



HAL
open science

Regularity of push-forward of Monge-Ampère measures

Eleonora Di Nezza, Charles Favre

► **To cite this version:**

Eleonora Di Nezza, Charles Favre. Regularity of push-forward of Monge-Ampère measures. *Annales de l'Institut Fourier*, 2019, 10.5802/aif.3233 . hal-01672332

HAL Id: hal-01672332

<https://hal.science/hal-01672332v1>

Submitted on 24 Dec 2017

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

REGULARITY OF PUSH-FORWARD OF MONGE-AMPÈRE MEASURES

ELEONORA DI NEZZA AND CHARLES FAVRE

ABSTRACT. We prove that the image under any dominant meromorphic map of the Monge-Ampère measure of a Hölder continuous quasi-psh function still possesses a Hölder potential. We also discuss the case of lower regularity.

Dedicated to Jean-Pierre Demailly for his 60th birthday

INTRODUCTION

Let (X, ω_X) be a compact Kähler manifold of dimension n normalized by the volume condition $\int_X \omega_X^n = 1$. We say that a potential $\varphi \in L^1(X)$ is ω_X -plurisubharmonic (ω_X -psh for short) if locally φ is the sum of a plurisubharmonic and a smooth function, and $\omega_X + dd^c\varphi \geq 0$ in the weak sense of currents, where $d = \partial + \bar{\partial}$ and $d^c = \frac{i}{2\pi}(\bar{\partial} - \partial)$ so that $dd^c = \frac{i}{\pi}\partial\bar{\partial}$. We denote by $\text{PSH}(X, \omega_X)$ the set of all ω_X -psh functions on X . Recall from [GZ07, Section 1] that the non-pluripolar Monge-Ampère measure of a function $\varphi \in \text{PSH}(X, \omega_X)$ is a positive measure defined as the increasing limit

$$(\omega_X + dd^c\varphi)^n = \lim_{j \rightarrow +\infty} \mathbf{1}_{\{\varphi > -j\}} (\omega_X + dd^c \max\{\varphi, -j\})^n$$

where the right hand side is defined using Bedford-Taylor intersection theory of bounded psh functions, see [BT76].

One of the main result of [GZ07] states that if μ is a probability measure on X which does not charge pluripolar sets, then there exists a unique (up to a constant) ω_X -psh function φ such that $\int_X (\omega_X + dd^c\varphi)^n = 1$ and

$$(1) \quad \mu = (\omega_X + dd^c\varphi)^n .$$

We denote by $\mathcal{E}(X, \omega_X)$ the set of all ω_X -psh functions whose non-pluripolar Monge-Ampère measure has full mass so that any solution to (1) belongs to $\mathcal{E}(X, \omega_X)$.

In the same paper, Guedj and Zeriahi introduced for any $p > 0$ the subset $\mathcal{E}^p(X, \omega_X)$ of $\mathcal{E}(X, \omega_X)$ consisting of all ω_X -psh functions satisfying the integrability condition $\varphi \in L^p((\omega_X + dd^c\varphi)^n)$. Since ω_X -psh functions are bounded from above it follows that

$$\mathcal{E}^p(X, \omega_X) \subset \mathcal{E}^q(X, \omega_X), \text{ for all } p > q.$$

Observe also that any ω_X -psh function lying in L^∞ belongs to the intersection of all $\mathcal{E}^p(X, \omega_X)$.

We shall say that a probability measure which does not charge pluripolar sets $\mu = (\omega_X + dd^c\varphi)^n$ is a Monge-Ampère measure having Hölder, continuous, L^∞ or \mathcal{E}^p potential for some $p > 0$ whenever φ is Hölder, continuous, L^∞ or belongs to the energy class $\mathcal{E}^p(X, \omega_X)$ respectively.

Let us now consider any dominant meromorphic map $f : X \dashrightarrow Y$ where (Y, ω_Y) is also a compact Kähler manifold of volume 1, and denote by m its complex dimension. Let Γ be a resolution of singularities of the graph of f . We obtain two surjective holomorphic

maps $\pi_1 : \Gamma \rightarrow X$ and $\pi_2 : \Gamma \rightarrow Y$ where π_1 is bimeromorphic so that Γ is a modification of a compact Kähler manifold. By Hironaka's Chow lemma, see e.g. [Pet94, Theorem 2.8] we may suppose that π_1 is a composition of blow-ups along smooth centers so that Γ is itself a compact Kähler manifold of complex dimension n . We fix any Kähler form ω_Γ on it.

One defines the push-forward under f of a measure μ not charging pluripolar sets as follows. Since π_1 is a bimeromorphism, there exist two closed analytic subsets $R \subset \Gamma$ and $V \subset X$ such that $\pi_1 : \Gamma \setminus R \rightarrow X \setminus V$ is a biholomorphism. One may thus set $\pi_1^* \mu$ to be the trivial extension through R of $(\pi_1)|_{\Gamma \setminus R}^* \mu$. This measure is again a probability measure which does not charge pluripolar sets.

We then define the probability measure $f_* \mu := (\pi_2)_* \pi_1^* \mu$. We observe that since f is dominant then π_2 is surjective hence the preimage of a pluripolar set in Y by π_2 is again pluripolar. By the preceding discussion, there exists $\psi \in \mathcal{E}(Y, \omega_Y)$ such that $f_* \mu = (\omega_Y + dd^c \psi)^m$.

Our main goal is to discuss the following question.

Problem 1. Suppose μ is a Monge-Ampère measure having Hölder, continuous, L^∞ or \mathcal{E}^p potential. Is it true that $f_* \mu$ is also a Monge-Ampère measure of a potential lying in the same class of regularity?

This problem is hard for Monge-Ampère measures having either continuous or L^∞ potentials since there is no known intrinsic characterization of these measures. For these classes of regularity even the case f is the identity map and $X = Y$ is still open (see for example [DGZ15, Question 15]).

Problem 2. Suppose μ is a probability measure on X not charging pluripolar sets and write $\mu = (\omega + dd^c \varphi)^n = (\omega' + dd^c \varphi')^n$ where ω, ω' are two Kähler forms of volume 1. Is it true that φ is continuous (resp. L^∞) iff φ' is?

An intrinsic characterization of Monge-Ampère measure of Hölder functions is given by [DDG+14], and of functions in the energy class \mathcal{E}^p in [GZ07, Theorem C] so that Problem 2 has a positive answer for these classes of regularity, see [DiN, Theorem 4.1]. Problem 1 remains though quite subtle. If we restrict our attention to the regularity in the \mathcal{E}^p energy classes, then the answer is no in general. Suppose that $\pi : X \rightarrow \mathbb{P}^2$ is the blow-up at some point $p \in \mathbb{P}^2$, and let $E = \pi^{-1}(p)$. It was observed by the first author in [DiN15, Proposition B] that there exists a probability measure $\mu = (\omega_X + dd^c \varphi)^2$ with $\varphi \in \mathcal{E}^1(X, \omega_X)$ but $\pi_* \mu = (\omega_{FS} + dd^c \psi)^2$ with $\psi \notin \mathcal{E}^1(\mathbb{P}^2, \omega_{FS})$, where ω_{FS} denotes the Fubini Study metric on \mathbb{P}^2 and ω_X is a Kähler form.

In this note we answer Problem 1 in two situations. We first treat the case μ is the Monge-Ampère of a Hölder function.

Theorem 1. *Let $f : X \dashrightarrow Y$ be any dominant meromorphic map between two compact Kähler manifolds. If μ is a Monge-Ampère measure having a Hölder potential with Hölder exponent α , then $f_* \mu$ is a Monge-Ampère measure having a Hölder potential with Hölder exponent bounded by $C\alpha^{\dim(X)}$ for some constant $C > 0$ depending only on f .*

We expect that the technics developed in the paper of Kołodziej-Nguyen [KolNg17] in the present volume allows one to extend the previous result to arbitrary compact hermitian manifolds.

Next we treat the case the image of the map has dimension 1.

Theorem 2. *Let $f : X \dashrightarrow Y$ be any dominant meromorphic map from a compact Kähler manifold to a compact Riemann surface. If μ is a Monge-Ampère measure having a Hölder, \mathcal{C}^0 , L^∞ , \mathcal{E}^p potential respectively, then $f_*\mu$ is a Monge-Ampère measure having a potential lying in the same regularity class.*

Motivations for studying this question come from the analysis of degenerating measures on families of projective manifolds developed in [F16]. Let us briefly recall the setting of that paper. Let \mathcal{X} be a smooth connected complex manifold of dimension $n + 1$, and $\pi : \mathcal{X} \rightarrow \mathbb{D}$ be a flat proper analytic map over the unit disk which is a submersion over the punctured disk and has connected fibers. We assume that \mathcal{X} is Kähler so that each fiber $X_t = \pi^{-1}(t)$ is also Kähler.

A tame family of Monge-Ampère measures is by definition a family of positive measures $\{\mu_t\}_{t \in \mathbb{D}}$ each supported on X_t that can be written under the form

$$\mu_t = p_*(T|_{X_t}^n),$$

where T is a positive closed $(1, 1)$ -current having local Hölder continuous potentials and defined on a complex manifold \mathcal{X}' which admits a proper bimeromorphic morphism $p : \mathcal{X}' \rightarrow \mathcal{X}$ which is an isomorphism over $X := \pi^{-1}(\mathbb{D}^*)$. It follows from [De93, Corollary 1.6] that the family of measures $\mu'_t := T|_{X_t}^n$ in \mathcal{X}' is continuous so that μ'_t converges to a positive measure μ'_0 supported on X'_0 as $t \rightarrow 0$. It follows that the convergence $\lim_{t \rightarrow 0} \mu_t = \mu_0$ also holds in \mathcal{X} .

As a corollary of the previous results we show the limiting measure μ_0 is of a very special kind:

Corollary 3. *Let $\{\mu_t\}_{t \in \mathbb{D}}$ be any tame family of Monge-Ampère measures, so that $\mu_t \rightarrow \mu_0$ as $t \rightarrow 0$.*

Then there exist a finite collection of closed subvarieties $\{V_i\}_{i=0, \dots, N}$ of X_0 and for each index i a positive measure ν_i supported on V_i such that

$$\mu_0 = \sum_{i=1}^N \nu_i$$

and ν_i is a Monge-Ampère measure on V_i having a Hölder potential.

In the previous statement, it may happen that V_i is singular, in which case it is understood that the pull-back of ν_i to a (Kähler) resolution of V_i is a Monge-Ampère measure having a Hölder continuous potential.

Acknowledgement. We thank Ahmed Zeriahi for useful discussions on these problems.

1. IMAGES OF MONGE-AMPÈRE MEASURES HAVING A HÖLDER POTENTIAL: PROOF OF THEOREM 1

As already mentioned, a dominant meromorphic map $f : X \dashrightarrow Y$ can be decomposed as $f = \pi_2 \circ \pi_1^{-1}$, where $\pi_1 : \Gamma \rightarrow X$ is holomorphic and bimeromorphic and $\pi_2 : \Gamma \rightarrow Y$ is a surjective holomorphic map. Recall that one can assume Γ to be Kähler, and that $f_*\mu := (\pi_2)_*\pi_1^*\mu$.

We first claim that if μ is the Monge-Ampère of a Hölder continuous function then $\pi_1^*\mu$ too. Let $\varphi \in \text{PSH}(X, \omega_X)$ be the Hölder potential such that $\mu = (\omega_X + dd^c\varphi)^n$. It then follows from Bedford and Taylor theory that $\pi_1^*\mu = (\pi_1^*\omega_X + dd^c\pi_1^*\varphi)^n$. Since $\pi_1^*\omega_X$ is a semipositive smooth form, there exists a positive constant $C > 0$ such that

$\pi_1^* \mu \leq (C\omega_\Gamma + dd^c \pi^* \varphi)^n$ where ω_Γ is a Kähler form on Γ , and [DDG+14, Theorem 4.3] implies that $\tilde{\mu} := \pi_1^* \mu$ is the Monge-Ampère measure of a Hölder continuous $C\omega_\Gamma$ -psh function. This proves the claim. We are then left to prove that $(\pi_2)_* \tilde{\mu}$ is the Monge-Ampère measure of a Hölder potential. This will be done in Lemma 7.

We first show that the push-forward of a smooth volume form has density in $L^{1+\varepsilon}$, for some constant $\varepsilon > 0$ depending only on f .

Proposition 4. *Let $f: X \rightarrow Y$ be a surjective holomorphic map. Then $f_* \omega_X^n = g \omega_Y^m$ with $g \in L^{1+\varepsilon}(\omega_Y^n)$, for some $\varepsilon > 0$.*

This result is basically [ST12, Proposition 3.2] (see also [Tosatti, Section 2]). We give nevertheless a detailed proof for reader's convenience. Pick any coherent ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$, and denote by $V(\mathcal{I}) = \text{supp}(\mathcal{O}_X/\mathcal{I})$ the closed analytic subvariety of X associated to \mathcal{I} . Let $\{U_i\}_{i=1}^N$ be a finite open covering of X by balls and $\{V_i\}_i$ be a subcovering such that $\overline{V}_i \subset U_i$. The analytic sheaf \mathcal{I} is globally generated on each U_i so that we can find holomorphic functions such that $\mathcal{I}|_{U_i} = (h_1^{(i)}, \dots, h_k^{(i)}) \cdot \mathcal{O}_{U_i}$. Let $\{\rho_i\}$ be a partition of unity subordinate to \overline{V}_i . We then define

$$(2) \quad \Phi_{\mathcal{I}} := \sum_{i=1}^N \rho_i \left(\sum_{j=1}^k |h_j^{(i)}|^2 \right).$$

Then $\Phi_{\mathcal{I}}: X \rightarrow \mathbb{R}_+$ is a smooth function which vanishes exactly on $V(\mathcal{I})$. Observe that if $\Phi_{\mathcal{I}}$ and $\Phi'_{\mathcal{I}}$ are defined using two different coverings, then there exists $C > 0$ such that

$$\frac{1}{C} \Phi'_{\mathcal{I}} \leq \Phi_{\mathcal{I}} \leq C \Phi'_{\mathcal{I}}.$$

In the sequel we shall abuse notation and not write the dependence of $\Phi_{\mathcal{I}}$ in terms of the local generators of the ideal sheaf. The logarithm of the obtained function is then well-defined up to a bounded function so that all statements in the next Lemma make sense.

Lemma 5. *Let $\mathcal{I}, \mathcal{J} \subseteq \mathcal{O}_X$ be two coherent ideal sheafs. The followings hold:*

- (i) *there exists $\varepsilon > 0$ such that $|\Phi_{\mathcal{I}}|^{-\varepsilon} \in L^1(X)$;*
- (ii) *if $\mathcal{I} \subseteq \mathcal{J}$ then $\Phi_{\mathcal{J}} \geq c \Phi_{\mathcal{I}}$ for some positive $c > 0$;*
- (iii) *if $V(\mathcal{J}) \subseteq V(\mathcal{I})$ then there exists $c, \theta > 0$ such that $\Phi_{\mathcal{J}} \geq c \Phi_{\mathcal{I}}^\theta$;*
- (iv) *given $f: X \rightarrow Y$ a holomorphic surjective map and a coherent ideal sheaf $\mathcal{J} \subseteq \mathcal{O}_Y$, then $\Phi_{f^* \mathcal{J}} = \Phi_{\mathcal{J}} \circ f$ (for a suitable choice of local generators of \mathcal{J} and $f^* \mathcal{J}$).*

Proof. Using a resolution of singularities of \mathcal{I} , one sees that the statement in (i) reduces to show that $|z_1|^{-\varepsilon}$ is locally integrable for some $\varepsilon > 0$, and this is the case if we choose ε small enough. The statements in (ii) and (iv) follow straightforward from the definition in (2). The statement in (iii) is a consequence of Łojasiewicz theorem, see e.g. [LJTR, Theorem 7.2]. \square

Lemma 6. *Let $f: X \rightarrow Y$ be a holomorphic surjective map and let $\mathcal{I} \subseteq \mathcal{O}_X$ be a coherent ideal sheaf. Then there exists a coherent ideal sheaf $\mathcal{J} \subseteq \mathcal{O}_Y$, and constants $c, \theta > 0$ such that for any $y \in Y$ we have*

$$\inf_{x \in f^{-1}(y)} \Phi_{\mathcal{I}} \geq c \Phi_{\mathcal{J}}^\theta$$

Proof. Let $\mathcal{I} \subseteq \mathcal{O}_Y$ be the coherent ideal sheaf of holomorphic functions vanishing on the set $f(V(\mathcal{I}))$ which is analytic since f is proper. Observe that $V(f^*\mathcal{I}) = f^{-1}(V(\mathcal{I})) \supset V(\mathcal{I})$, so that Lemma 5 ((iii) and (iv)) insure that there exist $c, \theta > 0$ such that

$$\Phi_{\mathcal{I}} \geq c\Phi_{f^*\mathcal{I}}^\theta = c(\Phi_{\mathcal{I}} \circ f)^\theta.$$

Hence the conclusion. \square

Proof of Proposition 4. Recall that Sard's theorem implies the existence of a closed subvariety $S \subsetneq Y$ such that $f: X \setminus f^{-1}(S) \rightarrow Y \setminus S$ is a submersion.

We first prove that $f_*\omega_X^n$ is absolutely continuous w.r.t. ω_Y^m . We need to check that $\omega_Y^m(E) = 0$ implies $f_*\omega_X^n(E) = 0$ for any Borel subset $E \subset Y$. As S and $f^{-1}(S)$ have volume zero one may assume that f is a submersion in which case the claim follows from Fubini's theorem.

Radon-Nikodym theorem now guarantees that $f_*\omega_X^n = g\omega_Y^m$ for some $0 \leq g \in L^1(Y)$. We want to show that the integral

$$\int_Y g^{1+\varepsilon}\omega_Y^m = \int_Y g^\varepsilon f_*\omega_X^n = \int_X (f^*g)^\varepsilon \omega_X^n$$

is finite for some $\varepsilon > 0$ small enough. Consider the smooth function $\phi(x) := \frac{f^*\omega_Y^m \wedge \omega_X^{n-m}}{\omega_X^n}(x)$, and set $\tilde{\phi}(y) := \inf_{x \in f^{-1}(y)} \phi(x)$ so that $\phi \geq f^*\tilde{\phi}$. We claim that for any $y \in Y$

$$(3) \quad g(y) \leq \frac{1}{\tilde{\phi}(y)}.$$

Let χ be a test function (i.e. a non negative smooth function) on Y , then

$$\begin{aligned} \int_Y \chi g \omega_Y^m &= \int_X f^*\chi \omega_X^n = \int_X \frac{f^*\chi}{\phi} f^*\omega_Y^m \wedge \omega_X^{n-m} \\ &\leq \int_X f^*\left(\frac{\chi}{\phi}\omega_Y^m\right) \wedge \omega_X^{n-m} \\ &\stackrel{\text{Fubini}}{=} C(f) \int_Y \frac{\chi}{\phi} \omega_Y^m \leq \int_Y \frac{\chi}{\phi} \omega_Y^m \end{aligned}$$

where $C(f)$ is the volume of a fiber over a generic point $y \in Y$, i.e. $C(f) = \int_{f^{-1}(y)} \omega_X^{n-m}$. The last inequality follows from the fact $C(f) \leq 1$ since $\int_X \omega_X^n = 1$. The claim is thus proved. Lemma 5 (i) and (iv) combined with Lemma 6 then insure that there exists $\varepsilon > 0$ such that $(f^*g)^\varepsilon \in L^1(\omega_X^n)$. \square

Theorem 1 is reduced to the following result which relies in an essential way on Proposition 4.

Proposition 7. *Suppose $f: X \rightarrow Y$ is a surjective holomorphic map between compact Kähler manifolds. If μ is a positive measure on X with Hölder continuous potentials, then $f_*\mu$ is a positive measure on Y with Hölder potentials.*

Observe that by multiplying ω_X by a suitable positive constant we may assume that $f^*\omega_Y \leq \omega_X$. The volume normalization is no longer satisfied but a positive multiple of μ is still the Monge-Ampère measure of a ω_X -psh Hölder continuous function. Write $f_*\mu = (\omega_Y + dd^c\psi)^m$ with $\psi \in \text{PSH}(Y, \omega_Y)$.

We claim that there exists $C > 0$, and $\varepsilon > 0$ such that for all $u \in \text{PSH}(Y, \omega_Y)$ with $\int_X u \omega_X^n = 0$

$$(4) \quad \int_Y \exp(-\varepsilon u) d(f_*\mu) \leq C .$$

Indeed, for any $u \in \text{PSH}(Y, \omega_Y)$ we have that

$$\int_Y e^{-\varepsilon u} d(f_*\mu) = \int_X e^{-\varepsilon(u \circ f)} d\mu .$$

Now the integral $\int_X e^{-\varepsilon(u \circ f)} d\mu$ is uniformly bounded by [DNS10, Theorem 1.1] since:

- μ has Hölder continuous potentials;
- $f^*\omega_Y \leq \omega_X$ hence $u \circ f \in \text{PSH}(X, \omega_X)$;
- and the set of functions in $\text{PSH}(X, \omega_X)$ such that $\int_X u \omega_X^n = 0$ is compact by [GZ05, Proposition 2.6].

This proves our claim. Using the terminology of [DN14] this means that $f_*\mu$ is *moderate*. It is worth mentioning that if [DGZ15, Question 16] holds true then the conclusion of Proposition 7 would follow immediately since any moderate measure would have a Hölder continuous potential. To get around this problem we use the characterization of measures with Hölder potentials given by Dinh and Nguyen.

Proof of Lemma 7. By [DN14, Lemma 3.3], $f_*\mu$ is the Monge-Ampère measure of a Hölder potential if and only if there exist $\tilde{c} > 1$ and $\tilde{\beta} \in (0, 1)$ such that

$$(5) \quad \int_Y |u - v| df_*\mu \leq \tilde{c} \max \left(\|u - v\|_{L^1(\omega_Y^n)}, \|u - v\|_{L^1(\omega_Y^n)}^{\tilde{\beta}} \right)$$

for all $u, v \in \text{PSH}(Y, \omega_Y)$. By assumption on μ we know there exist $c > 1$ and $\beta \in (0, 1)$ such that $\int_Y |u - v| df_*\mu = \int_X |f^*u - f^*v| d\mu$, and

$$(6) \quad \int_X |f^*u - f^*v| d\mu \leq c \max \left(\|f^*u - f^*v\|_{L^1(\omega_X^n)}, \|f^*u - f^*v\|_{L^1(\omega_X^n)}^\beta \right) .$$

Also, Proposition 4 gives

$$(7) \quad \int_X |f^*u - f^*v| \omega_X^n = \int_Y |u - v| g \omega_Y^n \leq \|g\|_{L^{1+\varepsilon}(\omega_Y^n)} \|u - v\|_{L^p(\omega_Y^n)}$$

where p is the conjugate exponent of $1 + \varepsilon$. Set $C_g := \|g\|_{L^{1+\varepsilon}(\omega_Y^n)} < +\infty$. Up to replace C_g with $C_g + 1$ we can assume that $C_g \geq 1$.

Denote by $m_u := \int_Y u \omega_Y^n$ and observe that $u' := u - m_u$, $v' := v - m_v$ satisfy $\int_X u' \omega_X^n = 0 = \int_X v' \omega_X^n$. Then the triangle inequality gives

$$(8) \quad \begin{aligned} \|u - v\|_{L^p(\omega_Y^n)} &= \left(\int_Y |(u' - v') + (m_u - m_v)|^p \omega_Y^n \right)^{1/p} \\ &\leq \|u' - v'\|_{L^p(\omega_Y^n)} + |m_u - m_v| \\ &\leq \|u' - v'\|_{L^p(\omega_Y^n)} + \|u - v\|_{L^1(\omega_Y^n)}. \end{aligned}$$

At this point, we make use of [DN14, Proposition 3.2] (that holds for normalized potentials) to replace the L^p -norm with the L^1 -norm. We then infer the existence of a constant $c' > 1$ such that

$$\|u' - v'\|_{L^p(\omega_Y^n)} \leq c' \max(1, -\log \|u' - v'\|_{L^1(\omega_Y^n)})^{\frac{p-1}{p}} \|u' - v'\|_{L^1(\omega_Y^n)}^{\frac{1}{p}} .$$

When $t := \|u' - v'\|_{L^1(\omega_Y^n)} \geq 1/e$ we clearly have

$$\|u' - v'\|_{L^p(\omega_Y^n)} \leq c' \|u' - v'\|_{L^1(\omega_Y^n)}^{\frac{1}{p}},$$

whereas for any integer $N \in \mathbb{N}^*$, there exists a constant $c_N > 0$ such that $-\log t \leq c_N t^{-1/N}$ when $t \leq 1/e$, hence

$$\|u' - v'\|_{L^p(\omega_Y^n)} \leq c'' \|u' - v'\|_{L^1(\omega_Y^n)}^{\frac{1}{p}(1 - \frac{p-1}{N})}.$$

As $\|u' - v'\|_{L^1(\omega_Y^n)} \leq 2\|u - v\|_{L^1(\omega_Y^n)}$, combining (6), (7) and (8) we get

$$\|f^*u - f^*v\|_{L^1(\mu)} \leq C \max \left(\|u - v\|_{L^1(\omega_Y^n)}^{\tilde{\beta}}, \|u - v\|_{L^1(\omega_Y^n)} \right),$$

with $\tilde{\beta} = \frac{\beta}{p} \left(1 - \frac{p-1}{N}\right)$. By [DN14, Lemma 3.3] $f_*\mu = (\omega_Y + dd^c\psi)^n$ where ψ is a Hölder continuous function.

To get a bound on the Hölder regularity of ψ , one argues as follows. First if $\mu = (\omega + dd^c\varphi)^n$ with φ a α -Hölder potential, and $\pi: \Gamma \rightarrow X$ is a proper modification, then $\pi^*\mu$ is dominated by a Monge-Ampère measure with α -Hölder potential, and [DDG+14, Proposition 3.3 (ii)] is satisfied with $b = 2\alpha/(\alpha + 2n)$ by [DDG+14, Theorem 4.3 (iii)]. Hence, following the proof of [DDG+14, Theorem], we see that $\pi^*\mu$ is a Monge-Ampère measure of a α_1 -Hölder continuous potential with $\alpha_1 < b/(n + 1)$ (see Remark below for more details about the latter statement).

By [DN14, Proposition 4.1], (6) holds with $\beta = \alpha_1^n/(2 + \alpha_1^n)$, and (5) is then satisfied for any $\tilde{\beta} < \beta/p$ so that $f_*\mu$ is a Monge-Ampère measure with $\tilde{\alpha}$ -Hölder potential for any $\tilde{\alpha} < 2\tilde{\beta}/(m + 1)$, see the discussion on [DN14, p.83]. Combining all these estimates we see that any

$$\tilde{\alpha} < \frac{\alpha^n}{p(m + 1)(\alpha/2 + n)^n(n + 1)^n}$$

works where p is the conjugate of the larger constant $\varepsilon > 0$ for which Proposition 4 holds. \square

Remark. We borrow notations from the proof of [DDG+14, Theorem A]. Fix $\alpha_1 < b/(n + 1)$ and choose $\varepsilon > 0$ such that $\alpha_1 \leq \alpha \leq \alpha_0 \leq b - \alpha_0(n + \varepsilon)$. By the previous arguments we know that condition (ii) in [DDG+14, Proposition 3.3] holds, i.e. for any $\phi \in \text{PSH}(\Gamma, \omega_\Gamma)$, we have $\|\rho_\delta\phi - \phi\|_{L^1(\pi^*\mu)} = O(\delta^b)$, where $b = 2\alpha/(\alpha + 2n)$. In particular, this gives

$$\pi^*\mu(E_0) \leq c_1\delta^{b-\alpha_0}.$$

Let $g \in L^1(\pi^*\mu)$ be defined as $g = 0$ on E_0 and $g = c$ on $\Gamma \setminus E_0$ where c is a positive constant such that $\pi^*\mu(\Gamma) = \int_\Gamma g d(\pi^*\mu)$. An easy computation gives that $c = \pi^*\mu(\Gamma)/\pi^*\mu(\Gamma \setminus E_0)$. Let $v \in \text{PSH}(\Gamma, \omega_\Gamma)$ be the bounded solution of the Monge-Ampère equation $(\omega_\Gamma + dd^c v)^n = g \cdot \pi^*\mu$. Observe that

$$\|1 - g\|_{L^1(\pi^*\mu)} = \int_{E_0} d\pi^*\mu + \int_{\Gamma \setminus E_0} |1 - c| d\pi^*\mu = 2 \int_{E_0} d\pi^*\mu \leq 2c_1\delta^{b-\alpha_0}.$$

Since $\pi^*\mu = (\omega_\Gamma + dd^c\tilde{\varphi})^n$ satisfies the $\mathcal{H}(\infty)$ property we can still apply [DZ, Theorem 1.1] and get

$$\|\tilde{\varphi} - v\|_{L^\infty} \leq c_3\delta^{\frac{b-\alpha_0}{n+\varepsilon}}.$$

The exact same arguments as in [DDG+14, Theorem A] then insure that the Hölder exponent of $\tilde{\varphi}$ is α_1 .

2. OVER A ONE-DIMENSIONAL BASE: PROOF OF THEOREM 2

In this section we treat Problem 1 in the case the base is a Riemann surface.

We start with the case of a surjective holomorphic map $f: X \rightarrow Y$ from a Kähler compact manifold to a compact Riemann surface.

Let $\mu = (\omega_X + dd^c\varphi)^n$ be a Monge-Ampère measure of a continuous ω_X -psh function φ . Suppose v_k, v is a family of ω_X -psh functions such that $v_k \rightarrow v$ in L^1 , then

$$\begin{aligned} \int_X v_k d\mu &= \int_X v_k (\omega_X + dd^c\varphi)^n \\ &= \int_X v_k \omega_X^n + \sum_{j=0}^{n-1} \int_X \varphi dd^c v_k \wedge \omega_X^j \wedge (\omega_X + dd^c\varphi)^{n-j-1} \rightarrow \int_X v d\mu \end{aligned}$$

by [De93, Corollary 1.6 (a)]. Observe that in the last equality we used the fact that

$$(\omega_X + dd^c\varphi)^n - \omega_X^n = \sum_{j=0}^{n-1} dd^c\varphi \wedge \omega_X^j \wedge (\omega_X + dd^c\varphi)^{n-j-1}$$

and Stokes' theorem.

Normalize the Kähler form on Y such that $\int \omega_Y = 1$, and pick any sequence $y_k \rightarrow y_\infty \in Y$. Let w_k be the solutions of the equations $\Delta w_k = \delta_{y_k} - \omega_Y$ with $\sup w_k = 0$ so that $w_k(y) - \log|y - y_k|$ is continuous in local coordinates near y_k . Write $f_*\mu = \omega_Y + dd^c\psi$ so that

$$\int_Y w_k d(f_*\mu) = \int_Y w_k \omega_Y + \int_Y \psi \Delta w_k = \psi(y_k) + \int_Y (w_k - \psi)\omega_Y.$$

Since $w_k \rightarrow w_\infty$ in L^p_{loc} for all $p < \infty$, Proposition 4 implies that $f^*w_k \rightarrow f^*w_\infty$ in the L^1 topology, so that the argument above gives $\int_Y w_k d(f_*\mu) = \int_X f^*w_k d\mu \rightarrow \int_X f^*w_\infty d\mu = \int_Y w_\infty d(f_*\mu)$. We then conclude that $\psi(y_k) \rightarrow \psi(y_\infty)$. Hence ψ is continuous.

Suppose then that μ is locally the Monge-Ampère of a bounded psh function, and pick any subharmonic function u defined in a neighborhood of a point $y \in Y$. Then f^*u is again psh in a neighborhood of $f^{-1}(y)$, and the standard Chern-Levine-Nirenberg inequalities imply that $f^*u \in L^1(\mu)$ so that $u \in L^1(f_*\mu)$ with a norm depending only on the L^1 -norm of u . It follows that $f_*\mu$ is locally the laplacian of a bounded subharmonic function.

Finally, assume $\mu = (\omega_X + dd^c\varphi)^n$ for some $\varphi \in \mathcal{E}^p(X, \omega_X)$. By [GZ07, Theorem C] this is equivalent to have that $\mathcal{E}^p(X, \omega_X) \subset L^p(\mu)$. Write as usual $f_*\mu = (\omega_Y + dd^c\psi)$ with $\psi \in \mathcal{E}(Y, \omega_Y)$.

We claim that $u \in \mathcal{E}^p(Y, \omega_Y)$ implies $f^*u \in \mathcal{E}^p(X, \omega_X)$. Indeed, without loss of generality we can assume that $\Omega := \omega_X - f^*\omega_Y$ is a Kähler form and by the multilinearity of the non-pluripolar product we have

$$\begin{aligned} \int_X |f^*u|^p (\omega_X + dd^c f^*u)^n &= \int_X |f^*u|^p (f^*\omega_Y + \Omega + dd^c f^*u)^n \\ &= \int_X |f^*u|^p (\Omega^n + (f^*\omega_Y + dd^c f^*u) \wedge \Omega^{n-1}) \end{aligned}$$

where the last identity follows from the fact that $(f^*\omega_Y + dd^c f^*u)^j = 0$ for $j > 1$. The term $\int_X |f^*u|^p \Omega^n$ is bounded thanks to the integrability properties of quasi-plurisubharmonic

functions w.r.t. volume forms [GZ17, Theorem 1.47]; while the term

$$\int_X |f^*u|^p (f^*\omega_Y + dd^c f^*u) \wedge \Omega^{n-1} = C(f) \int_Y |u|^p (\omega_Y + dd^c u)$$

is finite since $u \in \mathcal{E}^p(Y, \omega_Y)$. This proves the claim.

Now, given any $u \in \mathcal{E}^p(Y, \omega_Y)$ we have

$$\int_Y |u|^p d(f_*\mu) = \int_X |f^*u|^p d\mu < +\infty$$

since $f^*u \in \mathcal{E}^p(X, \omega_X) \subset L^p(\mu)$. The conclusion follows from [GZ07, Theorem C].

Consider now any dominant meromorphic map $f: X \dashrightarrow Y$ from a Kähler compact manifold to a compact Riemann surface. As above we decompose f such that $f_*\mu = (\pi_2)_*\pi_1^*\mu$ for any positive measure μ on X .

Assume that μ has continuous potentials. If we write $\mu = (\omega_X + dd^c\varphi)^n$ then $\pi_1^*\mu = (\pi_1^*\omega_X + dd^c\varphi \circ \pi)^n \leq (C\omega_\Gamma + dd^c\varphi \circ \pi)^n := \hat{\mu}$ where $\hat{\mu}$ has a continuous potential. This implies $f_*\mu \leq (\pi_2)_*\hat{\mu}$. Observe that by the previous arguments the measure $(\pi_2)_*\hat{\mu}$ has continuous potential. It follows that locally $f_*\mu = \Delta v \leq \Delta u$ where u, v are subharmonic functions. It follows that v is the sum of a continuous function and the opposite of a subharmonic (hence u.s.c.) function. Since it is also u.s.c we conclude to its continuity.

When μ has bounded potentials, the same argument applies noting that subharmonic functions are always bounded from above which implies v to be bounded.

Finally, we consider the case where μ is the Monge-Ampère measure of $\varphi \in \mathcal{E}^p(X, \omega_X)$. We first observe that given $v \in \mathcal{E}^p(\Gamma, \omega_\Gamma)$ we have $(\pi_1)_*v \in \mathcal{E}^p(X, \omega_X)$. Indeed,

$$\int_X |v \circ \pi^{-1}|^p (\omega_X + dd^c v \circ \pi^{-1})^n = \int_\Gamma |v|^p (\pi_1^*\omega_X + dd^c v)^n \leq \int_\Gamma |v|^p (C\omega_\Gamma + dd^c v)^n < +\infty.$$

This and the previous arguments give that if $u \in \mathcal{E}^p(Y, \omega_Y)$ then $f_*u = (\pi_1)_*\pi_2^*u \in \mathcal{E}^p(X, \omega_X)$, hence

$$\int_Y |u|^p df_*\mu = \int_X |u \circ f|^p d\mu < +\infty.$$

It follows from [GZ07, Theorem C] that $f_*\mu$ is the Monge-Ampère measure of a function in $\mathcal{E}^p(Y, \omega_Y)$.

3. THE CASE OF SUBMERSIONS

In this section we let $(X, \omega_X), (Y, \omega_Y)$ be two compact Kähler manifolds of dimension n and m , respectively and normalized such that $\int_X \omega_X^n = 1 = \int_Y \omega_Y^m$.

Proposition 8. *Let $f: X \rightarrow Y$ be a submersion. Then, $u \in \mathcal{E}^p(Y, \omega_Y)$ implies $f^*u \in \mathcal{E}^p(X, \omega_X)$. In particular, if a probability measure μ is the Monge-Ampère of a function in \mathcal{E}^p then also $f_*\mu$ has also a potential in \mathcal{E}^p .*

Proof. Since f is a submersion we can assume that there is a finite number of open neighbourhoods U_i such that $X \subset \bigcup_{j=0}^N U_j$, $f|_{U_j}(z, w) = z$ where $z = (z_1, \dots, z_m)$ and $w = (z_{m+1}, \dots, z_n)$. Moreover we can assume that on each U_j we have

$$\omega_X \leq C_j \frac{i}{2} (dz \wedge d\bar{z} + dw \wedge d\bar{w}), \quad \frac{i}{2} dz \wedge d\bar{z} \leq A_j f^*\omega_Y$$

where $A_j, C_j > 1$ and $dz \wedge d\bar{z}, dw \wedge d\bar{w}$ are short notations for $\sum_{j=1}^m dz_j \wedge d\bar{z}_j$ and $\sum_{k=m+1}^n dz_k \wedge d\bar{z}_k$, respectively. We then write

$$\begin{aligned} \int_X |f^*u|^p (\omega_X + dd^c f^*u)^n &\leq \sum_{j=1}^N \int_{U_j} |f^*u|^p \left(C_j \frac{i}{2} dz \wedge d\bar{z} + C_j \frac{i}{2} dw \wedge d\bar{w} + dd^c f^*u \right)^n \\ &\leq \sum_{j=1}^N \int_{U_j} |f^*u|^p \left(A'_j f^* \omega_Y + C_j \frac{i}{2} dw \wedge d\bar{w} + dd^c f^*u \right)^n \\ &= \sum_{j=1}^N \sum_{\ell=0}^n \int_{U_j} |f^*u|^p (A'_j f^* \omega_Y + dd^c f^*u)^\ell \wedge \left(C_j \frac{i}{2} dw \wedge d\bar{w} \right)^{n-\ell} \\ &= \sum_{j=1}^N \int_{U_j} |f^*u|^p (A'_j f^* \omega_Y + dd^c f^*u)^m \wedge \left(C_j \frac{i}{2} dw \wedge d\bar{w} \right)^{n-m}. \end{aligned}$$

The above integral is then finite because by assumption $u \in \mathcal{E}^p(Y, A\omega_Y)$ for any $A \geq 1$.

The last statement follows from the same arguments in the last part of the proof in the previous section. \square

4. TAME FAMILIES OF MONGE-AMPÈRE MEASURES: PROOF OF COROLLARY 3

Recall the setting from the introduction: \mathcal{X} is a smooth connected complex manifold of dimension $n + 1$, and $\pi: \mathcal{X} \rightarrow \mathbb{D}$ is a flat proper analytic map over the unit disk which is a submersion over the punctured disk and has connected fibers. We let $p: \mathcal{X}' \rightarrow \mathcal{X}$ be a proper bi-meromorphic map from a smooth complex manifold \mathcal{X}' which is an isomorphism over $\pi^{-1}(\mathbb{D}^*)$.

We let T be any closed positive $(1, 1)$ -current on \mathcal{X}' admitting local Hölder continuous potentials. Observe that by e.g. [De93, Corollary 1.6] we have

$$\mu'_t = dd^c \log |\pi \circ p - t| \wedge T^n \rightarrow \mu'_0 := dd^c \log |\pi \circ p| \wedge T^n.$$

Let us now analyze the structure of the positive measure $\mu_0 := p_* \mu'_0$. First observe that μ'_0 can be decomposed as a finite sum of positive measures $\mu'_E := (T|_E)^n$ where the sum is taken over all irreducible components E of \mathcal{X}'_0 . Each of these measures is locally the Monge-Ampère of a Hölder continuous psh function.

Write $V := p(E)$. Since E is irreducible, V is also an irreducible (possibly singular) subvariety of dimension ℓ . To conclude the proof it remains to show that $p_*(\mu'_E)$ is the Monge-Ampère measure of Hölder continuous function that is locally the sum of a smooth and psh function. More precisely, one needs to show that $p_*(\mu'_E)$ does not charge any proper algebraic subset of V , and given any resolution of singularities $\varpi: V' \rightarrow V$ the pull-back measure $\varpi^*(p_*(\mu'_E))$ can be locally written as $(dd^c u)^\ell$ where u is a Hölder psh function on V' .

This follows from Theorem 1 applied to any resolution of singularities V' of V and to any E' which admits a birational morphism $E' \rightarrow E$ such that the map $E' \rightarrow V'$ induced by p is also a morphism.

REFERENCES

- [BT76] Bedford, Eric; Taylor, Bert A. *The Dirichlet problem for the complex Monge-Ampère equation*. Invent. Math., 37 (1976), 1–44.

- [BJ16] Boucksom, Sébastien; Jonsson, Mattias. *Tropical and non-Archimedean limits of degenerating families of volume forms*. Journal de l'École polytechnique - Mathématiques, 4 (2017), 87–139.
- [CL06] Chambert-Loir, Antoine. *Mesures et équidistribution sur des espaces de Berkovich*. J. Reine Angew. Math. 595 (2006), 215–235.
- [CL06] Chambert-Loir, Antoine; Thuillier, Amaury. *Mesure de Mahler et équidistribution logarithmique*. Annales de l'Institut Fourier 59 (2009), 977–1014.
- [De93] Demailly, Jean-Pierre. *Monge-Ampère operators, Lelong numbers and intersection theory*. Complex Analysis and Geometry, Univ. Series in Math., edited by V. Ancona and A. Silva, Plenum Press, New-York (1993)
- [DDG+14] Demailly, Jean-Pierre; Dinew, Sławomir; Guedj, Vincent; Hiep, Pham Hoang; Kołodziej, Sławomir; Zeriahi, Ahmed. *Hölder continuous solutions to Monge-Ampère equations*. J. Eur. Math. Soc 16 (2014), 619 – 647.
- [DMF14] DeMarco, Laura; Faber, Xander. *Degenerations of complex dynamical systems*. Forum of Mathematics, Sigma. 2 (2014), e6, 36 pages.
- [DGZ15] Dinew, Sławomir; Guedj, Vincent; Zeriahi, Ahmed. *Open problems in pluripotential theory*. Complex Var. and Elliptic Eq. 61 (2016), 7, 902–930.
- [DZ] Dinew, Sławomir, Zhang, Zhou. *On stability and continuity of bounded solutions of degenerate complex Monge-Ampère equations over compact Kähler manifolds*. Adv. Math. 225, (2010), 367–388.
- [DiN] Di Nezza, Eleonora. *Stability of Monge-Ampère energy classes*. Journal of Geometric Analysis, 25 (2014), 4, 2565–2589.
- [DiN15] Di Nezza, Eleonora. *Finite Pluricomplex energy measures*. Potential Analysis, 44 (2015), 155 –167.
- [DNS10] Dinh, Tien Cuong; Nguyen, Viet Anh; Sibony, Nessim. *Exponential estimates for plurisubharmonic functions and stochastic dynamics*. J. Differential Geom. 84 (2010), 465 – 488.
- [DN14] Dinh, Tien Cuong; Nguyen, Viet Anh. *Characterization of Monge-Ampère measures with Hölder continuous potential* J. Functional Analysis 266 (2014), 67–84.
- [EGZ09] Eyssidieux, Philippe; Guedj, Vincent; Zeriahi, Ahmed. *Singular Kähler-Einstein metrics*. J. Amer. Math. Soc. 22 (2009), no. 3, 607 – 639.
- [F16] Favre, Charles. *Degeneration of endomorphisms of the complex projective space in the hybrid space*. [arXiv:1611.08490](https://arxiv.org/abs/1611.08490)
- [GZ05] Guedj, Vincent; Zeriahi, Ahmed. *Intrinsic capacities on compact Kähler manifolds*. J. Geom. Anal. 15 (2005), no. 4, 607– 639.
- [GZ07] Guedj, Vincent; Zeriahi, Ahmed. *The weighted Monge-Ampère energy of quasiplurisubharmonic functions*. J. Funct. Anal. 250 (2007), no. 2, 442 – 482.
- [GZ17] Guedj, Vincent; Zeriahi, Ahmed. *Degenerate Complex Monge-Ampère Equations*. EMS Tracts in Mathematics, 26, (2017).
- [KolNg17] Kołodziej, Sławomir; Ngoc Cuong Nguyen. *Hölder continuous solutions of the Monge-Ampère equation on compact hermitian manifolds*. In the present volume.
- [LJTR] M. Lejeune-Jalabert, Monique; Teissier, Bernard; Risler, Jean-Jacques. *Clôture intégrale des idéaux et équisingularité*. Annales de la Faculté des sciences de Toulouse : Mathématiques, Série 6 : Tome 17 (2008) no. 4 , p. 781–859.
- [Nak04] Nakayama, Noburu. *Zariski decomposition and abundance*. MSJ memoir 14 (2004).
- [Pet94] Peternell, Thomas. *Modifications*. In Grauert, Peternell, Remmert (Hrsg.) Several complex variables VII: Sheaf theoretic methods in complex analysis, Encyclopaedia of Mathematical Sciences, Springer Verlag, 1994.
- [ST12] Song, Jian; Tian, Gang. *Canonical measures and Kähler-Ricci flow*. J. Amer. Math. Soc. 25 (2012), no. 2 303–353.
- [Tosatti] Tosatti, Valentino. *Adiabatic limits of Ricci-flat Kähler metrics*. J. Differential Geom. 84 (2010), no. 2, 427–453.

IHES UNIVERSITÉ PARIS-SACLAY, 35 ROUTE DE CHARTRES, 91400 BURES SUR YVETTE, FRANCE
E-mail address: dinezza@ihes.fr

CMLS, ÉCOLE POLYTECHNIQUE, CNRS, UNIVERSITÉ PARIS-SACLAY, 91128 PALAISEAU CEDEX, FRANCE
E-mail address: charles.favre@polytechnique.edu