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# REGULARITY OF PUSH-FORWARD OF MONGE-AMPÈRE MEASURES

ELEONORA DI NEZZA AND CHARLES FAVRE

ABSTRACT. We prove that the image under any dominant meromorphic map of the Monge-Ampère measure of a Hölder continuous quasi-psh function still possesses a Hölder potential. We also discuss the case of lower regularity.

*Dedicated to Jean-Pierre Demailly for his 60th birthday*

## INTRODUCTION

Let  $(X, \omega_X)$  be a compact Kähler manifold of dimension  $n$  normalized by the volume condition  $\int_X \omega_X^n = 1$ . We say that a potential  $\varphi \in L^1(X)$  is  $\omega_X$ -plurisubharmonic ( $\omega_X$ -psh for short) if locally  $\varphi$  is the sum of a plurisubharmonic and a smooth function, and  $\omega_X + dd^c\varphi \geq 0$  in the weak sense of currents, where  $d = \partial + \bar{\partial}$  and  $d^c = \frac{i}{2\pi}(\bar{\partial} - \partial)$  so that  $dd^c = \frac{i}{\pi}\partial\bar{\partial}$ . We denote by  $\text{PSH}(X, \omega_X)$  the set of all  $\omega_X$ -psh functions on  $X$ . Recall from [GZ07, Section 1] that the non-pluripolar Monge-Ampère measure of a function  $\varphi \in \text{PSH}(X, \omega_X)$  is a positive measure defined as the increasing limit

$$(\omega_X + dd^c\varphi)^n = \lim_{j \rightarrow +\infty} \mathbf{1}_{\{\varphi > -j\}} (\omega_X + dd^c \max\{\varphi, -j\})^n$$

where the right hand side is defined using Bedford-Taylor intersection theory of bounded psh functions, see [BT76].

One of the main result of [GZ07] states that if  $\mu$  is a probability measure on  $X$  which does not charge pluripolar sets, then there exists a unique (up to a constant)  $\omega_X$ -psh function  $\varphi$  such that  $\int_X (\omega_X + dd^c\varphi)^n = 1$  and

$$(1) \quad \mu = (\omega_X + dd^c\varphi)^n .$$

We denote by  $\mathcal{E}(X, \omega_X)$  the set of all  $\omega_X$ -psh functions whose non-pluripolar Monge-Ampère measure has full mass so that any solution to (1) belongs to  $\mathcal{E}(X, \omega_X)$ .

In the same paper, Guedj and Zeriahi introduced for any  $p > 0$  the subset  $\mathcal{E}^p(X, \omega_X)$  of  $\mathcal{E}(X, \omega_X)$  consisting of all  $\omega_X$ -psh functions satisfying the integrability condition  $\varphi \in L^p((\omega_X + dd^c\varphi)^n)$ . Since  $\omega_X$ -psh functions are bounded from above it follows that

$$\mathcal{E}^p(X, \omega_X) \subset \mathcal{E}^q(X, \omega_X), \text{ for all } p > q.$$

Observe also that any  $\omega_X$ -psh function lying in  $L^\infty$  belongs to the intersection of all  $\mathcal{E}^p(X, \omega_X)$ .

We shall say that a probability measure which does not charge pluripolar sets  $\mu = (\omega_X + dd^c\varphi)^n$  is a Monge-Ampère measure having Hölder, continuous,  $L^\infty$  or  $\mathcal{E}^p$  potential for some  $p > 0$  whenever  $\varphi$  is Hölder, continuous,  $L^\infty$  or belongs to the energy class  $\mathcal{E}^p(X, \omega_X)$  respectively.

Let us now consider any dominant meromorphic map  $f : X \dashrightarrow Y$  where  $(Y, \omega_Y)$  is also a compact Kähler manifold of volume 1, and denote by  $m$  its complex dimension. Let  $\Gamma$  be a resolution of singularities of the graph of  $f$ . We obtain two surjective holomorphic

maps  $\pi_1 : \Gamma \rightarrow X$  and  $\pi_2 : \Gamma \rightarrow Y$  where  $\pi_1$  is bimeromorphic so that  $\Gamma$  is a modification of a compact Kähler manifold. By Hironaka's Chow lemma, see e.g. [Pet94, Theorem 2.8] we may suppose that  $\pi_1$  is a composition of blow-ups along smooth centers so that  $\Gamma$  is itself a compact Kähler manifold of complex dimension  $n$ . We fix any Kähler form  $\omega_\Gamma$  on it.

One defines the push-forward under  $f$  of a measure  $\mu$  not charging pluripolar sets as follows. Since  $\pi_1$  is a bimerorphism, there exist two closed analytic subsets  $R \subset \Gamma$  and  $V \subset X$  such that  $\pi_1 : \Gamma \setminus R \rightarrow X \setminus V$  is a biholomorphism. One may thus set  $\pi_1^* \mu$  to be the trivial extension through  $R$  of  $(\pi_1)|_{\Gamma \setminus R}^* \mu$ . This measure is again a probability measure which does not charge pluripolar sets.

We then define the probability measure  $f_* \mu := (\pi_2)_* \pi_1^* \mu$ . We observe that since  $f$  is dominant then  $\pi_2$  is surjective hence the preimage of a pluripolar set in  $Y$  by  $\pi_2$  is again pluripolar. By the preceding discussion, there exists  $\psi \in \mathcal{E}(Y, \omega_Y)$  such that  $f_* \mu = (\omega_Y + dd^c \psi)^m$ .

Our main goal is to discuss the following question.

**Problem 1.** Suppose  $\mu$  is a Monge-Ampère measure having Hölder, continuous,  $L^\infty$  or  $\mathcal{E}^p$  potential. Is it true that  $f_* \mu$  is also a Monge-Ampère measure of a potential lying in the same class of regularity?

This problem is hard for Monge-Ampère measures having either continuous or  $L^\infty$  potentials since there is no known intrinsic characterization of these measures. For these classes of regularity even the case  $f$  is the identity map and  $X = Y$  is still open (see for example [DGZ15, Question 15]).

**Problem 2.** Suppose  $\mu$  is a probability measure on  $X$  not charging pluripolar sets and write  $\mu = (\omega + dd^c \varphi)^n = (\omega' + dd^c \varphi')^n$  where  $\omega, \omega'$  are two Kähler forms of volume 1. Is it true that  $\varphi$  is continuous (resp.  $L^\infty$ ) iff  $\varphi'$  is?

An intrinsic characterization of Monge-Ampère measure of Hölder functions is given by [DDG+14], and of functions in the energy class  $\mathcal{E}^p$  in [GZ07, Theorem C] so that Problem 2 has a positive answer for these classes of regularity, see [DiN, Theorem 4.1]. Problem 1 remains though quite subtle. If we restrict our attention to the regularity in the  $\mathcal{E}^p$  energy classes, then the answer is no in general. Suppose that  $\pi : X \rightarrow \mathbb{P}^2$  is the blow-up at some point  $p \in \mathbb{P}^2$ , and let  $E = \pi^{-1}(p)$ . It was observed by the first author in [DiN15, Proposition B] that there exists a probability measure  $\mu = (\omega_X + dd^c \varphi)^2$  with  $\varphi \in \mathcal{E}^1(X, \omega_X)$  but  $\pi_* \mu = (\omega_{FS} + dd^c \psi)^2$  with  $\psi \notin \mathcal{E}^1(\mathbb{P}^2, \omega_{FS})$ , where  $\omega_{FS}$  denotes the Fubini Study metric on  $\mathbb{P}^2$  and  $\omega_X$  is a Kähler form.

In this note we answer Problem 1 in two situations. We first treat the case  $\mu$  is the Monge-Ampère of a Hölder function.

**Theorem 1.** *Let  $f : X \dashrightarrow Y$  be any dominant meromorphic map between two compact Kähler manifolds. If  $\mu$  is a Monge-Ampère measure having a Hölder potential with Hölder exponent  $\alpha$ , then  $f_* \mu$  is a Monge-Ampère measure having a Hölder potential with Hölder exponent bounded by  $C\alpha^{\dim(X)}$  for some constant  $C > 0$  depending only on  $f$ .*

We expect that the technics developed in the paper of Kołodziej-Nguyen [KolNg17] in the present volume allows one to extend the previous result to arbitrary compact hermitian manifolds.

Next we treat the case the image of the map has dimension 1.

**Theorem 2.** *Let  $f : X \dashrightarrow Y$  be any dominant meromorphic map from a compact Kähler manifold to a compact Riemann surface. If  $\mu$  is a Monge-Ampère measure having a Hölder,  $C^0$ ,  $L^\infty$ ,  $\mathcal{E}^p$  potential respectively, then  $f_*\mu$  is a Monge-Ampère measure having a potential lying in the same regularity class.*

Motivations for studying this question come from the analysis of degenerating measures on families of projective manifolds developed in [F16]. Let us briefly recall the setting of that paper. Let  $\mathcal{X}$  be a smooth connected complex manifold of dimension  $n + 1$ , and  $\pi : \mathcal{X} \rightarrow \mathbb{D}$  be a flat proper analytic map over the unit disk which is a submersion over the punctured disk and has connected fibers. We assume that  $\mathcal{X}$  is Kähler so that each fiber  $X_t = \pi^{-1}(t)$  is also Kähler.

A tame family of Monge-Ampère measures is by definition a family of positive measures  $\{\mu_t\}_{t \in \mathbb{D}}$  each supported on  $X_t$  that can be written under the form

$$\mu_t = p_*(T|_{X_t}^n),$$

where  $T$  is a positive closed  $(1, 1)$ -current having local Hölder continuous potentials and defined on a complex manifold  $\mathcal{X}'$  which admits a proper bimeromorphic morphism  $p : \mathcal{X}' \rightarrow \mathcal{X}$  which is an isomorphism over  $X := \pi^{-1}(\mathbb{D}^*)$ . It follows from [De93, Corollary 1.6] that the family of measures  $\mu'_t := T|_{X_t}^n$  in  $\mathcal{X}'$  is continuous so that  $\mu'_t$  converges to a positive measure  $\mu'_0$  supported on  $X'_0$  as  $t \rightarrow 0$ . It follows that the convergence  $\lim_{t \rightarrow 0} \mu_t = \mu_0$  also holds in  $\mathcal{X}$ .

As a corollary of the previous results we show the limiting measure  $\mu_0$  is of a very special kind:

**Corollary 3.** *Let  $\{\mu_t\}_{t \in \mathbb{D}}$  be any tame family of Monge-Ampère measures, so that  $\mu_t \rightarrow \mu_0$  as  $t \rightarrow 0$ .*

*Then there exist a finite collection of closed subvarieties  $\{V_i\}_{i=0, \dots, N}$  of  $X_0$  and for each index  $i$  a positive measure  $\nu_i$  supported on  $V_i$  such that*

$$\mu_0 = \sum_{i=1}^N \nu_i$$

*and  $\nu_i$  is a Monge-Ampère measure on  $V_i$  having a Hölder potential.*

In the previous statement, it may happen that  $V_i$  is singular, in which case it is understood that the pull-back of  $\nu_i$  to a (Kähler) resolution of  $V_i$  is a Monge-Ampère measure having a Hölder continuous potential.

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## 1. IMAGES OF MONGE-AMPÈRE MEASURES HAVING A HÖLDER POTENTIAL: PROOF OF THEOREM 1

As already mentioned, a dominant meromorphic map  $f : X \dashrightarrow Y$  can be decomposed as  $f = \pi_2 \circ \pi_1^{-1}$ , where  $\pi_1 : \Gamma \rightarrow X$  is holomorphic and bimeromorphic and  $\pi_2 : \Gamma \rightarrow Y$  is a surjective holomorphic map. Recall that one can assume  $\Gamma$  to be Kähler, and that  $f_*\mu := (\pi_2)_*\pi_1^*\mu$ .

We first claim that if  $\mu$  is the Monge-Ampère of a Hölder continuous function then  $\pi_1^*\mu$  too. Let  $\varphi \in \text{PSH}(X, \omega_X)$  be the Hölder potential such that  $\mu = (\omega_X + dd^c\varphi)^n$ . It then follows from Bedford and Taylor theory that  $\pi_1^*\mu = (\pi_1^*\omega_X + dd^c\pi_1^*\varphi)^n$ . Since  $\pi_1^*\omega_X$  is a semipositive smooth form, there exists a positive constant  $C > 0$  such that

$\pi_1^* \mu \leq (C\omega_\Gamma + dd^c \pi^* \varphi)^n$  where  $\omega_\Gamma$  is a Kähler form on  $\Gamma$ , and [DDG+14, Theorem 4.3] implies that  $\tilde{\mu} := \pi_1^* \mu$  is the Monge-Ampère measure of a Hölder continuous  $C\omega_\Gamma$ -psh function. This proves the claim. We are then left to prove that  $(\pi_2)_* \tilde{\mu}$  is the Monge-Ampère measure of a Hölder potential. This will be done in Lemma 7.

We first show that the push-forward of a smooth volume form has density in  $L^{1+\varepsilon}$ , for some constant  $\varepsilon > 0$  depending only on  $f$ .

**Proposition 4.** *Let  $f: X \rightarrow Y$  be a surjective holomorphic map. Then  $f_* \omega_X^n = g \omega_Y^m$  with  $g \in L^{1+\varepsilon}(\omega_Y^n)$ , for some  $\varepsilon > 0$ .*

This result is basically [ST12, Proposition 3.2] (see also [Tosatti, Section 2]). We give nevertheless a detailed proof for reader's convenience. Pick any coherent ideal sheaf  $\mathcal{I} \subset \mathcal{O}_X$ , and denote by  $V(\mathcal{I}) = \text{supp}(\mathcal{O}_X/\mathcal{I})$  the closed analytic subvariety of  $X$  associated to  $\mathcal{I}$ . Let  $\{U_i\}_{i=1}^N$  be a finite open covering of  $X$  by balls and  $\{V_i\}_i$  be a subcovering such that  $\overline{V}_i \subset U_i$ . The analytic sheaf  $\mathcal{I}$  is globally generated on each  $U_i$  so that we can find holomorphic functions such that  $\mathcal{I}|_{U_i} = (h_1^{(i)}, \dots, h_k^{(i)}) \cdot \mathcal{O}_{U_i}$ . Let  $\{\rho_i\}$  be a partition of unity subordinate to  $\overline{V}_i$ . We then define

$$(2) \quad \Phi_{\mathcal{I}} := \sum_{i=1}^N \rho_i \left( \sum_{j=1}^k |h_j^{(i)}|^2 \right).$$

Then  $\Phi_{\mathcal{I}}: X \rightarrow \mathbb{R}_+$  is a smooth function which vanishes exactly on  $V(\mathcal{I})$ . Observe that if  $\Phi_{\mathcal{I}}$  and  $\Phi'_{\mathcal{I}}$  are defined using two different coverings, then there exists  $C > 0$  such that

$$\frac{1}{C} \Phi'_{\mathcal{I}} \leq \Phi_{\mathcal{I}} \leq C \Phi'_{\mathcal{I}}.$$

In the sequel we shall abuse notation and not write the dependence of  $\Phi_{\mathcal{I}}$  in terms of the local generators of the ideal sheaf. The logarithm of the obtained function is then well-defined up to a bounded function so that all statements in the next Lemma make sense.

**Lemma 5.** *Let  $\mathcal{I}, \mathcal{J} \subseteq \mathcal{O}_X$  be two coherent ideal sheafs. The followings hold:*

- (i) *there exists  $\varepsilon > 0$  such that  $|\Phi_{\mathcal{I}}|^{-\varepsilon} \in L^1(X)$ ;*
- (ii) *if  $\mathcal{I} \subseteq \mathcal{J}$  then  $\Phi_{\mathcal{J}} \geq c \Phi_{\mathcal{I}}$  for some positive  $c > 0$ ;*
- (iii) *if  $V(\mathcal{J}) \subseteq V(\mathcal{I})$  then there exists  $c, \theta > 0$  such that  $\Phi_{\mathcal{J}} \geq c \Phi_{\mathcal{I}}^\theta$ ;*
- (iv) *given  $f: X \rightarrow Y$  a holomorphic surjective map and a coherent ideal sheaf  $\mathcal{J} \subseteq \mathcal{O}_Y$ , then  $\Phi_{f^* \mathcal{J}} = \Phi_{\mathcal{J}} \circ f$  (for a suitable choice of local generators of  $\mathcal{J}$  and  $f^* \mathcal{J}$ ).*

*Proof.* Using a resolution of singularities of  $\mathcal{I}$ , one sees that the statement in (i) reduces to show that  $|z_1|^{-\varepsilon}$  is locally integrable for some  $\varepsilon > 0$ , and this is the case if we choose  $\varepsilon$  small enough. The statements in (ii) and (iv) follow straightforward from the definition in (2). The statement in (iii) is a consequence of Łojasiewicz theorem, see e.g. [LJTR, Theorem 7.2].  $\square$

**Lemma 6.** *Let  $f: X \rightarrow Y$  be a holomorphic surjective map and let  $\mathcal{I} \subseteq \mathcal{O}_X$  be a coherent ideal sheaf. Then there exists a coherent ideal sheaf  $\mathcal{J} \subseteq \mathcal{O}_Y$ , and constants  $c, \theta > 0$  such that for any  $y \in Y$  we have*

$$\inf_{x \in f^{-1}(y)} \Phi_{\mathcal{I}} \geq c \Phi_{\mathcal{J}}^\theta$$

*Proof.* Let  $\mathcal{I} \subseteq \mathcal{O}_Y$  be the coherent ideal sheaf of holomorphic functions vanishing on the set  $f(V(\mathcal{I}))$  which is analytic since  $f$  is proper. Observe that  $V(f^*\mathcal{I}) = f^{-1}(V(\mathcal{I})) \supset V(\mathcal{I})$ , so that Lemma 5 ((iii) and (iv)) insure that there exist  $c, \theta > 0$  such that

$$\Phi_{\mathcal{I}} \geq c\Phi_{f^*\mathcal{I}}^\theta = c(\Phi_{\mathcal{I}} \circ f)^\theta.$$

Hence the conclusion.  $\square$

*Proof of Proposition 4.* Recall that Sard's theorem implies the existence of a closed subvariety  $S \subsetneq Y$  such that  $f: X \setminus f^{-1}(S) \rightarrow Y \setminus S$  is a submersion.

We first prove that  $f_*\omega_X^n$  is absolutely continuous w.r.t.  $\omega_Y^m$ . We need to check that  $\omega_Y^m(E) = 0$  implies  $f_*\omega_X^n(E) = 0$  for any Borel subset  $E \subset Y$ . As  $S$  and  $f^{-1}(S)$  have volume zero one may assume that  $f$  is a submersion in which case the claim follows from Fubini's theorem.

Radon-Nikodym theorem now guarantees that  $f_*\omega_X^n = g\omega_Y^m$  for some  $0 \leq g \in L^1(Y)$ . We want to show that the integral

$$\int_Y g^{1+\varepsilon}\omega_Y^m = \int_Y g^\varepsilon f_*\omega_X^n = \int_X (f^*g)^\varepsilon \omega_X^n$$

is finite for some  $\varepsilon > 0$  small enough. Consider the smooth function  $\phi(x) := \frac{f^*\omega_Y^m \wedge \omega_X^{n-m}}{\omega_X^n}(x)$ , and set  $\tilde{\phi}(y) := \inf_{x \in f^{-1}(y)} \phi(x)$  so that  $\phi \geq f^*\tilde{\phi}$ . We claim that for any  $y \in Y$

$$(3) \quad g(y) \leq \frac{1}{\tilde{\phi}(y)}.$$

Let  $\chi$  be a test function (i.e. a non negative smooth function) on  $Y$ , then

$$\begin{aligned} \int_Y \chi g \omega_Y^m &= \int_X f^*\chi \omega_X^n = \int_X \frac{f^*\chi}{\phi} f^*\omega_Y^m \wedge \omega_X^{n-m} \\ &\leq \int_X f^*\left(\frac{\chi}{\phi}\omega_Y^m\right) \wedge \omega_X^{n-m} \\ &\stackrel{\text{Fubini}}{=} C(f) \int_Y \frac{\chi}{\phi} \omega_Y^m \leq \int_Y \frac{\chi}{\phi} \omega_Y^m \end{aligned}$$

where  $C(f)$  is the volume of a fiber over a generic point  $y \in Y$ , i.e.  $C(f) = \int_{f^{-1}(y)} \omega_X^{n-m}$ . The last inequality follows from the fact  $C(f) \leq 1$  since  $\int_X \omega_X^n = 1$ . The claim is thus proved. Lemma 5 (i) and (iv) combined with Lemma 6 then insure that there exists  $\varepsilon > 0$  such that  $(f^*g)^\varepsilon \in L^1(\omega_X^n)$ .  $\square$

Theorem 1 is reduced to the following result which relies in an essential way on Proposition 4.

**Proposition 7.** *Suppose  $f: X \rightarrow Y$  is a surjective holomorphic map between compact Kähler manifolds. If  $\mu$  is a positive measure on  $X$  with Hölder continuous potentials, then  $f_*\mu$  is a positive measure on  $Y$  with Hölder potentials.*

Observe that by multiplying  $\omega_X$  by a suitable positive constant we may assume that  $f^*\omega_Y \leq \omega_X$ . The volume normalization is no longer satisfied but a positive multiple of  $\mu$  is still the Monge-Ampère measure of a  $\omega_X$ -psh Hölder continuous function. Write  $f_*\mu = (\omega_Y + dd^c\psi)^m$  with  $\psi \in \text{PSH}(Y, \omega_Y)$ .

We claim that there exists  $C > 0$ , and  $\varepsilon > 0$  such that for all  $u \in \text{PSH}(Y, \omega_Y)$  with  $\int_X u \omega_X^n = 0$

$$(4) \quad \int_Y \exp(-\varepsilon u) d(f_*\mu) \leq C .$$

Indeed, for any  $u \in \text{PSH}(Y, \omega_Y)$  we have that

$$\int_Y e^{-\varepsilon u} d(f_*\mu) = \int_X e^{-\varepsilon(u \circ f)} d\mu .$$

Now the integral  $\int_X e^{-\varepsilon(u \circ f)} d\mu$  is uniformly bounded by [DNS10, Theorem 1.1] since:

- $\mu$  has Hölder continuous potentials;
- $f^*\omega_Y \leq \omega_X$  hence  $u \circ f \in \text{PSH}(X, \omega_X)$ ;
- and the set of functions in  $\text{PSH}(X, \omega_X)$  such that  $\int_X u \omega_X^n = 0$  is compact by [GZ05, Proposition 2.6].

This proves our claim. Using the terminology of [DN14] this means that  $f_*\mu$  is *moderate*. It is worth mentioning that if [DGZ15, Question 16] holds true then the conclusion of Proposition 7 would follow immediately since any moderate measure would have a Hölder continuous potential. To get around this problem we use the characterization of measures with Hölder potentials given by Dinh and Nguyen.

*Proof of Lemma 7.* By [DN14, Lemma 3.3],  $f_*\mu$  is the Monge-Ampère measure of a Hölder potential if and only if there exist  $\tilde{c} > 1$  and  $\tilde{\beta} \in (0, 1)$  such that

$$(5) \quad \int_Y |u - v| df_*\mu \leq \tilde{c} \max \left( \|u - v\|_{L^1(\omega_Y^n)}, \|u - v\|_{L^1(\omega_Y^n)}^{\tilde{\beta}} \right)$$

for all  $u, v \in \text{PSH}(Y, \omega_Y)$ . By assumption on  $\mu$  we know there exist  $c > 1$  and  $\beta \in (0, 1)$  such that  $\int_Y |u - v| df_*\mu = \int_X |f^*u - f^*v| d\mu$ , and

$$(6) \quad \int_X |f^*u - f^*v| d\mu \leq c \max \left( \|f^*u - f^*v\|_{L^1(\omega_X^n)}, \|f^*u - f^*v\|_{L^1(\omega_X^n)}^\beta \right) .$$

Also, Proposition 4 gives

$$(7) \quad \int_X |f^*u - f^*v| \omega_X^n = \int_Y |u - v| g \omega_Y^n \leq \|g\|_{L^{1+\varepsilon}(\omega_Y^n)} \|u - v\|_{L^p(\omega_Y^n)}$$

where  $p$  is the conjugate exponent of  $1 + \varepsilon$ . Set  $C_g := \|g\|_{L^{1+\varepsilon}(\omega_Y^n)} < +\infty$ . Up to replace  $C_g$  with  $C_g + 1$  we can assume that  $C_g \geq 1$ .

Denote by  $m_u := \int_Y u \omega_Y^n$  and observe that  $u' := u - m_u$ ,  $v' := v - m_v$  satisfy  $\int_X u' \omega_X^n = 0 = \int_X v' \omega_X^n$ . Then the triangle inequality gives

$$(8) \quad \begin{aligned} \|u - v\|_{L^p(\omega_Y^n)} &= \left( \int_Y |(u' - v') + (m_u - m_v)|^p \omega_Y^n \right)^{1/p} \\ &\leq \|u' - v'\|_{L^p(\omega_Y^n)} + |m_u - m_v| \\ &\leq \|u' - v'\|_{L^p(\omega_Y^n)} + \|u - v\|_{L^1(\omega_Y^n)}. \end{aligned}$$

At this point, we make use of [DN14, Proposition 3.2] (that holds for normalized potentials) to replace the  $L^p$ -norm with the  $L^1$ -norm. We then infer the existence of a constant  $c' > 1$  such that

$$\|u' - v'\|_{L^p(\omega_Y^n)} \leq c' \max(1, -\log \|u' - v'\|_{L^1(\omega_Y^n)})^{\frac{p-1}{p}} \|u' - v'\|_{L^1(\omega_Y^n)}^{\frac{1}{p}} .$$

When  $t := \|u' - v'\|_{L^1(\omega_Y^n)} \geq 1/e$  we clearly have

$$\|u' - v'\|_{L^p(\omega_Y^n)} \leq c' \|u' - v'\|_{L^1(\omega_Y^n)}^{\frac{1}{p}},$$

whereas for any integer  $N \in \mathbb{N}^*$ , there exists a constant  $c_N > 0$  such that  $-\log t \leq c_N t^{-1/N}$  when  $t \leq 1/e$ , hence

$$\|u' - v'\|_{L^p(\omega_Y^n)} \leq c'' \|u' - v'\|_{L^1(\omega_Y^n)}^{\frac{1}{p}(1 - \frac{p-1}{N})}.$$

As  $\|u' - v'\|_{L^1(\omega_Y^n)} \leq 2\|u - v\|_{L^1(\omega_Y^n)}$ , combining (6), (7) and (8) we get

$$\|f^*u - f^*v\|_{L^1(\mu)} \leq C \max \left( \|u - v\|_{L^1(\omega_Y^n)}^{\tilde{\beta}}, \|u - v\|_{L^1(\omega_Y^n)} \right),$$

with  $\tilde{\beta} = \frac{\beta}{p} \left(1 - \frac{p-1}{N}\right)$ . By [DN14, Lemma 3.3]  $f_*\mu = (\omega_Y + dd^c\psi)^n$  where  $\psi$  is a Hölder continuous function.

To get a bound on the Hölder regularity of  $\psi$ , one argues as follows. First if  $\mu = (\omega + dd^c\varphi)^n$  with  $\varphi$  a  $\alpha$ -Hölder potential, and  $\pi: \Gamma \rightarrow X$  is a proper modification, then  $\pi^*\mu$  is dominated by a Monge-Ampère measure with  $\alpha$ -Hölder potential, and [DDG+14, Proposition 3.3 (ii)] is satisfied with  $b = 2\alpha/(\alpha + 2n)$  by [DDG+14, Theorem 4.3 (iii)]. Hence, following the proof of [DDG+14, Theorem], we see that  $\pi^*\mu$  is a Monge-Ampère measure of a  $\alpha_1$ -Hölder continuous potential with  $\alpha_1 < b/(n+1)$  (see Remark below for more details about the latter statement).

By [DN14, Proposition 4.1], (6) holds with  $\beta = \alpha_1^n/(2 + \alpha_1^n)$ , and (5) is then satisfied for any  $\tilde{\beta} < \beta/p$  so that  $f_*\mu$  is a Monge-Ampère measure with  $\tilde{\alpha}$ -Hölder potential for any  $\tilde{\alpha} < 2\tilde{\beta}/(m+1)$ , see the discussion on [DN14, p.83]. Combining all these estimates we see that any

$$\tilde{\alpha} < \frac{\alpha^n}{p(m+1)(\alpha/2 + n)^n(n+1)^n}$$

works where  $p$  is the conjugate of the larger constant  $\varepsilon > 0$  for which Proposition 4 holds.  $\square$

**Remark.** We borrow notations from the proof of [DDG+14, Theorem A]. Fix  $\alpha_1 < b/(n+1)$  and choose  $\varepsilon > 0$  such that  $\alpha_1 \leq \alpha \leq \alpha_0 \leq b - \alpha_0(n + \varepsilon)$ . By the previous arguments we know that condition (ii) in [DDG+14, Proposition 3.3] holds, i.e. for any  $\phi \in \text{PSH}(\Gamma, \omega_\Gamma)$ , we have  $\|\rho_\delta\phi - \phi\|_{L^1(\pi^*\mu)} = O(\delta^b)$ , where  $b = 2\alpha/(\alpha + 2n)$ . In particular, this gives

$$\pi^*\mu(E_0) \leq c_1\delta^{b-\alpha_0}.$$

Let  $g \in L^1(\pi^*\mu)$  be defined as  $g = 0$  on  $E_0$  and  $g = c$  on  $\Gamma \setminus E_0$  where  $c$  is a positive constant such that  $\pi^*\mu(\Gamma) = \int_\Gamma g d(\pi^*\mu)$ . An easy computation gives that  $c = \pi^*\mu(\Gamma)/\pi^*\mu(\Gamma \setminus E_0)$ . Let  $v \in \text{PSH}(\Gamma, \omega_\Gamma)$  be the bounded solution of the Monge-Ampère equation  $(\omega_\Gamma + dd^c v)^n = g \cdot \pi^*\mu$ . Observe that

$$\|1 - g\|_{L^1(\pi^*\mu)} = \int_{E_0} d\pi^*\mu + \int_{\Gamma \setminus E_0} |1 - c| d\pi^*\mu = 2 \int_{E_0} d\pi^*\mu \leq 2c_1\delta^{b-\alpha_0}.$$

Since  $\pi^*\mu = (\omega_\Gamma + dd^c\tilde{\varphi})^n$  satisfies the  $\mathcal{H}(\infty)$  property we can still apply [DZ, Theorem 1.1] and get

$$\|\tilde{\varphi} - v\|_{L^\infty} \leq c_3\delta^{\frac{b-\alpha_0}{n+\varepsilon}}.$$

The exact same arguments as in [DDG+14, Theorem A] then insure that the Hölder exponent of  $\tilde{\varphi}$  is  $\alpha_1$ .



## 2. OVER A ONE-DIMENSIONAL BASE: PROOF OF THEOREM 2

In this section we treat Problem 1 in the case the base is a Riemann surface.

We start with the case of a surjective holomorphic map  $f: X \rightarrow Y$  from a Kähler compact manifold to a compact Riemann surface.

Let  $\mu = (\omega_X + dd^c\varphi)^n$  be a Monge-Ampère measure of a continuous  $\omega_X$ -psh function  $\varphi$ . Suppose  $v_k, v$  is a family of  $\omega_X$ -psh functions such that  $v_k \rightarrow v$  in  $L^1$ , then

$$\begin{aligned} \int_X v_k d\mu &= \int_X v_k (\omega_X + dd^c\varphi)^n \\ &= \int_X v_k \omega_X^n + \sum_{j=0}^{n-1} \int_X \varphi dd^c v_k \wedge \omega_X^j \wedge (\omega_X + dd^c\varphi)^{n-j-1} \rightarrow \int_X v d\mu \end{aligned}$$

by [De93, Corollary 1.6 (a)]. Observe that in the last equality we used the fact that

$$(\omega_X + dd^c\varphi)^n - \omega_X^n = \sum_{j=0}^{n-1} dd^c\varphi \wedge \omega_X^j \wedge (\omega_X + dd^c\varphi)^{n-j-1}$$

and Stokes' theorem.

Normalize the Kähler form on  $Y$  such that  $\int \omega_Y = 1$ , and pick any sequence  $y_k \rightarrow y_\infty \in Y$ . Let  $w_k$  be the solutions of the equations  $\Delta w_k = \delta_{y_k} - \omega_Y$  with  $\sup w_k = 0$  so that  $w_k(y) - \log|y - y_k|$  is continuous in local coordinates near  $y_k$ . Write  $f_*\mu = \omega_Y + dd^c\psi$  so that

$$\int_Y w_k d(f_*\mu) = \int_Y w_k \omega_Y + \int_Y \psi \Delta w_k = \psi(y_k) + \int_Y (w_k - \psi)\omega_Y.$$

Since  $w_k \rightarrow w_\infty$  in  $L^p_{\text{loc}}$  for all  $p < \infty$ , Proposition 4 implies that  $f^*w_k \rightarrow f^*w_\infty$  in the  $L^1$  topology, so that the argument above gives  $\int_Y w_k d(f_*\mu) = \int_X f^*w_k d\mu \rightarrow \int_X f^*w_\infty d\mu = \int_Y w_\infty d(f_*\mu)$ . We then conclude that  $\psi(y_k) \rightarrow \psi(y_\infty)$ . Hence  $\psi$  is continuous.

Suppose then that  $\mu$  is locally the Monge-Ampère of a bounded psh function, and pick any subharmonic function  $u$  defined in a neighborhood of a point  $y \in Y$ . Then  $f^*u$  is again psh in a neighborhood of  $f^{-1}(y)$ , and the standard Chern-Levine-Nirenberg inequalities imply that  $f^*u \in L^1(\mu)$  so that  $u \in L^1(f_*\mu)$  with a norm depending only on the  $L^1$ -norm of  $u$ . It follows that  $f_*\mu$  is locally the laplacian of a bounded subharmonic function.

Finally, assume  $\mu = (\omega_X + dd^c\varphi)^n$  for some  $\varphi \in \mathcal{E}^p(X, \omega_X)$ . By [GZ07, Theorem C] this is equivalent to have that  $\mathcal{E}^p(X, \omega_X) \subset L^p(\mu)$ . Write as usual  $f_*\mu = (\omega_Y + dd^c\psi)$  with  $\psi \in \mathcal{E}(Y, \omega_Y)$ .

We claim that  $u \in \mathcal{E}^p(Y, \omega_Y)$  implies  $f^*u \in \mathcal{E}^p(X, \omega_X)$ . Indeed, without loss of generality we can assume that  $\Omega := \omega_X - f^*\omega_Y$  is a Kähler form and by the multilinearity of the non-pluripolar product we have

$$\begin{aligned} \int_X |f^*u|^p (\omega_X + dd^c f^*u)^n &= \int_X |f^*u|^p (f^*\omega_Y + \Omega + dd^c f^*u)^n \\ &= \int_X |f^*u|^p (\Omega^n + (f^*\omega_Y + dd^c f^*u) \wedge \Omega^{n-1}) \end{aligned}$$

where the last identity follows from the fact that  $(f^*\omega_Y + dd^c f^*u)^j = 0$  for  $j > 1$ . The term  $\int_X |f^*u|^p \Omega^n$  is bounded thanks to the integrability properties of quasi-plurisubharmonic

functions w.r.t. volume forms [GZ17, Theorem 1.47]; while the term

$$\int_X |f^*u|^p (f^*\omega_Y + dd^c f^*u) \wedge \Omega^{n-1} = C(f) \int_Y |u|^p (\omega_Y + dd^c u)$$

is finite since  $u \in \mathcal{E}^p(Y, \omega_Y)$ . This proves the claim.

Now, given any  $u \in \mathcal{E}^p(Y, \omega_Y)$  we have

$$\int_Y |u|^p d(f_*\mu) = \int_X |f^*u|^p d\mu < +\infty$$

since  $f^*u \in \mathcal{E}^p(X, \omega_X) \subset L^p(\mu)$ . The conclusion follows from [GZ07, Theorem C].

Consider now any dominant meromorphic map  $f: X \dashrightarrow Y$  from a Kähler compact manifold to a compact Riemann surface. As above we decompose  $f$  such that  $f_*\mu = (\pi_2)_*\pi_1^*\mu$  for any positive measure  $\mu$  on  $X$ .

Assume that  $\mu$  has continuous potentials. If we write  $\mu = (\omega_X + dd^c\varphi)^n$  then  $\pi_1^*\mu = (\pi_1^*\omega_X + dd^c\varphi \circ \pi)^n \leq (C\omega_\Gamma + dd^c\varphi \circ \pi)^n := \hat{\mu}$  where  $\hat{\mu}$  has a continuous potential. This implies  $f_*\mu \leq (\pi_2)_*\hat{\mu}$ . Observe that by the previous arguments the measure  $(\pi_2)_*\hat{\mu}$  has continuous potential. It follows that locally  $f_*\mu = \Delta v \leq \Delta u$  where  $u, v$  are subharmonic functions. It follows that  $v$  is the sum of a continuous function and the opposite of a subharmonic (hence u.s.c.) function. Since it is also u.s.c we conclude to its continuity.

When  $\mu$  has bounded potentials, the same argument applies noting that subharmonic functions are always bounded from above which implies  $v$  to be bounded.

Finally, we consider the case where  $\mu$  is the Monge-Ampère measure of  $\varphi \in \mathcal{E}^p(X, \omega_X)$ . We first observe that given  $v \in \mathcal{E}^p(\Gamma, \omega_\Gamma)$  we have  $(\pi_1)_*v \in \mathcal{E}^p(X, \omega_X)$ . Indeed,

$$\int_X |v \circ \pi^{-1}|^p (\omega_X + dd^c v \circ \pi^{-1})^n = \int_\Gamma |v|^p (\pi_1^*\omega_X + dd^c v)^n \leq \int_\Gamma |v|^p (C\omega_\Gamma + dd^c v)^n < +\infty.$$

This and the previous arguments give that if  $u \in \mathcal{E}^p(Y, \omega_Y)$  then  $f_*u = (\pi_1)_*\pi_2^*u \in \mathcal{E}^p(X, \omega_X)$ , hence

$$\int_Y |u|^p df_*\mu = \int_X |u \circ f|^p d\mu < +\infty.$$

It follows from [GZ07, Theorem C] that  $f_*\mu$  is the Monge-Ampère measure of a function in  $\mathcal{E}^p(Y, \omega_Y)$ .

### 3. THE CASE OF SUBMERSIONS

In this section we let  $(X, \omega_X), (Y, \omega_Y)$  be two compact Kähler manifolds of dimension  $n$  and  $m$ , respectively and normalized such that  $\int_X \omega_X^n = 1 = \int_Y \omega_Y^m$ .

**Proposition 8.** *Let  $f: X \rightarrow Y$  be a submersion. Then,  $u \in \mathcal{E}^p(Y, \omega_Y)$  implies  $f^*u \in \mathcal{E}^p(X, \omega_X)$ . In particular, if a probability measure  $\mu$  is the Monge-Ampère of a function in  $\mathcal{E}^p$  then also  $f_*\mu$  has also a potential in  $\mathcal{E}^p$ .*

*Proof.* Since  $f$  is a submersion we can assume that there is a finite number of open neighbourhoods  $U_i$  such that  $X \subset \bigcup_{j=0}^N U_j$ ,  $f|_{U_j}(z, w) = z$  where  $z = (z_1, \dots, z_m)$  and  $w = (z_{m+1}, \dots, z_n)$ . Moreover we can assume that on each  $U_j$  we have

$$\omega_X \leq C_j \frac{i}{2} (dz \wedge d\bar{z} + dw \wedge d\bar{w}), \quad \frac{i}{2} dz \wedge d\bar{z} \leq A_j f^*\omega_Y$$

where  $A_j, C_j > 1$  and  $dz \wedge d\bar{z}, dw \wedge d\bar{w}$  are short notations for  $\sum_{j=1}^m dz_j \wedge d\bar{z}_j$  and  $\sum_{k=m+1}^n dz_k \wedge d\bar{z}_k$ , respectively. We then write

$$\begin{aligned} \int_X |f^*u|^p (\omega_X + dd^c f^*u)^n &\leq \sum_{j=1}^N \int_{U_j} |f^*u|^p \left( C_j \frac{i}{2} dz \wedge d\bar{z} + C_j \frac{i}{2} dw \wedge d\bar{w} + dd^c f^*u \right)^n \\ &\leq \sum_{j=1}^N \int_{U_j} |f^*u|^p \left( A'_j f^* \omega_Y + C_j \frac{i}{2} dw \wedge d\bar{w} + dd^c f^*u \right)^n \\ &= \sum_{j=1}^N \sum_{\ell=0}^n \int_{U_j} |f^*u|^p (A'_j f^* \omega_Y + dd^c f^*u)^\ell \wedge \left( C_j \frac{i}{2} dw \wedge d\bar{w} \right)^{n-\ell} \\ &= \sum_{j=1}^N \int_{U_j} |f^*u|^p (A'_j f^* \omega_Y + dd^c f^*u)^m \wedge \left( C_j \frac{i}{2} dw \wedge d\bar{w} \right)^{n-m}. \end{aligned}$$

The above integral is then finite because by assumption  $u \in \mathcal{E}^p(Y, A\omega_Y)$  for any  $A \geq 1$ .

The last statement follows from the same arguments in the last part of the proof in the previous section.  $\square$

#### 4. TAME FAMILIES OF MONGE-AMPÈRE MEASURES: PROOF OF COROLLARY 3

Recall the setting from the introduction:  $\mathcal{X}$  is a smooth connected complex manifold of dimension  $n + 1$ , and  $\pi: \mathcal{X} \rightarrow \mathbb{D}$  is a flat proper analytic map over the unit disk which is a submersion over the punctured disk and has connected fibers. We let  $p: \mathcal{X}' \rightarrow \mathcal{X}$  be a proper bi-meromorphic map from a smooth complex manifold  $\mathcal{X}'$  which is an isomorphism over  $\pi^{-1}(\mathbb{D}^*)$ .

We let  $T$  be any closed positive  $(1, 1)$ -current on  $\mathcal{X}'$  admitting local Hölder continuous potentials. Observe that by e.g. [De93, Corollary 1.6] we have

$$\mu'_t = dd^c \log |\pi \circ p - t| \wedge T^n \rightarrow \mu'_0 := dd^c \log |\pi \circ p| \wedge T^n.$$

Let us now analyze the structure of the positive measure  $\mu_0 := p_* \mu'_0$ . First observe that  $\mu'_0$  can be decomposed as a finite sum of positive measures  $\mu'_E := (T|_E)^n$  where the sum is taken over all irreducible components  $E$  of  $\mathcal{X}'_0$ . Each of these measures is locally the Monge-Ampère of a Hölder continuous psh function.

Write  $V := p(E)$ . Since  $E$  is irreducible,  $V$  is also an irreducible (possibly singular) subvariety of dimension  $\ell$ . To conclude the proof it remains to show that  $p_*(\mu'_E)$  is the Monge-Ampère measure of Hölder continuous function that is locally the sum of a smooth and psh function. More precisely, one needs to show that  $p_*(\mu'_E)$  does not charge any proper algebraic subset of  $V$ , and given any resolution of singularities  $\varpi: V' \rightarrow V$  the pull-back measure  $\varpi^*(p_*(\mu'_E))$  can be locally written as  $(dd^c u)^\ell$  where  $u$  is a Hölder psh function on  $V'$ .

This follows from Theorem 1 applied to any resolution of singularities  $V'$  of  $V$  and to any  $E'$  which admits a birational morphism  $E' \rightarrow E$  such that the map  $E' \rightarrow V'$  induced by  $p$  is also a morphism.

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