



**HAL**  
open science

# TOPOLOGICAL PHASE TRANSITION III: SOLAR SURFACE ERUPTIONS AND SUNSPOTS

Tian Ma, Shouhong Wang

► **To cite this version:**

Tian Ma, Shouhong Wang. TOPOLOGICAL PHASE TRANSITION III: SOLAR SURFACE ERUPTIONS AND SUNSPOTS. 2018. hal-01672381v3

**HAL Id: hal-01672381**

**<https://hal.science/hal-01672381v3>**

Preprint submitted on 19 Sep 2018

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# TOPOLOGICAL PHASE TRANSITION III: SOLAR SURFACE ERUPTIONS AND SUNSPOTS

TIAN MA AND SHOUHONG WANG

ABSTRACT. This paper is aimed to provide a new theory for the formation of the solar surface eruptions and sunspots. The key ingredient of the study is the new *anti-diffusive effect of heat*, based on the recently developed statistical theory of heat by the authors [3]. The anti-diffusive effect of heat states that due to the higher rate of photon absorption and emission of the particles with higher energy levels, the photon flux will move toward to the higher temperature regions from the lower temperature regions. This anti-diffusive effect of heat leads to a modified *law of heat transfer*, which includes a reversed heat flux counteracting the heat diffusion. It is this anti-diffusive effect of heat and thereby the modified law of heat transfer that lead to the temperature blow-up and consequently the formation of sunspots, solar eruptions, and solar prominences. This anti-diffusive effect of heat may be utilized to design a plasma instrument, directly converting solar energy into thermal energy. This may likely offer a new form of fuel much more efficient than the photovoltaic devices.

## CONTENTS

1. Introduction	2
2. Sun's Surface Fluid Dynamics	5
2.1. Astronomical phenomenon on the Sun's surface	5
2.2. Anti-diffusive effect of heat	7
2.3. Sun's surface fluid dynamical equations	8
3. Theory on Formation of Sunspots and Solar Eruptions	10
3.1. Blow-up theorem	10
3.2. Sun's surface eruptions and sunspots	13

---

*Date:* September 19, 2018.

*Key words and phrases.* topological phase transition, sunspot, solar flare, solar prominence, solar eruption, temperature blow-up, statistical theory of heat, energy level formula of temperature, photon number entropy formula, anti-diffusive effect of heat, Fourier law, Fick law, heat transfer, photon absorption, photon radiation.

The work was supported in part by the US National Science Foundation (NSF), the Office of Naval Research (ONR), and by the Chinese National Science Foundation.

3.3. Period of solar eruptions	15
3.4. Non blow-up condition of temperature	16
References	18

## 1. INTRODUCTION

The main objective of this paper is to provide a new theory for the formation of the solar surface eruptions and sunspots. This is part of the research program initiated recently by the authors on theory and applications of topological phase transitions, including

- (1) quantum phase transitions [4],
- (2) formation of galactic spiral structure [5],
- (3) boundary-layer separation of fluid flows, and
- (4) interior separation of fluid flows.

The Sun is mainly made up of hydrogen and helium, and is entirely in a gaseous state. Solar structure is divided into two parts: the interior and the atmosphere. The atmosphere is composed of three distinct parts: the photosphere, the chromosphere and the corona, and the most visible light comes from the photosphere. Astronomical observations reveal that the solar flare, the solar prominence and the sunspots are important phenomena for the solar atmosphere. Solar flares refer to sudden flashes in Sun's surface brightness. A solar prominence is a large, bright gaseous feature extending outward from the Sun's surface, usually in a loop shape. Sunspots are the areas on Sun's surface, which are darker than their surrounding areas. Astronomical observations show that solar flares, prominences and sunspots are intimately related. The theory established in this paper verifies this claim, and explains their relations.

The main ingredients of the paper are as follows.

*First*, the most important ingredient of the study present in this paper is the *law of heat transfer* based on the recently developed statistical theory of heat by the authors [3]. In this theory, we derived the energy level temperature formula, showing that the temperature is essentially the average energy level of system particles. We also obtained the photon number entropy formula, demonstrating that the entropy is the number of photons in the gap between system particles, and the physical carrier of heat is the photons.

Another important component of the theory is the vibratory mechanism of photon absorption and radiation:

*a particle can only absorb and radiate photons while experiencing vibratory motion. The higher the frequency of the vibration of the particle, the larger the absorbing and radiating energy. The vibration or irregular motion of particles in a system is caused by collisions between particles and by absorbing and radiating photons.*

This mechanism shows immediately that for particles in high speed vibration and irregular motion, the rate of photon emission and absorption increases, leading to the number density of photons to increase, and further causing the particle energy levels to elevate. Hence, the photon absorption and emission induce the concentration of temperature, which we call the *anti-diffusive effect of heat*:

(1.1) *Due to the higher rate of photon absorption and emission of the particles with higher energy levels, the photon flux will move toward the higher temperature regions from the lower temperature regions.*

By the Stefan-Boltzmann law, the reversed heat flux measuring the anti-diffusive effect is expressed as

$$(1.2) \quad \left( \frac{dT}{dt} \right)_{ADE} = \beta_0 T^4,$$

where  $\beta_0$  is the heat effect coefficient. Then by the Fourier law, we derive the following law for heat transfer for the solar atmosphere:

$$(1.3) \quad \frac{\partial T}{\partial t} + (u \cdot \nabla)T = \kappa \Delta T + \beta_0 T^4 + \beta_1 (\mathbf{E}^2 + \mathbf{H}^2).$$

Here on the right-hand side, the first term represents the usual diffusion of heat, the last term is the heat source due to the solar electromagnetic fields. Importantly the second term represents the anti-diffusive effect of heat, and it is this anti-diffusive effect that leads to the formation of sunspots, the solar flares and the prominences.

*Second*, the full model governing Sun's surface plasma fluid combines the fluid dynamical equations, the above new heat equation (1.3), and the Maxwell equations. One key component of the theory for the formation of sunspots and solar eruptions is to prove a blow-up theorem, Theorem 3.1. This theorem shows that there exist  $x_0 \in \Omega$  and  $t_0 > 0$ , such that the temperature  $T$  blows up at  $(x_0, t_0)$  with blow-up time estimated as

$$(1.4) \quad t_0 = \frac{|\Omega|^3}{3a^3\beta_0}, \quad a = \int_{\Omega} T_0(x) dx,$$

where  $T_0$  is the initial value of temperature,  $\Omega = S^2 \times (r_0, r_1)$ ,  $r_0$  is the solar radius, and  $r_1 = r_0 + h$  with the thickness of solar atmosphere  $h$ .

*Third*, the sunspots can now be clearly explained by the anti-diffusive effect of heat and the temperature blow-up that we just mentioned. We summarize this explanation as follows:

- (1) Due to thermal fluctuations, the temperature in the solar atmosphere is nonhomogeneous, leading to elevated temperature in some local areas. The anti-diffusive effect of heat then shows that the higher temperature regions absorb more photons from their surrounding areas, leading to their temperature decreasing, and consequently generating sunspots;
- (2) The anti-diffusive effect of heat makes the temperature around the sunspot areas increasing rapidly, generating the temperature blow-up, and leading to solar eruptions. In fact, we deduce from the temperature blow-up at  $(x_0, t_0)$  that

$$(1.5) \quad \lim_{t \rightarrow t_0} \left| \frac{du(x_0, t)}{dt} \right| = \infty,$$

which represents the high speed gas explosion and particle ejections, with ejection direction given by

$$(1.6) \quad \vec{r} = - \lim_{t \rightarrow t_0} \frac{\nabla T(x_0, t)}{|\nabla T(x_0, t)|}.$$

- (3) It is clear that the temperature blow-up generates solar flares.
- (4) By the Maxwell equations, it is clear that the eruption described by (1.5) generates very strong electromagnetic radiation in the  $\vec{r}$  direction.
- (5) The eruption (1.5) leads also to a huge current jet  $J = \rho_e u$  in the  $\vec{r}$  direction, which, in view of the Ampère law, gives rise to violent magnetic loops, perpendicular to the direction  $\vec{r}$ , leading to the solar prominences.
- (6) Astronomical observations show that sunspots and solar flares occur periodically in an 11-year cycle. The blow-up time (1.4) links the initial temperature  $T_0$ , the solar eruption period, and the anti-diffusive effect coefficient  $\beta_0$ . Such a link is applicable to all stars. For the Sun, we can easily estimate

$$\beta_0 = 2.85 \times 10^{-22} / (K^3 \cdot s).$$

*Fourth*, the anti-diffusive effect of heat (1.1) may provide a new source of fuel. Basically, in a high temperature plasma system where free electrons are abundant, photons in the low temperature area experience the anti-diffusive effect and can move to the high temperature

region, further raising the temperature in the high temperature region. The conversion of the thermal energy in the high-temperature region offers an alternative fuel source, which we call plasma fuel. The Sun is a natural source of photons for the low-temperature regions required for such a process to continue. This direct way of converting solar energy into a new form of fuel will be much more efficient than the photovoltaic devices.

In view of the non blow-up condition (3.30), the following condition is required for the anti-diffusive effect of heat to dominate the normal heat conduction:

$$(1.7) \quad \kappa < \frac{16}{9} \beta_0 |\Omega|^{1/15} \left[ \int_{\Omega} T_0^5 dx \right]^{3/5},$$

where  $T_0$  is the initial temperature,  $\kappa$  is the heat conduction coefficient, and  $\beta_0$  is the anti-diffusive effect coefficient. Both  $\kappa$  and  $\beta_0$  depend on the material. Therefore, for a given plasma system, the larger the domain and the higher the initial temperature, the stronger the anti-diffusive effect. Consequently an effective plasma device for converting solar energy into a fuel source can be achieved by increasing the domain size and/or the initial temperature.

The paper is organized as follows. Section 2 introduces the anti-diffusive effect of heat and the model for the Sun's surface fluid dynamics. Section 3 proves the blow-up theorem, derives a non blow-up condition, and provides the detailed explanation of the formation of solar surface eruptions, solar flares, and solar prominences.

## 2. SUN'S SURFACE FLUID DYNAMICS

**2.1. Astronomical phenomenon on the Sun's surface.** The Sun is a star which we are most familiar with. The solar mass is about  $2 \times 10^{30}$ kg, and the Sun's radius is  $7 \times 10^5$ km. Its average density is  $\rho = 1.4g/cm^3$  and, for comparison, we recall that the density of water is  $\rho = 1g/cm^3$ . The Sun consists mainly of hydrogen (94%) and helium (6%), and is entirely in a gaseous state. Hence it can be viewed a gaseous fluid ball.

Solar structure is divided into two parts: the interior and the atmosphere. The atmosphere is composed of three distinct parts: the photosphere, the chromosphere and the corona. The photosphere is in the bottom layer, the chromosphere is in the middle and the corona is in the outer of the solar atmosphere. The most visible light come from the photosphere, and its temperature is ranged in  $4000K \sim 7000K$ . The chromospheric temperature is  $1.5 \times 10^4K$ , and the corona has the

highest temperature at about  $2 \times 10^6 K$ . But the corona density is very low, at  $10^{-9}$  times the density of the earth's atmosphere. Based on astronomical observations, the following are known important phenomena for the solar atmosphere:

1) *Solar flare*. It is a very attracting event which occurs in Sun's atmosphere, and is a sudden flash in Sun's surface brightness. Solar flares affect all three layers: photosphere, chromosphere, and corona, and are often accompanied by a coronal mass ejection, or by an erupting prominence. When the plasma medium is heated to near  $T = 5 \times 10^6 K$ , flares are powered by the sudden release of huge energy, together with very strong electromagnetic radiations and high speed (near the speed of light) particle eruptions. The active period of solar flares follows the 11-year cycle, called the solar cycle. See Figure 2.1 for solar flares.

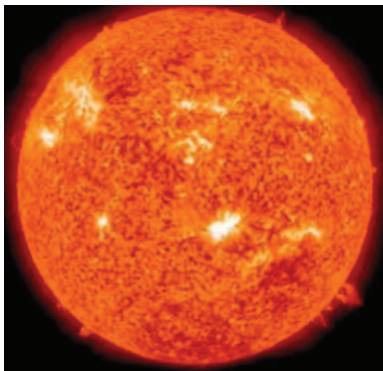


FIGURE 2.1. Solar flares

2) *Solar prominence*. A prominence is a large, bright gaseous feature extending outward from the Sun's surface, usually in a loop shape; see Figure 2.2. Prominences occur often along with solar flares, are anchored to the Sun's surface in the photosphere, and extend outward into the Sun's corona, reaching as high as thousands of kilometers.

3) *Sunspots*. They are temporary phenomena on the Sun's surface that appear as spots darker than the surrounding areas. The sunspot regions have lower surface temperature. Sunspots occur in an approximately 11-year solar cycle, the same as that of the solar flares.

Usually, sunspots accompany secondary phenomena such as coronal loops, prominences and solar flares. Most solar flares and coronal mass ejections originate in active regions around sunspots.

Astronomical observations show that solar flares, prominences and sunspots are intimately related. The theory that we established in this paper verifies clearly this claim, and explains their relations.

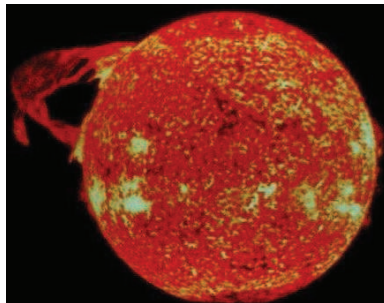


FIGURE 2.2

2.2. **Anti-diffusive effect of heat.** In [3], the authors developed a statistical theory of heat, consisting of the following main ingredients:

1). *Energy level temperature formulas.* These formulas are derived from the well-known Maxwell-Boltzmann, the Fermi-Dirac, and the Bose-Einstein distributions, and they show that the temperature is essentially the average energy level of system particles.

2). *Photon number entropy formula.* This formula shows that entropy is the number of photons in the gap between system particles, and the physical carrier of heat is the photons.

3). *Vibratory mechanism of photon absorption and radiation.* A particle can only absorb and radiate photons while experiencing vibratory motion. The higher the frequency of the vibration of the particle, the larger the absorbing and radiating energy. The vibration or irregular motion of the particles in a system is caused by collisions between particles and by absorbing and radiating photons.

4). *Law of heat transfer.* For particles in high speed vibration and irregular motion, the rate of photon emission and absorption increases, leading to the number density of photons to increase, and further causing the particle energy levels to elevate. Hence, the photon absorption and emission induce the concentration of the temperature, which we call the *anti-diffusive effect of heat*:

(2.1) *Due to the higher rate of photon absorption and emission of the particles with higher energy levels, the photon flux will move toward the higher temperature regions from the lower temperature regions.*

On the other hand, we know that temperature obeys the Fourier law, i.e. photons are dictated by the Fick law. Hence, the heat transfer follows the balance between the anti-diffusive effect (2.1) and the Fick



diffusion law for the system photons, i.e. the law of heat transfer can be expressed as

$$(2.2) \quad \frac{dT}{dt} = \kappa\Delta T + Q + \text{the concentration rate of photons},$$

where  $\kappa\Delta T$  represents the diffusion term,  $Q$  is the heat resource.

**Remark 2.1.** In the classical heat conduction theory, the law of heat transfer is written as

$$(2.3) \quad \frac{dT}{dt} = \kappa\Delta T + Q,$$

which is different from (2.2). In fact, in the case where the temperature is not very high, or the system medium is not plasma, the third term in the right-hand side of (2.2) is very small and can be ignored. However, for the Sun's surface plasma fluid, this term will play a crucial role for the appearance of the solar flares.  $\square$

Based on the anti-diffusive effect of heat (2.1), the concentration of photons is a reversed process of heat diffusion. Hence, by the Stefan-Boltzmann law, the third term on the right-hand side of (2.2) should be proportional to  $T^4$ :

$$(2.4) \quad \text{the concentrating rate of photons} = \beta_0 T^4.$$

Equivalently the anti-diffusive effect of heat is expressed as

$$\left(\frac{dT}{dt}\right)_{ADE} = \beta_0 T^4,$$

where  $\beta_0$  is the anti-diffusive effect coefficient.

In fact, it is the anti-diffusive effect of heat (2.1) that leads to the formation of sunspots, and it is the relation (2.4) that yields the solar flares and prominences.

**2.3. Sun's surface fluid dynamical equations.** Sun's surface fluid is composed of gaseous plasma. The state functions describing solar flares and prominences are: the velocity field  $u$  of the plasma fluid, the temperature  $T$ , the electromagnetic fields  $\mathbf{E}$  and  $\mathbf{H}$ . Hence, the dynamical model governing Sun's surface fluid is the three groups of equations coupling the Navier-Stokes equations, the heat equation, and the Maxwell equations.

We start with the spatial domain given by

$$\Omega = S^2 \times (r_0, r_1),$$

where  $S^2$  is the two-dimensional unit sphere,  $r_0$  is the solar radius, and  $r_1 = r_0 + h$  with the thickness of solar atmosphere  $h$ . We now describe the three set of equations, and their coupling.

1). The Navier-Stokes equations are written as

$$(2.5) \quad \rho \left[ \frac{\partial u}{\partial t} + (u \cdot \nabla)u \right] = \mu \Delta u - \nabla p + f,$$

where  $\rho$  is the mass density,  $p$  is the pressure, and  $f$  is the force density. Because the fluid is plasma, each particle is charged. Hence, the force field  $f$  includes

$$f = \text{charge force} + \text{Lorentz force} + \text{thermal force}.$$

Based on classical electromagnetic theory,

$$\begin{aligned} \text{charge force} &= \rho_e \mathbf{E}, \\ \text{Lorentz force} &= J \times \mathbf{H}, \end{aligned}$$

where  $\rho_e$  is the effective charge density in the plasma,  $J = \rho_e u$  is the effective current density. By theory of thermodynamics, we have

$$\text{thermal force} = -g\mathbf{k}\rho(1 - \alpha T),$$

where  $g$  is the solar gravitational constant,  $\alpha$  is the thermal expansion coefficient, and  $\mathbf{k}$  is the radial unit vector.

Then equations (2.5) are expressed as

$$(2.6) \quad \rho \left[ \frac{\partial u}{\partial t} + (u \cdot \nabla)u \right] = \mu \Delta u - \nabla p + \rho_e \mathbf{E} + \rho_e u \times \mathbf{H} - g\vec{k}\rho(1 - \alpha T).$$

Also, (2.6) is complemented with the continuous equation

$$(2.7) \quad \frac{\partial \rho}{\partial t} + \text{div}(\rho u) = 0.$$

2). In view of (2.2) and (2.4), the heat equation is given by

$$(2.8) \quad \frac{\partial T}{\partial t} + (u \cdot \nabla)T = \kappa \Delta T + Q + \beta_0 T^4,$$

where  $\kappa$  is the heat conduction coefficient, and  $Q$  is the heat source excited by the solar electromagnetic fields, written as

$$Q = \beta_1(\mathbf{E}^2 + \mathbf{H}^2).$$

Thus, the heat equation (2.8) becomes

$$(2.9) \quad \frac{\partial T}{\partial t} + (u \cdot \nabla)T = \kappa \Delta T + \beta_0 T^4 + \beta_1(\mathbf{E}^2 + \mathbf{H}^2).$$

3). The Maxwell equations read

$$\begin{aligned}
(2.10) \quad & \mu_0 \frac{\partial \mathbf{H}}{\partial t} = -\text{curl } \mathbf{E}, \\
& \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \text{curl } \mathbf{H} - J, \quad J = \rho_e u, \\
& \text{div } \mathbf{H} = 0, \\
& \text{div } \mathbf{E} = \rho_e,
\end{aligned}$$

where  $\mu_0$  is the magnetic permeability, and  $\varepsilon_0$  is the dielectric constant.

4). *Model for Sun's surface plasma fluid.* Combing the fluid dynamical equations (2.6)–(2.7), the heat equation (2.9), and the Maxwell equations (2.10), we derive the model governing Sun's surface plasma fluid as follows

$$\begin{aligned}
(2.11) \quad & \rho \left[ \frac{\partial u}{\partial t} + (u \cdot \nabla) u \right] = \mu \Delta u - \nabla p \\
& + \rho_e (\mathbf{E} + u \times \mathbf{H}) - g \mathbf{k} \rho (1 - \alpha T),
\end{aligned}$$

$$(2.12) \quad \frac{\partial T}{\partial t} + (u \cdot \nabla) T = \kappa \Delta T + \beta_0 T^4 + \beta_1 (\mathbf{E}^2 + \mathbf{H}^2),$$

$$(2.13) \quad \frac{\partial \mathbf{H}}{\partial t} = -\frac{1}{\mu_0} \text{curl } \mathbf{E},$$

$$(2.14) \quad \frac{\partial \mathbf{E}}{\partial t} = \frac{1}{\varepsilon_0} \text{curl } \mathbf{H} - \frac{1}{\varepsilon_0} \rho_e u,$$

$$(2.15) \quad \text{div } \mathbf{H} = 0,$$

$$(2.16) \quad \text{div } \mathbf{E} = \rho_e,$$

$$(2.17) \quad \frac{\partial \rho}{\partial t} = -\text{div}(\rho u).$$

The system (2.11)–(2.17) constitutes the basis to establish Sun's electromagnetic eruption theory.

### 3. THEORY ON FORMATION OF SUNSPOTS AND SOLAR ERUPTIONS

**3.1. Blow-up theorem.** From the mathematical point of view, the phenomena of solar flares and prominences correspond to the blow-up of the solutions of the equations (2.11)–(2.17). Blow-up is mathematical property that a solution  $\Phi(x, t) = (u, T, \mathbf{H}, \mathbf{E})$  is called to blow-up at  $(x_0, t_0)$  if

$$(3.1) \quad \lim_{t \rightarrow t_0} |\Phi(x_0, t)| = \infty,$$

where  $|\Phi|^2 = u^2 + T^2 + \mathbf{E}^2 + \mathbf{H}^2$ .

Hence, the blow-up theorem for the system (2.11)–(2.17) introduced in the following is crucial for us to understand the Sun's electromagnetic eruptions. For simplicity, consider the case where  $\rho$  is a constant, and (2.17) becomes

$$(3.2) \quad \operatorname{div} u = 0.$$

Consider the initial and boundary conditions:

$$(3.3) \quad u_n|_{\partial\Omega} = 0, \quad \frac{\partial u_\tau}{\partial n}\Big|_{\partial\Omega} = 0, \quad \frac{\partial T}{\partial n}\Big|_{\partial\Omega} = 0,$$

$$(3.4) \quad \Phi = (u, T, \mathbf{H}, \mathbf{E})|_{t=0} = \Phi_0,$$

where  $n, \tau$  are the unit normal and tangent vectors on  $\partial\Omega$ . Then we have the following theorem.

**Theorem 3.1** (Blow-up theorem). *Let  $\Phi(x, t)$  be a solution of (2.11)–(2.16) and (3.2), and satisfy the initial and boundary conditions (3.3) and (3.4). If the initial value  $\Phi_0$  is bounded, i.e.*

$$\sup_{\Omega} |\Phi_0(x)| < \infty,$$

*then there exist  $x_0 \in \Omega$  and  $t_0 > 0$ , such that the temperature  $T$  blows up at  $(x_0, t_0)$ , and consequently  $\Phi(x, t)$  blows-up  $(x_0, t_0)$  as well, i.e. (3.1) holds true.*

*Proof.* Take the integration on both sides of (2.12):

$$(3.5) \quad \frac{d}{dt} \int_{\Omega} T dx = \int_{\Omega} [\kappa \Delta T - (u \cdot \nabla)T + \beta_0 T^4 + \beta_1(\mathbf{E}^2 + \mathbf{H}^2)] dx.$$

In view of (3.2) and (3.3), by the Gauss formula, we have

$$\begin{aligned} \int_{\Omega} \Delta T dx &= \int_{\partial\Omega} \frac{\partial T}{\partial n} dx = 0, \\ \int_{\Omega} (u \cdot \nabla)T dx &= - \int_{\Omega} T \operatorname{div} u dx + \int_{\partial\Omega} T u_n dS = 0. \end{aligned}$$

Then (3.5) becomes

$$(3.6) \quad \frac{d}{dt} \int_{\Omega} T dx = \int_{\Omega} [\beta_0 T^4 + \beta_1(\mathbf{E}^2 + \mathbf{H}^2)] dx.$$

Physically,  $T > 0$ , which can also be proved using the same method as in the paper [1]. Hence, we have

$$(3.7) \quad \int_{\Omega} |T| dx = \int_{\Omega} T dx > 0.$$

In addition, by the anti-Hölder inequality (see [2]), for any  $0 < p < 1$  and  $q = p/(p - 1)$  we have

$$(3.8) \quad \int_{\Omega} |fg| dx \geq \left[ \int_{\Omega} |f|^p dx \right]^{1/p} \left[ \int_{\Omega} |g|^q dx \right]^{1/q}.$$

Take  $f = T^4$ ,  $g = 1$ ,  $p = 1/4$  and  $q = -1/3$ , and note (3.7), then (3.8) becomes

$$(3.9) \quad \int_{\Omega} T^4 dx \geq \frac{1}{|\Omega|^3} \left[ \int_{\Omega} T dx \right]^4.$$

Recall the comparison theorem of differential equations: For two non-negative functions  $f_1$  and  $f_2$  satisfying

$$f_1(t) \leq f_2(t), \quad \forall t \geq 0,$$

we consider the initial value problems

$$\begin{aligned} \frac{dx_1}{dt} &= f_1(t) \quad \text{with } x_1(0) = a, \\ \frac{dx_2}{dt} &= f_2(t) \quad \text{with } x_2(0) = a, \end{aligned}$$

where  $a \geq 0$ . Their solutions  $x_1(t)$  and  $x_2(t)$  satisfy

$$x_1(t) \leq x_2(t), \quad \forall t \geq 0.$$

Then, consider the equation

$$(3.10) \quad \frac{d}{dt} \int_{\Omega} T dx = \frac{\beta_0}{|\Omega|^3} \left[ \int_{\Omega} T dx \right]^4.$$

Based on the comparison theorem, for the solution  $\int_{\Omega} T dx$  of (3.6) and the solution  $\int_{\Omega} T_1 dx$  of (3.10) with the initial value conditions

$$(3.11) \quad \int_{\Omega} T dx \Big|_{t=0} = a,$$

$$(3.12) \quad \int_{\Omega} T_1 dx \Big|_{t=0} = a,$$

we deduce from (3.9) that

$$(3.13) \quad \int_{\Omega} T_1 dx \leq \int_{\Omega} T dx, \quad \forall t \geq 0.$$

Denote  $y = \int_{\Omega} T_1 dx$ , then equation (3.10) with the initial value condition (3.12) is written as

$$(3.14) \quad \begin{aligned} \frac{dy}{dt} &= k\beta_0 y^4, \\ y(0) &= a, \end{aligned}$$

where  $k = 1/|\Omega|^3$ . It is easy to see that the solution of (3.14) is

$$(3.15) \quad y = \frac{a}{[1 - 3ka^3\beta_0 t]^{1/3}}, \quad k = 1/|\Omega|^3.$$

Again, by (3.13) we have,

$$(3.16) \quad y \leq \int_{\Omega} T dx,$$

where  $T$  is the solution of (2.12). By (3.15), we see that

$$\lim_{t \rightarrow t_0} y(t) = \infty, \quad t_0 = \frac{1}{3ka^3\beta_0}.$$

Then we deduce from (3.16) that

$$(3.17) \quad \lim_{t \rightarrow t_0} \int_{\Omega} T dx = \infty,$$

which implies this theorem holds true.  $\square$

**3.2. Sun's surface eruptions and sunspots.** The blow-up theorem, Theorem 3.1, provides a solid mathematical foundation for the solar eruption theory developed in this paper. It shows that the eruptions are typically topological phase transitions at the blow-up point  $t_0 = 1/3ka^3\beta_0$ , at which the state functions  $\Phi(x, t) = (u, T, \mathbf{H}, \mathbf{E})$  tend to infinite, depicting the huge and complicated explosions as high speed mass ejections, sudden flares of flashlight, very strong radiations, and large spurts of loop shaped magnetic energy.

Based on Theorem 3.1 and the anti-diffusive effect of heat (2.1), we discuss the sunspot and solar eruption problems in the following.

1). *Sunspots.* We know that sunspots are the regions on Sun's surface, and possess the following two main characteristics:

- (1) the temperature in sport is lower than the surrounding areas, and consequently the brightness is darker; and
- (2) it often accompanies secondary events as the solar flares and prominences – the solar eruptions.

The above two characteristics can be well explained as follows by the anti-diffusive effect of heat (2.1) and the blow-up of the temperature:

- (1) Due to thermal fluctuations, the temperature in the solar atmosphere is nonhomogeneous, leading to elevated temperature in some local places. Based on the anti-diffusive effect, the regions with higher temperature absorb more photons from their surrounding areas, leading to the temperature decreasing, and consequently generating the sunspots;

- (2) When the sunspots appear, the anti-diffusive effect (2.4) makes the temperature in the regions around sunspots rapidly increasing to generate the blow-up as described in Theorem 3.1, which causes solar eruptions.

2). *Solar flares.* In the proof of Theorem 3.1, we derived (3.17), which implies that there exists a point  $x_0 \in \Omega$  such that

$$(3.18) \quad \lim_{t \rightarrow t_0} T(x_0, t) = \infty \quad \text{with } t_0 = \frac{1}{3ka^3\beta_0}.$$

It is the blow-up (3.18) that generates the solar flare. In addition, it also induces a chain of tragical variations in the velocity field  $u$  and the electromagnetic fields  $\mathbf{H}$  and  $\mathbf{E}$ , resulting in the coronal mass ejections, the strong radiations, and the solar prominences.

3). *Coronal mass ejections.* Equation (2.11) dictates the behavior of mass ejections. When the temperature  $T$  blows up at  $(x_0, t_0)$  as given in (3.18), the maximal forces acting on the particles near  $x_0$  are just  $\nabla p$ , as  $0 < \alpha T < 1$  in (2.11). Hence, in the neighborhood of  $(x_0, t_0)$ , (2.11) can be approximatively expressed as

$$(3.19) \quad \frac{du}{dt} = -\frac{1}{\rho} \nabla p.$$

By the gaseous equation of state:

$$p = R\rho T/m,$$

where  $R$  is the gas constant and  $m$  is the particle mass, the equation (3.19) is written as

$$(3.20) \quad \frac{du}{dt} = -\frac{R}{m} \nabla T.$$

By (3.18) we have

$$\lim_{t \rightarrow t_0} |\nabla T(x_0, t)| = \infty.$$

Therefore we deduce from (3.20) that

$$(3.21) \quad \lim_{t \rightarrow t_0} \left| \frac{du(x_0, t)}{dt} \right| = \infty.$$

The equality (3.21) represents the high speed gas explosion and particle ejections. The ejection direction is

$$(3.22) \quad \vec{r} = -\lim_{t \rightarrow t_0} \frac{\nabla T(x_0, t)}{|\nabla T(x_0, t)|}.$$

4). *Strong radiations.* The blow up (3.21) of the velocity causes the formation of the high speed jet of charged particles, inducing the

electromagnetic eruptions. By the Maxwell equations (2.13) and (2.14), we have

$$(3.23) \quad \begin{aligned} \frac{\partial^2 \mathbf{E}}{\partial t^2} + \frac{1}{\varepsilon_0 \mu_0} \text{curl}^2 \mathbf{E} &= -\frac{\rho_e}{\varepsilon_0} \frac{\partial u}{\partial t}, \\ \frac{\partial^2 \mathbf{H}}{\partial t^2} + \frac{1}{\varepsilon_0 \mu_0} \text{curl}^2 \mathbf{H} &= \frac{\rho_e}{\varepsilon_0} u. \end{aligned}$$

In view of (3.21) and (3.22), the equations (3.23) generate very strong electromagnetic radiation in the  $\vec{r}$  direction.

5). *Solar prominences.* Consider the equation (2.14), which can be approximatively expressed as (i.e. the Ampère law):

$$(3.24) \quad \text{curl } \mathbf{H} = \rho_e u.$$

By (3.21) and (3.22), we have

$$(3.25) \quad \rho_e u \cdot \vec{r} \gg 1 \quad \text{at } (x_0, t_0),$$

which represents the huge current jet in the  $\vec{r}$  direction.

Then, by the Ampère law (3.24), the strong current of (3.25) gives rise to violent magnetic loops, perpendicular to the direction  $\vec{r}$  in (3.22), as shown in Figure 3.1. As the current orientation  $\vec{r}$  is not in the normal direction of solar surface, the erupting magnetic loops are the solar prominences that we observe.

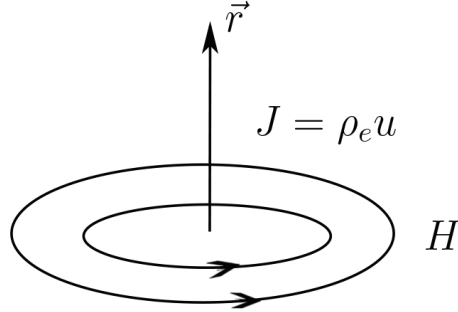


FIGURE 3.1. Huge current jet  $J$  generates a violent magnetic loop, which is the solar prominence.

**3.3. Period of solar eruptions.** Astronomical observations show that sunspots and solar flares occur periodically in an 11-year cycle. The temperature solution (3.15) provides also the formula of solar eruption period, which is rewritten as

$$(3.26) \quad \int_{\Omega} T dx = \frac{a}{[1 - 3ka^3 \beta_0 t]^{1/3}},$$



where  $k = 1/|\Omega|^3$ , and  $a$  is the initial value

$$a = \int_{\Omega} T(x, 0) dx.$$

When a solar eruption ends, the temperature distribution in the solar chromosphere is almost homogenous. Let  $T_0$  be the homogenous temperature, i.e.  $T(x, 0) = T_0$ . Then the initial value  $a = T_0|\Omega|$ , and (3.26) becomes

$$(3.27) \quad \int_{\Omega} T dx = \frac{|\Omega|T_0}{[1 - 3T_0^3\beta_0 t]^{1/3}}.$$

It is easy to deduce from (3.27) that the time  $t_p$  for the next eruption to occur satisfies

$$1 - 3T_0^3\beta_0 t_p = 0.$$

Hence the period  $t_p$  is

$$(3.28) \quad t_p = \frac{1}{3T_0^3\beta_0},$$

where  $\beta_0$  represents the heat effect coefficient, depending on the material, and  $T_0$  is the initial average temperature.

The period formula (3.28) is also applicable to all stars, because it was found that other stars also have the eruption phenomena. For the Sun,

$$t_p = 11 \text{ years} = 3.469 \times 10^8 s,$$

and  $T_0$  is the average temperature of the chromosphere:

$$T_0 = 1.5 \times 10^4 K.$$

Therefore, the heat effect coefficient for solar gaseous plasma is derived from (3.28) as follows

$$(3.29) \quad \beta_0 = 2.85 \times 10^{-22} / (K^3 \cdot s).$$

**Remark 3.2.** The heat effect coefficients for non-plasma materials are much smaller than the value in (3.29).  $\square$

**3.4. Non blow-up condition of temperature.** In this section, we demonstrate that under the Dirichlet boundary condition for the temperature  $T^1$ , the *non blow-up condition* for temperature in domain  $\Omega$

---

<sup>1</sup>The Dirichlet boundary condition for  $T$  amounts to saying that there is heat exchange of the system with outside. Hence there is no contradiction between the non blow-up condition and the blow-up theorem 3.1, where the Neumann boundary condition for  $T$  is used.

is

$$(3.30) \quad \kappa > \frac{16}{9}\beta_0|\Omega|^{1/15} \left[ \int_{\Omega} T_0^5 dx \right]^{3/5},$$

where  $T_0$  is the initial temperature,  $\kappa$  is the heat conduction coefficient, and  $\beta_0$  is the heat effect coefficient. Note that  $\beta_0$  is negligible for non-plasma material. Therefore, in the usual cases we often encounter, both  $T_0$  and  $\beta_0$  are small, so that no temperature blow-up occurs.

To verify (3.30), we consider the heat conduction equation with anti-diffusive effect of heat:

$$(3.31) \quad \begin{aligned} \frac{\partial T}{\partial t} &= \kappa \Delta T + \beta_0 T^4, \quad x \in \Omega \subset \mathbb{R}^3, \\ T|_{\partial\Omega} &= 0. \end{aligned}$$

Multiplying both sides of (3.31) by  $T$  and taking integral, we have

$$(3.32) \quad \frac{1}{2} \frac{d}{dt} \int_{\Omega} T^2 dx = \int_{\Omega} [-\kappa |\nabla T|^2 + \beta_0 T^5] dx.$$

By the Sobolev inequality (see e.g. p.23 in [2]):

$$\left[ \int_{\Omega} |\nabla T|^2 dx \right]^{1/2} \geq \frac{n(n-2)}{2(n-1)} \left[ \int_{\Omega} T^{\frac{2n}{n-2}} dx \right]^{\frac{n-2}{2n}},$$

where  $n$  is the dimension of space<sup>2</sup>. In this paper  $n = 3$ , and we have

$$(3.33) \quad \int_{\Omega} |\nabla T|^2 dx \geq \frac{9}{16} \left[ \int_{\Omega} T^6 dx \right]^{1/3}.$$

In addition, by the Hölder inequality, we derive

$$\|T\|_{L^6} \geq \frac{1}{|\Omega|^{1/30}} \|T\|_{L^5}.$$

Hence

$$(3.34) \quad \left[ \int_{\Omega} T^6 dx \right]^{1/3} \geq \frac{1}{|\Omega|^{1/15}} \left[ \int_{\Omega} T^5 dx \right]^{2/5}.$$

It follows from (3.33) and (3.34) that

$$\int_{\Omega} |\nabla T|^2 dx \geq \frac{9}{16|\Omega|^{1/15}} \left[ \int_{\Omega} T^5 dx \right]^{2/5}.$$

---

<sup>2</sup>We remark here that by the above inequality and the Hölder inequality, we have

$$\|\nabla u\|_{L^p} \geq \frac{c}{|\Omega|^{(N^*-q)/N^*q}} \|u\|_{L^q} \quad \text{for } n > p, N^* = \frac{np}{n-p}.$$

Then we deduce from (3.32) that

$$(3.35) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Omega} T^2 dx &\leq -\frac{9\kappa}{16|\Omega|^{1/15}} \left[ \int_{\Omega} T^5 dx \right]^{2/5} + \beta_0 \int_{\Omega} T^5 dx \\ &\leq -\left[ \frac{9\kappa}{16|\Omega|^{1/15}} - \beta_0 \left( \int_{\Omega} T^5 dx \right)^{3/5} \right] \left[ \int_{\Omega} T^5 dx \right]^{2/5}. \end{aligned}$$

It is clear that if

$$(3.36) \quad \kappa \geq \frac{16}{9} \beta_0 |\Omega|^{1/15} \left[ \int_{\Omega} T^5 dx \right]^{3/5} + \varepsilon, \quad \forall t > 0,$$

where  $\varepsilon > 0$  is arbitrarily small, then we have

$$(3.37) \quad \lim_{t \rightarrow \infty} \int_{\Omega} T^2 dx = 0.$$

In fact, by the inequality

$$\left[ \int_{\Omega} T^5 dx \right]^{2/5} \geq C_0 \left[ \int_{\Omega} T^2 dx \right],$$

and by the Gronwall inequality, we deduce from (3.35) that

$$(3.38) \quad \int_{\Omega} T^2 dx \leq e^{-C_0 \int_0^t \alpha(\tau) d\tau},$$

where by (3.36)

$$\alpha = \frac{9\kappa}{16|\Omega|^{1/15}} - \beta_0 \left[ \int_{\Omega} T^5 dx \right]^{3/5} \geq \frac{9\varepsilon}{16|\Omega|^{1/15}}.$$

Then, (3.38) implies that (3.37) holds true, and consequently the heat system (3.31) has no blow-up. Physically, (3.36) means that (3.30) is the condition to ensure no temperature blow-up.

## REFERENCES

- [1] C. FOIAS, O. MANLEY, AND R. TEMAM, *Attractors for the Bénard problem: existence and physical bounds on their fractal dimension*, *Nonlinear Anal.*, 11 (1987), pp. 939–967.
- [2] T. MA, *Theory and Methods of Partial Differential Equations (in Chinese)*, Beijing, Science Press, 2011.
- [3] T. MA AND S. WANG, *Statistical theory of heat*, Hal preprint: hal-01578634, (2017).
- [4] —, *Topological Phase Transitions I: Quantum Phase Transitions*, Hal preprint: hal-01651908, (2017).
- [5] —, *Topological Phase Transitions II: Spiral Structure of Galaxies*, Hal preprint: hal-01671178, (2017).

(Ma) DEPARTMENT OF MATHEMATICS, SICHUAN UNIVERSITY, CHENGDU, P.  
R. CHINA

*Email address:* `matian56@sina.com`

(Wang) DEPARTMENT OF MATHEMATICS, INDIANA UNIVERSITY, BLOOMING-  
TON, IN 47405

*Email address:* `showang@indiana.edu`