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# Efficient decoding of random errors for quantum expander codes

**Antoine Grospellier** & Anthony Leverrier & Omar Fawzi

10 Novembre 2017





Alice



Bob

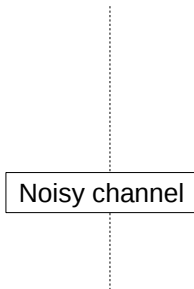
$m$   
k Q-bits

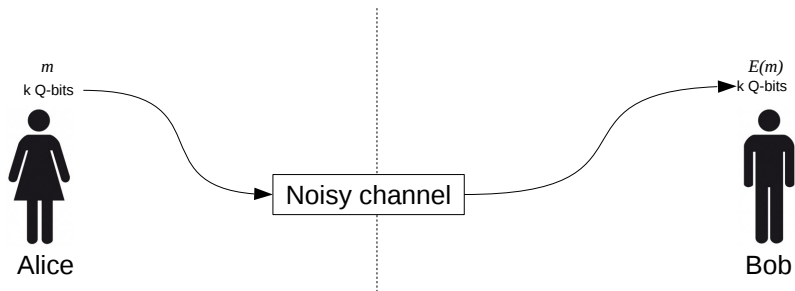


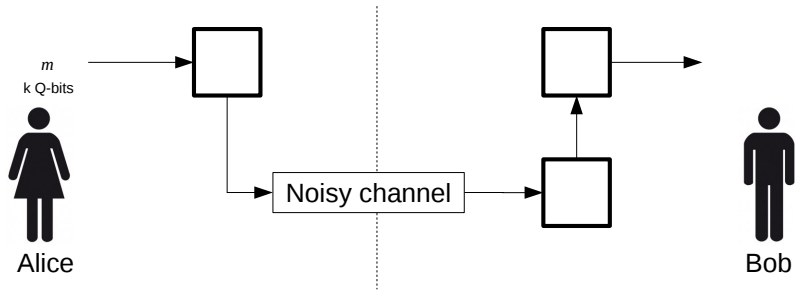
Alice

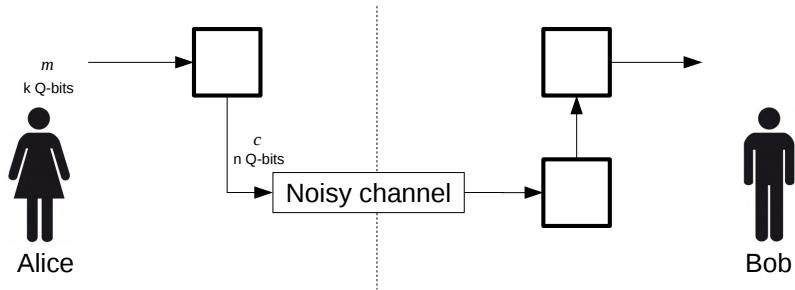


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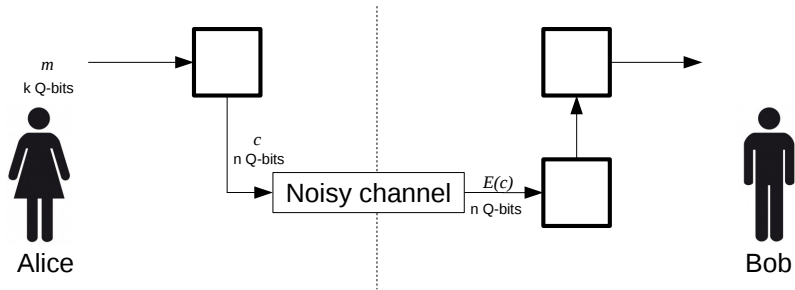


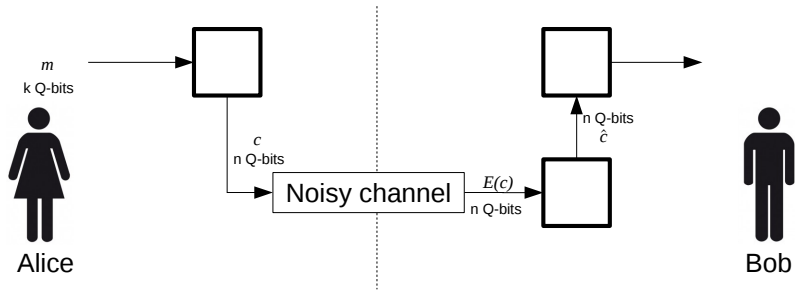


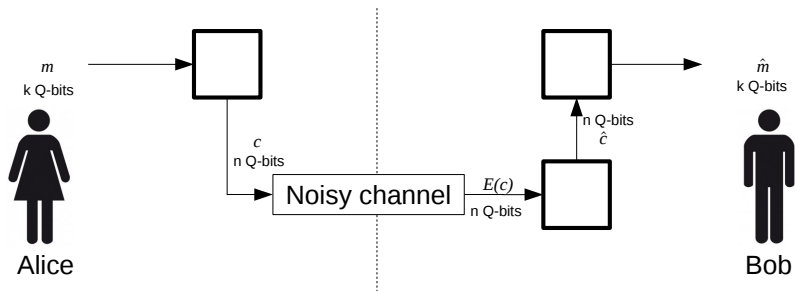












# Content of the talk

The hypergraph product of an expander code :

- is an LDPC quantum code
- has a constant rate
- has a minimal distance :  $d = \Theta(\sqrt{n})$

The decoding algorithm :

- has a capacity of correction :  $\Theta(\sqrt{n})$
- **corrects the error with high probability for the depolarizing channel**

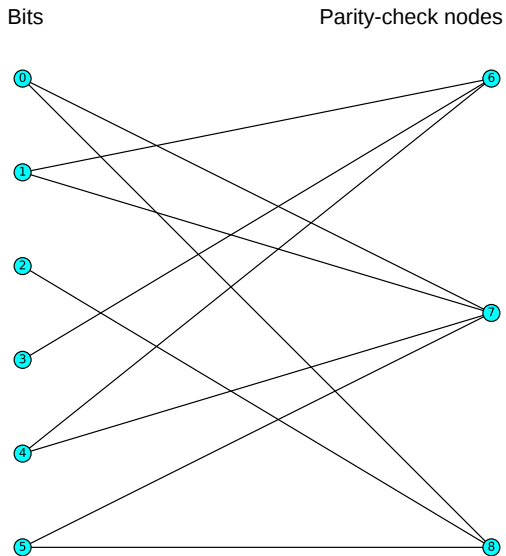
# Content of the talk

- 1 Classical expander codes
- 2 Quantum expander codes
- 3 Our contribution

# Plan

- 1 Classical expander codes
- 2 Quantum expander codes
- 3 Our contribution

# Tanner graph of a code



## Theorem [Sipser & Spielman, '96]

We can construct a good family  $(\mathcal{C}_n)_{n \in \mathbb{N}}$  of  $[n, k, d]$ -error correcting codes. Here "Good" means :

- This family is LDPC
- $k$  and  $d$  are linear in  $n$
- There exists an efficient correcting algorithm

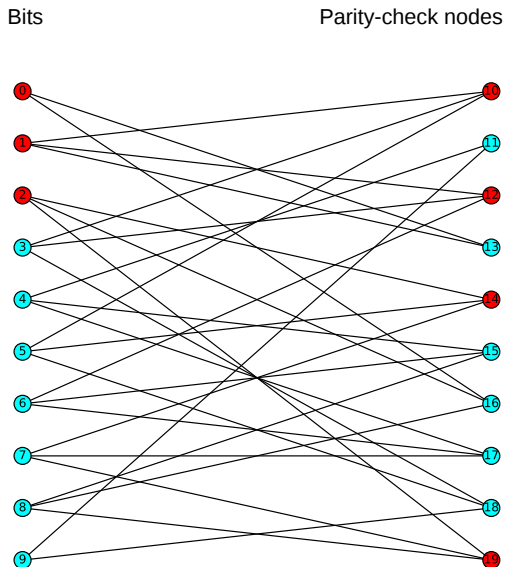
## Remark

Good expander codes can be found efficiently by picking a random biregular graph



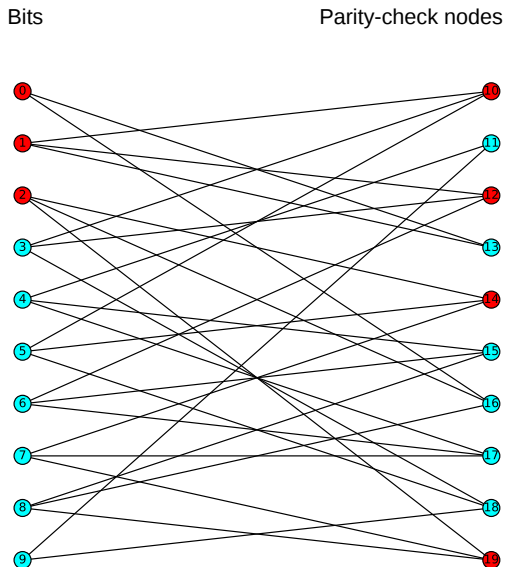
# Decoding algorithm

- Error :  
 $e_0 = \{0, 1, 2\}$
- Unsatisfied check-nodes (syndrome) :  
 $\{10, 12, 14, 19\}$
- Satisfied check-nodes :  
 $\{11, 13, 15, 16, 17, 18\}$

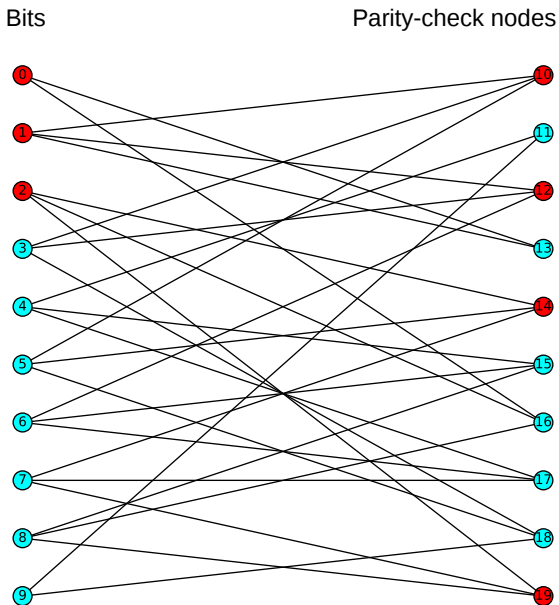


# Decoding algorithm

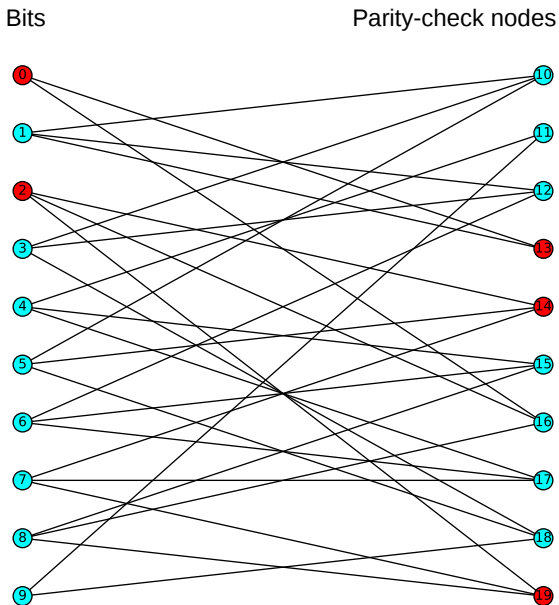
- INPUT : syndrome
- The error  $e_0$  is unknown
- OUTPUT :  $e$   
a set of bits
- GOAL :  $e = e_0$
- The algorithm flips a bit when it decreases the syndrome



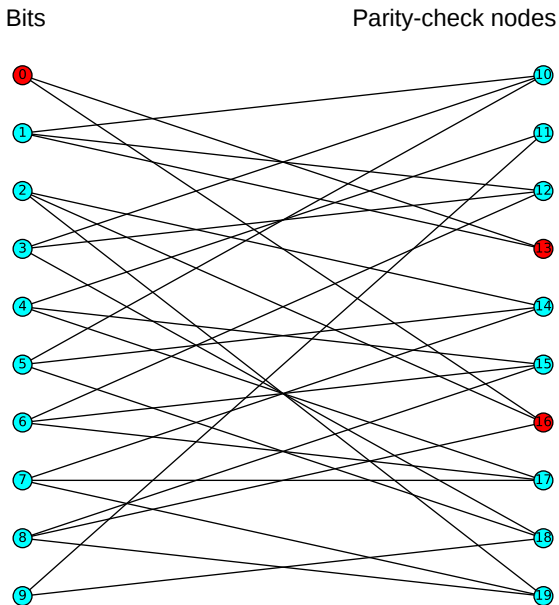
# Decoding algorithm : first example



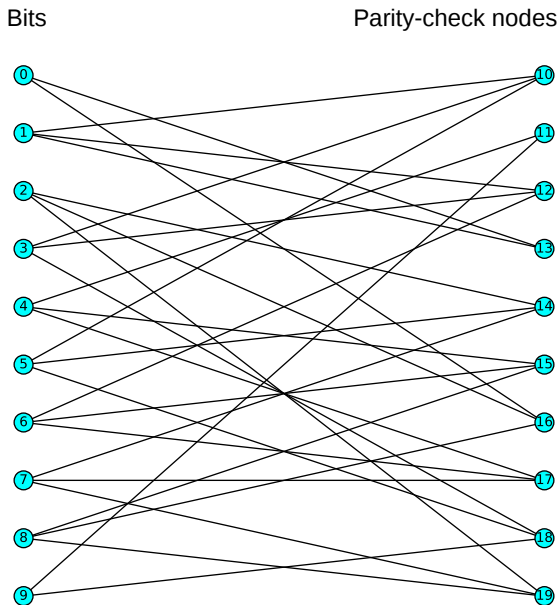
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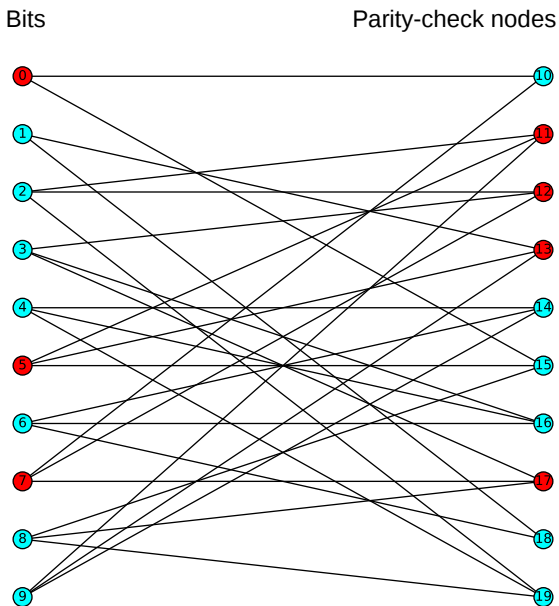
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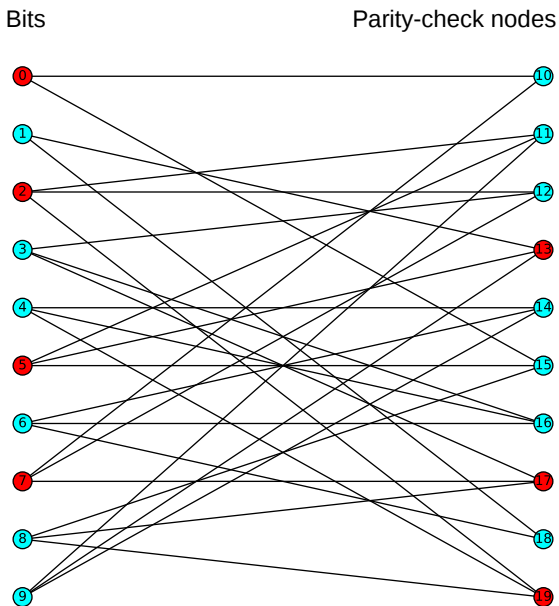
# Decoding algorithm : first example



# Decoding algorithm : second example

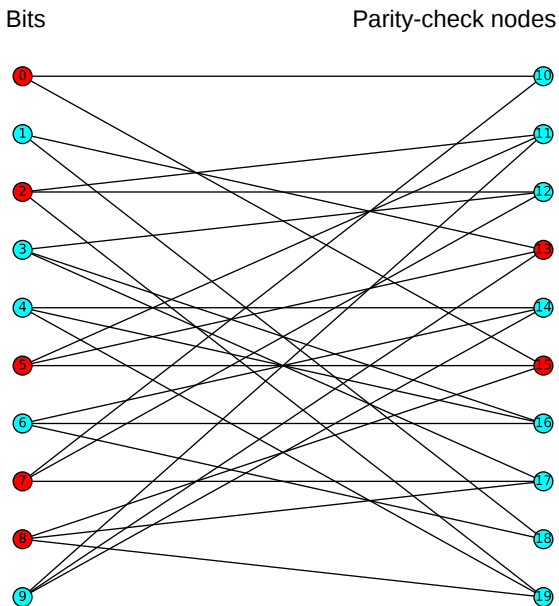


# Decoding algorithm : second example

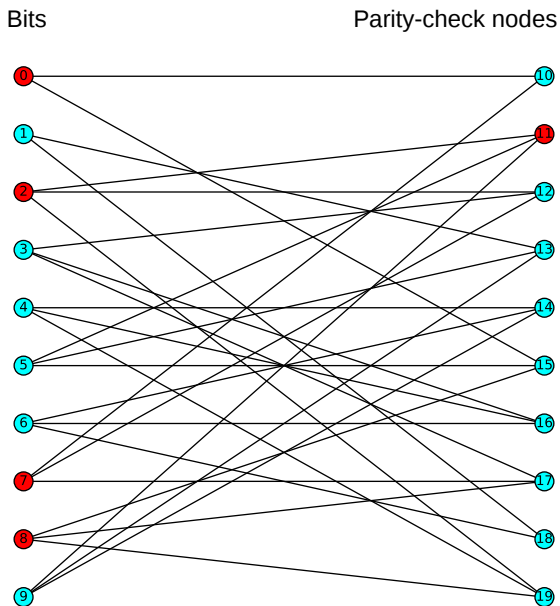




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# Decoding algorithm : second example



# Plan

- 1 Classical expander codes
- 2 Quantum expander codes
- 3 Our contribution

## Hypergraph product [Tillich & Zémor, '09]

Using a classical code  $\mathcal{C}$ , we can construct a  $[[n, k, d]]$ -CSS code with :

- $k = \Theta(n)$
- $d = \Theta(\sqrt{n})$  : we can correct any error on  $d$  Q-bits

# From classical to quantum error correcting codes

## Hypergraph product [Tillich & Zémor, '09]

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- $k = \Theta(n)$
- $d = \Theta(\sqrt{n})$  : we can correct any error on  $d$  Q-bits

## Theorem [Leverrier & Tillich & Zémor, '15]

For the hypergraph product of an expander code ( $\epsilon < 1/6$ ) :

There is an efficient decoding algorithm for this code.

This algorithm corrects any error of size  $\leq \Theta(\sqrt{n})$

This algorithm is very close to the algorithm of Sipser and Spielman

# Plan

- 1 Classical expander codes
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# Our work

- **Question** : What happens for random errors of size  $\Theta(n)$ ?
- **Depolarizing channel** : each Q-bit has an X-type error (resp. Y,Z-type error) with probability  $p$  independently

# Our work

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## Theorem : what we proved

For a probability of error  $p < 10^{-16}$  :

$$\lim_{n \rightarrow +\infty} \mathbb{P}(\mathcal{A} \text{ corrects the error}) = 1$$



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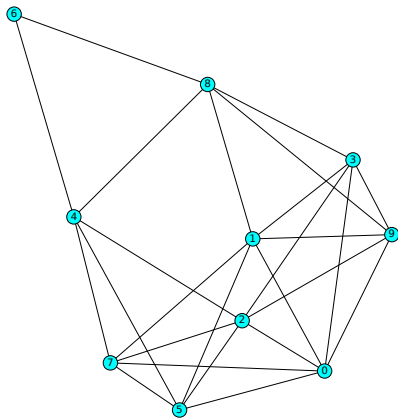
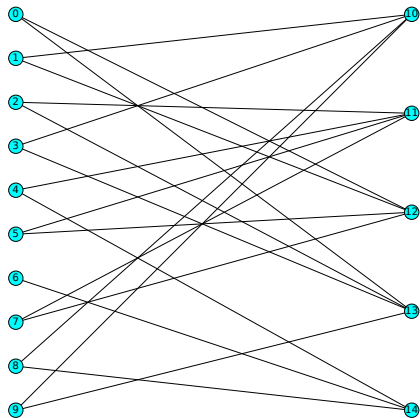
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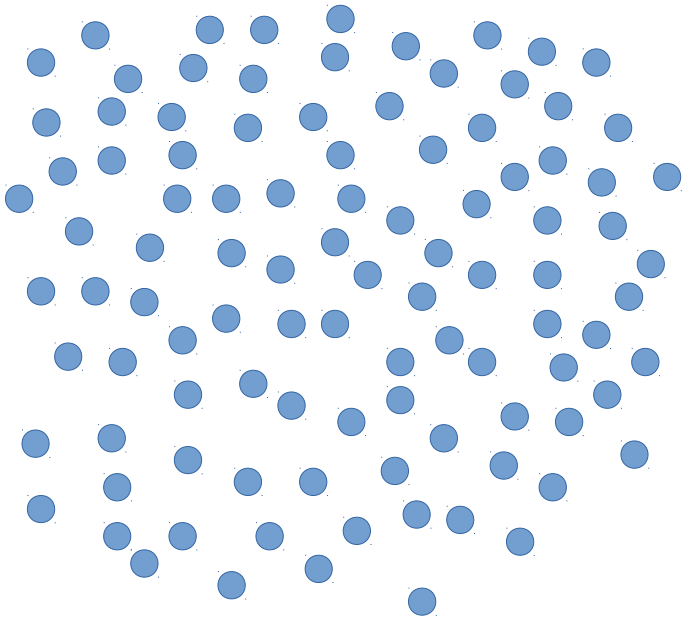
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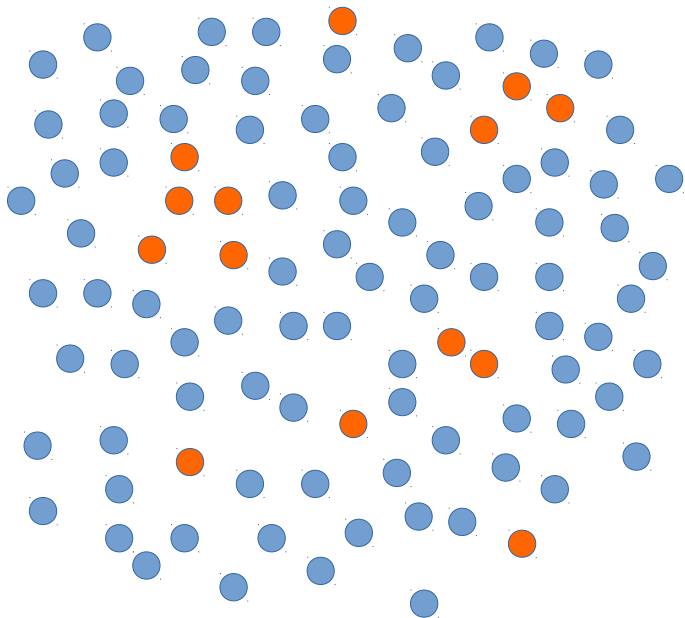
**Idea** : The algorithm is local :

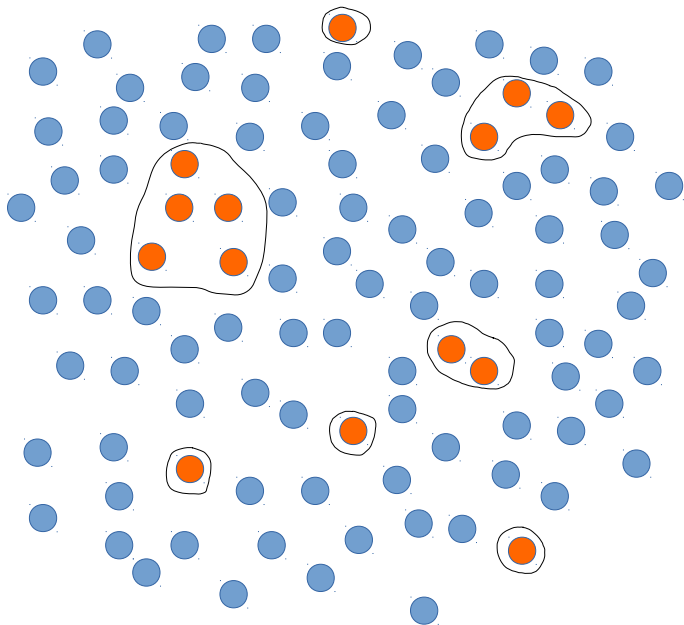
- If two errors are far they don't interact
- The initial error can be decomposed in clusters of size  $O(\ln(n))$
- If there is no cluster of size  $\Theta(\sqrt{n})$  during the algorithm, the error will be corrected

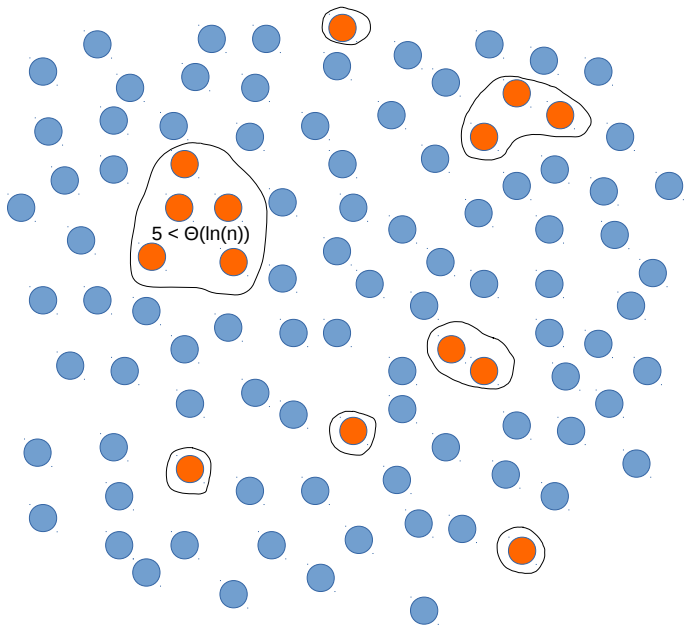
# Locality of the algorithm : the adjacency graph











In the following, we will say whp  $P$  (with high probability the property  $P$  holds) if :  $\lim_{n \rightarrow +\infty} \mathbb{P}(P) = 1$

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## Percolation Theorem

For a probability of error  $p < \frac{1}{d-1}$ , whp :

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### **Good news :**

The algorithm corrects any error of size  $\leq \Theta(\sqrt{n})$

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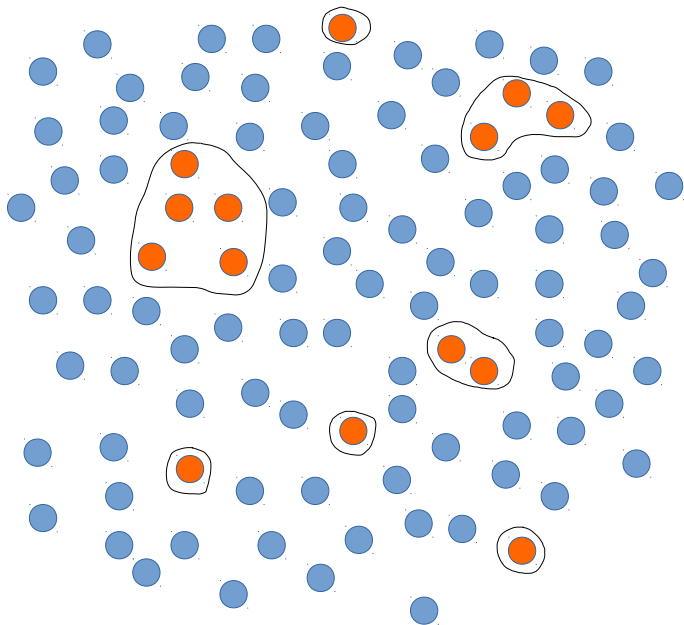
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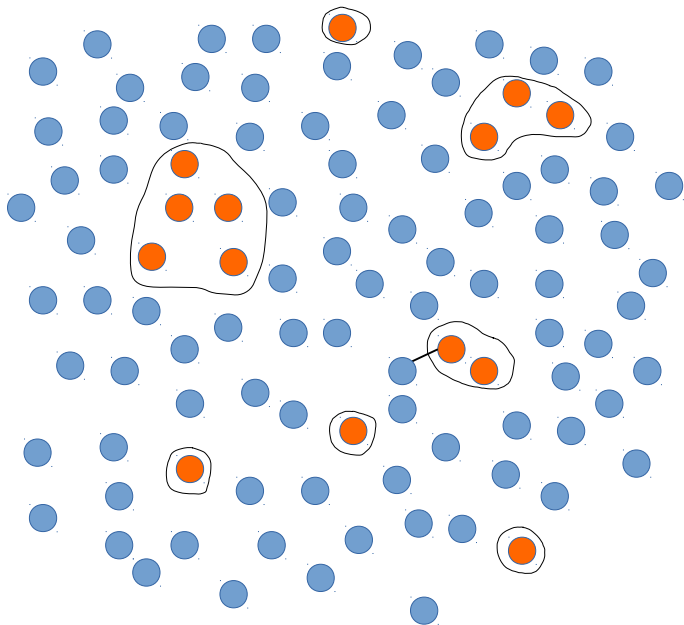
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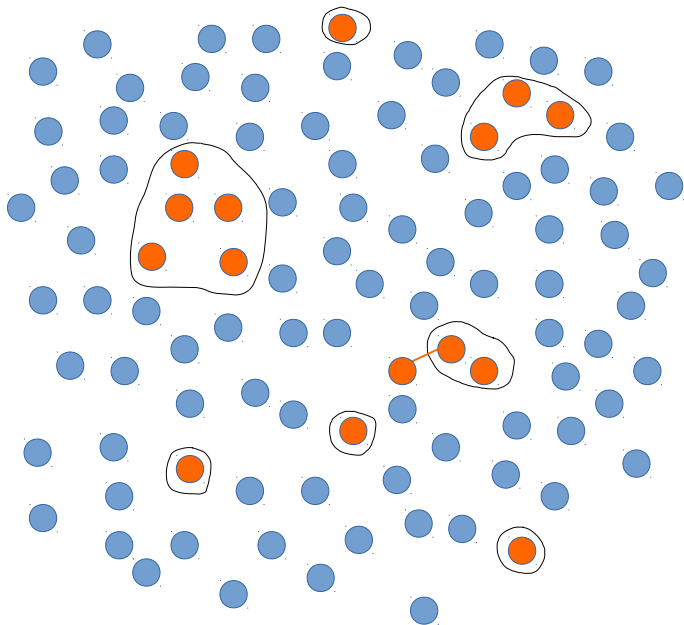
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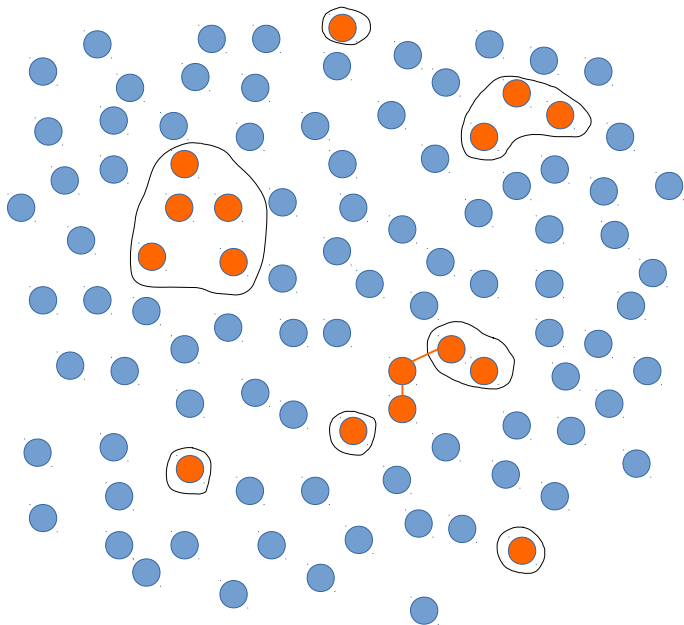
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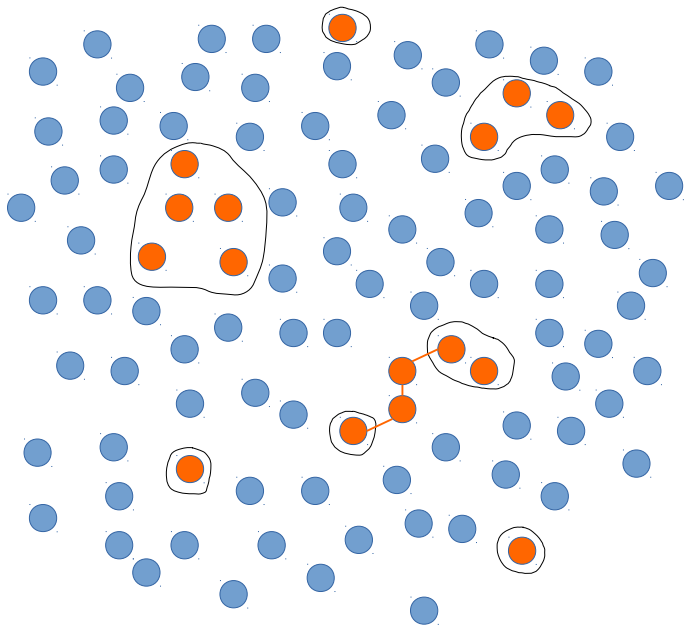
**Problem** : Some clusters can merge during the decoding











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- if  $|X \cap E(p)| \geq 1 \times |X|$  then  $|X| < \Theta(\ln(n))$

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## Percolation theorem, generalisation

$\forall \alpha > 0$ , if  $p < cst(\alpha, d)$ , whp :

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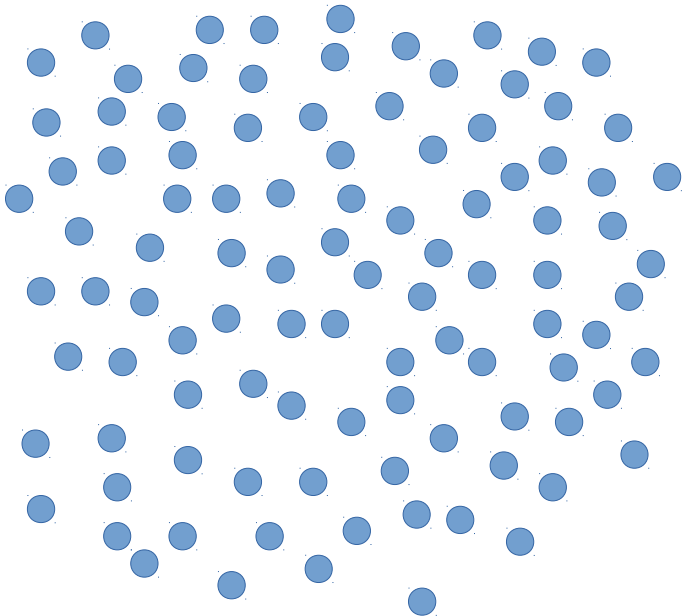
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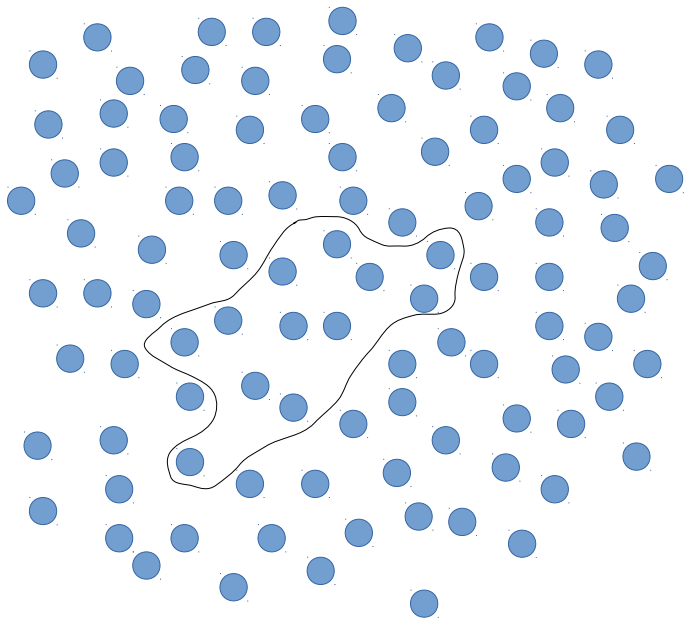
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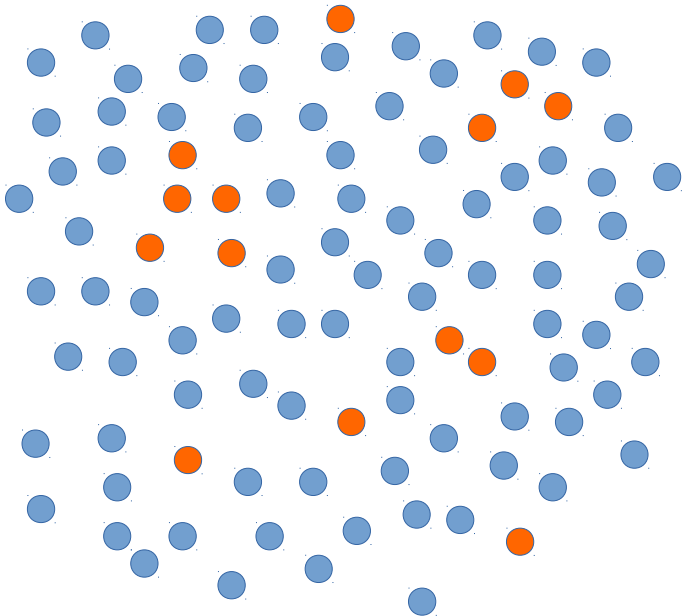
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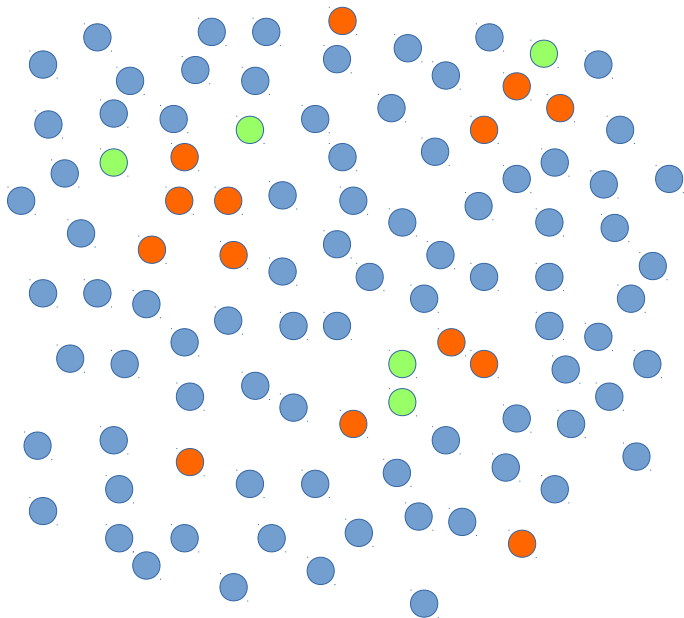
## Complexity of the decoding algorithm

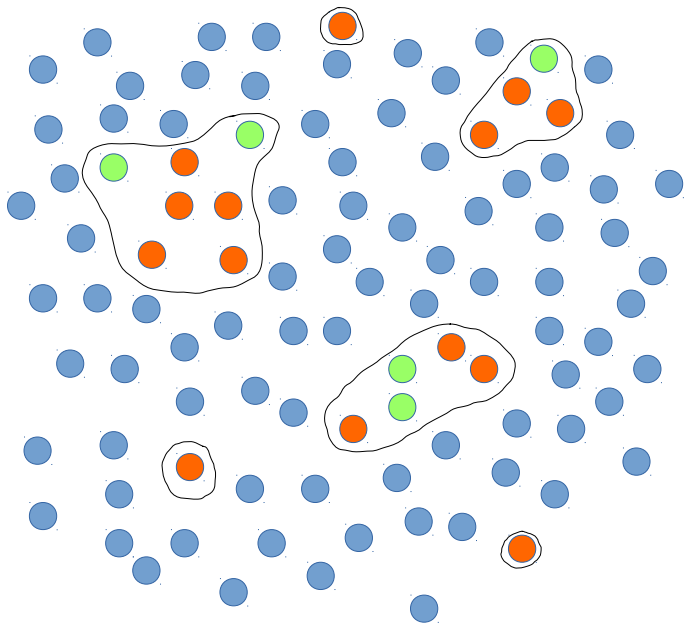
The number of flips is linear in the size of the initial error



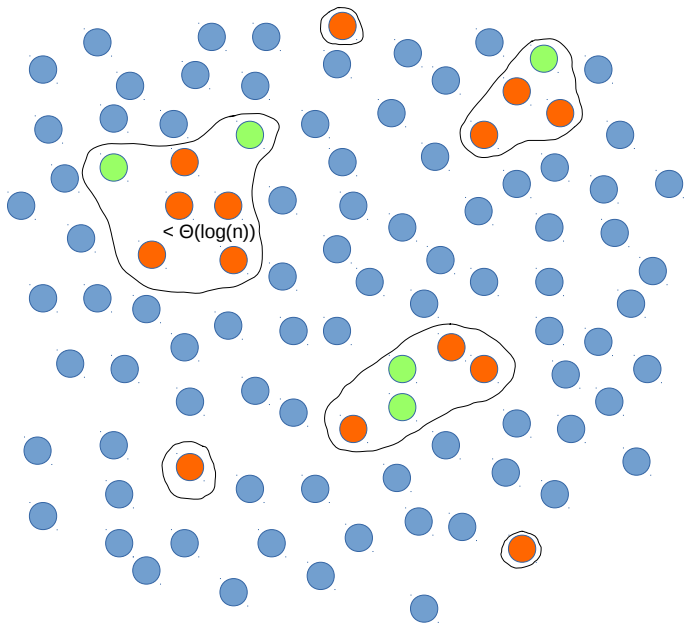












## Theorem : what we proved

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### Ideas to improve this bound :

- 1 Improve the bound in the percolation theorem :  
What is the critical probability ?
- 2 Restrict the proof to interesting clusters :
  - \* The diameter of an interesting cluster is  $\leq \Theta(\ln(\ln(n)))$
  - \* The number of edges inside an interesting cluster is large  
(ideas from bootstrap percolation)

# Conclusion

The hypergraph product of an expander code :

- is an LDPC quantum code
- has a constant rate
- has a minimal distance :  $d = \Theta(\sqrt{n})$

The decoding algorithm :

- has a capacity of correction :  $\Theta(\sqrt{n})$
- corrects the error with high probability for the depolarizing channel

Futur work :

- improve our bound
- run simulations
- apply this result to fault tolerant quantum computation (Gottesman)

Thank you for your attention

# A motivation : fault-tolerant quantum computation

## Threshold Theorem [Ben-Or & Aharonov, '97]

We can simulate a quantum circuit with perfect gates by a circuit with noisy gates of size **quasi-linear**

## Theorem : what we proved

For a probability of error  $p < 10^{-16}$  :

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## What we hope to prove using [Gottesman, '13]

We can simulate a quantum circuit with perfect gates by a circuit with noisy gates of size **linear**

$$n = 10, m = 5, d_1 = 2, d_2 = \frac{n \times d_1}{m} = 4$$

0

1

2

3

4

5

6

7

8

9

$$n = 10, m = 5, d_1 = 2, d_2 = \frac{n \times d_1}{m} = 4$$

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11

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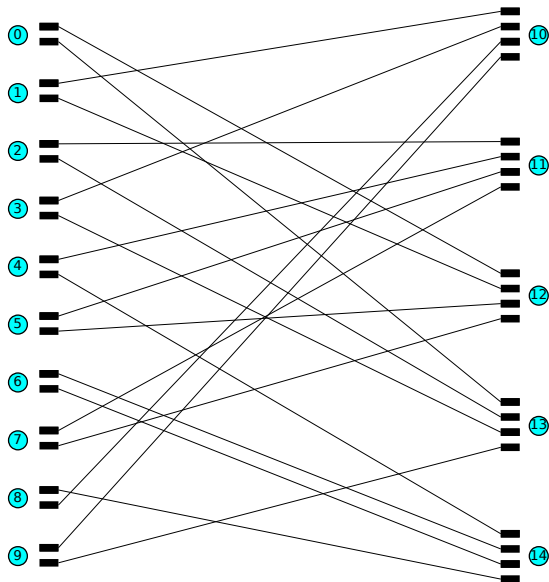
13

14

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