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REGULARITY OF PUSH-FORWARD OF MONGE-AMPÈRE MEASURES

ELEONORA DI NEZZA AND CHARLES FAVRE

ABSTRACT. We prove that the image under any dominant meromorphic map of the Monge-Ampère measure of a Hölder continuous quasi-psh function still possesses a Hölder potential. We also discuss the case of lower regularity.

Dedicated to Jean-Pierre Demailly for his 60th birthday

INTRODUCTION

Let (X, ω_X) be a compact Kähler manifold of dimension n normalized by the volume condition $\int_X \omega_X^n = 1$. We say that a potential $\varphi \in L^1(X)$ is ω_X -plurisubharmonic (ω_X -psh for short) if locally φ is the sum of a plurisubharmonic and a smooth function, and $\omega_X + dd^c\varphi \geq 0$ in the weak sense of currents, where $d = \partial + \bar{\partial}$ and $d^c = \frac{i}{2\pi}(\bar{\partial} - \partial)$ so that $dd^c = \frac{i}{\pi}\partial\bar{\partial}$. We denote by $\text{PSH}(X, \omega_X)$ the set of all ω_X -psh functions on X . Recall from [GZ07, Section 1] that the non-pluripolar Monge-Ampère measure of a function $\varphi \in \text{PSH}(X, \omega_X)$ is a positive measure defined as the increasing limit

$$(\omega_X + dd^c\varphi)^n = \lim_{j \rightarrow +\infty} \mathbf{1}_{\{\varphi > -j\}} (\omega_X + dd^c \max\{\varphi, -j\})^n$$

where the right hand side is defined using Bedford-Taylor intersection theory of bounded psh functions, see [BT76].

One of the main result of [GZ07] states that if μ is a probability measure on X which does not charge pluripolar sets, then there exists a unique (up to a constant) ω_X -psh function φ such that $\int_X (\omega_X + dd^c\varphi)^n = 1$ and

$$(1) \quad \mu = (\omega_X + dd^c\varphi)^n .$$

We denote by $\mathcal{E}(X, \omega_X)$ the set of all ω_X -psh functions whose non-pluripolar Monge-Ampère measure has full mass so that any solution to (1) belongs to $\mathcal{E}(X, \omega_X)$.

In the same paper, Guedj and Zeriahi introduced for any $p > 0$ the subset $\mathcal{E}^p(X, \omega_X)$ of $\mathcal{E}(X, \omega_X)$ consisting of all ω_X -psh functions satisfying the integrability condition $\varphi \in L^p((\omega_X + dd^c\varphi)^n)$. Since ω_X -psh functions are bounded from above it follows that

$$\mathcal{E}^p(X, \omega_X) \subset \mathcal{E}^q(X, \omega_X), \text{ for all } p > q.$$

Observe also that any ω_X -psh function lying in L^∞ belongs to the intersection of all $\mathcal{E}^p(X, \omega_X)$.

We shall say that a probability measure which does not charge pluripolar sets $\mu = (\omega_X + dd^c\varphi)^n$ is a Monge-Ampère measure having Hölder, continuous, L^∞ or \mathcal{E}^p potential for some $p > 0$ whenever φ is Hölder, continuous, L^∞ or belongs to the energy class $\mathcal{E}^p(X, \omega_X)$ respectively.

Let us now consider any dominant meromorphic map $f : X \dashrightarrow Y$ where (Y, ω_Y) is also a compact Kähler manifold of volume 1, and denote by m its complex dimension. Let Γ be a resolution of singularities of the graph of f . We obtain two surjective holomorphic

maps $\pi_1 : \Gamma \rightarrow X$ and $\pi_2 : \Gamma \rightarrow Y$ where π_1 is bimeromorphic so that Γ is a modification of a compact Kähler manifold. By Hironaka's Chow lemma, see e.g. [Pet94, Theorem 2.8] we may suppose that π_1 is a composition of blow-ups along smooth centers so that Γ is itself a compact Kähler manifold of complex dimension n . We fix any Kähler form ω_Γ on it.

One defines the push-forward under f of a measure μ not charging pluripolar sets as follows. Since π_1 is a bimerorphism, there exist two closed analytic subsets $R \subset \Gamma$ and $V \subset X$ such that $\pi_1 : \Gamma \setminus R \rightarrow X \setminus V$ is a biholomorphism. One may thus set $\pi_1^* \mu$ to be the trivial extension through R of $(\pi_1)|_{\Gamma \setminus R}^* \mu$. This measure is again a probability measure which does not charge pluripolar sets.

We then define the probability measure $f_* \mu := (\pi_2)_* \pi_1^* \mu$. We observe that since f is dominant then π_2 is surjective hence the preimage of a pluripolar set in Y by π_2 is again pluripolar. By the preceding discussion, there exists $\psi \in \mathcal{E}(Y, \omega_Y)$ such that $f_* \mu = (\omega_Y + dd^c \psi)^m$.

Our main goal is to discuss the following question.

Problem 1. Suppose μ is a Monge-Ampère measure having Hölder, continuous, L^∞ or \mathcal{E}^p potential. Is it true that $f_* \mu$ is also a Monge-Ampère measure of a potential lying in the same class of regularity?

This problem is hard for Monge-Ampère measures having either continuous or L^∞ potentials since there is no known intrinsic characterization of these measures. For these classes of regularity even the case f is the identity map and $X = Y$ is still open (see for example [DGZ15, Question 15]).

Problem 2. Suppose μ is a probability measure on X not charging pluripolar sets and write $\mu = (\omega + dd^c \varphi)^n = (\omega' + dd^c \varphi')^n$ where ω, ω' are two Kähler forms of volume 1. Is it true that φ is continuous (resp. L^∞) iff φ' is?

An intrinsic characterization of Monge-Ampère measure of Hölder functions is given by [DDG+14], and of functions in the energy class \mathcal{E}^p in [GZ07, Theorem C] so that Problem 2 has a positive answer for these classes of regularity, see [DiN, Theorem 4.1]. Problem 1 remains though quite subtle. If we restrict our attention to the regularity in the \mathcal{E}^p energy classes, then the answer is no in general. Suppose that $\pi : X \rightarrow \mathbb{P}^2$ is the blow-up at some point $p \in \mathbb{P}^2$, and let $E = \pi^{-1}(p)$. It was observed by the first author in [DiN15, Proposition B] that there exists a probability measure $\mu = (\omega_X + dd^c \varphi)^2$ with $\varphi \in \mathcal{E}^1(X, \omega_X)$ but $\pi_* \mu = (\omega_{FS} + dd^c \psi)^2$ with $\psi \notin \mathcal{E}^1(\mathbb{P}^2, \omega_{FS})$, where ω_{FS} denotes the Fubini Study metric on \mathbb{P}^2 and ω_X is a Kähler form.

In this note we answer Problem 1 in two situations. We first treat the case μ is the Monge-Ampère of a Hölder function.

Theorem 1. *Let $f : X \dashrightarrow Y$ be any dominant meromorphic map between two compact Kähler manifolds. If μ is a Monge-Ampère measure having a Hölder potential with Hölder exponent α , then $f_* \mu$ is a Monge-Ampère measure having a Hölder potential with Hölder exponent bounded by $C\alpha^{\dim(X)}$ for some constant $C > 0$ depending only on f .*

We expect that the technics developed in the paper of Kołodziej-Nguyen [KolNg17] in the present volume allows one to extend the previous result to arbitrary compact hermitian manifolds.

Next we treat the case the image of the map has dimension 1.

Theorem 2. *Let $f : X \dashrightarrow Y$ be any dominant meromorphic map from a compact Kähler manifold to a compact Riemann surface. If μ is a Monge-Ampère measure having a Hölder, C^0 , L^∞ , \mathcal{E}^p potential respectively, then $f_*\mu$ is a Monge-Ampère measure having a potential lying in the same regularity class.*

Motivations for studying this question come from the analysis of degenerating measures on families of projective manifolds developed in [F16]. Let us briefly recall the setting of that paper. Let \mathcal{X} be a smooth connected complex manifold of dimension $n + 1$, and $\pi : \mathcal{X} \rightarrow \mathbb{D}$ be a flat proper analytic map over the unit disk which is a submersion over the punctured disk and has connected fibers. We assume that \mathcal{X} is Kähler so that each fiber $X_t = \pi^{-1}(t)$ is also Kähler.

A tame family of Monge-Ampère measures is by definition a family of positive measures $\{\mu_t\}_{t \in \mathbb{D}}$ each supported on X_t that can be written under the form

$$\mu_t = p_*(T|_{X_t}^n),$$

where T is a positive closed $(1, 1)$ -current having local Hölder continuous potentials and defined on a complex manifold \mathcal{X}' which admits a proper bimeromorphic morphism $p : \mathcal{X}' \rightarrow \mathcal{X}$ which is an isomorphism over $X := \pi^{-1}(\mathbb{D}^*)$. It follows from [De93, Corollary 1.6] that the family of measures $\mu'_t := T|_{X_t}^n$ in \mathcal{X}' is continuous so that μ'_t converges to a positive measure μ'_0 supported on X'_0 as $t \rightarrow 0$. It follows that the convergence $\lim_{t \rightarrow 0} \mu_t = \mu_0$ also holds in \mathcal{X} .

As a corollary of the previous results we show the limiting measure μ_0 is of a very special kind:

Corollary 3. *Let $\{\mu_t\}_{t \in \mathbb{D}}$ be any tame family of Monge-Ampère measures, so that $\mu_t \rightarrow \mu_0$ as $t \rightarrow 0$.*

Then there exist a finite collection of closed subvarieties $\{V_i\}_{i=0, \dots, N}$ of X_0 and for each index i a positive measure ν_i supported on V_i such that

$$\mu_0 = \sum_{i=1}^N \nu_i$$

and ν_i is a Monge-Ampère measure on V_i having a Hölder potential.

In the previous statement, it may happen that V_i is singular, in which case it is understood that the pull-back of ν_i to a (Kähler) resolution of V_i is a Monge-Ampère measure having a Hölder continuous potential.

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1. IMAGES OF MONGE-AMPÈRE MEASURES HAVING A HÖLDER POTENTIAL: PROOF OF THEOREM 1

As already mentioned, a dominant meromorphic map $f : X \dashrightarrow Y$ can be decomposed as $f = \pi_2 \circ \pi_1^{-1}$, where $\pi_1 : \Gamma \rightarrow X$ is holomorphic and bimeromorphic and $\pi_2 : \Gamma \rightarrow Y$ is a surjective holomorphic map. Recall that one can assume Γ to be Kähler, and that $f_*\mu := (\pi_2)_*\pi_1^*\mu$.

We first claim that if μ is the Monge-Ampère of a Hölder continuous function then $\pi_1^*\mu$ too. Let $\varphi \in \text{PSH}(X, \omega_X)$ be the Hölder potential such that $\mu = (\omega_X + dd^c\varphi)^n$. It then follows from Bedford and Taylor theory that $\pi_1^*\mu = (\pi_1^*\omega_X + dd^c\pi_1^*\varphi)^n$. Since $\pi_1^*\omega_X$ is a semipositive smooth form, there exists a positive constant $C > 0$ such that

$\pi_1^* \mu \leq (C\omega_\Gamma + dd^c \pi^* \varphi)^n$ where ω_Γ is a Kähler form on Γ , and [DDG+14, Theorem 4.3] implies that $\tilde{\mu} := \pi_1^* \mu$ is the Monge-Ampère measure of a Hölder continuous $C\omega_\Gamma$ -psh function. This proves the claim. We are then left to prove that $(\pi_2)_* \tilde{\mu}$ is the Monge-Ampère measure of a Hölder potential. This will be done in Lemma 7.

We first show that the push-forward of a smooth volume form has density in $L^{1+\varepsilon}$, for some constant $\varepsilon > 0$ depending only on f .

Proposition 4. *Let $f: X \rightarrow Y$ be a surjective holomorphic map. Then $f_* \omega_X^n = g \omega_Y^m$ with $g \in L^{1+\varepsilon}(\omega_Y^n)$, for some $\varepsilon > 0$.*

This result is basically [ST12, Proposition 3.2] (see also [Tosatti, Section 2]). We give nevertheless a detailed proof for reader's convenience. Pick any coherent ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$, and denote by $V(\mathcal{I}) = \text{supp}(\mathcal{O}_X/\mathcal{I})$ the closed analytic subvariety of X associated to \mathcal{I} . Let $\{U_i\}_{i=1}^N$ be a finite open covering of X by balls and $\{V_i\}_i$ be a subcovering such that $\overline{V}_i \subset U_i$. The analytic sheaf \mathcal{I} is globally generated on each U_i so that we can find holomorphic functions such that $\mathcal{I}|_{U_i} = (h_1^{(i)}, \dots, h_k^{(i)}) \cdot \mathcal{O}_{U_i}$. Let $\{\rho_i\}$ be a partition of unity subordinate to \overline{V}_i . We then define

$$(2) \quad \Phi_{\mathcal{I}} := \sum_{i=1}^N \rho_i \left(\sum_{j=1}^k |h_j^{(i)}|^2 \right).$$

Then $\Phi_{\mathcal{I}}: X \rightarrow \mathbb{R}_+$ is a smooth function which vanishes exactly on $V(\mathcal{I})$. Observe that if $\Phi_{\mathcal{I}}$ and $\Phi'_{\mathcal{I}}$ are defined using two different coverings, then there exists $C > 0$ such that

$$\frac{1}{C} \Phi'_{\mathcal{I}} \leq \Phi_{\mathcal{I}} \leq C \Phi'_{\mathcal{I}}.$$

In the sequel we shall abuse notation and not write the dependence of $\Phi_{\mathcal{I}}$ in terms of the local generators of the ideal sheaf. The logarithm of the obtained function is then well-defined up to a bounded function so that all statements in the next Lemma make sense.

Lemma 5. *Let $\mathcal{I}, \mathcal{J} \subseteq \mathcal{O}_X$ be two coherent ideal sheafs. The followings hold:*

- (i) *there exists $\varepsilon > 0$ such that $|\Phi_{\mathcal{I}}|^{-\varepsilon} \in L^1(X)$;*
- (ii) *if $\mathcal{I} \subseteq \mathcal{J}$ then $\Phi_{\mathcal{J}} \geq c \Phi_{\mathcal{I}}$ for some positive $c > 0$;*
- (iii) *if $V(\mathcal{J}) \subseteq V(\mathcal{I})$ then there exists $c, \theta > 0$ such that $\Phi_{\mathcal{J}} \geq c \Phi_{\mathcal{I}}^\theta$;*
- (iv) *given $f: X \rightarrow Y$ a holomorphic surjective map and a coherent ideal sheaf $\mathcal{J} \subseteq \mathcal{O}_Y$, then $\Phi_{f^* \mathcal{J}} = \Phi_{\mathcal{J}} \circ f$ (for a suitable choice of local generators of \mathcal{J} and $f^* \mathcal{J}$).*

Proof. Using a resolution of singularities of \mathcal{I} , one sees that the statement in (i) reduces to show that $|z_1|^{-\varepsilon}$ is locally integrable for some $\varepsilon > 0$, and this is the case if we choose ε small enough. The statements in (ii) and (iv) follow straightforward from the definition in (2). The statement in (iii) is a consequence of Łojasiewicz theorem, see e.g. [LJTR, Theorem 7.2]. \square

Lemma 6. *Let $f: X \rightarrow Y$ be a holomorphic surjective map and let $\mathcal{I} \subseteq \mathcal{O}_X$ be a coherent ideal sheaf. Then there exists a coherent ideal sheaf $\mathcal{J} \subseteq \mathcal{O}_Y$, and constants $c, \theta > 0$ such that for any $y \in Y$ we have*

$$\inf_{x \in f^{-1}(y)} \Phi_{\mathcal{I}} \geq c \Phi_{\mathcal{J}}^\theta$$

Proof. Let $\mathcal{I} \subseteq \mathcal{O}_Y$ be the coherent ideal sheaf of holomorphic functions vanishing on the set $f(V(\mathcal{I}))$ which is analytic since f is proper. Observe that $V(f^*\mathcal{I}) = f^{-1}(V(\mathcal{I})) \supset V(\mathcal{I})$, so that Lemma 5 ((iii) and (iv)) insure that there exist $c, \theta > 0$ such that

$$\Phi_{\mathcal{I}} \geq c\Phi_{f^*\mathcal{I}}^\theta = c(\Phi_{\mathcal{I}} \circ f)^\theta.$$

Hence the conclusion. \square

Proof of Proposition 4. Recall that Sard's theorem implies the existence of a closed subvariety $S \subsetneq Y$ such that $f: X \setminus f^{-1}(S) \rightarrow Y \setminus S$ is a submersion.

We first prove that $f_*\omega_X^n$ is absolutely continuous w.r.t. ω_Y^m . We need to check that $\omega_Y^m(E) = 0$ implies $f_*\omega_X^n(E) = 0$ for any Borel subset $E \subset Y$. As S and $f^{-1}(S)$ have volume zero one may assume that f is a submersion in which case the claim follows from Fubini's theorem.

Radon-Nikodym theorem now guarantees that $f_*\omega_X^n = g\omega_Y^m$ for some $0 \leq g \in L^1(Y)$. We want to show that the integral

$$\int_Y g^{1+\varepsilon}\omega_Y^m = \int_Y g^\varepsilon f_*\omega_X^n = \int_X (f^*g)^\varepsilon \omega_X^n$$

is finite for some $\varepsilon > 0$ small enough. Consider the smooth function $\phi(x) := \frac{f^*\omega_Y^m \wedge \omega_X^{n-m}}{\omega_X^n}(x)$, and set $\tilde{\phi}(y) := \inf_{x \in f^{-1}(y)} \phi(x)$ so that $\phi \geq f^*\tilde{\phi}$. We claim that for any $y \in Y$

$$(3) \quad g(y) \leq \frac{1}{\tilde{\phi}(y)}.$$

Let χ be a test function (i.e. a non negative smooth function) on Y , then

$$\begin{aligned} \int_Y \chi g \omega_Y^m &= \int_X f^*\chi \omega_X^n = \int_X \frac{f^*\chi}{\phi} f^*\omega_Y^m \wedge \omega_X^{n-m} \\ &\leq \int_X f^*\left(\frac{\chi}{\phi}\omega_Y^m\right) \wedge \omega_X^{n-m} \\ &\stackrel{\text{Fubini}}{=} C(f) \int_Y \frac{\chi}{\phi} \omega_Y^m \leq \int_Y \frac{\chi}{\phi} \omega_Y^m \end{aligned}$$

where $C(f)$ is the volume of a fiber over a generic point $y \in Y$, i.e. $C(f) = \int_{f^{-1}(y)} \omega_X^{n-m}$. The last inequality follows from the fact $C(f) \leq 1$ since $\int_X \omega_X^n = 1$. The claim is thus proved. Lemma 5 (i) and (iv) combined with Lemma 6 then insure that there exists $\varepsilon > 0$ such that $(f^*g)^\varepsilon \in L^1(\omega_X^n)$. \square

Theorem 1 is reduced to the following result which relies in an essential way on Proposition 4.

Proposition 7. *Suppose $f: X \rightarrow Y$ is a surjective holomorphic map between compact Kähler manifolds. If μ is a positive measure on X with Hölder continuous potentials, then $f_*\mu$ is a positive measure on Y with Hölder potentials.*

Observe that by multiplying ω_X by a suitable positive constant we may assume that $f^*\omega_Y \leq \omega_X$. The volume normalization is no longer satisfied but a positive multiple of μ is still the Monge-Ampère measure of a ω_X -psh Hölder continuous function. Write $f_*\mu = (\omega_Y + dd^c\psi)^m$ with $\psi \in \text{PSH}(Y, \omega_Y)$.

We claim that there exists $C > 0$, and $\varepsilon > 0$ such that for all $u \in \text{PSH}(Y, \omega_Y)$ with $\int_X u \omega_X^n = 0$

$$(4) \quad \int_Y \exp(-\varepsilon u) d(f_*\mu) \leq C .$$

Indeed, for any $u \in \text{PSH}(Y, \omega_Y)$ we have that

$$\int_Y e^{-\varepsilon u} d(f_*\mu) = \int_X e^{-\varepsilon(u \circ f)} d\mu .$$

Now the integral $\int_X e^{-\varepsilon(u \circ f)} d\mu$ is uniformly bounded by [DNS10, Theorem 1.1] since:

- μ has Hölder continuous potentials;
- $f^*\omega_Y \leq \omega_X$ hence $u \circ f \in \text{PSH}(X, \omega_X)$;
- and the set of functions in $\text{PSH}(X, \omega_X)$ such that $\int_X u \omega_X^n = 0$ is compact by [GZ05, Proposition 2.6].

This proves our claim. Using the terminology of [DN14] this means that $f_*\mu$ is *moderate*. It is worth mentioning that if [DGZ15, Question 16] holds true then the conclusion of Proposition 7 would follow immediately since any moderate measure would have a Hölder continuous potential. To get around this problem we use the characterization of measures with Hölder potentials given by Dinh and Nguyen.

Proof of Lemma 7. By [DN14, Lemma 3.3], $f_*\mu$ is the Monge-Ampère measure of a Hölder potential if and only if there exist $\tilde{c} > 1$ and $\tilde{\beta} \in (0, 1)$ such that

$$(5) \quad \int_Y |u - v| df_*\mu \leq \tilde{c} \max \left(\|u - v\|_{L^1(\omega_Y^n)}, \|u - v\|_{L^1(\omega_Y^n)}^{\tilde{\beta}} \right)$$

for all $u, v \in \text{PSH}(Y, \omega_Y)$. By assumption on μ we know there exist $c > 1$ and $\beta \in (0, 1)$ such that $\int_Y |u - v| df_*\mu = \int_X |f^*u - f^*v| d\mu$, and

$$(6) \quad \int_X |f^*u - f^*v| d\mu \leq c \max \left(\|f^*u - f^*v\|_{L^1(\omega_X^n)}, \|f^*u - f^*v\|_{L^1(\omega_X^n)}^\beta \right) .$$

Also, Proposition 4 gives

$$(7) \quad \int_X |f^*u - f^*v| \omega_X^n = \int_Y |u - v| g \omega_Y^n \leq \|g\|_{L^{1+\varepsilon}(\omega_Y^n)} \|u - v\|_{L^p(\omega_Y^n)}$$

where p is the conjugate exponent of $1 + \varepsilon$. Set $C_g := \|g\|_{L^{1+\varepsilon}(\omega_Y^n)} < +\infty$. Up to replace C_g with $C_g + 1$ we can assume that $C_g \geq 1$.

Denote by $m_u := \int_Y u \omega_Y^n$ and observe that $u' := u - m_u$, $v' := v - m_v$ satisfy $\int_X u' \omega_X^n = 0 = \int_X v' \omega_X^n$. Then the triangle inequality gives

$$(8) \quad \begin{aligned} \|u - v\|_{L^p(\omega_Y^n)} &= \left(\int_Y |(u' - v') + (m_u - m_v)|^p \omega_Y^n \right)^{1/p} \\ &\leq \|u' - v'\|_{L^p(\omega_Y^n)} + |m_u - m_v| \\ &\leq \|u' - v'\|_{L^p(\omega_Y^n)} + \|u - v\|_{L^1(\omega_Y^n)}. \end{aligned}$$

At this point, we make use of [DN14, Proposition 3.2] (that holds for normalized potentials) to replace the L^p -norm with the L^1 -norm. We then infer the existence of a constant $c' > 1$ such that

$$\|u' - v'\|_{L^p(\omega_Y^n)} \leq c' \max(1, -\log \|u' - v'\|_{L^1(\omega_Y^n)})^{\frac{p-1}{p}} \|u' - v'\|_{L^1(\omega_Y^n)}^{\frac{1}{p}} .$$

When $t := \|u' - v'\|_{L^1(\omega_Y^n)} \geq 1/e$ we clearly have

$$\|u' - v'\|_{L^p(\omega_Y^n)} \leq c' \|u' - v'\|_{L^1(\omega_Y^n)}^{\frac{1}{p}},$$

whereas for any integer $N \in \mathbb{N}^*$, there exists a constant $c_N > 0$ such that $-\log t \leq c_N t^{-1/N}$ when $t \leq 1/e$, hence

$$\|u' - v'\|_{L^p(\omega_Y^n)} \leq c'' \|u' - v'\|_{L^1(\omega_Y^n)}^{\frac{1}{p}(1 - \frac{p-1}{N})}.$$

As $\|u' - v'\|_{L^1(\omega_Y^n)} \leq 2\|u - v\|_{L^1(\omega_Y^n)}$, combining (6), (7) and (8) we get

$$\|f^*u - f^*v\|_{L^1(\mu)} \leq C \max \left(\|u - v\|_{L^1(\omega_Y^n)}^{\tilde{\beta}}, \|u - v\|_{L^1(\omega_Y^n)} \right),$$

with $\tilde{\beta} = \frac{\beta}{p} \left(1 - \frac{p-1}{N}\right)$. By [DN14, Lemma 3.3] $f_*\mu = (\omega_Y + dd^c\psi)^n$ where ψ is a Hölder continuous function.

To get a bound on the Hölder regularity of ψ , one argues as follows. First if $\mu = (\omega + dd^c\varphi)^n$ with φ a α -Hölder potential, and $\pi: \Gamma \rightarrow X$ is a proper modification, then $\pi^*\mu$ is dominated by a Monge-Ampère measure with α -Hölder potential, and [DDG+14, Proposition 3.3 (ii)] is satisfied with $b = 2\alpha/(\alpha + 2n)$ by [DDG+14, Theorem 4.3 (iii)]. Hence, following the proof of [DDG+14, Theorem], we see that $\pi^*\mu$ is a Monge-Ampère measure of a α_1 -Hölder continuous potential with $\alpha_1 < b/(n + 1)$ (see Remark below for more details about the latter statement).

By [DN14, Proposition 4.1], (6) holds with $\beta = \alpha_1^n/(2 + \alpha_1^n)$, and (5) is then satisfied for any $\tilde{\beta} < \beta/p$ so that $f_*\mu$ is a Monge-Ampère measure with $\tilde{\alpha}$ -Hölder potential for any $\tilde{\alpha} < 2\tilde{\beta}/(m + 1)$, see the discussion on [DN14, p.83]. Combining all these estimates we see that any

$$\tilde{\alpha} < \frac{\alpha^n}{p(m + 1)(\alpha/2 + n)^n(n + 1)^n}$$

works where p is the conjugate of the larger constant $\varepsilon > 0$ for which Proposition 4 holds. \square

Remark. We borrow notations from the proof of [DDG+14, Theorem A]. Fix $\alpha_1 < b/(n + 1)$ and choose $\varepsilon > 0$ such that $\alpha_1 \leq \alpha \leq \alpha_0 \leq b - \alpha_0(n + \varepsilon)$. By the previous arguments we know that condition (ii) in [DDG+14, Proposition 3.3] holds, i.e. for any $\phi \in \text{PSH}(\Gamma, \omega_\Gamma)$, we have $\|\rho_\delta\phi - \phi\|_{L^1(\pi^*\mu)} = O(\delta^b)$, where $b = 2\alpha/(\alpha + 2n)$. In particular, this gives

$$\pi^*\mu(E_0) \leq c_1\delta^{b-\alpha_0}.$$

Let $g \in L^1(\pi^*\mu)$ be defined as $g = 0$ on E_0 and $g = c$ on $\Gamma \setminus E_0$ where c is a positive constant such that $\pi^*\mu(\Gamma) = \int_\Gamma g d(\pi^*\mu)$. An easy computation gives that $c = \pi^*\mu(\Gamma)/\pi^*\mu(\Gamma \setminus E_0)$. Let $v \in \text{PSH}(\Gamma, \omega_\Gamma)$ be the bounded solution of the Monge-Ampère equation $(\omega_\Gamma + dd^c v)^n = g \cdot \pi^*\mu$. Observe that

$$\|1 - g\|_{L^1(\pi^*\mu)} = \int_{E_0} d\pi^*\mu + \int_{\Gamma \setminus E_0} |1 - c| d\pi^*\mu = 2 \int_{E_0} d\pi^*\mu \leq 2c_1\delta^{b-\alpha_0}.$$

Since $\pi^*\mu = (\omega_\Gamma + dd^c\tilde{\varphi})^n$ satisfies the $\mathcal{H}(\infty)$ property we can still apply [DZ, Theorem 1.1] and get

$$\|\tilde{\varphi} - v\|_{L^\infty} \leq c_3\delta^{\frac{b-\alpha_0}{n+\varepsilon}}.$$

The exact same arguments as in [DDG+14, Theorem A] then insure that the Hölder exponent of $\tilde{\varphi}$ is α_1 .

2. OVER A ONE-DIMENSIONAL BASE: PROOF OF THEOREM 2

In this section we treat Problem 1 in the case the base is a Riemann surface.

We start with the case of a surjective holomorphic map $f: X \rightarrow Y$ from a Kähler compact manifold to a compact Riemann surface.

Let $\mu = (\omega_X + dd^c\varphi)^n$ be a Monge-Ampère measure of a continuous ω_X -psh function φ . Suppose v_k, v is a family of ω_X -psh functions such that $v_k \rightarrow v$ in L^1 , then

$$\begin{aligned} \int_X v_k d\mu &= \int_X v_k (\omega_X + dd^c\varphi)^n \\ &= \int_X v_k \omega_X^n + \sum_{j=0}^{n-1} \int_X \varphi dd^c v_k \wedge \omega_X^j \wedge (\omega_X + dd^c\varphi)^{n-j-1} \rightarrow \int_X v d\mu \end{aligned}$$

by [De93, Corollary 1.6 (a)]. Observe that in the last equality we used the fact that

$$(\omega_X + dd^c\varphi)^n - \omega_X^n = \sum_{j=0}^{n-1} dd^c\varphi \wedge \omega_X^j \wedge (\omega_X + dd^c\varphi)^{n-j-1}$$

and Stokes' theorem.

Normalize the Kähler form on Y such that $\int \omega_Y = 1$, and pick any sequence $y_k \rightarrow y_\infty \in Y$. Let w_k be the solutions of the equations $\Delta w_k = \delta_{y_k} - \omega_Y$ with $\sup w_k = 0$ so that $w_k(y) - \log|y - y_k|$ is continuous in local coordinates near y_k . Write $f_*\mu = \omega_Y + dd^c\psi$ so that

$$\int_Y w_k d(f_*\mu) = \int_Y w_k \omega_Y + \int_Y \psi \Delta w_k = \psi(y_k) + \int_Y (w_k - \psi)\omega_Y.$$

Since $w_k \rightarrow w_\infty$ in L^p_{loc} for all $p < \infty$, Proposition 4 implies that $f^*w_k \rightarrow f^*w_\infty$ in the L^1 topology, so that the argument above gives $\int_Y w_k d(f_*\mu) = \int_X f^*w_k d\mu \rightarrow \int_X f^*w_\infty d\mu = \int_Y w_\infty d(f_*\mu)$. We then conclude that $\psi(y_k) \rightarrow \psi(y_\infty)$. Hence ψ is continuous.

Suppose then that μ is locally the Monge-Ampère of a bounded psh function, and pick any subharmonic function u defined in a neighborhood of a point $y \in Y$. Then f^*u is again psh in a neighborhood of $f^{-1}(y)$, and the standard Chern-Levine-Nirenberg inequalities imply that $f^*u \in L^1(\mu)$ so that $u \in L^1(f_*\mu)$ with a norm depending only on the L^1 -norm of u . It follows that $f_*\mu$ is locally the laplacian of a bounded subharmonic function.

Finally, assume $\mu = (\omega_X + dd^c\varphi)^n$ for some $\varphi \in \mathcal{E}^p(X, \omega_X)$. By [GZ07, Theorem C] this is equivalent to have that $\mathcal{E}^p(X, \omega_X) \subset L^p(\mu)$. Write as usual $f_*\mu = (\omega_Y + dd^c\psi)$ with $\psi \in \mathcal{E}(Y, \omega_Y)$.

We claim that $u \in \mathcal{E}^p(Y, \omega_Y)$ implies $f^*u \in \mathcal{E}^p(X, \omega_X)$. Indeed, without loss of generality we can assume that $\Omega := \omega_X - f^*\omega_Y$ is a Kähler form and by the multilinearity of the non-pluripolar product we have

$$\begin{aligned} \int_X |f^*u|^p (\omega_X + dd^c f^*u)^n &= \int_X |f^*u|^p (f^*\omega_Y + \Omega + dd^c f^*u)^n \\ &= \int_X |f^*u|^p (\Omega^n + (f^*\omega_Y + dd^c f^*u) \wedge \Omega^{n-1}) \end{aligned}$$

where the last identity follows from the fact that $(f^*\omega_Y + dd^c f^*u)^j = 0$ for $j > 1$. The term $\int_X |f^*u|^p \Omega^n$ is bounded thanks to the integrability properties of quasi-plurisubharmonic

functions w.r.t. volume forms [GZ17, Theorem 1.47]; while the term

$$\int_X |f^*u|^p (f^*\omega_Y + dd^c f^*u) \wedge \Omega^{n-1} = C(f) \int_Y |u|^p (\omega_Y + dd^c u)$$

is finite since $u \in \mathcal{E}^p(Y, \omega_Y)$. This proves the claim.

Now, given any $u \in \mathcal{E}^p(Y, \omega_Y)$ we have

$$\int_Y |u|^p d(f_*\mu) = \int_X |f^*u|^p d\mu < +\infty$$

since $f^*u \in \mathcal{E}^p(X, \omega_X) \subset L^p(\mu)$. The conclusion follows from [GZ07, Theorem C].

Consider now any dominant meromorphic map $f: X \dashrightarrow Y$ from a Kähler compact manifold to a compact Riemann surface. As above we decompose f such that $f_*\mu = (\pi_2)_*\pi_1^*\mu$ for any positive measure μ on X .

Assume that μ has continuous potentials. If we write $\mu = (\omega_X + dd^c\varphi)^n$ then $\pi_1^*\mu = (\pi_1^*\omega_X + dd^c\varphi \circ \pi)^n \leq (C\omega_\Gamma + dd^c\varphi \circ \pi)^n := \hat{\mu}$ where $\hat{\mu}$ has a continuous potential. This implies $f_*\mu \leq (\pi_2)_*\hat{\mu}$. Observe that by the previous arguments the measure $(\pi_2)_*\hat{\mu}$ has continuous potential. It follows that locally $f_*\mu = \Delta v \leq \Delta u$ where u, v are subharmonic functions. It follows that v is the sum of a continuous function and the opposite of a subharmonic (hence u.s.c.) function. Since it is also u.s.c we conclude to its continuity.

When μ has bounded potentials, the same argument applies noting that subharmonic functions are always bounded from above which implies v to be bounded.

Finally, we consider the case where μ is the Monge-Ampère measure of $\varphi \in \mathcal{E}^p(X, \omega_X)$. We first observe that given $v \in \mathcal{E}^p(\Gamma, \omega_\Gamma)$ we have $(\pi_1)_*v \in \mathcal{E}^p(X, \omega_X)$. Indeed,

$$\int_X |v \circ \pi^{-1}|^p (\omega_X + dd^c v \circ \pi^{-1})^n = \int_\Gamma |v|^p (\pi_1^*\omega_X + dd^c v)^n \leq \int_\Gamma |v|^p (C\omega_\Gamma + dd^c v)^n < +\infty.$$

This and the previous arguments give that if $u \in \mathcal{E}^p(Y, \omega_Y)$ then $f_*u = (\pi_1)_*\pi_2^*u \in \mathcal{E}^p(X, \omega_X)$, hence

$$\int_Y |u|^p df_*\mu = \int_X |u \circ f|^p d\mu < +\infty.$$

It follows from [GZ07, Theorem C] that $f_*\mu$ is the Monge-Ampère measure of a function in $\mathcal{E}^p(Y, \omega_Y)$.

3. THE CASE OF SUBMERSIONS

In this section we let $(X, \omega_X), (Y, \omega_Y)$ be two compact Kähler manifolds of dimension n and m , respectively and normalized such that $\int_X \omega_X^n = 1 = \int_Y \omega_Y^m$.

Proposition 8. *Let $f: X \rightarrow Y$ be a submersion. Then, $u \in \mathcal{E}^p(Y, \omega_Y)$ implies $f^*u \in \mathcal{E}^p(X, \omega_X)$. In particular, if a probability measure μ is the Monge-Ampère of a function in \mathcal{E}^p then also $f_*\mu$ has also a potential in \mathcal{E}^p .*

Proof. Since f is a submersion we can assume that there is a finite number of open neighbourhoods U_i such that $X \subset \bigcup_{j=0}^N U_j$, $f|_{U_j}(z, w) = z$ where $z = (z_1, \dots, z_m)$ and $w = (z_{m+1}, \dots, z_n)$. Moreover we can assume that on each U_j we have

$$\omega_X \leq C_j \frac{i}{2} (dz \wedge d\bar{z} + dw \wedge d\bar{w}), \quad \frac{i}{2} dz \wedge d\bar{z} \leq A_j f^*\omega_Y$$

where $A_j, C_j > 1$ and $dz \wedge d\bar{z}, dw \wedge d\bar{w}$ are short notations for $\sum_{j=1}^m dz_j \wedge d\bar{z}_j$ and $\sum_{k=m+1}^n dz_k \wedge d\bar{z}_k$, respectively. We then write

$$\begin{aligned} \int_X |f^*u|^p (\omega_X + dd^c f^*u)^n &\leq \sum_{j=1}^N \int_{U_j} |f^*u|^p \left(C_j \frac{i}{2} dz \wedge d\bar{z} + C_j \frac{i}{2} dw \wedge d\bar{w} + dd^c f^*u \right)^n \\ &\leq \sum_{j=1}^N \int_{U_j} |f^*u|^p \left(A'_j f^* \omega_Y + C_j \frac{i}{2} dw \wedge d\bar{w} + dd^c f^*u \right)^n \\ &= \sum_{j=1}^N \sum_{\ell=0}^n \int_{U_j} |f^*u|^p (A'_j f^* \omega_Y + dd^c f^*u)^\ell \wedge \left(C_j \frac{i}{2} dw \wedge d\bar{w} \right)^{n-\ell} \\ &= \sum_{j=1}^N \int_{U_j} |f^*u|^p (A'_j f^* \omega_Y + dd^c f^*u)^m \wedge \left(C_j \frac{i}{2} dw \wedge d\bar{w} \right)^{n-m}. \end{aligned}$$

The above integral is then finite because by assumption $u \in \mathcal{E}^p(Y, A\omega_Y)$ for any $A \geq 1$.

The last statement follows from the same arguments in the last part of the proof in the previous section. \square

4. TAME FAMILIES OF MONGE-AMPÈRE MEASURES: PROOF OF COROLLARY 3

Recall the setting from the introduction: \mathcal{X} is a smooth connected complex manifold of dimension $n + 1$, and $\pi: \mathcal{X} \rightarrow \mathbb{D}$ is a flat proper analytic map over the unit disk which is a submersion over the punctured disk and has connected fibers. We let $p: \mathcal{X}' \rightarrow \mathcal{X}$ be a proper bi-meromorphic map from a smooth complex manifold \mathcal{X}' which is an isomorphism over $\pi^{-1}(\mathbb{D}^*)$.

We let T be any closed positive $(1, 1)$ -current on \mathcal{X}' admitting local Hölder continuous potentials. Observe that by e.g. [De93, Corollary 1.6] we have

$$\mu'_t = dd^c \log |\pi \circ p - t| \wedge T^n \rightarrow \mu'_0 := dd^c \log |\pi \circ p| \wedge T^n.$$

Let us now analyze the structure of the positive measure $\mu_0 := p_* \mu'_0$. First observe that μ'_0 can be decomposed as a finite sum of positive measures $\mu'_E := (T|_E)^n$ where the sum is taken over all irreducible components E of \mathcal{X}'_0 . Each of these measures is locally the Monge-Ampère of a Hölder continuous psh function.

Write $V := p(E)$. Since E is irreducible, V is also an irreducible (possibly singular) subvariety of dimension ℓ . To conclude the proof it remains to show that $p_*(\mu'_E)$ is the Monge-Ampère measure of Hölder continuous function that is locally the sum of a smooth and psh function. More precisely, one needs to show that $p_*(\mu'_E)$ does not charge any proper algebraic subset of V , and given any resolution of singularities $\varpi: V' \rightarrow V$ the pull-back measure $\varpi^*(p_*(\mu'_E))$ can be locally written as $(dd^c u)^\ell$ where u is a Hölder psh function on V' .

This follows from Theorem 1 applied to any resolution of singularities V' of V and to any E' which admits a birational morphism $E' \rightarrow E$ such that the map $E' \rightarrow V'$ induced by p is also a morphism.

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