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STABILISATION OF WAVE EQUATIONS ON THE TORUS WITH ROUGH DAMPINGS

by

N. Burq & P. Gérard

Abstract. — For the damped wave equation on a compact manifold with *continuous* dampings, the geometric control condition is necessary and sufficient for uniform stabilisation. In this article, on the two dimensional torus, in the special case where $a(x) = \sum_{j=1}^N a_j 1_{x \in R_j}$ (R_j are polygons), we give a very simple necessary and sufficient geometric condition for uniform stabilisation. We also propose a natural generalization of the geometric control condition which makes sense for L^∞ dampings. We show that this condition is always necessary for uniform stabilisation (for any compact (smooth) manifold and any L^∞ damping), and we prove that it is sufficient in our particular case on \mathbb{T}^2 (and for our particular dampings).

Résumé. — Pour l'équation des ondes amortie sur une variété compacte, dans le cas d'un amortissement *continu*, la condition de contrôle géométrique est nécessaire et suffisante pour la stabilisation uniforme. Dans cet article, sur le tore \mathbb{T}^2 et dans le cas où $a(x) = \sum_{j=1}^N a_j 1_{x \in R_j}$ (R_j sont des polygones), nous exhibons une condition géométrique nécessaire et suffisante très simple. Nous proposons aussi une généralisation naturelle de la condition de contrôle géométrique, pour un amortissement seulement L^∞ . Cette généralisation est toujours nécessaire pour la stabilisation uniforme (sur toute variété compacte régulière), et nous démontrons dans cet article qu'elle est suffisante dans notre cas particulier du tore \mathbb{T}^2 (et pour nos fonctions d'amortissement particulières).

1. Notations and main results

Let (M, g) be a (smooth) compact Riemannian manifold endowed with the metric g , Δ_g the Laplace operator on functions on M and for $a \in L^\infty(M)$, let us consider the damped wave (or Klein-Gordon) equation

$$(1.1) \quad (\partial_t^2 - \Delta + a(x)\partial_t + m)u = 0, \quad (u|_{t=0}, \partial_t u|_{t=0}) = (u_0, u_1) \in H^1(M) \times L^2(M),$$

where $m \geq 0$. If $a \geq 0$ a.e. it is well known that the energy

$$(1.2) \quad E(u)(t) = \int_M (|\nabla_g u|_g^2 + |\partial_t u|^2 + m|u|^2) d\text{vol}_g$$

is decaying and satisfies

$$E(u)(t) = E(u)(0) - \int_0^t \int_M 2a(x)|\partial_t u|^2 d\text{vol}_g.$$

We shall say that the *uniform stabilisation* holds for the damping a if one of the following equivalent properties holds

1. There exists a rate $f(t)$ such that $\lim_{t \rightarrow +\infty} f(t) = 0$ and for any $(u_0, u_1) \in H^1(M) \times L^2(M)$,

$$E(u)(t) \leq f(t)E(u)(0).$$

2. There exists $C, c > 0$ such that for any $(u_0, u_1) \in H^1(M) \times L^2(M)$,

$$E(u)(t) \leq Ce^{-ct}E(u)(0).$$

3. There exists $T > 0$ and $c > 0$ such that for any $(u_0, u_1) \in H^1(M) \times L^2(M)$, if u is the solution to the damped wave equation (1.1), then

$$E(u)(0) \leq C \int_0^T \int_M 2a(x)|\partial_t u|^2 d\text{vol}_g.$$

4. There exists $T > 0$ and $c > 0$ such that for any $(u_0, u_1) \in H^1(M) \times L^2(M)$, if u is the solution to the undamped wave equation

$$(1.3) \quad (\partial_t^2 - \Delta + m)u = 0, \quad (u|_{t=0}, \partial_t u|_{t=0}) = (u_0, u_1) \in H^1(M) \times L^2(M)$$

then

$$E(u)(0) \leq C \int_0^t \int_M 2a(x)|\partial_t u|^2 d\text{vol}_g.$$

The following result is classical (see the works by Rauch Taylor [30, 31], Babich-Popov [3], Babich-Ulin [4], Ralston [29], Bardos-Lebeau-Rauch [5], Burq-Gérard [9], Koch-Tataru [19])

Theorem 1. — *Let $m \geq 0$. Assume that the damping a is continuous. For $\rho_0 = (x_0, \xi_0) \in S^*M$ denote by $\gamma_{\rho_0}(s)$ the geodesic starting from x_0 in (co)-direction ξ_0 . Then the damping a stabilizes uniformly the wave equation iff the following geometric condition is satisfied*

$$(GCC) \quad \exists T, c > 0; \inf_{\rho_0 \in S^*M} \int_0^T a(\gamma_{\rho_0}(s)) ds \geq c.$$

When the damping a is no more continuous but merely L^∞ , the same propagation of singularities methods (that we shall recall below) show that the following strong geometric condition

$$(SGCC) \quad \exists T, c > 0; \forall \rho_0 \in S^*M, \exists s \in (0, T), \exists \delta > 0; a \geq c \text{ a.e. on } B(\gamma_{\rho_0}(s), \delta).$$

is *sufficient* for uniform stabilisation. On the other hand, the same approach also shows that the following weak geometric condition

$$(WGCC) \quad \exists T > 0; \forall \rho_0 \in S^*M, \exists s \in (0, T); \gamma_{\rho_0}(s) \in \text{supp}(a)$$

where $\text{supp}(a)$ is the support (in the distributional sense) of a , is *necessary* for uniform stabilisation. Though the question appears to be very natural, until the present work, the only known case in between was essentially an (unpublished) example of Lebeau where $M = \mathbb{S}^d$ and a is the characteristic function of the half-sphere. In this case, uniform stabilisation holds (see Zhu [35] for a proof and a generalization of this result).

Theorem 2 (Lebeau, unpublished). — *On the d -dimensional sphere,*

$$\mathbb{S}^d = \{x = (x_0, \dots, x_d) \in \mathbb{R}^{d+1}; \|x\| = 1\},$$

uniform stabilisation holds for the characteristic function of the half sphere $\mathbb{S}_+^d = \{x = (x_0, \dots, x_d) \in \mathbb{R}^{d+1}; \|x\| = 1, x_0 > 0\}$.

Remark 1.1. — Notice that in this case, all the geodesics enter the interior of the support of a , and hence fulfill the (SGCC) requirements, except the family of geodesics included in the boundary of the support of a , the $d - 1$ dimensional sphere,

$$\partial\mathbb{S}_+^d = \{x = (x_0, \dots, x_d) \in \mathbb{R}^{d+1}; \|x\| = 1, x_0 = 0\}.$$

When the manifold is a two dimensional torus (rational or irrational) and the damping a is a linear combination of characteristic functions of polygons, i.e. there exists N , R_j , $j = 1, \dots, N$ (disjoint and non necessarily vertical) polygons and $0 < a_j$, $j = 1, \dots, N$ such that

$$(1.4) \quad a(x) = \sum_{j=1}^N a_j 1_{x \in R_j},$$

then another natural simple geometric condition is the following:

Assumption 1.2. — *Assume that the manifold is a two dimensional torus $\mathbb{T}^2 = \mathbb{R}^2 / A\mathbb{Z} \times B\mathbb{Z}$, $a; B > 0$. Assume that there exists $T > 0$ such that all geodesics (straight lines) of length T either encounters the interior of one of the polygons or follows for some time one of the sides of a polygon R_{j_1} on the left and for some time one of the sides of a polygon R_{j_2} (possibly the same) on the right.*

Our main result is the following

Theorem 3. — *The damping a stabilizes uniformly the wave equation if and only if Assumption 1.2 is satisfied.*

Corollary 1.3. — *Stabilisation holds for the examples 1.a and 1.d of figure 1 but not for examples 1.b, 1.c and 1.e*

Remark 1.4. — Stabilisation implies that exact controlability holds for some finite $T > 0$. However our proof relies on a contradiction argument and resolvent estimates. It gives no geometric interpretation for this controlability time. This is this contradiction argument which allows us to avoid a particularly delicate regime at the edge of the uncertainty principle (see Section 2.2). Giving a geometric interpretation of the time necessary for control would require dealing with this regime (see [8]).

The plan of the paper is the following: In Section 2, we focus on the model case of the left checkerboard in Figure 1. We first reduce the question of uniform stabilisation to the proof of an observation estimate for high frequency solutions of Helmholtz equations. We proceed by contradiction and perform a micro-localization which shows that the only obstruction is the vertical geodesic in the middle of the board. Then we prove a non concentration estimate which shows that solutions of Helmholtz equations (quasi-modes) cannot concentrate too fast on this trajectory. This is essentially the only point in the proof which is specific to the torus. Finally, by means of a second micro-localization with respect to this vertical geodesic, we obtain a contradiction. In Section 3, we show how the general case can be reduced to this

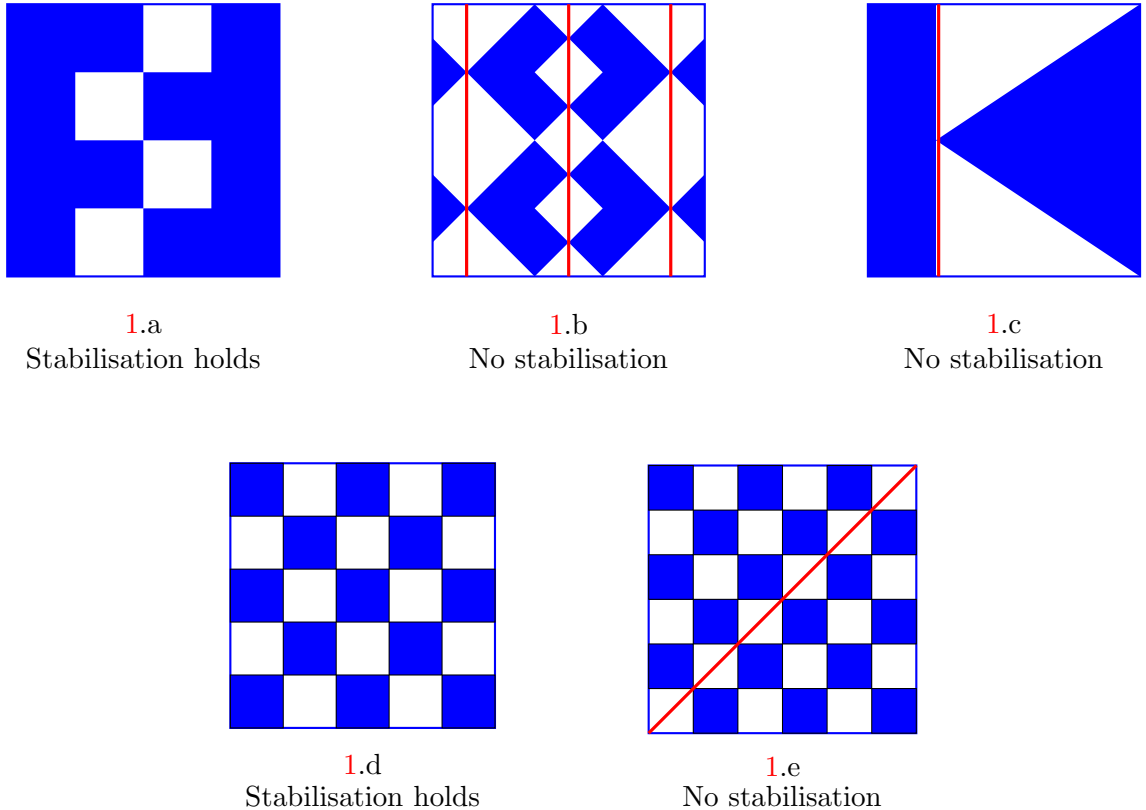


FIGURE 1. Checkerboards: the damping a is equal to 1 in the blue region, 0 elsewhere. For all these examples (WGCC) is satisfied but not (SGCC)

model case. For the convenience of the reader, we gathered in appendices a few quite classical results. In a first appendix, we introduce a generalized version of (GCC) that makes sense for $a \in L^\infty$ and which is equivalent to Assumption 1.2 in our particular case. We prove that this generalized geometric control condition is always necessary (on any Riemannian manifold and for any damping $0 \leq a \in L^\infty$) and we conjecture that it is always sufficient. In a second appendix, we recall the link between resolvent estimates and stabilisation.

The *second micro-localization* procedure has a well established history starting with the works by Laurent [20, 21], Kashiwara-Kawai [18], Sjöstrand [33], Lebeau [23] in the analytic context, (see also Bony-Lerner [6] in the C^∞ framework and Sjöstrand-Zworski [34] in the semi-classical setting) and in the framework of defect measures by Fermanian [13], Miller [24, 25, 26], Nier [28], Fermanian-Gerard [14, 15]. Notice that most of these previous works in the framework of measures dealt with lagrangian or involutive sub-manifolds, and it is worth comparing our contribution with these previous works, in particular [28, 2]. Here we are interested in the wave equation while the authors in [28, 2] were interested in the Schrödinger equation, and (compared to [2]) we are dealing with worse quasi-modes ($o(h)$ instead of $o(h^2)$). Another difference is that we perform a second microlocalization along a symplectic submanifold (namely $\{(x = 0, y, \xi = 0, \eta) \in T^*\mathbb{T}^2\}$), while they consider an isotropic submanifold $\{x = 0\}$ in [28] or $\{(x', x'', \xi' = 0, \xi'') \in T^*\mathbb{T}^d\}$ in [2]. On the other

hand, a feature shared by the present work and [28, 2] is that in all cases the analysis requires to work at the edges of uncertainty principle and use refinements of some exotic Weyl-Hörmander classes ($S^{1,1}$ in [28], $S^{0,0}$ in [2] and $S^{1/2,1/2}$ in the present work), see [17] and Léautaud-Lerner [22] for related work.

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2. The model case of a checkerboard

In this section we prove Theorem 3 for the following model on the two dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/(2\mathbb{Z})^2$. We shall later microlocally reduce the general case to this model.

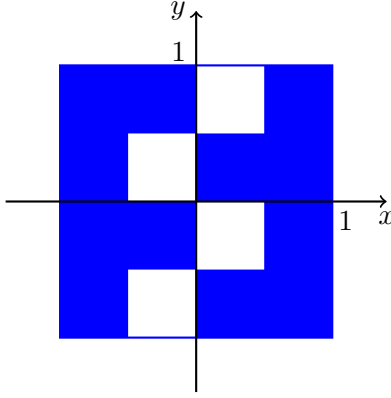


FIGURE 2. The checkerboard: a microlocal model where the damping a is equal to 1 in the blue region, 0 elsewhere

2.1. First micro-localization. — According to Proposition B.2, we need to prove (B.5)

$$(B.5) \quad \begin{aligned} & \exists h_0 > 0; \forall 0 < h < h_0, \forall (u, f) \in H^2(M) \times L^2(M), (h^2 \Delta + 1)u = f, \\ & \|u\|_{L^2(M)} \leq C \left(\|a^{1/2}u\|_{L^2} + \frac{1}{h} \|f\|_{L^2} \right). \end{aligned}$$

To prove this estimate we argue by contradiction. This gives a sequence $(h_n) \rightarrow 0$, and (u_n, f_n) such that

$$(h_n^2 \Delta + 1)u_n = f_n, \quad \|u_n\|_{L^2} = 1, \|a^{1/2}u_n\|_{L^2} = o(1)_{n \rightarrow +\infty}, \|f_n\|_2 = o(h_n)_{n \rightarrow +\infty}.$$

Extracting a subsequence, we can assume that the sequence (u_n) has a semi classical measure ν on $T^*\mathbb{T}^2$ which is supported in the characteristic set

$$\{(X, \Xi) \in S^*\mathbb{T}^2; \|\Xi\|^2 = 1\}.$$

Furthermore, this measure has total mass 1 and is invariant by the bicharacteristic flow:

$$\Xi \cdot \nabla_X \nu = 0.$$

We refer to [7, Section 3] for a proof of these results in a very similar context. Also, since the only two bicharacteristics which do not enter the interior of the set where $a = 1$ are

$$\{(x = 0, \xi = 0, \eta = \pm 1)\},$$

we know that ν is supported on the union of these two bicharacteristics.

2.2. A priori non concentration estimate. — In this section we show that (u_n) cannot concentrate on too small neighbourhoods around $\{x = 0\}$.

Let us recall that $\|(h_n^2\Delta + 1)u_n\|_{L^2} = o(h_n)$. Define

$$(2.1) \quad \epsilon(h_n) = \max \left(h_n^{1/6}, \left(\|(h_n^2\Delta + 1)u_n\|/h_n \right)^{1/6} \right),$$

so that

$$(2.2) \quad h_n^{-1}\epsilon^{-6}(h_n)\|(h_n^2\Delta + 1)u_n\|_{L^2} \leq 1, \quad \lim_{n \rightarrow +\infty} \epsilon(h_n) = 0.$$

The purpose of this section is to prove

Proposition 2.1. — *Assume that $\|u_n\|_{L^2} = \mathcal{O}(1)$, and (2.2) holds. Then there exists $C > 0$ such that*

$$\forall n \in \mathbb{N}, \|u_n\|_{L^2(\{|x| \leq h_n^{1/2}\epsilon^{-2}(h_n)\})} \leq C\epsilon^{1/2}(h_n).$$

The proposition follows from the following one dimensional propagation estimate (see [10] for related estimates)

Proposition 2.2. — *There exists $C > 0, h_0$ such for any $0 < h < h_0, 1 \leq \beta \leq h^{-\frac{1}{2}}$, and any (u, f) solutions of*

$$(h^2\Delta + 1)u = f,$$

$$(2.3) \quad \|u\|_{L^\infty(\{|x| \leq \beta h^{\frac{1}{2}}\})} \leq C\beta^{-\frac{1}{2}}h^{-\frac{1}{4}} \left(\|u\|_{L^2(\{\beta h^{\frac{1}{2}} \leq |x| \leq 2\beta h^{\frac{1}{2}}\})} + h^{-1}\beta^2\|f\|_{L^2(\{|x| \leq 2\beta h^{\frac{1}{2}}\})} \right).$$

Let us first show that Proposition 2.1 follows from Proposition 2.2. Indeed, choosing $\beta = \epsilon^{-3}(h)$, Hölder's inequality gives

$$(2.4) \quad \|u\|_{L^2(\{|x| \leq h^{\frac{1}{2}}\epsilon^{-2}(h)\})} \leq h^{\frac{1}{4}}\epsilon^{-1}(h)\|u\|_{L^\infty(\{|x| \leq h^{\frac{1}{2}}\epsilon^{-3}(h)\})} \\ \leq C\epsilon^{\frac{1}{2}}(h) \left(\|u\|_{L^2(\{h^{\frac{1}{2}}\epsilon^{-3}(h) \leq |x| \leq 2h^{\frac{1}{2}}\epsilon^{-3}(h)\})} + h^{-1}\epsilon^{-6}(h)\|f\|_{L^2(\{|x| \leq 2\beta h^{\frac{1}{2}}\})} \right) \\ \leq C\epsilon^{\frac{1}{2}}(h) \left(\|u\|_{L^2} + h^{-1}\epsilon^{-6}(h)\|f\|_{L^2} \right) \leq 2C\epsilon^{\frac{1}{2}}(h),$$

where in the last inequality we used (2.2). Now we can prove Proposition 2.2. We perform first Fourier transform with respect to the y variable and reduce the analysis to proving the same estimate (with constants uniform with respect to the η parameter) for solution of

$$(h^2\partial_x^2 + 1 - h^2\eta^2)u = f.$$

We change variables $x = \beta h^{\frac{1}{2}}z$, and it is enough to prove, for solutions of

$$(h\beta^{-2}\partial_z^2 + 1 - h^2\eta^2)v = g,$$

$$(2.5) \quad \|v\|_{L^\infty(\{|z| \leq 1\})} \leq C \left(\|v\|_{L^2(\{1 \leq |z| \leq 2\})} + h^{-1}\beta^2\|g\|_{L^2(\{|z| \leq 2\})} \right).$$

Finally, this latter estimate follows (with $\tau = \beta^2 h^{-1}(1 - h^2 \eta^2)$) from the following result which is generalization of [10, Proposition 3.2].

Lemma 2.3. — *There exists $C > 0$ such that, for any $\tau \in \mathbb{R}$ and any solution (v, k) on $(-2, 2)$ of*

$$(\partial_z^2 + \tau)v = k,$$

then

$$\|v\|_{L^\infty(-1,1)} \leq C \left(\|v\|_{L^2(\{1 \leq |z| \leq 2\})} + \frac{1}{\sqrt{1 + |\tau|}} \|k\|_{L^1(-2,2)} \right),$$

Let $\chi \in C_0^\infty(-2, 2)$ equal to 1 on $(-1, 1)$. Then $u = \chi v$ satisfies

$$(2.6) \quad (\partial_z^2 + \tau)u = \chi k + 2\partial_z(\chi'v) - \chi''v.$$

We distinguish two regimes.

– elliptic regime, $\tau \leq -1$. Then, multiplying by u and integrating by parts gives

$$(2.7) \quad \|\partial_z u\|_{L^2(-2,2)}^2 + |\tau| \|u\|_{L^2(-2,2)}^2 \\ = - \left(\chi k + 2\partial_z(\chi'v) - \chi''v, u \right)_{L^2} = - \left(\chi k - \chi''v, u \right)_{L^2} + 2 \left(\chi'v, \partial_z u \right)_{L^2},$$

which implies

$$(2.8) \quad \|\partial_z u\|_{L^2(-2,2)}^2 + |\tau| \|u\|_{L^2(-2,2)}^2 \\ \leq C \left(\|k\|_{L^1(-2,2)} \|u\|_{L^\infty} + \|v\|_{L^2(\{1 \leq |z| \leq 2\})} (\|u\|_{L^2(\{1 \leq |z| \leq 2\})} + \|\partial_z u\|_{L^2(-2,2)}) \right),$$

and the one-dimensional Gagliardo-Nirenberg inequality

$$\|u\|_{L^\infty} \leq C \|\partial_z u\|_{L^2}^{1/2} \|u\|_{L^2}^{1/2}$$

allows to conclude in this regime.

– hyperbolic regime, $\tau \geq -1$. Let $\sigma = \sqrt{\tau} \in \mathbb{R}^+ \cup i[0, 1]$. The solution of (2.6) is

$$u(x) = \int_{y=-2}^x e^{-i\sigma(x-y)} \int_{z=-2}^y e^{i\sigma(y-z)} g(z) dz dy \\ = \int_{z=-2}^x g(z) \int_{y=z}^x e^{i\sigma(2y-x-z)} dy dz,$$

where $g = \chi k - \chi''v + 2\partial_z(\chi'v) = g_1 + \partial_z g_2$. Since, for $x, z \in [-2, 2]$,

$$\left| \int_{y=z}^x e^{i\sigma(2y-x-z)} dy \right| \leq \frac{C}{1 + |\sigma|},$$

the contribution of g_1 is uniformly bounded by

$$\frac{C}{1 + |\tau|} (\|\chi k\|_{L^1(-2,2)} + \|v\|_{L^1(\{1 \leq |z| \leq 2\})}).$$

Integrating by parts in the integral involving $\partial_z g_2$, we see that similarly, the contribution of $\partial_z g_2$ is bounded by

$$C \|\chi'v\|_{L^1(-2,2)}.$$

2.3. Second micro-localization. — In this section we develop the tools required to understand the concentration properties of our sequence (u_n) on the symplectic sub-manifold $\{x = 0, \xi = 0\}$ of the phase space $T^*\mathbb{T}^2$.

2.3.1. Symbols and operators. — Let S^m be the class of smooth functions of the variables $(X, \Xi, z, \zeta) \in \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}$ which have compact supports with respect to the (X, Ξ) variables and are polyhomogeneous of degree m with respect to the (z, ζ) variables, with limits in the radial direction

$$\lim_{r \rightarrow +\infty} \frac{1}{r^m} a \left(X, \Xi, \frac{(rz, r\zeta)}{\|(z, \zeta)\|} \right) = \tilde{a} \left(X, \Xi, \frac{(z, \zeta)}{\|(z, \zeta)\|} \right).$$

When $m = 0$, via the change of variables

$$(z, \zeta) \mapsto (\tilde{z}, \tilde{\zeta}) = \frac{(z, \zeta)}{\sqrt{1 + |z|^2 + |\zeta|^2}},$$

such functions are identified with smooth compactly supported functions on $\mathbb{R}_{(X, \Xi)}^4 \times \overline{B(0, 1)}_{\tilde{z}, \tilde{\zeta}}$, where $\overline{B(0, 1)}$ denotes the closed unit ball in \mathbb{R}^2 .

Let $\epsilon(h)$ satisfying

$$\lim_{h \rightarrow 0} \epsilon(h) = 0, \quad \epsilon(h) \geq h^{1/2}.$$

In order to perform the second micro-localization around the sub-manifold given by the equations $x = 0, \xi = 0$, we define, for $a \in S^m$,

$$\text{Op}(a) = a \left(X, hD_X, \frac{\epsilon(h)}{h^{1/2}}x, \epsilon(h)h^{1/2}D_x \right),$$

where $X = (x, y)$, $\Xi = (\xi, \eta)$. Notice that this quantification is the usual one [17], associated to the symbol

$$a \left(X, h\Xi, \frac{\epsilon(h)}{h^{1/2}}x, \epsilon(h)h^{1/2}\xi \right).$$

A simple calculation shows that since $\epsilon(h) \geq h^{1/2}$, the latter symbol belongs to the class $S((1 + \epsilon^2(h)h^{-1}x^2 + \epsilon^2(h)h\xi^2)^{m/2}, g)$ of the Weyl-Hörmander calculus [17] for the metric

$$(2.9) \quad g = \frac{\epsilon^2(h)}{h} \frac{dx^2}{1 + \epsilon^2(h)h^{-1}x^2 + \epsilon^2(h)h\xi^2} + \epsilon^2(h)h \frac{d\xi^2}{1 + \epsilon^2(h)h^{-1}x^2 + \epsilon^2(h)h\xi^2} + \frac{dy^2}{1 + y^2 + h^2\eta^2} + h^2 \frac{d\eta^2}{1 + y^2 + h^2\eta^2}.$$

As a consequence, we deduce that the operators such defined enjoy good properties and we have a good symbolic calculus, namely for all $a \in S^0$, the operator $\text{Op}(a)$ is bounded on $L^2(\mathbb{R}^2)$ uniformly with respect to h , and

$$\forall a \in S^p, b \in S^q, ab \in S^{p+q} \text{ and } \text{Op}(a)\text{Op}(b) = \text{Op}(ab) + \epsilon^2(h)r,$$

where $r \in \text{Op}(S^{p+q-1})$, and

$$\forall a \in S^0, a \geq 0 \Rightarrow \exists C > 0; \mathbf{Re}(\text{Op}(a)) \geq -C\epsilon^2(h), \|\mathbf{Im}(\text{Op}(a))\| \leq C\epsilon^2(h).$$

2.3.2. *Definition of the second semi-classical measures.* — In this Section, we consider a sequence (u_n) of functions on the two dimensional torus \mathbb{T}^2 such that

$$(2.10) \quad (h_n^2 \Delta + 1)u_n = \mathcal{O}(1)_{L^2},$$

We identify u_n with a periodic function on \mathbb{R}^2 . Now, using the symbolic calculus properties in Section 2.3.1, we can extract a subsequence (still denoted by (u_n)) such that there exists a measure $\tilde{\mu}$ on $T^*\mathbb{T}^2 \times \overline{N} - \overline{N}$ denotes the sphere compactification of $N = \mathbb{R}_{z,\zeta}^2$ — such that, for any symbol $a \in S^0$,

$$\lim_{n \rightarrow +\infty} \left(\text{Op}_{h_n}(a)u_n, u_n \right)_{L^2} = \langle \tilde{\mu}, \tilde{a} \rangle,$$

where the continuous function, \tilde{a} on $T^*\mathbb{R}^2 \times \overline{N}$ is naturally defined in the interior by the value of the symbol a and on the sphere at infinity by

$$\tilde{a}(x, y, \xi, \eta, \tilde{z}, \tilde{\zeta}) = \lim_{r \rightarrow +\infty} a(x, y, \xi, \eta, r\tilde{z}, r\tilde{\zeta}),$$

(which exists because a is polyhomogeneous of degree 0). The measure $\tilde{\mu}$ is of course periodic, and hence defines naturally a measure μ on $T^*\mathbb{T}^2 \times \overline{N}$, and using (2.10), it is easy to see that there is no loss of mass at infinity in the Ξ variable:

$$(2.11) \quad \mu(T^*\mathbb{T}^2 \times \overline{N}) = \lim_{n \rightarrow +\infty} \|u_n\|_{L^2(\mathbb{T}^2)}^2.$$

2.3.3. *Properties of the second semi-classical measure.* — In this section, we turn to the sequence constructed in Section 2.1 and study refined properties of the second semi-classical measure constructed above, for the choice $\epsilon(h)$ given by (2.1). Notice that compared to (2.10) the sequence considered here satisfies the stronger

$$(h_n^2 \Delta + 1)u_n = o(h_n)_{L^2}.$$

Proposition 2.4. — *The measure μ satisfies the following properties.*

1. The measure μ has total mass $1 = \liminf_{n \rightarrow +\infty} \|u_n\|_{L^2}^2$ (h_n - oscillation)
2. The projection of the measure μ on the (x, y, ξ, η) variables is the measure ν of Section 2.1. Consequently (elliptic regularity), the measure μ is supported on the set

$$\{(x, y, \xi, \eta, z, \zeta); x = 0, \xi = 0, \eta = \pm 1\}$$

3. The measure μ is supported on the sphere at infinity in the (z, ζ) variables.
4. The measure μ vanishes 2-microlocally on the right on $\{x = 0, y \in (0, \frac{1}{2}) \cup (-1, -\frac{1}{2})\}$ and 2-microlocally on the left on $\{x = 0, y \in (\frac{1}{2}, 1) \cup (-\frac{1}{2}, 0)\}$, namely,

$$(2.12) \quad \begin{aligned} \mu(\{(x, y, \xi, \eta, z, \zeta); x = 0, y \in (0, 1/2) \cup (-1, -1/2), z > 0\}) &= 0 \\ \mu(\{(x, y, \xi, \eta, z, \zeta); x = 0, y \in (-1/2, 0) \cup (1/2, 1), z < 0\}) &= 0 \end{aligned}$$

5. According to point 3 above, if we identify the sphere at infinity in the (z, ζ) variables with \mathbb{S}^1 by means of the choice of variables $z = r \cos(\theta), \zeta = r \sin(\theta), r \rightarrow +\infty$, the measure μ can be seen as a measure in $(x, y, \xi, \eta, \theta)$ variables, supported on $x = 0, \xi = 0, \eta = \pm 1$. In this coordinate system, we have

$$(2.13) \quad (\eta \partial_y - \sin^2(\theta) \partial_\theta) \mu = 0.$$

Proof. — The proof of point 1 follows from (2.11). To prove point 2, we just remark that the choice of test functions $a(x, \Xi, z, \zeta) = a(X, \Xi)$ shows that the direct image $\pi_*(\mu)$ of μ by the map

$$\pi : (X, \Xi, z, \zeta) \mapsto (X, \Xi),$$

is actually the (first) semi-classical measure ν constructed in Section 2.1, and consequently, this property follows from Section 2.1. To prove point 3, we recall that from Proposition 2.1, we have that for any $\chi \in C_0^\infty$, bounded by 1 and supported in $(-A, A)$

$$(2.14) \quad \|\chi(h_n^{-1/2}\epsilon(h_n)x)u_n\|_{L^2}^2 \leq \|u_n\|_{L^2(\{|x| \leq Ah^{1/2}\epsilon^{-1}(h)\})}^2 \leq \|u_n\|_{L^2(\{|x| \leq h^{1/2}\epsilon^{-2}(h)\})}^2 \Rightarrow \langle \mu, \chi(z) \rangle = 0.$$

To prove point 4, recall from Figure 2 that the damping a is equal to 1 on $(0, \frac{1}{2}) \times (0, \frac{1}{2})$ and that

$$\|au_n\|_{L^2} = \|a^{1/2}u_n\|_{L^2} = o(1)_{n \rightarrow +\infty}.$$

Let $\psi \in C^\infty(\mathbb{R})$ supported in $\{1 < r\}$ and equal to 1 for $r \geq 2$. Let $\chi \in C_0^\infty(-1, 1)$ equal to 1 on $(-\frac{1}{2}, \frac{1}{2})$. Given $\delta > 0$, consider the symbol

$$b(x, y, \xi, \eta, z, \zeta) = \chi(2x)\chi(4y-1)\chi(\xi)\chi(\eta-1)\psi\left(\frac{z}{\delta|\zeta|}\right)\psi(z^2 + \zeta^2)$$

On the other hand, since $\chi(2x)\chi(4y-1)$ is supported on $(-\frac{1}{2}, \frac{1}{2})_x \times (0, \frac{1}{2})_y$ and since $\psi(\frac{z}{\delta|\zeta|})$ is supported in $z > 0$, we infer that the range of $\text{Op}_{h_n}(b)$ is supported in the domain $(0, \frac{1}{2})_x \times (0, \frac{1}{2})_y$ and consequently

$$(2.15) \quad (\text{Op}_{h_n}(b)u_n, u_n) = (1_{x \in (0, \frac{1}{2})}1_{y \in (0, \frac{1}{2})}\text{Op}_{h_n}(b)u_n, u_n) = (\text{Op}_{h_n}(b)u_n, 1_{x \in (0, \frac{1}{2})}1_{y \in (0, \frac{1}{2})}u_n) \\ = (\text{Op}_{h_n}(b)u_n, 1_{x \in (0, \frac{1}{2})}1_{y \in (0, \frac{1}{2})}au_n) = o(1)_{n \rightarrow +\infty}.$$

This implies

$$\mu\left(\left\{(x, y, \xi, \eta, z, \zeta); x = 0, y \in \left(0, \frac{1}{2}\right), z \geq 2\delta|\zeta|\right\}\right) = 0.$$

Taking $\delta > 0$ arbitrarily small, we deduce that on the (z, ζ) sphere at infinity which contains the support of μ , we have

$$\mu\left(\left\{(x, y, \xi, \eta, z, \zeta); x = 0, y \in \left(0, \frac{1}{2}\right), z > 0\right\}\right) = 0.$$

The other properties in (2.12) follow similarly.

To prove the last property, we write

$$(2.16) \quad \frac{1}{2ih_n} \left[h_n^2 \Delta + 1, \text{Op}_{h_n}(a) \right] \\ = \text{Op}_{h_n}((\xi \partial_x + \eta \partial_y + \zeta \partial_z)a) - i \frac{h_n}{2} \text{Op}_{h_n}(\Delta_{x,y}a) - i \frac{h_n}{2} (\epsilon(h_n)h_n^{-1/2})^2 \text{Op}_{h_n}(\partial_z^2 a).$$

Since unfolding the bracket shows that, as $n \rightarrow \infty$,

$$\frac{1}{2ih_n} \left(\left[h_n^2 \Delta + 1, \text{Op}_{h_n}(a) \right] u_n, u_n \right) \rightarrow 0,$$

we get

$$(2.17) \quad o(1)_{n \rightarrow \infty} = \left(\text{Op}_{h_n}((\xi \partial_x + \eta \partial_y + \zeta \partial_z)a) u_n, u_n \right).$$

Let us compute the limit on the sphere at infinity of $(\xi \partial_x + \eta \partial_y + \zeta \partial_z)a$. We denote by \tilde{a} the function a in the r, θ coordinate system. In this system of coordinates, the operator $\zeta \partial_z$ reads

$$-\sin^2(\theta) \partial_\theta + r \cos(\theta) \sin(\theta) \partial_r.$$

Now we use that, for a polyhomogeneous symbol a of *degree* 0, the main part of a at infinity does not depend on r . As a consequence, the symbol $r \partial_r a$ is polyhomogeneous of *degree* -1 . Therefore we get, for any polyhomogeneous symbol a of degree 0,

$$(2.18) \quad \zeta \partial_z a|_{\mathbb{S}^1} = \lim_{r \rightarrow +\infty} (-\sin^2(\theta) \partial_\theta + r \cos(\theta) \sin(\theta) \partial_r) \tilde{a}(x, y, \xi, \eta, r, \theta) \\ = -\sin^2(\theta) \partial_\theta \lim_{r \rightarrow +\infty} \tilde{a}(x, y, \xi, \eta, r, \theta).$$

Since the measure $\tilde{\mu}$ is supported in $\xi = 0$, equation (2.13) follows from (2.17). \square

We can now conclude the contradiction argument, and end the proof of the resolvent estimate (B.5). Notice that the two fixed points for the flow of

$$\dot{\theta} = -\sin^2(\theta)$$

are given by $\theta = 0(\pi)$. We want to show that the measure $\tilde{\mu}$ vanishes identically to get a contradiction with point 1 in Proposition 2.4. For $(x = 0, \xi = 0, y_0, \eta_0 = \pm 1, \theta_0)$ in the support of $\tilde{\mu}$, let us denote by $\phi_s(\theta_0)$ the solution of

$$\frac{d}{ds} \phi_s(\theta_0) = -\sin^2(\phi_s(\theta_0)), \quad \phi_0(\theta_0) = \theta_0,$$

so that $\phi_s(\theta_0) = \text{Arccotan}(s + \cotan(\theta_0))$. From the invariance (2.13) of the measure $\tilde{\mu}$, we deduce that

$$\forall s \in \mathbb{R}, (x = 0, y_s = y_0 + s\eta_0 \pmod{2\pi}, \xi = 0, \eta_0, \theta_s = \phi_s(\theta_0)) \in \text{supp}(\tilde{\mu}).$$

Consequently, if $\theta_0 \in [0, \pi)$, there exists $s > 0$ such that $y_s \in (0, \frac{1}{2}) \pmod{2\pi}$ while $\theta_s \in [0, \frac{\pi}{2})$, while, if $\theta_0 \in [-\pi, 0)$, there exists $s > 0$ such that $y_s \in (-\frac{1}{2}, 0) \pmod{2\pi}$ while $\theta_s \in [-\pi, -\frac{\pi}{2})$. This is impossible according to (2.12).

3. Back to the general case

Let us work on the torus $\mathbb{T}^2 = \mathbb{R}^2/A\mathbb{Z} \times B\mathbb{Z}$ with $A > 0, B > 0$. Since the irrational directions $\Xi = (A\xi, B\eta); \xi/\eta \notin \mathbb{Q}$ correspond to dense geodesics, and since a is bounded from below on an open set, we deduce that the measure ν defined in Section 2.1 is supported — in the Ξ variables — on the set of rational directions

$$\Xi = (A\xi, B\eta) ; \xi/\eta \in \mathbb{Q},$$

satisfying moreover the elliptic regularity condition,

$$|\Xi|^2 = 1.$$

Notice that this set is countable. Therefore the closed set Σ of directions Ξ for which there exists a point (X, Ξ) in the support of ν is at most countable. Since a nonempty perfect subset of \mathbb{R}^d is not countable — see e.g. [32, p.64],

Σ must contain an isolated point Ξ_0 , which can be written as

$$(3.1) \quad \Xi_0 = \frac{1}{\sqrt{n^2 A^2 + m^2 B^2}}(nA, mB), \quad \Xi_0^\perp = \frac{1}{\sqrt{n^2 A^2 + m^2 B^2}}(-mB, nA),$$

where the integers n, m may be chosen to have $\gcd 1$. The change of coordinates in \mathbb{R}^2 ,

$$(3.2) \quad F : (x, y) \mapsto X = F(x, y) = x\Xi_0^\perp + y\Xi_0,$$

is orthogonal and hence $-\Delta_X = D_x^2 + D_y^2$.

We have the following simple lemma (see [12, Lemma 2.7]), which can be deduced from an elementary calculation.

Lemma 3.1. — *Suppose that Ξ_0 and F are given by (3.1) and (3.2). If $u = u(x, y)$ is periodic with respect to $A\mathbb{Z} \times B\mathbb{Z}$ then $F^*u := u \circ F$ satisfies*

$$(3.3) \quad F^*u(x + ka, y + \ell b) = F^*u(x, y - k\gamma), \quad k, \ell \in \mathbb{Z}, \quad (x, y) \in \mathbb{R}^2,$$

where, for fixed $p, q \in \mathbb{Z}$ such that $qn - pm = 1$,

$$a = \frac{AB}{\sqrt{n^2 A^2 + m^2 B^2}}, \quad b = \sqrt{n^2 A^2 + m^2 B^2}, \quad \gamma = -\frac{pnA^2 + qmB^2}{\sqrt{n^2 A^2 + m^2 B^2}}.$$

When $B/A = r/s \in \mathbb{Q}$, $r, s \in \mathbb{Z} \setminus \{0\}$, then

$$(3.4) \quad F^*u(x + k\tilde{a}, y + \ell b) = F^*u(x, y), \quad k, \ell \in \mathbb{Z}, \quad (x, y) \in \mathbb{R}^2,$$

for $\tilde{a} = (n^2 s^2 + m^2 r^2)a$.

In this new coordinate system, we know that there exists x_0 such that $(x_0, y_0, 0, 1)$ is in the support of the measure $F^*\mu$. By translation invariance, we can assume that $x_0 = 0$. Since $(\xi\partial_x + \eta\partial_y)F^*\mu = 0$, we infer that actually the whole line $(x_0 = 0, \mathbb{R}(\bmod 2\pi), 0, 1)$ belongs to the support of $F^*\mu$. If this bicharacteristic curve enters the interior of the support of a (i.e. encounters a point in a neighborhood of which a is bounded away from 0), then by propagation, no point of this bicharacteristic curve lies in the support of μ which gives a contradiction. On the other hand since assumption 1.2 is satisfied, we know that there exists two (at least) polygons R_1, R_2 so that the right side of R_1 is $\{0\} \times [\alpha, \beta]$ while the left side of R_2 is $\{0\} \times [\gamma, \delta]$. We may shrink these polygons to rectangles having the same property.

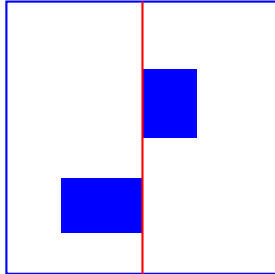


FIGURE 3. The microlocal model: on the left the rectangle R_1 , on the right the rectangle R_2 , in the middle the bicharacteristic in the support of μ

In other words, we are microlocally reduced to the study of the checkerboard in Figure 2. Notice that the change of variables we used in Lemma 3.1 does not keep periodicity with respect to the x variables but transforms it into some pseudo-periodicity condition (see (3.3)). However, for the study of the checkerboard model in Section 2, we only used periodicity with respect to the y variables — which is preserved. The rest of the contradiction argument follows the same lines as in Section 2.

A

Generalized geometric condition

For a general Riemannian manifold and a general damping function $a \in L^\infty(M)$, a natural substitute of (GCC) is the following generalized geometric condition.

$$(GGCC) \quad \exists T, c > 0 : \liminf_{\epsilon \rightarrow 0} \inf_{\rho_0 \in S^*M} \frac{1}{\text{Vol}(\Gamma_{\rho_0, \epsilon, T})} \int_{\Gamma_{\rho_0, \epsilon, T}} a(x) dx \geq c,$$

where $\Gamma_{\rho_0, \epsilon, T}$ is an ϵ neighborhood in M of the geodesic segment $\{\gamma_{\rho_0}(s), s \in (0, T)\}$. At first glance, (GGCC) might seem to be a strong condition. We shall prove below that it cannot be relaxed as, on any manifold and for any $a \in L^\infty(M)$, it is a necessary condition for uniform stabilisation. On the other hand, it might seem to be difficult to fulfill. However, we also prove below that in the case of two dimensional tori it is equivalent to Assumption 1.2. We conjecture that on a general manifold and for general $a \in L^\infty$, uniform stabilisation holds if and only if (GGCC) holds. The results in this article show that it is indeed the case on two dimensional tori, if a satisfies (1.4).

A.1. The generalized geometric condition is necessary for stabilisation. —

Theorem 4. — *Assume that (GGCC) does not hold. Then uniform stabilisation does not hold.*

Proof. — The proof of this results relies on geometric optics constructions (with complex phases) for the wave equation by Ralston [29, Section 2.2]. For completeness, we give a simple self contained proof on tori. The proof in the general case follows from using Ralston's constructions. In a first step, we construct approximate solutions to the wave equation localized on geodesics. The idea is to define oscillating solutions (phase and symbol) by constructing the germs on the bicharacteristic curve. In our context, we only need crude first order constructions (Ralston constructs solutions to infinite order).

Let $M = \mathbb{T}^d$. Let $\rho_0 = (x_0, t_0, \xi_0, \tau_0)$ a point in the characteristic variety of the wave equation

$$\text{Char} = \{(x, t, \xi, \tau) \in T^*(\mathbb{T}^d) : |\tau|^2 = |\xi|^2 = 1\}.$$

Assume that $\tau_0 = 1$ (the case $\tau_0 = -1$ being similar). Let γ be the bicharacteristic curve issued from ρ_0 . We can parametrize γ by time so that

$$\gamma(t) = (t, x(t) = x_0 - (t - t_0)\xi_0, 1, \xi_0).$$

Performing a rotation and a translation, we can assume that $x_0 = 0, \xi_0 = (1, 0, \dots, 0), t_0 = 0$. We now write $x = (x_1, x')$ and seek approximate solutions of the wave equation under the form

$$(A.1) \quad u_h(t, x) = e^{\frac{i}{h}\psi(t, x)} \sigma(t, x, h),$$

where $\sigma(t=0) = \sigma_0 \in C_0^\infty(\mathbb{R}^d)$ has sufficiently small compact support near 0. Function σ has compact support of fixed size and can be periodized and hence viewed as a function on $\mathbb{R}_t \times \mathbb{T}^d$. Applying the operator $\partial_t^2 - \Delta_x$ we get

$$(A.2) \quad \begin{aligned} (\partial_t^2 - \Delta_x)u_h &= -\frac{1}{h^2} ((\partial_t\psi)^2 - \nabla_x\psi \cdot \nabla_x\psi) \sigma e^{\frac{i}{h}\psi} \\ &+ \frac{i}{h} (2\partial_t\psi\partial_t\sigma + \sigma\partial_t^2\psi - 2\nabla_x\psi\nabla_x\sigma - \sigma\Delta_x\psi) e^{\frac{i}{h}\psi} \\ &+ [(\partial_t^2 - \Delta_x)\sigma] e^{\frac{i}{h}\psi} \end{aligned}$$

We now choose ψ canceling the term in front of h^{-2} in (A.2) modulo a $\mathcal{O}(|(x_1+t, x')|^3 |\sigma| e^{-\frac{\text{Im}\psi}{h}})$ error by setting

$$(A.3) \quad \psi(t, x) = x_1 + t + i((x_1 + t)^2 + g(t)|x'|^2),$$

with g solving

$$2ig'(t) + 4g^2(t) = 0 \Rightarrow g(t) = \frac{g(0)}{1 - 2itg(0)}, \quad g(0) := 1.$$

Notice in particular that

$$(A.4) \quad \text{Re}(g(t)) = \frac{1}{1 + 4t^2} > 0$$

and we cancel the term in front of h^{-1} in (A.2) modulo a $\mathcal{O}(|(x_1 + t, x')| |\nabla\sigma| e^{-\frac{\text{Im}\psi}{h}})$ error by choosing σ solving the equation

$$(\partial_t - \partial_{x_1})\sigma - i(d-1)g(t)\sigma = 0 \Rightarrow \sigma(t, x_1, x') = \frac{\sigma_0(x_1 + t, x')}{(1 - 2it)^{\frac{d-1}{2}}}.$$

As a consequence, we get with these choices

$$(A.5) \quad (\partial_t^2 - \Delta_x)u_h = \mathcal{O}\left(h^{-2}|(x_1 + t, x')|^3 + h^{-1}|(x_1 + t, x')| + 1\right) e^{-\frac{1}{h}((x_1+t)^2 + \text{Re}(g)(t)|x'|^2)}$$

Lemma A.1. — Let $v_h(t, x) = h^{1-\frac{d}{4}}u_h(t, x)$ (with the choice $g(0) = 1 = \sigma_0(0)$) which, once periodized, can be viewed as smooth functions on $\mathbb{R} \times \mathbb{T}^d$. Then

—

$$\left(\|(v_h(t), \partial_t v_h(t))\|_{H^1(\mathbb{T}^d) \times L^2(\mathbb{T}^d)} \right)^2 = 2\sqrt{\frac{\pi}{2}} + \mathcal{O}(h^{1/2}),$$

where the error term is uniform for t in a compact set.

— The solution, w_h of the wave equation in $\mathbb{R} \times \mathbb{T}^d$

$$(\partial_t^2 - \Delta_x)w_h = 0, \quad w_h|_{t=0} = v_h|_{t=0}, \quad \partial_t w_h|_{t=0} = \partial_t v_h|_{t=0}$$

satisfies

$$(A.6) \quad \begin{aligned} &\forall T < +\infty, \exists C > 0, h_0 > 0; \forall 0 < h < h_0, \\ &\|w_h - v_h, \partial_t(w_h - v_h)\|_{L^\infty(-T, T; H^1(\mathbb{T}^d) \times L^2(\mathbb{T}^d))} \leq Ch^{1/2}. \end{aligned}$$

Proof. — To prove the first part, we notice that

$$\begin{aligned}
(\text{A.7}) \quad \|\nabla_x v_h\|_{L^2}^2 &= \|h^{-1}v_h\|_{L^2}^2 + \mathcal{O}(\|h^{-1}|(x_1+t, x')|v_h\|_{L^2}^2) + \mathcal{O}(h^{2-\frac{d}{2}}\|e^{\frac{i}{h}\psi(t,x)}\nabla_x\sigma\|_{L^2}^2) \\
&= h^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\frac{2}{h}((x_1+t)^2 + \mathbf{Re}g(t)|x'|^2)} |\sigma(t, x)|^2 dx + \mathcal{O}(h) \\
&= \int_{\mathbb{R}^d} e^{-2(y_1^2 + \mathbf{Re}g(t)|y'|^2)} \frac{|\sigma_0(0)|^2}{|1-2it|^{d-1}} dy + \mathcal{O}(h) \\
&= \frac{|1-2it|^{-(d-1)}}{\mathbf{Re}(g(t))^{\frac{d-1}{2}}} \int_{\mathbb{R}^d} e^{-2(y_1^2 + |y'|^2)} |\sigma_0(0)|^2 dy + \mathcal{O}(h)
\end{aligned}$$

and we conclude by noticing that $\mathbf{Re}(g(t)) = \frac{1}{1+4t^2}$. A similar computation holds for $\|\partial_t v_h\|_{L^2}^2$. To prove the second part, we notice that according to (A.5), we have

$$\begin{aligned}
(\text{A.8}) \quad \|(\partial_t^2 - \Delta_x)v_h\|_{L^2} \\
= h^{1-\frac{d}{4}} \mathcal{O}\left(\|(h^{-2}|(x_1+t, x')|^3 + h^{-1}|(x_1+t, x')| + 1)e^{-\frac{1}{h}((x_1+t)^2 + \mathbf{Re}(g(t)x'^2)}\|_{L^2}\right) = \mathcal{O}(h^{\frac{1}{2}}).
\end{aligned}$$

Hence, $w_h - v_h$ is solution of the linear wave equation with r.h.s. bounded in $L_{loc}^\infty(\mathbb{R}_t; L^2(\mathbb{T}^d))$ by $\mathcal{O}(h^{\frac{1}{2}})$, with 0 initial data, which proves (A.6). \square

To complete the proof of Theorem 4, we are going to test the observation estimates (1.3) of such sequences of solutions. Hence, we assume that (GGCC) does not hold. Fix $T > 0$. Then there exists $\eta_n = (x_n, \xi_n) \in S^*\mathbb{T}^d$, $\epsilon_n \rightarrow 0$ such that, with

$$\lim_{n \rightarrow +\infty} \kappa_n = 0, \quad \kappa_n := \frac{1}{\epsilon_n^{d-1}} \int_{\Gamma_{\eta_n, \epsilon_n, T}} a(x) dx.$$

Let $t_n = 0$. Let $\rho_n = (x_n, t_n = 0, \xi_n, \tau_n = 1)$, and $w_{h,n}$ be the solution of the wave equation constructed in Lemma A.1, with initial point ρ_n (notice that we use that the family of solutions which depends on two parameters h and the initial point in the cotangent bundle is uniformly controlled with respect to this latter parameter). Since, according to Lemma A.1, we have

$$\|(w_{h,n}, \partial_t w_{h,n})|_{t=0}\|_{H^1 \times L^2} = \sqrt{2} \sqrt{\frac{\pi}{2}}^{\frac{d}{2}} + o(1)_{n \rightarrow +\infty},$$

to show that uniform stabilisation does not hold, it is now enough to show that for a properly chosen sequence $h_n \rightarrow 0$

$$(\text{A.9}) \quad \lim_{n \rightarrow +\infty} \int_0^T a(x) |\partial_t w_{h_n, n}|^2 dx dt = 0$$

According to (A.6), it is equivalent to prove,

$$(\text{A.10}) \quad \lim_{n \rightarrow +\infty} \int_0^T a(x) |\partial_t v_{h_n, n}|^2 dx dt = 0.$$

Extracting a subsequence, we can assume that the sequence of initial points ρ_n converges to $\rho = (x_0, t_0 = 0, \xi_0, \tau_0 = 1)$. The only point we shall use about our approximate solutions is an upper bound which follows on tori from (A.1), (A.3) and (A.4), and in the general case from the constructions in [29, Section 2.2]

Lemma A.2. — Let M, g be a Riemannian compact manifold. Then the approximate solutions constructed above satisfy

$$(A.11) \quad \begin{aligned} & \forall T < +\infty, \exists C, \alpha > 0; \forall n, \forall h > 0, \forall (x, t) \in \mathbb{T}^d \times [0, T], \\ & |\partial_t v_{h,n}(t, x)| \leq Ch^{-\frac{d}{4}} e^{-\alpha \frac{d_g(x, x_n(t))^2}{h}} \end{aligned}$$

where $x_n(t)$ is the point at distance t on the geodesic starting at time 0 from x_n in the direction $-\xi_n$ at time 0 — recall that on tori, $x_n(t) = x_n - t\xi_n$, $\|\xi_n\| = 1$.

Corollary A.3. — Let $\Gamma_{\rho_n, T}$ be the image on M of the geodesic starting at $s = 0$ from x_n in direction $-\xi_n$ and of length T . Then there exists $C > 0, \alpha > 0$ such that

$$(A.12) \quad \int_0^T |\partial_t v_{h,n}(x, t)|^2 dt \leq Ch^{-\frac{d-1}{2}} e^{-\alpha \frac{\text{dist}(x, \Gamma_{\rho_n, T})^2}{h}}.$$

Since v_{h_n} is exponentially decaying at fixed distance from the geodesic starting from x_n in the direction ξ_n , we can reduce the estimation (A.12) to a (fixed) neighborhood of the geodesic starting from x_0 in the direction ξ_0 . We can now choose a (n -dependent) coordinate system such that $x = 0$ and the geodesic Γ_n is given near $x = 0$ is by

$$(A.13) \quad \Gamma_{\rho_n, T} = \{x_n(t) = (t \in I, x'_n)\} \subset \mathbb{R}^d,$$

so that $\text{dist}(x, \Gamma_{\rho_n, T}) \geq c|x'|$. Then (A.12) follows from (A.11). We have according to (A.12)

$$(A.14) \quad \begin{aligned} & \int_0^T \int_M a(x) |\partial_t w_{h_n}|^2 dx dt = \int_0^T \int_M a(x) |\partial_t v_{h_n}|^2 dx dt + \mathcal{O}(h) \\ & \leq C \int_M a(x) h_n^{-\frac{d-1}{2}} e^{-\alpha \frac{\text{dist}(x, \Gamma_{\rho_n, T})^2}{h_n}} dx \\ & \leq \int_{\Gamma_{\rho_n, \epsilon_n, T}} a(x) h_n^{-\frac{d-1}{2}} e^{-\alpha \frac{\text{dist}(x, \Gamma_{\rho_n, T})^2}{h_n}} dx + \int_0^T \int_{\Gamma_{\rho_n, \epsilon_n, T}^c} a(x) h_n^{-\frac{d-1}{2}} e^{-\alpha \frac{\text{dist}(x, \Gamma_{\rho_n, T})^2}{h_n}} dx + \mathcal{O}(h). \end{aligned}$$

Using that $a \in L^\infty$, we get, using again the coordinate system (A.13), that the contribution of the second term above is bounded by $Ce^{-\alpha \frac{\epsilon_n^2}{h_n}}$. On the other hand, the contribution of the first term is bounded by

$$Ch_n^{-\frac{d-1}{2}} \int_{\Gamma_{\rho_n, \epsilon_n, T}} a(x) dx \leq \kappa_n \left(\frac{\epsilon_n^2}{h_n} \right)^{\frac{d-1}{2}}.$$

We now choose $h_n = \kappa_n^{\frac{1}{d-1}} \epsilon_n^2 \rightarrow 0$ such that

$$(A.15) \quad \frac{\epsilon_n^2}{h_n} = \kappa_n^{-\frac{1}{d-1}} \rightarrow +\infty, \quad \kappa_n \left(\frac{\epsilon_n^2}{h_n} \right)^{\frac{d-1}{2}} = \kappa_n^{\frac{1}{2}} \rightarrow 0.$$

This choice implies

$$\int_0^T \int_M a(x) |\partial_t w_{h_n}|^2 dx dt = o(1)_{n \rightarrow +\infty},$$

which contradicts (1.3) because the energy of the initial data $(w_{h_n}, \partial_t w_{h_n})$ is constant and nonzero. This completes the proof of Theorem 4. \square

A.2. Assumption 1.2 and (GGCC). — It turns out that on tori and for dampings a satisfying (1.4), Assumption 1.2 and (GGCC) are equivalent:

Proposition A.4. — *On a two dimensional torus \mathbb{T}^2 , if the damping a satisfies (1.4), then (GGCC) is equivalent to Assumption 1.2.*

Proof. — Since Assumption 1.2 implies uniform stabilisation (Theorem 3) which in turn implies (GGCC) (Theorem 4), it is enough to show that (GGCC) implies Assumption 1.2. Assume (GGCC). If Assumption 1.2 was not satisfied, then there would, for any $T > 0$ exist a geodesic curve γ of length T which either does not encounter $\mathcal{R} = \cup_{j=1}^N \overline{R}_j$, or does encounter \mathcal{R} only at corners, or encounter \mathcal{R} , only on the left (or only on the right). In the first case, then by compactness, the geodesic curve remains at distance ϵ_0 of \mathcal{R} , and consequently for $0 < \epsilon < \epsilon_0$, then

$$\int_{\Gamma_{\rho_0, \epsilon, T}} a(x) dx = 0.$$

In the second case (see checkerboard in Figure 1.b), by compactness, the geodesic curve encounters only a finite number of corners, and consequently ($d = 2$)

$$\int_{\Gamma_{\rho_0, \epsilon, T}} a(x) dx = \mathcal{O}(\epsilon^d),$$

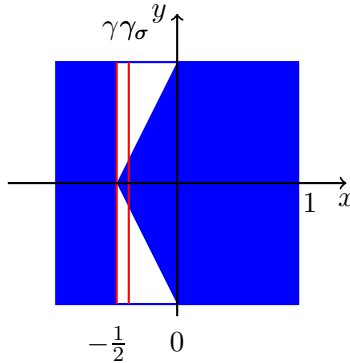
while

$$\text{Vol}(\Gamma_{\rho_0, \epsilon, T}) \sim C\epsilon^{d-1}$$

which implies that (GGCC) does not hold. In the last case (see the right checkerboard in Figure 1.c), let us consider the family of geodesics $\gamma_\sigma = \{\gamma_{\rho_\sigma}(s), s \in (0, T)\}$, $\sigma \in [0, 1)$, parallel on the right to $\gamma_0 = \{\gamma_{\rho_0}(s), s \in (0, T)\}$ (i.e. if $\rho_0 = (X_0, \Xi_0)$, then $\rho_\sigma = X_0 + \sigma \Xi_0^\perp$, where Ξ_0^\perp is the unit vector orthogonal to Ξ_0 , pointing on the right of γ_0). Since on the right γ_0 encounters no side of any rectangle R_j , it may encounter only (finitely many) corner points. As a consequence, for any $\sigma > 0$ sufficiently small, and $0 < \epsilon \ll \sigma$,

$$\frac{1}{\text{Vol}(\Gamma_{\rho_0, \epsilon, T})} \int_{\Gamma_{\rho_0, \epsilon, T}} a(x) dx \sim c\sigma,$$

letting $\sigma \rightarrow 0$ shows that (GGCC) does not hold. □



B

Resolvent estimates and stabilisation

In this appendix, we collect a few classical results on resolvent estimates.

B.1. Resolvent estimates and stabilisation. — It is classical [16] that stabilisation or observability of a self adjoint evolution system is equivalent to resolvent estimates (see also [11, 27, 1]). We prove only the fact that resolvent estimates imply stabilisation.

Proposition B.1. — Consider a semi-group e^{tA} on a Hilbert space H , with infinitesimal generator A defined on $D(A)$. Assume there exists $C > 0$ such that the resolvent of A , $(A - \lambda)^{-1}$ exists for $\mathbf{Re}\lambda \geq -\delta$ and satisfies

$$\exists C > 0; \forall \lambda \in \mathbb{C}^\delta = \{z \in \mathbb{C}; \mathbf{Re}z \geq -\delta\}, \|(A - \lambda)^{-1}\|_{\mathcal{L}(H)} \leq C.$$

Then there exists $M > 0$ such that for any $t > 0$

$$\|e^{tA}\|_{\mathcal{L}(H)} \leq Me^{-\delta t}.$$

Proof. — For $u_0 \in D(A)$, and $\chi \in C^\infty(\mathbb{R})$ equal to 0 for $t \leq -1$ and to 1 for $t \geq 0$, consider

$$u(t) = \chi(t)e^{t(A-\omega)}u_0.$$

For ω large enough, u belongs to $L^\infty(\mathbb{R}; H)$ and satisfies

$$(\partial_t + \omega - A)u(t) = \chi'(t)e^{t(A-\omega)}u_0 =: v(t).$$

Taking Fourier transforms in the time variable, we get

$$(B.1) \quad (i\tau + \omega - A)\widehat{u}(\tau) = \widehat{v}(\tau).$$

Since $v(t)$ is supported in $t \in [-1, 0]$, the r.h.s. in (B.1) is holomorphic and bounded in any domain

$$\mathbb{C}_\alpha = \{\tau \in \mathbb{C} : \mathbf{Im}\tau \geq \alpha, \alpha \in \mathbb{R}\}.$$

From the assumption on the resolvent, we deduce that \widehat{u} admits an holomorphic extension to $\{\tau : \mathbf{Im}\tau \leq \delta + \omega\}$ which satisfies

$$\|\widehat{u}(\tau)\|_H \leq C\|\widehat{v}(\tau)\|_H.$$

We deduce that

$$(B.2) \quad \|e^{(\omega+\delta)t}u\|_{L^2(\mathbb{R}_t; H)} = \|\widehat{u}(\tau + i(\omega + \delta))\|_{L^2(\mathbb{R}_\tau; H)} \leq C\|\widehat{v}(\tau + i(\omega + \delta))\|_{L^2(\mathbb{R}_\tau; H)} \\ \leq C\|e^{(\omega+\delta)t}v\|_{L^2(\mathbb{R}_t; H)} \leq C'\|u_0\|_H.$$

This implies exponential decay of $e^{tA}u_0$ in the L_t^2 norm, with the weight $e^{\delta t}$. Now consider $w(t) := \chi(t - T)e^{tA}u_0$, which satisfies

$$(\partial_t - A)w = \chi'(t - T)e^{tA}u_0, w|_{t=T-1} = 0.$$

From Duhamel formula, we deduce

$$w(T) = \int_{T-1}^T e^{(T-s)A}\chi'(t - T)e^{sA}u_0 ds,$$

and consequently (recall that the semigroup norm is locally bounded in time)

$$\begin{aligned}
\text{(B.3)} \quad \|w(T)\|_H &\leq \int_{T-1}^T \|e^{(T-s)A} \chi'(t-T) e^{sA} u_0\|_H ds \\
&\leq C \sup_{\sigma \in [0,1]} \|e^{\sigma A}\|_{\mathcal{L}(H)} \int_{T-1}^T \|e^{sA} u_0\|_H ds \leq C' e^{-\delta T} \|e^{\delta s} e^{sA} u_0\|_{L^2(T-1, T); H} \\
&\leq C'' e^{-\delta T} \|u_0\|_H.
\end{aligned}$$

□

B.2. Semi-groups for damped wave equations. — The solution to (1.1) is given very classically by

$$\begin{pmatrix} u \\ \partial_t u \end{pmatrix} = e^{tA} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}.$$

where

$$A = \begin{pmatrix} 0 & \text{Id} \\ \Delta & -a \end{pmatrix}$$

is defined on $H^1(M) \times L^2(M)$ with domain $H^2(M) \times H^1(M)$. When $m > 0$, since

$$E(u) = \|u\|_{H^1}^2 + \|\partial_t u\|_{L^2}^2,$$

to study the decay of the energy, we can apply directly the characterization given by Proposition B.1. When $m = 0$, the semi-group e^{tA} is no more a contraction semi-group on $H^1 \times L^2$ (because the energy (1.2) does not control the H^1 norm). For $s = 1, 2$, $\dot{H}^s = H^s(M)/\mathbb{R}$ the quotient space of $H^s(M)$ by the constant functions, endowed with the norm

$$\|\dot{u}\|_{\dot{H}^1} = \|\nabla u\|_{L^2}, \quad \|\dot{u}\|_{\dot{H}^2} = \|\Delta u\|_{L^2}.$$

We define the operator

$$\dot{A} = \begin{pmatrix} 0 & \Pi \\ \dot{\Delta} & -a \end{pmatrix}$$

on $\dot{H}^1 \times L^2$ with domain $\dot{H}^2 \times H^1$, where Π is the canonical projection $H^1 \rightarrow \dot{H}^1$ and $\dot{\Delta}$ is defined by

$$\dot{\Delta} \dot{u} = \Delta u$$

(independent of the choice of $u \in \dot{u}$). The operator \dot{A} is maximal dissipative and hence defines a semi-group of contractions on $\dot{\mathcal{H}} = \dot{H}^1 \times L^2$. Indeed for $U = \begin{pmatrix} \dot{u} \\ v \end{pmatrix}$,

$$\mathbf{Re}(\dot{A}U, U)_{\dot{\mathcal{H}}} = \mathbf{Re}(\nabla u, \nabla v)_{L^2} + (\Delta u - av, n)_{L^2} = -(av, v)_{L^2},$$

and

$$\begin{aligned}
\text{(B.4)} \quad (\dot{A} - \text{Id}) \begin{pmatrix} \dot{u} \\ v \end{pmatrix} = \begin{pmatrix} \dot{f} \\ g \end{pmatrix} &\Leftrightarrow \Pi v - \dot{u} = \dot{f}, \dot{\Delta} \dot{u} - (a+1)v = g \\
&\Leftrightarrow \Pi v - \dot{u} = \dot{f}, \Delta v - (1+a)v = g + \Delta f \in H^{-1}(M)
\end{aligned}$$

and we can solve this equation by variational theory. Notice that this shows that the resolvent $(\dot{A} - \text{Id})^{-1}$ is well defined and continuous from $\dot{H}^1 \times L^2$ to $\dot{H}^2 \times H^1$. Since the injection $\dot{H}^1 \times L^2$ to $\dot{H}^2 \times H^1$ is compact (this follows from identifying \dot{H}^n with the kernel of the linear

form $u \mapsto \int_M u$), we get that $(A - \text{Id})^{-1}$ is compact on $\dot{\mathcal{H}}$. On the other hand, it is very easy to show that for $(u_0, u_1) \in H^1 \times L^2$,

$$\begin{pmatrix} \Pi & 0 \\ 0 & \text{Id} \end{pmatrix} e^{tA} = e^{t\dot{A}} \begin{pmatrix} \Pi & 0 \\ 0 & \text{Id} \end{pmatrix},$$

and consequently, stabilisation is equivalent to the exponential decay (in norm) of $e^{t\dot{A}}$ (and consequently, according to Proposition B.1 equivalent to resolvent estimates for \dot{A}).

B.3. Reduction to high frequency observation estimates. — In this section, we show that for $m \geq 0$, stabilisation is equivalent to semi-classical observation estimates (see [27]).

Proposition B.2. — *Assume that $0 \leq a \in L^\infty$ is non trivial ($\int_M a > 0$). Then stabilisation holds for (1.1) if and only if*

$$(B.5) \quad \begin{aligned} & \exists h_0 > 0; \forall 0 < h < h_0, \forall (u, f) \in H^2(M) \times L^2(M), (h^2 \Delta + 1)u = f, \\ & \|u\|_{L^2(M)} \leq C(\|a^{1/2}u\|_{L^2} + \frac{1}{h}\|f\|_{L^2}). \end{aligned}$$

We prove the proposition for $m = 0$. The proof for $m > 0$ is similar (slightly simpler). From Proposition B.1, stabilisation is equivalent to the fact that the resolvent $(\dot{A} - \lambda)^{-1}$ is bounded on \mathbb{C}^δ . Since \dot{A} is maximal dissipative, its resolvent is defined (and bounded) on any domain $\mathbb{C}^{-\epsilon}$ ($\epsilon > 0$). We deduce that it is equivalent to prove that it is uniformly bounded on $i\mathbb{R}$ (and consequently by perturbation on a δ neighborhood of $i\mathbb{R}$). Since

$$(\dot{A} - \lambda) = (1 + (1 - \lambda)(\dot{A} - 1)^{-1})(\dot{A} - 1),$$

and $(\dot{A} - 1)^{-1}$ is compact on \mathcal{H} (see Section B.2), the operator $(1 + (1 - \lambda)(\dot{A} - 1)^{-1})$ is Fredholm with index 0 and consequently, $\dot{A} - \lambda$ is invertible iff it is injective. As a consequence, stabilisation is equivalent to the following *a priori* estimates

$$(B.6) \quad \exists C > 0; \forall \lambda \in \mathbb{R}, U \in \dot{H}^2 \times H^1, F \in \dot{H}^1 \times L^2, (\dot{A} - i\lambda)U = F \Rightarrow \|U\|_{\mathcal{H}} \leq C\|F\|_{\mathcal{H}}.$$

B.3.1. High frequency resolvent estimates imply stabilisation. — We argue by contradiction. We assume (B.5) holds and assume that (B.6) does not hold. Then there exists sequences $(\lambda_n), (U_n), (F_n)$ such that

$$(\dot{A} - i\lambda_n)U_n = F_n, \quad \|U_n\|_{\mathcal{H}} > n\|F_n\|_{\mathcal{H}}.$$

Since $U_n \neq 0$, we can assume $\|U_n\|_{\mathcal{H}} = 1$. Extracting subsequences we can also assume that $\lambda_n \rightarrow \lambda \in \mathbb{R} \cup \{\pm\infty\}$ as $n \rightarrow \infty$. We write

$$U_n = \begin{pmatrix} \dot{u}_n \\ v_n \end{pmatrix}, F_n = \begin{pmatrix} \dot{f}_n \\ g_n \end{pmatrix},$$

and distinguish according to three cases

– Zero frequency: $\lambda = 0$. In this case, we have

$$\dot{A}U_n = o(1)_{\mathcal{H}} \Leftrightarrow \Pi v_n = o(1)_{\dot{H}^1}, \quad \Delta u_n - a v_n = o(1)_{L^2}.$$

We deduce that there exists $c_n \in \mathbb{C}$ such that

$$v_n - c_n = o(1)_{H^1}, \quad \Delta u_n - a c_n = o(1)_{L^2}.$$

But

$$\int_M \Delta u_n = 0 \Rightarrow c_n \int_M a = o(1) \Rightarrow c_n = o(1).$$

As a consequence, we get $v_n = o(1)_{L^2}$ and $\Delta u_n = o(1)_{L^2} \Rightarrow \dot{u}_n = o(1)_{\dot{H}^1}$. This contradicts $\|U_n\|_{\mathcal{H}} = 1$.

– Low frequency: $\lambda \in \mathbb{R}^*$. In this case, we have

$$(\dot{A} - i\lambda)U_n = o(1)_{\mathcal{H}} \Leftrightarrow \Pi v_n - i\lambda \dot{u}_n = o(1)_{\dot{H}^1}, \quad \Delta u_n - (i\lambda + a)v_n = o(1)_{L^2}.$$

We deduce

$$\Delta v_n - i\lambda(a + i\lambda)v_n = o(1)_{L^2} + \Delta(o(1)_{\dot{H}^1}) = o(1)_{H^{-1}}.$$

Since (v_n) is bounded in L^2 , from this equation, we deduce that Δv_n is bounded in H^{-1} and consequently v_n is bounded in H^1 . Extracting another subsequence, we can assume that v_n converges in L^2 to v which satisfies

$$\Delta v + \lambda^2 v - i\lambda a v = 0.$$

Taking the imaginary part of the scalar product with v in L^2 gives (since $\lambda \neq 0$) $\int_M a|v|^2 = 0$, and consequently $av = 0$ which implies that v is an eigenfunction of the Laplace operator. But since the zero set of non trivial eigenfunctions has Lebesgue measure 0 in M , $av = 0$ implies that $v = 0$ (and consequently $v_n = o(1)_{L^1}$). Now, we have

$$\Delta u_n = (i\lambda + a)v_n + o(1)_{L^2} = o(1)_{L^2} \Rightarrow u_n = o(1)_{\dot{H}^1}.$$

This contradicts $\|U_n\|_{\mathcal{H}} = 1$.

– High frequency $\lambda_n \rightarrow \pm\infty$. We study the case $\lambda_n \rightarrow +\infty$ — the other case is obtained by considering $\overline{U_n}$. Let $h_n = \lambda_n^{-1}$.

$$(B.7) \quad (\dot{A} - i\lambda_n)U_n = o(1)_{\mathcal{H}} \Leftrightarrow -i\lambda_n \dot{u}_n + \Pi v_n = o(h_n)_{\dot{H}^1}, \quad \Delta u_n - (i\lambda_n + a)v_n = o(1)_{L^2}$$

$$(B.8) \quad \Leftrightarrow \dot{u}_n = -ih_n \Pi v_n + o(h_n)_{\dot{H}^1}, \quad (h_n^2 \Delta + 1 - ih_n a)v_n = o(h_n)_{L^2} + o(h_n^2)_{H^{-1}}$$

To conclude in this regime, we need

Lemma B.3. — *The observation inequality (B.5) implies the more general*

$$(B.9) \quad \begin{aligned} & \exists h_0 > 0; \forall 0 < h < h_0, \forall (u, f) \in H^2(M) \times L^2(M), (h^2 \Delta + 1)u = f_1 + f_2, \\ & \|h \nabla_x u\|_{L^2} + \|u\|_{L^2(M)} \leq C(\|a^{1/2} u\|_{L^2} + \frac{1}{h} \|f_1\|_{L^2} + \frac{1}{h^2} \|f_2\|_{H^{-1}}). \end{aligned}$$

Proof. — Let $P_h^\pm = h^2 \Delta + 1 \pm iha$ defined on L^2 with domain H^2 . Writing

$$P_h^\pm = (1 + (2 \pm iha)(h^2 \Delta - 1)^{-1})(h^2 \Delta - 1),$$

and since $(h^2 \Delta - 1)^{-1}$ is compact on L^2 , we deduce that $(1 + (2 \pm iha)(h^2 \Delta - 1)^{-1})$ is Fredholm with index 0, hence P_h^\pm is invertible iff it is injective. On the other hand we have

$$h\|a^{1/2} u\|_{L^2}^2 = \pm \mathbf{Im}(P_h^\pm u, u)_{L^2} \leq \|P_h^\pm u\|_{L^2} \|u\|_{L^2},$$

which combined with (B.5) implies $((h^2\Delta + 1)u = P_h^\pm u \mp ihau)$

$$(B.10) \quad \|u\|_{L^2}^2 \leq C(\|a^{1/2}u\|_{L^2}^2 + \frac{1}{h^2}(\|P_h^\pm u\|_{L^2}^2 + h^2\|au\|_{L^2}^2)) \\ \leq \frac{C'}{h}\|P_h^\pm u\|_{L^2}\|u\|_{L^2} + \frac{C'}{h^2}\|P_h^\pm u\|_{L^2}^2 \Rightarrow \|u\|_{L^2} \leq \frac{C''}{h}\|P_h^\pm u\|_{L^2}$$

Since

$$(B.11) \quad \|\|u\|_{L^2}^2 - \|h\nabla_x u\|_{L^2}^2\| = |\mathbf{Re}(P_h^\pm u, u)_{L^2}| \leq \|P_h^\pm u\|_{L^2}\|u\|_{L^2},$$

We deduce that P_h^\pm is injective hence bijective from H^2 to L^2 with inverse bounded by C''/h from L^2 to L^2 and by C/h^2 from L^2 to H^1 . We now proceed by duality to obtain (B.9). The adjoint of P_h^\pm is P_h^\mp and is consequently bounded from H^{-1} to L^2 by C/h^2 . Using again the identity (B.11) we get that

$$P_h^\pm u = f_1 + f_2 \Rightarrow \|h\nabla_x u\|_{L^2} + \|u\|_{L^2(M)} \leq \frac{C}{h}\|f_1\|_{L^2} + \frac{C}{h^2}\|f_2\|_{H^{-1}}.$$

Finally

$$(h^2\Delta + 1)u = f_1 + f_2 \Rightarrow P_h^+ u = iahu + f_1 + f_2,$$

and we get

$$(B.12) \quad \|h\nabla_x u\|_{L^2} + \|u\|_{L^2(M)} \leq C\left(\frac{1}{h}\|ihau + f_1\|_{L^2} + \frac{1}{h^2}\|f_2\|_{H^{-1}}\right) \\ \leq C'\left(\|a^{1/2}u\|_{L^2} + \frac{1}{h}\|f_1\|_{L^2} + \frac{1}{h^2}\|f_2\|_{H^{-1}}\right)$$

□

We now come back to our sequence satisfying (B.8). From (B.9), (B.8) implies

$$\|h_n \nabla_x v_n\|_{L^2} + \|v_n\|_{L^2} = o(1)_{n \rightarrow +\infty},$$

and in turn

$$\|\nabla_x u_n\|_{L^2} = o(1)_{n \rightarrow +\infty}.$$

This contradicts $\|U_n\|_{\mathcal{H}} = 1$.

B.3.2. Stabilisation imply resolvent estimates. — Consider now $U = \begin{pmatrix} \dot{u} \\ v \end{pmatrix}, F = \begin{pmatrix} \dot{f} \\ g \end{pmatrix}$ such that

$$(\dot{A} - i\lambda)U = F \Leftrightarrow -i\lambda\dot{u} + \Pi v = \dot{f} \text{ and } (\Delta v + \lambda^2 - i\lambda a)v = i\lambda g + \Delta f.$$

From (B.5) with $h = \lambda^{-1}$, we get

$$\|v\|_{L^2} + \|h\nabla_x v\|_{L^2} \leq C\|g\|_{L^2} + C\|\Delta f\|_{H^{-1}} \leq C(\|g\|_{L^2} + C\|\nabla_x f\|_{L^2}),$$

and also

$$\|\nabla_x u\|_{L^2} = h\|\nabla_x(v - f)\| \leq C(\|g\|_{L^2} + C\|\nabla_x f\|_{L^2}).$$

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