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Symmetries of the backward heat equation with potential and interest rate models

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Abstract

We compute the isovector algebra of the Hamilton-Jacobi-Bellman equation when the potential belongs to a class that strictly includes quadratic potentials, and then determine a canonical basis for it.

This setting allows us to parametrize canonically the important class of one factor interest rate models. *To cite this article: P.Lescot, H.Quintard, C. R. Acad. Sci. Paris, Ser. I 340 (2005).*

R esum e

Sym etries de l' equation de la chaleur r etrograde avec potentiel et mod eles de taux d'int er et. Nous calculons l'alg ebre des isovecteurs de l' equation de Hamilton-Jacobi-Bellman lorsque le potentiel appartient   une certaine classe qui inclut strictement celle des potentiels quadratiques, et en d eterminons ensuite une base canonique.

Ce cadre nous permet de paramétrer canoniquement l'importante classe des modèles affines de taux d'intérêt à un facteur. *Pour citer cet article : P.Lescot, H.Quintard, C. R. Acad. Sci. Paris, Ser. I 340 (2005).*

1 Introduction

We determine the Lie algebra $\mathcal{H}_{(C,D)}$ of pure isovectors for the backward heat equation

$$\theta^2 \frac{\partial \eta}{\partial t} = -\frac{\theta^4}{2} \frac{\partial^2 \eta}{\partial q^2} + V\eta, \quad (\mathcal{E}_V)$$

or more precisely for the related (HJB) equation (\mathcal{E}'_V) obtained by setting $S = -\theta^2 \ln(\eta)$, with potential $V = \frac{C}{q^2} + Dq^2$; this generalizes previously known results ([7], [8]) corresponding to $C = D = 0$, and to $D = 0, C > 0$. It turns out that, for fixed D , whichever $C \neq 0$, $\mathcal{H}_{C,D}$ is a certain 4-dimensional subalgebra \mathcal{P}_D of $\mathcal{H}_{0,D}$, which is itself 6-dimensional ; furthermore, \mathcal{P}_D and $\mathcal{H}_{0,D}$ possess canonical bases continuous in D and compatible with the inclusions $\mathcal{P}_D \subset \mathcal{H}_{0,D}$.

This potential appears naturally in the study of one-factor interest rate models: each such model can be parametrized by a Bernstein process for V .

The contents of §4 first appeared in the second author's Master's Thesis (Rouen, 2011).

Complete proofs of these results will appear along with other material in [9].

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2 Context and previous results about the heat equation with potential

First we need to recall some definitions and properties of some useful tools. For more details the reader could refer to [5] and [3].

Let E be a vector space of dimension n , and U an open subset of E .

Definition 2.1 *A differential form of degree p , $p \in \mathbb{N}$ on U is a smooth application from U to $\Lambda^p E^*$ where $\Lambda^p E^*$ is the space of linear alternating p -forms. We call $\Omega^p(U)$ the set of those forms and we set $\Omega(U) = \bigoplus_{p=0}^{\infty} \Omega^p(U)$.*

Theorem 2.2 *There is a unique linear application $d : \Omega(U) \rightarrow \Omega(U)$ with the following properties :*

- (1) $d(\Omega^p(U)) \subset \Omega^{p+1}(U)$;
- (2) on $\Omega^0(U)$, d is the differential on functions ;
- (3) if $\alpha, \beta \in \Omega(U)$, $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg(\alpha)} \alpha \wedge d\beta$;
- (4) $d \circ d = 0$.

This application is called the exterior differential.

Definition 2.3 *Let U and V be two open subsets of vector spaces M and N and $f \in C^\infty(U, V)$. The inverse image by f of $\alpha \in \Omega(V)$, noted $f^*\alpha$ is the form on U defined by : $(f^*\alpha)_x = {}^t(T_x f) \cdot \alpha_{f(x)}$, where $T_x f$ is the linear tangent application to f at point x .*

Definition 2.4 *The Lie derivative associated with a vector field X is the linear application $\mathcal{L}_X : \Omega^p(U) \rightarrow \Omega^p(U)$, $\alpha \mapsto \left(\frac{d}{dt}\right)(\phi_t^* \alpha)|_{t=0}$ where ϕ_t is the local one-parameter group associated with X .*

$\phi_t^ \alpha$ is a differential form.*

Theorem 2.5 *The operator \mathcal{L}_X is characterized by the following properties :*

- (1) *if $f \in \mathcal{C}^\infty(U)$, $\mathcal{L}_X f = df(X) = X \cdot f$;*
- (2) *$\mathcal{L}_X \circ d = d \circ \mathcal{L}_X$;*
- (3) *for all α and β differential forms we have $\mathcal{L}_X(\alpha \wedge \beta) = \mathcal{L}_X \alpha \wedge \beta + \alpha \wedge \mathcal{L}_X \beta$ (i.e. \mathcal{L}_X is a derivation of the algebra $\Omega(U)$).*

Now we recall the steps of the Harrisson-Estabrook method to find the symmetries of a PDE system (E) of order p on U . We will consider every partial derivative of order i , $i < p$ as a new variable (and call m the number of those new variables), that gives a certain number of new PDEs. The differential system of all those PDEs and (E) will be called (E'). Then we find a family of differential forms such that their vanishing is equivalent to (E').

We call I the closed ideal spanned by those differential forms. The goal is to find a submanifold of dimension n in the space of dimension $m + n$ called M where the differential forms vanish.

Definition 2.6 *An isovector N of (E) is a vector field on M such that $\mathcal{L}_N(I) \subset I$.*

Theorem 2.7 [3]

- (1) *The isovectors of (E) form a Lie algebra.*
- (2) *This Lie algebra is independent of the new variables up to an isomorphism.*

Let's study the equation (\mathcal{E}_V) with those tools.

The first step is to transform this equation in an Hamilton-Jacobi-Bellman equation. To perform that we proceed as in [8], setting $S = -\gamma^2 \ln(\eta)$. This gives the equation :

$$-\frac{\partial S}{\partial t} + \frac{1}{2} \left(\frac{\partial S}{\partial q} \right)^2 - V - \frac{\gamma^2}{2} \frac{\partial^2 S}{\partial q^2} = 0. \quad (\mathcal{E}'_V)$$

We denote by \mathcal{G}_V the isovector algebra for that equation (see [8], (3.2), p.209).

A typical element N of \mathcal{G}_V is then of the form $N = N^q \frac{\partial}{\partial q} + N^t \frac{\partial}{\partial t} + N^S \frac{\partial}{\partial S} + N^B \frac{\partial}{\partial B} + N^E \frac{\partial}{\partial E}$.

The N^* are dermined by the formulas (3.9') et (3.22)-(3.29) in the proof of Theorem 3.3 in [8] which can be summarized as :

$$\begin{aligned} N^q &= \frac{1}{2}T'_N(t)q + l(t), & N^t &= T_N(t), & N^S &= h(q, t, S), \\ N^B &= \frac{1}{2}T'_N(t)B - \frac{\partial h}{\partial q} + B \frac{\partial h}{\partial S}, & & & & \\ N^E &= -\left(\frac{1}{2}T''_N(t)q + l'(t)\right)B - T'_N(t)E - \frac{\partial h}{\partial t} + E \frac{\partial h}{\partial S}, & & & & \end{aligned} \quad (1)$$

where p , l and ϕ satisfy :

$$\begin{aligned} h(q, t, S) &= \gamma^2 p(q, t) e^{\frac{1}{\gamma^2} S} - \phi(q, t), & \phi(q, t) &= \frac{1}{4}T''_N q^2 + l'q - \sigma(t), \\ \gamma^2 \frac{\partial p}{\partial t} &= -\frac{\gamma^4}{2} \frac{\partial^2 p}{\partial q^2} + pV, & & & & \\ -\frac{\partial \phi}{\partial t} + T_N \frac{\partial V}{\partial t} + \left(\frac{1}{2}T'_N q + l\right) \frac{\partial V}{\partial q} - \frac{\gamma^2}{2} \frac{\partial^2 \phi}{\partial q^2} + T'_N V &= 0. & & & & \end{aligned} \quad (2)$$

We set $\mathcal{H}_V = \{N \in \mathcal{G}_V \mid \frac{\partial N^S}{\partial S} = 0\}$ (for $V \equiv 0$, \mathcal{H}_V is \mathcal{H} from [7]). The elements of \mathcal{H}_V are termed "pure isovectors".

Let $e^{\mu N}$ map (t, q, S, B, E) to $(t_\mu, q_\mu, S_\mu, B_\mu, E_\mu)$, and $\eta_\mu(t_\mu, q_\mu) = e^{-\frac{1}{\gamma^2} S_\mu}$;

then η_μ is a solution of (\mathcal{E}_V) . Setting $e^{\mu \tilde{N}(\mu)} := \eta_\mu$, we obtain :

$$\tilde{N}\eta(t, q) = -N^t \frac{\partial \eta}{\partial t} - N^q \frac{\partial \eta}{\partial q} - N^S \frac{1}{\gamma^2} \eta \quad (3)$$

(see [8], p.216, and, in the context of the Black-Scholes equation, [6]).

Lemma 2.8 \mathcal{H}_V is a Lie subalgebra of \mathcal{G}_V (see [7], p.214) and $N \mapsto -\tilde{N}$ is a Lie algebra morphism, in particular $\tilde{\mathcal{H}}_V := \{\tilde{N} \mid N \in \mathcal{H}_V\}$ is a Lie algebra.

3 Parametrization of a one-factor affine model

In this section, we will show that we one-factor affine model can be parametrized by a Schrödinger process for $V(t, q) = \frac{C}{q^2} + Dq^2$.

First we recall the definition of a Schrödinger process. A very general definition was given in [11] under the name of Bernstein process. For more clarity we shall use the definition from a more recent paper [10] :

Definition 3.1 *A Schrödinger process between two Borelian probability measures μ_0, μ_T on \mathbb{R} associated to a solution η of (\mathcal{E}'_V) is a process $(X_t)_{t \in [0, T]}$ verifying that for $u = \gamma^2 \frac{1}{\eta} \frac{\partial \eta}{\partial q}$:*

- (1) $X_t = X_0 + \int_0^t u(s, X_s) ds + \gamma^2 B_t$ where B_t is a brownian motion ;
- (2) X_0 (resp X_T) has law μ_0 (resp μ_T) ;
- (3) $\mathcal{E}[\int_0^T [|1/2|u(t, X_t)|^2 + V(t, X_t)] dt] < \infty$.

According to theorems 2.2 and 2.4 of [10], if we take μ_0 et μ_T two borelian probability measures on \mathbb{R} , there exists a unique solution η of (\mathcal{E}'_V) and a unique process X that satisfy the previous definition.

We say that $(X_t)_{t \geq 0}$ is a Schrödinger process if for all $T > 0$, $(X_t)_{t \in [0, T]}$ is a Schrödinger process on $[0, T]$ for the law μ_0 of X_0 and the law μ_T of X_T .

An *one-factor affine interest rate model* is characterized by the instantaneous rate $r(t)$, satisfying the following stochastic differential equation :

$$dr(t) = \sqrt{\alpha r(t) + \beta} dw(t) + (\phi - \lambda r(t)) dt \quad (4)$$

under the risk-neutral probability Q ($\alpha = 0$ corresponds to the Vasicek model, and $\beta = 0$ corresponds to the Cox–Ingersoll–Ross model ; cf. [4]).

Assuming $\alpha > 0$, let us set $\tilde{\phi} =_{def} \phi + \frac{\lambda\beta}{\alpha}$ and $\delta =_{def} \frac{4\tilde{\phi}}{\alpha}$, and we also assume that $\tilde{\phi} \geq 0$. Let us then set $X_t = \alpha r(t) + \beta$ and $z(t) = \sqrt{X_t}$.

Theorem 3.2 *Consider the stopping time $T = \inf\{t > 0 | X_t = 0\}$; $T = +\infty$ a.s. for $\delta \geq 2$, and $T < +\infty$ a.s. for $\delta < 2$, and there exists a Schrödinger process $y(t)$ for $\gamma = \frac{\alpha}{2}$ and the potential $V(t, q) = \frac{C}{q^2} + Dq^2$, where $C := \frac{\alpha^2}{8}(\tilde{\phi} - \frac{\alpha}{4})(\tilde{\phi} - \frac{3\alpha}{4}) = \frac{\alpha^4}{128}(\delta - 1)(\delta - 3)$, and $D := \frac{\lambda^2}{8}$ such that $\forall t \in [0, T[$ $z(t) = y(t)$. In particular, for $\delta \geq 2$, z itself is a Bernstein process.*

4 Isoectors in the case $V(t, q) = \frac{C}{q^2} + Dq^2$

Theorem 3.2 motivates us to find the explicit form of the isoectors for the potential : $V(t, q) = \frac{C}{q^2} + Dq^2$.

We shall denote $\mathcal{H}_{(C,D)} = \mathcal{H}_V$, $\mathcal{P}_D = \mathcal{H}_{(0,D)}$, $\tilde{\mathcal{H}}_{(C,D)} = \tilde{\mathcal{H}}_V$ and $\tilde{\mathcal{P}}_D = \tilde{\mathcal{H}}_{(0,D)}$.

Theorem 4.1 *For $C \neq 0$, $\mathcal{H}_{(C,D)} = \mathcal{H}_{(1,D)} \simeq \mathcal{H}_{(1,0)}$ has dimension 4 ; for $C = 0$, $\mathcal{H}_{(C,D)} \simeq \mathcal{H}_{(0,0)}$ has dimension 6. Furthermore these Lie algebras possess canonical bases, continuous in D for fixed C , and compatible with the inclusions $\mathcal{H}_{(C,D)} \subseteq \mathcal{H}_{(0,D)}$.*

Corollary 4.2 *The isoector algebra \mathcal{H}_V associated with V has dimension 6 if and only if $\tilde{\phi} \in \{\frac{\alpha}{4}, \frac{3\alpha}{4}\}$, i.e. $\delta \in \{1, 3\}$; in the opposite case, it has dimension 4.*

In particular the isoector algebra associated with the affine model has dimension 6 if and only if $\delta \in \{1, 3\}$; in the opposite case, it has dimension 4.

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