \frac{3\varepsilon}{5},
\end{aligned} \tag{39}$$

where we use (30), (31), (35) and (38) in the second inequality, and recalling (33).

Now let u_ε be an element of $\mathbb{W}_2$ such that $|u_\varepsilon| \le H_\varepsilon$ and that achieves the minimum in (36), that is

$$\beta_\varepsilon = \mathbb{P}^0\left[\bigcup_{m=1}^{\ell_\varepsilon}\left\{V^0\,V^1\,V^0\circ\theta\,V^1\circ\theta\ldots V^0\circ\theta^{m-1}\,V^0\circ\theta^{m-1}\text{ is a minimal erasing word of }u_\varepsilon\right\}\right],$$

and define the events

$$\check{\mathcal{D}}_i = \bigcup_{m=1}^{\ell_\varepsilon}\left\{V^0\circ\theta^{\tilde{\tau}_i}\,V^1\circ\theta^{\tilde{\tau}_i}\ldots V^0\circ\theta^{\tilde{\tau}_i+m-1}\,V^0\circ\theta^{\tilde{\tau}_i+m-1}\text{ is a minimal erasing word of }u_\varepsilon\right\},\quad i\in\mathbb{N}.$$

From assumption (IID), the events $\check{\mathcal{D}}_i, i \in \mathbb{N}$, are iid of probability β_ε. On another hand, on the event $\{\tau_1(Y) \le T_\varepsilon\}$, for any $0 \le k \le K_\varepsilon$,

$$\left|U^{[Y]}_{\tau_1(Y)+k}\circ\theta^{-k}\right| \le |Y| + 2k + \tau_1(Y) \le 2r + 2K_\varepsilon + T_\varepsilon = H_\varepsilon.$$

Thus, as $Q_\phi\left(V^0\circ\theta^{\tilde{\tau}_i}\,V^1\circ\theta^{\tilde{\tau}_i}\ldots V^0\circ\theta^{\tilde{\tau}_{i+1}-1}\,V^1\circ\theta^{\tilde{\tau}_{i+1}-1}\right) = \emptyset$ for all i, the sub-additivity of ϕ and an immediate induction entail that $\left|U^{[Y]}_{\tilde{\tau}_i+k}\circ\theta^{-k}\right| \le H_\varepsilon$ for all $i \ge 1$. Therefore, for any $k \le K_\varepsilon$ and any $i \in \mathbb{N}_+$, by the very definition of β_ε we have that $\mathbb{P}^0\left[\theta^k\mathcal{D}_i^k(Y)\right] \ge \mathbb{P}^0\left[\check{\mathcal{D}}_i\right] = \beta_\varepsilon$, and in turn by independence of the $\check{\mathcal{D}}_i$'s, that for all $k \le K_\varepsilon$,

$$\mathbb{P}^0\left[\left(\bigcap_{i=1}^{M_\varepsilon}\overline{\theta^k\mathcal{D}_i^k(Y)}\right)\cap\{\tau_1(Y)\le T_\varepsilon\}\right] \le \prod_{i=1}^{M_\varepsilon}\mathbb{P}^0\left[\overline{\check{\mathcal{D}}_i}\right] = (1-\beta_\varepsilon)^{M_\varepsilon}. \tag{40}$$

All the same, on the event $\{\tau_1(Y) \leq T_\varepsilon\}$, for any $0 \leq k \leq K_\varepsilon$ we have that

$$\left| U^{[\emptyset]}_{\tau_1(Y)+k} \circ \theta^{-k} \right| \leq 2k + \tau_1(Y) \leq H_\varepsilon,$$

thus we can conclude similarly that

$$\mathbb{P}^0 \left[\left(\bigcap_{i=1}^{M_\varepsilon} \overline{\theta^k \mathcal{D}^k_i(\emptyset)} \right) \cap \{\tau_1(Y) \leq T_\varepsilon\} \right] \leq (1-\beta_\varepsilon)^{M_\varepsilon}.$$

Injecting this together with (40) and (37) in (39) entails that, for any $n > N_\varepsilon$,

$$\mathbb{P}^0 \left[\overline{\bigcap_{k=0}^{\infty} \bigcup_{l=0}^{n} \mathscr{A}_l(Y) \cap \theta^k \mathscr{A}_{l+k}(Y)} \right] < \varepsilon,$$

which concludes the proof. ■

We now prove the uniqueness of the solution using the following forward coupling result,

Proposition 13. *Suppose that (IID) and (H3) holds. Let Y and Y^* be two elements of $\mathscr{Y}_2^\infty$. Then there is forward coupling between $\left(U_n^{[Y]}\right)_{n\in\mathbb{N}}$ and $\left(U_n^{[Y^*]}\right)_{n\in\mathbb{N}}$.*

Proof. We aim at proving that the stopping time

$$\rho(Y, Y^*) := \inf \left\{ n \geq 0 : U_l^{[Y]} = U_l^{[Y^*]} \text{ for all } l \geq n \right\}$$

is a.s. finite, that is

$$\lim_{n\to\infty} \mathbb{P}^0 \left[\rho(Y, Y^*) \leq n\right] = 1. \tag{41}$$

Observe that, as the two recursions $\left(U_n^{[Y]}\right)_{n\in\mathbb{N}}$ and $\left(U_n^{[Y^*]}\right)_{n\in\mathbb{N}}$ are driven by the same input, they coalesce as soon as they meet for the first time. Hence, (41) holds true in particular if

$$\lim_{n\to\infty} \mathbb{P}^0 \left[\bigcup_{l=0}^{n} \left\{ U_l^{[Y]} = U_l^{[Y^*]} = \emptyset \right\} \right] = \lim_{n\to\infty} \mathbb{P}^0 \left[\bigcup_{l=0}^{n} \mathscr{A}_l(Y) \cap \mathscr{A}_l(Y^*) \right] = 1. \tag{42}$$

From Proposition 12, the latter holds true whenever we replace Y^* by $U_0^{[Z]} \circ \theta^{-k}$ for any finite $\mathbb{W}_2$-valued r.v. Z and any $k \in \mathbb{N}$. The proof of (42) for any finite Y^* is analog. ■

Consequently,

Theorem 3. *If the policy ϕ is sub-additive and assumptions (IID) and (H3) hold, there exists a unique solution U to (17) in $\mathscr{Y}_2^\infty$, to which all sequences $\left(U_n^{[Y]}\right)_{n\in\mathbb{N}}$, for $Y \in \mathscr{Y}_2^\infty$, converge with strong backwards coupling.*

Proof. Fix a r.v. $Y \in \mathscr{Y}_2^\infty$, and let r be such that $Y \in \mathscr{Y}_2^r$. From Proposition 12, (25) holds true. Thus, as (H1') subsumes assumption (IID), we can apply Proposition 8: Y converges with strong backwards coupling, and thereby also in the forward sense, to a stationary sequence $(U \circ \theta^n)_{n\in\mathbb{N}}$, where $U \in \mathscr{Y}_2^\infty$. Now, Proposition 13 entails in particular that any couple of such stationary sequences $(U \circ \theta^n)_{n\in\mathbb{N}}$ and $(U^* \circ \theta^n)_{n\in\mathbb{N}}$ couple, and therefore coincide almost surely. Thus there exists a unique solution U to (17) in $\mathscr{Y}_2^\infty$. ■

4.6 Stationary perfect ϕ-matchings

Let us summarize the results of Section 4 thus far: a unique (bounded, even) stationary buffer content exists whenever G is non-bipartite, ϕ is sub-additive (which is the case for FCFM, LCFM, ML and U), and in either one of the following cases:

- (H1') holds for $\mu \in \textsc{Ncond}(G)$, and (H2) holds (which is true in particular if G is separable or $\phi =$ LCFM - see Proposition 7) together with (29) for any $r \in \mathbb{N}_+$ - see Theorem 2;
- (IID) holds for $\mu \in \textsc{Ncond}(G)$, and (H3) holds true (which, from Proposition 11, is the case if ϕ is FCFM or ML, or if ϕ is separable) - see Theorem 3.

With these results in hands, we now address the problem of constructing a stationary bi-infinite perfect matching on the original time scale,

Proposition 14. *Suppose that G is non-bipartite, ϕ is sub-additive, and either one of the following is true:*

- *(H1), (H1'), (H1"), (H2) and (29) hold for any $r \in \mathbb{N}_+$;*
- *(IID) and (H3) hold.*

Then there exists exactly two bi-infinite perfect matchings under ϕ.

Proof. In both cases, (H1) and (H1") hold true, so we can construct two stationary ergodic quadruples $\mathscr{Q}^1 := \left(\Omega^0, \mathscr{F}^0, \mathbb{P}^1, \theta\right)$ and $\bar{\mathscr{Q}} := \left(\bar{\Omega}, \bar{\mathscr{F}}, \bar{\mathbb{P}}, \bar{\theta}\right)$ analogously to $\mathscr{Q}^0$, as follows:

- $\mathbb{P}^1$ is the image measure on Ω^0 of the sequence $((V_{2_n-1}, \Sigma_{2n-1}, V_{2n}, \Sigma_{2n}))_{n\in\mathbb{Z}}$;
- $\bar{\Omega} = (\mathcal{V} \times \mathcal{S})^{\mathbb{Z}}$, $\bar{\mathscr{F}}$ is the Borel sigma-algebra on $\bar{\Omega}$, $\bar{\mathbb{P}}$ is the image measure of $((V_n, \Sigma_n))_{n\in\mathbb{Z}}$ on $\bar{\Omega}$ and the shift $\bar{\theta}$ is defined by $\bar{\theta}\left((\bar{\omega}_n)_{n\in\mathbb{Z}}\right) = (\bar{\omega}_{n-1})_{n\in\mathbb{Z}}$ for all samples $(\bar{\omega}_n)_{n\in\mathbb{Z}}$.

In words, $\bar{\mathscr{Q}}$ is the ergodic quadruple corresponding to the canonical space of the original input, and $\mathscr{Q}^0$ (respectively, $\mathscr{Q}^1$) corresponds to the canonical space of the input of pairs started at even (resp., odd) times in the original time scale.

Our aim is to construct a ϕ-matching on the original quadruple $\bar{\mathscr{Q}}$. First observe that in both cases, (H1') is satisfied, and there exists on $\mathscr{Q}^0$ (from Theorem 8 in the first case, Theorem 3 in the second), a unique even θ-stationary buffer content $U^0 \in \mathscr{Y}_2^{\infty}$, to which all sequences $\left(U_n^{[Y]}\right)_{n\in\mathbb{N}}$ for $Y \in \mathscr{Y}_2^{\infty}$ converge with strong backwards coupling, and such that $\mathbb{P}^0\left[U^0 = \emptyset\right] > 0$.

We can apply the exact same arguments on $\mathscr{Q}^1$, leading to the existence of a unique θ-stationary even buffer content U^1 on that space, that is such that $\mathbb{P}^1\left[U^1 = \emptyset\right] > 0$. Now observe that we can identify Ω^0 to $\bar{\Omega}$ *via* the one-to-one relation

$$\begin{cases} \Omega^0 & \longleftrightarrow \quad \bar{\Omega} \\ \left((v_n^0, \sigma_n^0, v_n^1, \sigma_n^1)\right)_{n\in\mathbb{Z}} & \longleftrightarrow \quad ((v_n, \sigma_n))_{n\in\mathbb{Z}} \text{ such that} \\ & \qquad (v_{2n}, \sigma_{2n}) = (v_n^0, \sigma_n^0) \text{ and } (v_{2n+1}, \sigma_{2n+1}) = (v_n^1, \sigma_n^1),\ n \in \mathbb{Z}. \end{cases}$$

Up to this bijective transformation, we can also identify θ to $\bar{\theta} \circ \bar{\theta}$, and construct two different buffer contents $\left(W_n^0\right)_{n\in\mathbb{Z}}$ and $\left(W_n^1\right)_{n\in\mathbb{Z}}$ on $\bar{\Omega}$, as follows:

$$\begin{cases} W_{2n}^0 & = U^0 \circ \bar{\theta}^{2n} \\ W_{2n+1}^0 & = \left(U^0 \circ \bar{\theta}^{2n}\right) \odot_\phi \left(V^0 \circ \bar{\theta}^{2n}, \Sigma^0 \circ \bar{\theta}^{2n}\right) \end{cases} \qquad n \in \mathbb{Z},\ \bar{\mathbb{P}} - \text{ a.s.}; \tag{43}$$

$$\begin{cases} W^1_{2n} &= \left(U^1 \circ \bar{\theta}^{2n-1}\right) \odot_\phi \left(V^0 \circ \bar{\theta}^{2n-1}, \Sigma^0 \circ \bar{\theta}^{2n-1}\right) \\ W^1_{2n+1} &= U^1 \circ \bar{\theta}^{2n+1}, \end{cases} \qquad n \in \mathbb{Z}, \bar{\mathbb{P}} - \text{ a.s.}, \tag{44}$$

which correspond respectively to a buffer content sequence that is stationary on $\mathbb{W}_2$ at even times, and to a buffer content sequence that is stationary on $\mathbb{W}_2$ at odd times. By construction, both sequences $\left(W^0_n\right)_{n\in\mathbb{Z}}$ and $\left(W^1_n\right)_{n\in\mathbb{Z}}$ have infinitely many construction points (the first one at even times, the second at odd times), therefore we can construct from each one, a unique perfect ϕ-matching on $\bar{\Omega}$, the first one depleting at even times and the second one at odd times. The proof is complete. ■

An example of the above construction for a separable graph of order 3 is given in Example 5.

Example 5. Consider the following separable compatibility graph on $\mathcal{V} = \{1, 2, \dots, 6\}$,

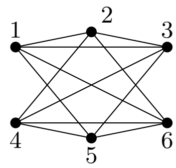

Figure 7: A separable graph of order 3.

Set $\bar{\Omega} := \{\bar{\omega}_1, \dots, \bar{\omega}_6\}$, where

$$\begin{cases} \bar{\omega}_1 &= ...142356\mathbf{1}42356..., \\ \bar{\omega}_2 &= ...423561\mathbf{4}23561..., \\ \bar{\omega}_3 &= ...235614\mathbf{2}35614..., \\ \bar{\omega}_4 &= ...356142\mathbf{3}56142..., \\ \bar{\omega}_5 &= ...561423\mathbf{5}61423..., \\ \bar{\omega}_6 &= ...614235\mathbf{6}14235...., \end{cases} \tag{45}$$

in which the 0-coordinate is marked in bold. Equipped with the power-set $\bar{\mathscr{F}}$ of $\bar{\Omega}$, the shift $\bar{\theta}$ defined by $\bar{\theta}\omega_i = \bar{\theta}\omega_{i+1}$ for $i \leq 5$ and $\bar{\theta}\omega_6 = \bar{\omega}_1$ and $\bar{\mathbb{P}}$ the uniform probability on $\bar{\Omega}$ (which correspond to μ the uniform probability on $\mathcal{V}$), it is immediate that $\bar{\mathscr{Q}} = \left(\bar{\Omega}, \bar{\mathscr{F}}, \bar{\mathbb{P}}, \bar{\theta}\right)$ is a stationary ergodic quadruple. Following the construction in the proof of Proposition 14, the canonical space of the "paired" input at even and odd times is given by $\Omega^0 = \{\omega^0_1, \omega^0_2, \omega^0_3\}$, where (emphasizing again the 0-coordinate in bold)

$$\begin{cases} \omega^0_1 &= ...14\ 23\ 56\ \mathbf{14}\ 23\ 56\ ..., \\ \omega^0_2 &= ...23\ 56\ 14\ \mathbf{23}\ 56\ 14\ ..., \\ \omega^0_3 &= ...56\ 14\ 23\ \mathbf{56}\ 14\ 23\ ... \end{cases}$$

in a way that at time 0, a 1-item and then a 4-item enter the system for the sample ω^0_1, a 2-item and then a 3-item for ω^0_2, and a 5-item and then a 6-item for ω^0_3. All the same, the canonical space of the paired input at odd times is $\Omega^1 = \{\omega^1_1, \omega^1_2, \omega^1_3\}$, where

$$\begin{cases} \omega^1_1 &= ...42\ 35\ 61\ \mathbf{42}\ 35\ 61\ ..., \\ \omega^1_2 &= ...35\ 61\ 42\ \mathbf{35}\ 61\ 42\ ..., \\ \omega^1_3 &= ...61\ 42\ 35\ \mathbf{61}\ 42\ 35\ ... \end{cases}$$

Whenever furnished with the uniform probability, both quadruples $\mathscr{Q}^0$ and $\mathscr{Q}^1$ thereby obtained are stationary and ergodic. Set $\phi = \text{FCFM}$. It is immediate to observe that both words 142356 and 423561 are strong erasing words for (G, ϕ) (assertion (i) of Proposition 7) and that (29) holds. Thus from Theorem 8 there exists a unique stationary buffer content U^0 on $\mathscr{Q}^0$ and a unique buffer content U^1 on $\mathscr{Q}^1$, which are respectively given by

$$U^0(\omega^0_1) = \emptyset, \quad U^0(\omega^0_2) = 14, \quad U^0(\omega^0_3) = \emptyset,$$
$$U^1(\omega^1_1) = \emptyset, \quad U^1(\omega^1_2) = \emptyset, \quad U^1(\omega^1_3) = \emptyset.$$

We then construct two buffer content sequences $\left(W^0_n\right)_{n\in\mathbb{Z}}$ and $\left(W^1_n\right)_{n\in\mathbb{Z}}$ on $\bar{\mathscr{Q}}$ from (43) and (44), whose valuation e.g. at sample point $\bar\omega_1$ are respectively given by

$$\begin{cases} W^0_{6n}(\bar\omega_1) &= \emptyset, \quad W^0_{6n+1}(\bar\omega_1) = 1, \quad W^0_{6n+2}(\bar\omega_1) = 14,\\ W^0_{6n+3}(\bar\omega_1) &= 4, \quad W^0_{6n+4}(\bar\omega_1) = \emptyset, \quad W^0_{6n+5}(\bar\omega_1) = 5\,; \quad n\in\mathbb{Z},\end{cases}$$

$$\begin{cases} W^1_{6n}(\bar\omega_1) &= 6, \quad W^1_{6n+1}(\bar\omega_1) = \emptyset, \quad W^1_{6n+2}(\bar\omega_1) = 4,\\ W^1_{6n+3}(\bar\omega_1) &= \emptyset, \quad W^1_{6n+4}(\bar\omega_1) = 3, \quad W^1_{6n+5}(\bar\omega_1) = \emptyset\,; \quad n\in\mathbb{Z}.\end{cases}$$

Both $\left(W^0_n\right)_{n\in\mathbb{Z}}$ and $\left(W^1_n\right)_{n\in\mathbb{Z}}$ determine uniquely a bi-infinite perfect matching on $\mathscr{Q}^0$. These two matchings are represented in Figure 8.

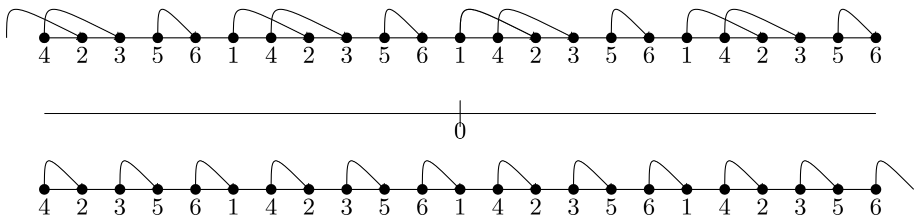

Figure 8: The two stationary matchings corresponding to the graph of Figure 7 and the input (45)
.

4.7 Perfect FCFM-matchings in reverse time

To conclude, let us comeback to the FCFM model, for which bi-infinite perfect matchings have an interesting property.

First, observe that we can complete the "exchange" mechanism introduced in the definition of the backwards and the forwards chains in Section 3.2, using construction points as follows: start from a construction point, and then replace all items from left to right by the copy of the class of their match, on the fly, as soon as they are matched. We illustrate this procedure in Figure 9, by the completion of the exchanges over two perfectly matched blocks, for the compatibility graph of Figure 1 and the arrival scenario of Figure 2.

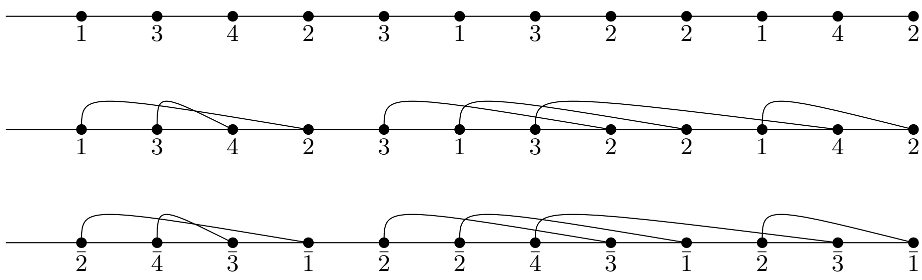

Figure 9: Top: the arrival scenario of Figure 2 (augmented with a 2 item). Middle: the two corresponding blocks, perfectly matched in FCFM with the compatibility graph of Figure 1. Bottom: completion of the exchanges by matchings.

Now observe the following: after completion of the exchanges on any perfectly matched block by FCFM, for any arrival scenario, by reading the arrivals on the matched block from right to left, we see nothing but a FCFM matching of the items of classes in $\overline{\mathcal{V}}$. To prove this, let the four nodes i, j, k and

ℓ be such that in G, $i{-}k$, $j{-}k$ and $i{-}\ell$, and suppose that, after the exchange, four copies $\bar{i}$, $\bar{j}$, $\bar{k}$ and $\bar{\ell}$ are read in that order, in reverse time, i.e. from right to left. Let us also assume that the FCFM rule in reverse time is violated on this quadruple: then the $\bar{k}$ item is matched with the $\bar{j}$ item while the $\bar{i}$ item is still unmatched, and then the latter item is matched with the $\bar{\ell}$ item. This occurs if and only if, in direct time, the four items of classes i, j, k and ℓ arrive in that order, and the k item choses the j item over the i item for its match, and then the unmatched i item is matched with the ℓ item. This violates in turn the FCFM policy, according to which the k item should have been matched with the i item instead of the j item. Hence the assertion above: over any perfectly matched block in FCFM, the block of exchanged items read in reverse time is also perfectly matched in FCFM - see the bottom display of Figure 9.

Now assume that the conditions of Proposition 14 are satisfied for ϕ = FCFM. Then there exist exactly two bi-infinite perfect FCFM-matchings of the input. Generalizing the above observation to all perfectly matched blocks on $\mathbb{Z}$, we conclude that there exist exactly two perfect FCFM-matchings of the exchanged items in reverse time, corresponding respectively to the two aforementioned perfect FCFM-matchings in direct time, after complete exchanges over blocks, read from right to left.

References

[1] I. Adan and G. Weiss. Exact FCFS matching rates for two infinite multi-type sequences. *Operations Research*, **60**(2):475–489, 2012.

[2] I Adan, A Busic, J. Mairesse and G. Weiss. Reversibility and further properties of the FCFM Bipartite matching model. ArXiv math.PR 1507.05939.

[3] F. Baccelli and P. Brémaud. *Elements of Queueing Theory* (2nd ed.). Springer, 2002.

[4] A. Busić, V. Gupta, and J. Mairesse. Stability of the bipartite matching model. *Adv. Appl. Probab.*, **45**(2):351–378, 2013.

[5] A. A. Borovkov. *Asymptotic Methods in Queueing Theory.* J. Wiley, New york, 1984.

[6] A. A. Borovkov and S. Foss. Stochastic Recursive Sequences and their Generalizations. *Siberian Adv. in Math.*, **2**(1), 16–81, 1992.

[7] A. A. Borovkov and S. Foss. Two ergodicity criteria for stochastically recursive sequences. *Acta Applic. Math.*, **34**, 125–134, 1994.

[8] A. Brandt, P. Franken and B. Lisek *Stationary Stochastic Models.* Akademie-Verlag/Wiley, 1990.

[9] R. Caldentey, E.H. Kaplan, and G. Weiss. FCFS infinite bipartite matching of servers and customers. *Adv. Appl. Probab*, **41**(3), 695–730, 2009.

[10] M. Crandall and L. Tartar. Some relations between non-expansive and order preserving mappings. *Proc. of the American Math. Soc.*, **78**(3), 385–390, 1980.

[11] I. Gurvich and A. Ward. On the dynamic control of matching queues. *Stochastic Systems*, **4**(2), 1–45, 2014.

[12] F.P. Kelly. *Reversibility and Stochastic Networks*, Wiley, 1979.

[13] R.M. Loynes. The stability of queues with non-independent interarrivals and service times. *Proceedings of the Cambridge Philosophical Society*, **58**, 497–520, 1962.

[14] J. Mairesse and P. Moyal. Stability of the stochastic matching model. *Journal of Applied Probability* **53**(4), 1064–1077, 2016.

[15] P. Moyal and O. Perry. On the Instability of matching queues. *Annals of Applied Probability*, 2017. ArXiv math.PR 1511.04282.

[16] J.G. Propp and D.B. Wilson. Exact sampling with coupled Markov chains and applications to stastical mechanics. *Random structures and Algorithms* **9**(1-2): 223–252, 1996.

[17] Ph. Robert. *Stochastic networks and queues*. Springer-Verlag, 2003.