Quantum expander codes

Antoine Grospellier, Anthony Leverrier, Omar Fawzi

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Imaginons notre futur

Content of the talk

The hypergraph product of an expander code :

- is an LDPC quantum code
- has a constant rate
- has a minimal distance : $d = \Theta(\sqrt{n})$

The decoding algorithm :

- has a capacity of correction : $\Theta(\sqrt{n})$
- **corrects the error with high probability for the depolarizing channel**

Content of the talk

1. Classical expander codes
2. Quantum expander codes
3. My contribution

Plan

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2. Quantum expander codes
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Definition : classical error correcting codes

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- $\mathcal{C}$ is a k-dimensional subspace of $(\mathbb{Z}/2\mathbb{Z})^n$

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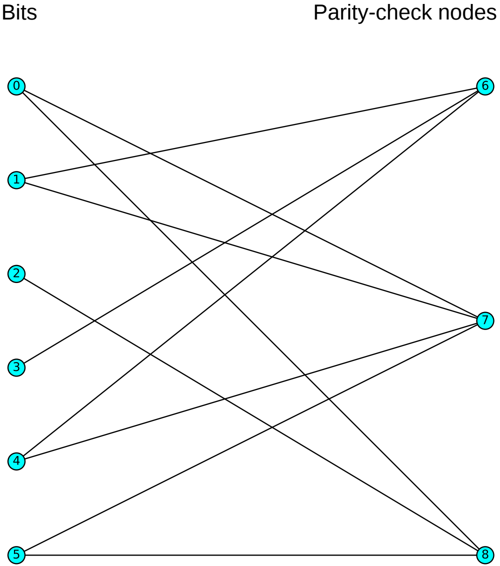
Definition : parity check matrix of a code

$H \in \mathcal{M}_{n-k,n}$ is a parity check matrix for $\mathcal{C}$ if :

$$c \in \mathcal{C} \Leftrightarrow H \cdot c = 0$$

Tanner graph of a code

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$



Classical expander codes [Sipser & Spielman, '96]

Theorem [Sipser & Spielman, '96]

We can construct a good family $(\mathcal{C}_n)_{n\in\mathbb{N}}$ of $[n, k, d]$-error correcting codes. Here "Good" means :

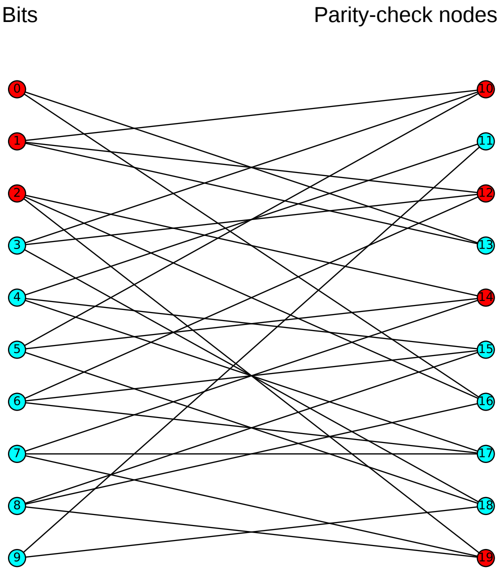
- This family is LDPC
- k and d are linear in n
- There exists an efficient correcting algorithm

Remark

Good expander codes can be found efficiently by picking a random biregular graph

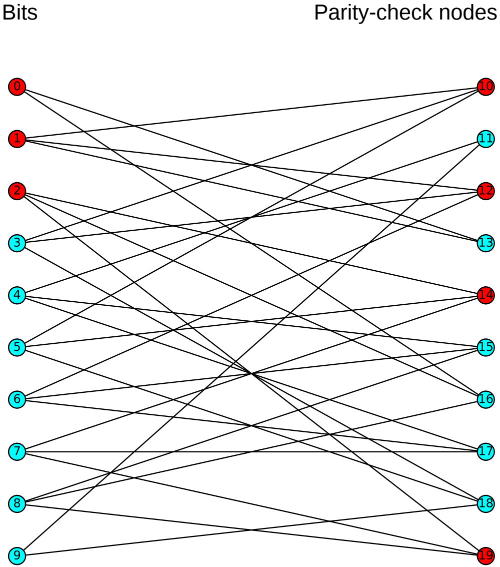
Decoding algorithm

- Error :
  $e_0 = \{0, 1, 2\}$
- Unsatisfied check-nodes (syndrome) :
  $\{10, 12, 14, 19\}$
- Satisfied check-nodes :
  $\{11, 13, 15, 16, 17, 18\}$

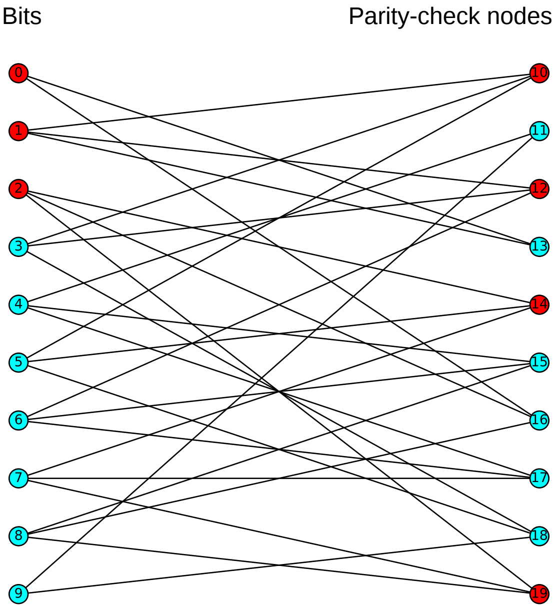


Decoding algorithm

- INPUT : syndrome
- The error e_0 is unknown
- OUTPUT : e
  a set of bits
- GOAL : $e = e_0$
- The algorithm flips a bit when it decreases the syndrome



Decoding algorithm : first example



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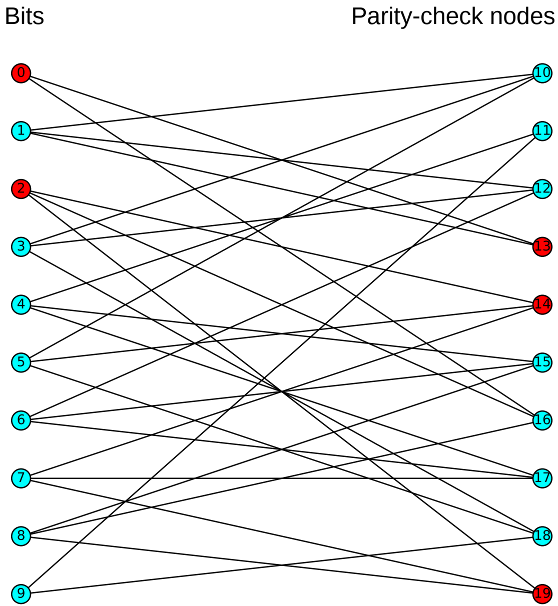

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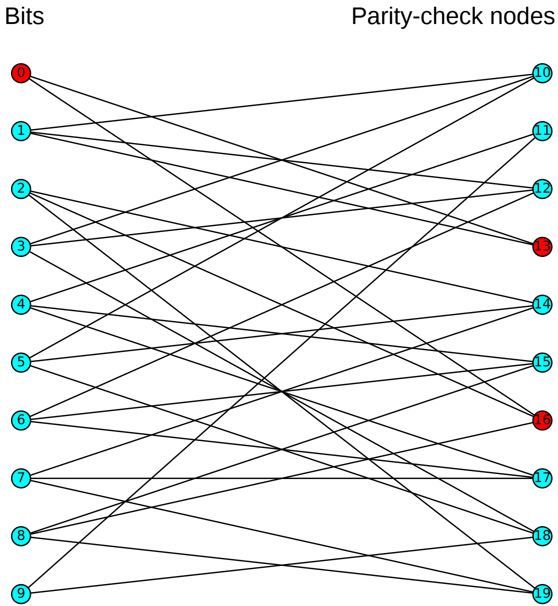

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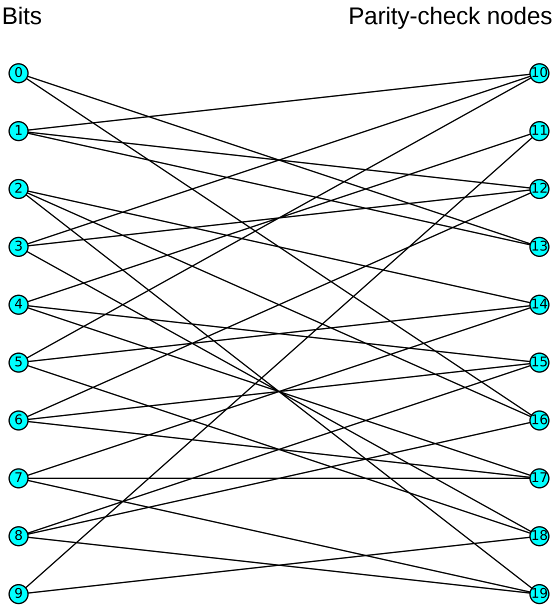

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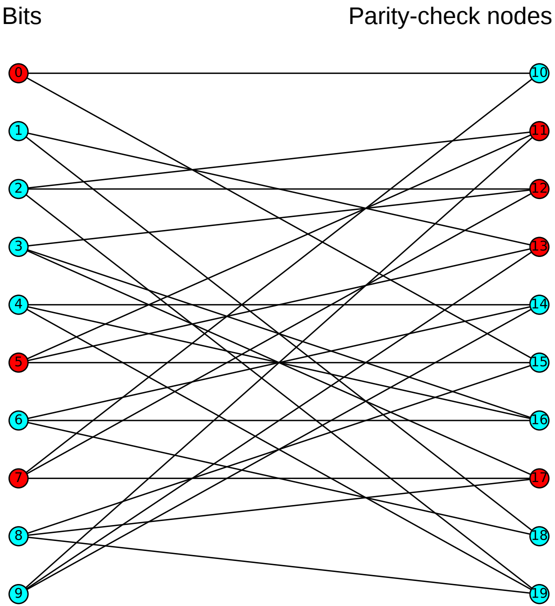

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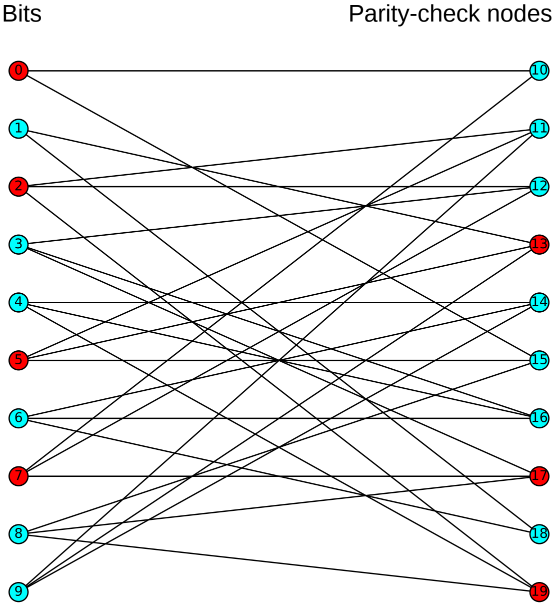

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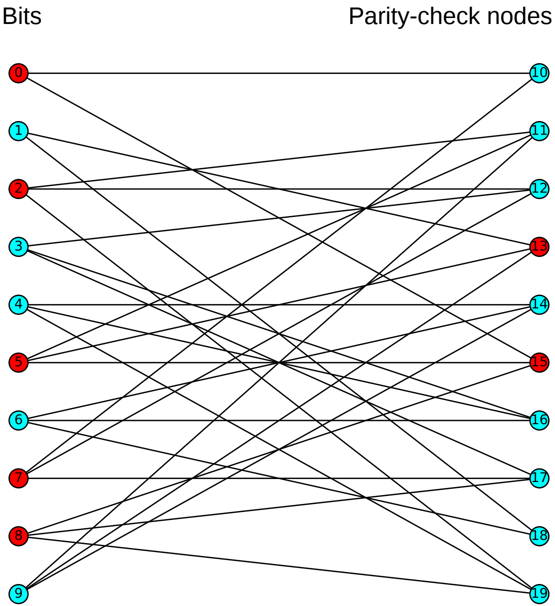

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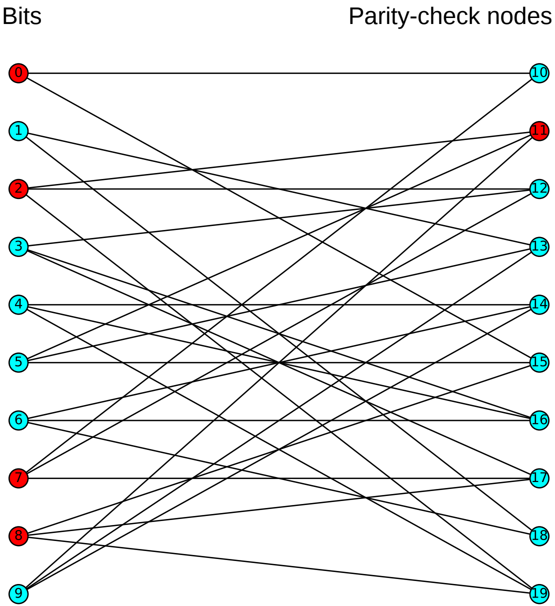

Plan

1. Classical expander codes

2. Quantum expander codes

3. My contribution

From classical to quantum error correcting codes

- Bit : $b \in \mathbb{Z}/2\mathbb{Z}$

- Q-bit : $|\psi\rangle \in \mathbb{C}^2$, $\| \, |\psi\rangle \, \|_2 = 1$

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From classical to quantum error correcting codes

- Bit : $b \in \mathbb{Z}/2\mathbb{Z}$
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- Classical error : Flip

X-Pauli error : $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

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- A $[\![n, k]\!]$-code is a 2^k-dimensional subspace of $\mathbb{C}^{2^n}$
- Quantum error :
  X-Pauli error
  Z-Pauli error

Z-Pauli error : $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

From classical to quantum error correcting codes

CSS codes : Calderbank, Shor and Steane

We can construct a quantum error correcting code using $\mathcal{C}_X$ and $\mathcal{C}_Z$ two classical error correcting codes such that $\mathcal{C}_X^\perp \subseteq \mathcal{C}_Z$

Remark

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Hypergraph product [Tillich & Zémor, '09]

Using a classical code $\mathcal{C}$, we can construct a $[\![n, k, d]\!]$-CSS code with :

- $k = \Theta(n)$
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Theorem [Leverrier & Tillich & Zémor, '15]

For the hypergraph product of an expander code ($\epsilon < 1/6$) :
There is an efficient decoding algorithm for this code.
This algorithm corrects any error of size $\leq \Theta(\sqrt{n})$
This algorithm is very close to the algorithm of Sipser and Spielman

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Our work

- **Question :** What happens for random errors of size $\Theta(n)$?
- **Depolarizing channel :** each Q-bit has an X-type error (resp. Z-type error) with probabilty p independently

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Idea : The algorithm is local :

- If two errors are far they don't interact
- The initial error can be decomposed in clusters of size $O(ln(n))$
- If there is no cluster of size $\Theta(\sqrt{n})$ during the algorithm, the error will be corrected

A motivation : fault-tolerant quantum computation

Threshold Theorem [Ben-Or & Aharonov, '97]

We can simulate a quantum circuit with perfect gates by a circuit with noisy gates of size **quasi-linear**

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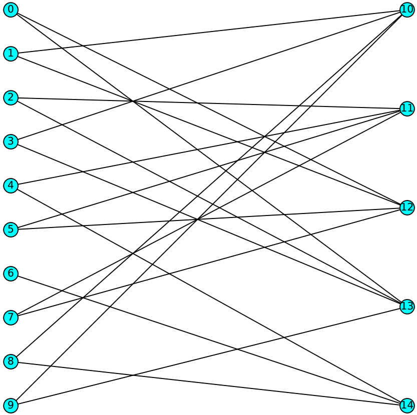
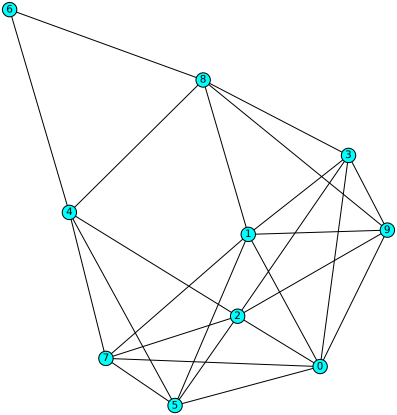
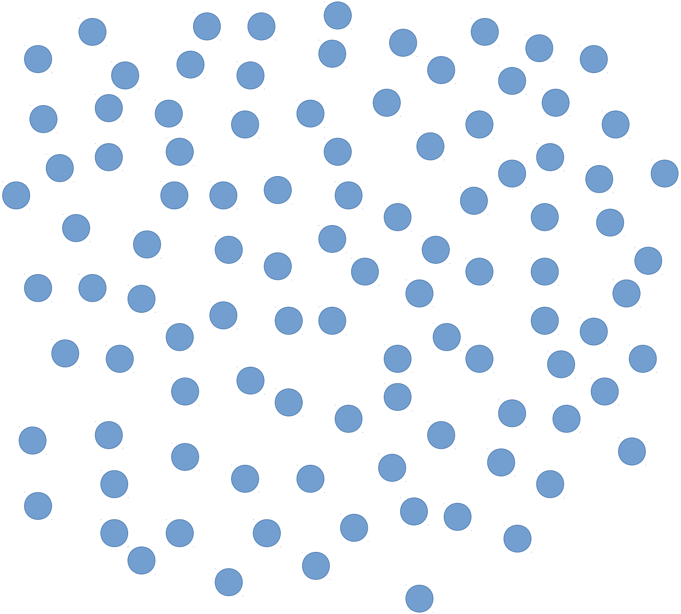
What we hope to prove using [Gottesman, '13]

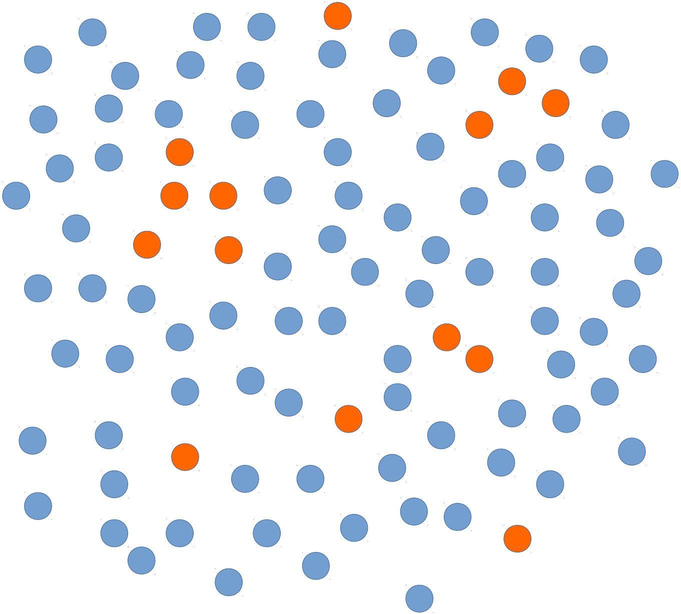
We can simulate a quantum circuit with perfect gates by a circuit with noisy gates of size **linear**

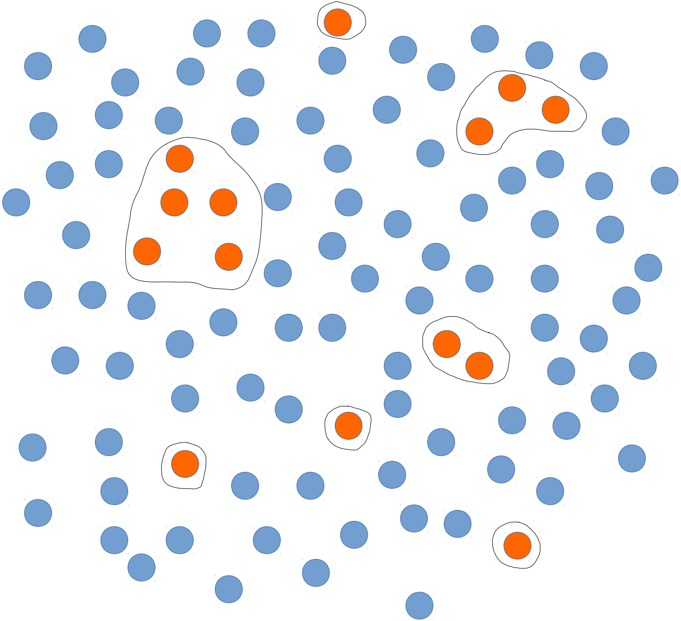
Locality of the algorithm : the bit-graph

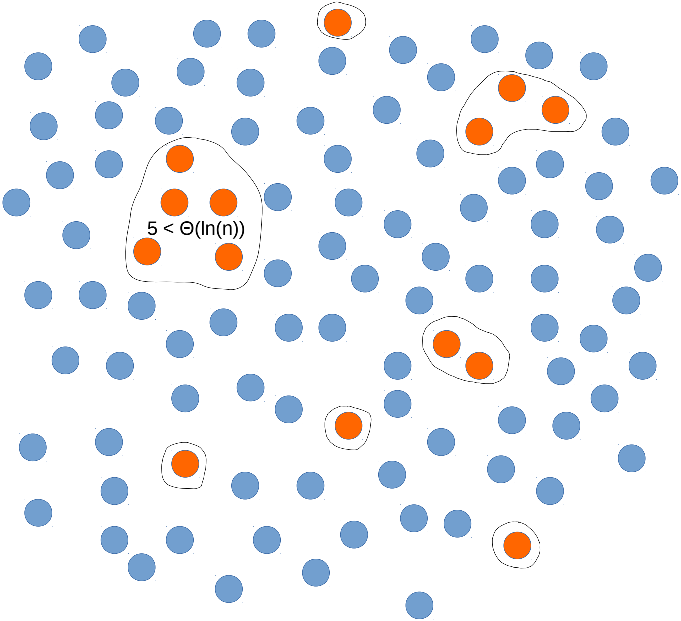











5 < Θ(ln(n))

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Percolation Theorem

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The algorithm corrects any error of size $\leq \Theta(\sqrt{n})$
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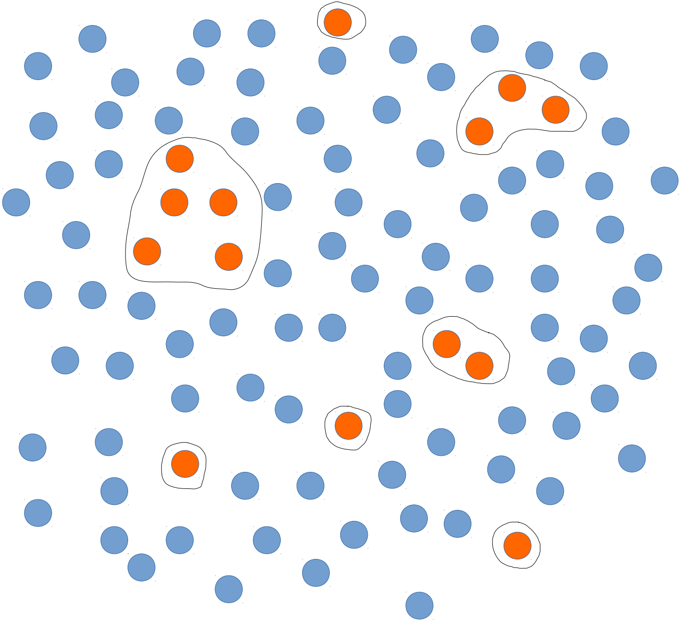
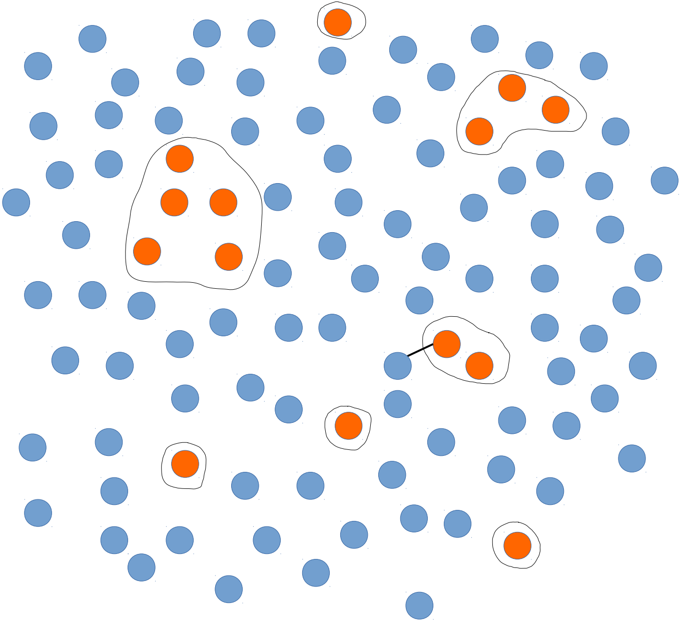
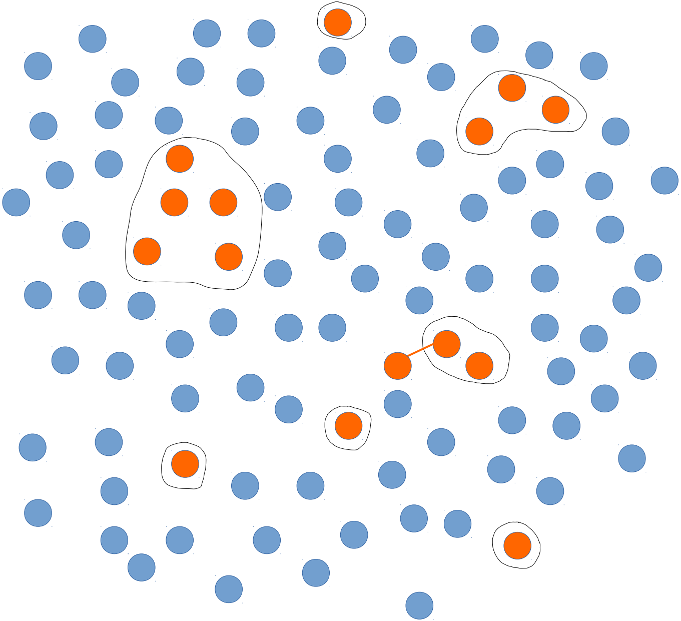
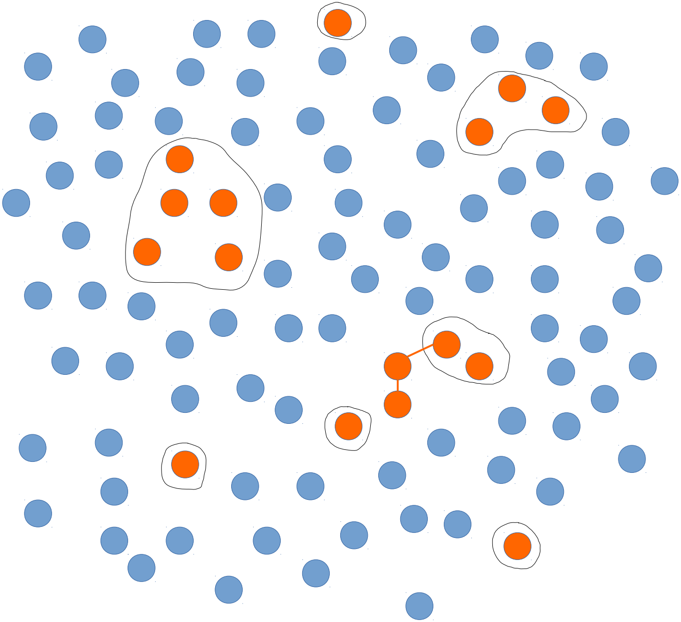
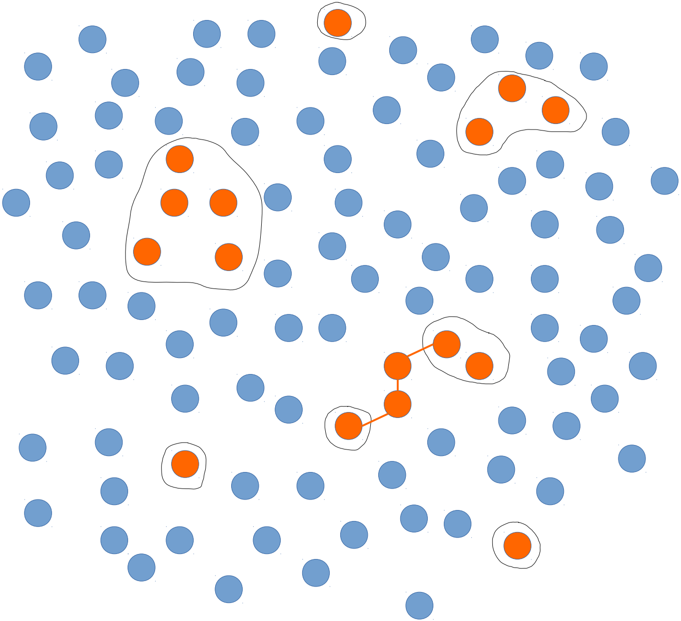
Problem : Some clusters can merge during the decoding











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$\forall \alpha > 0$, if $p < cst(\alpha, d)$, whp :

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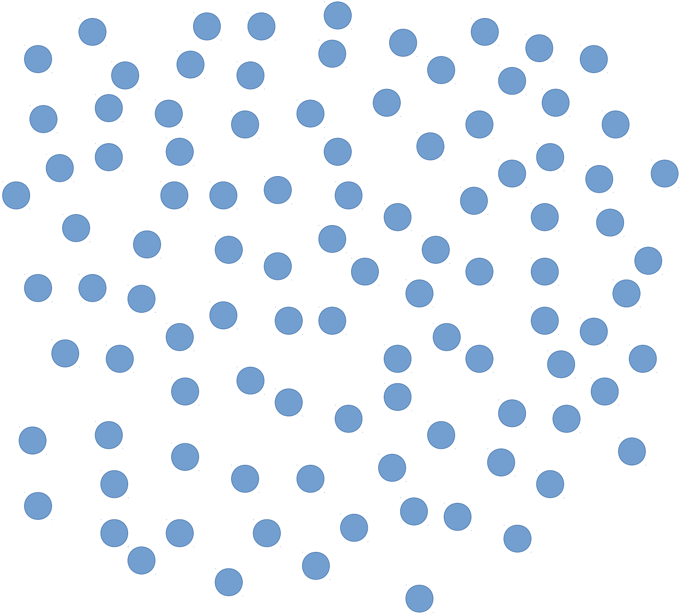
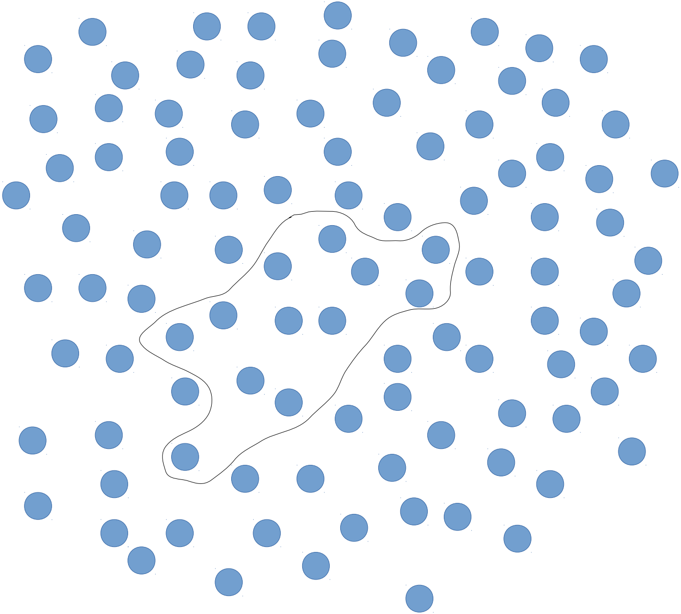
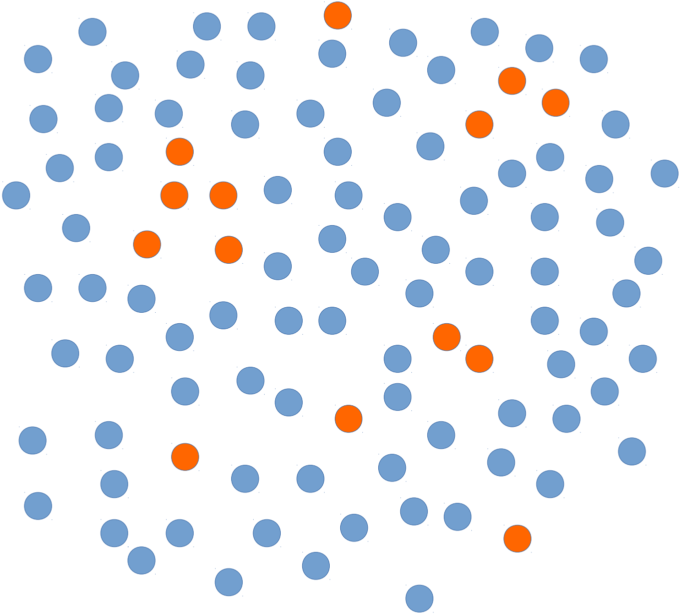
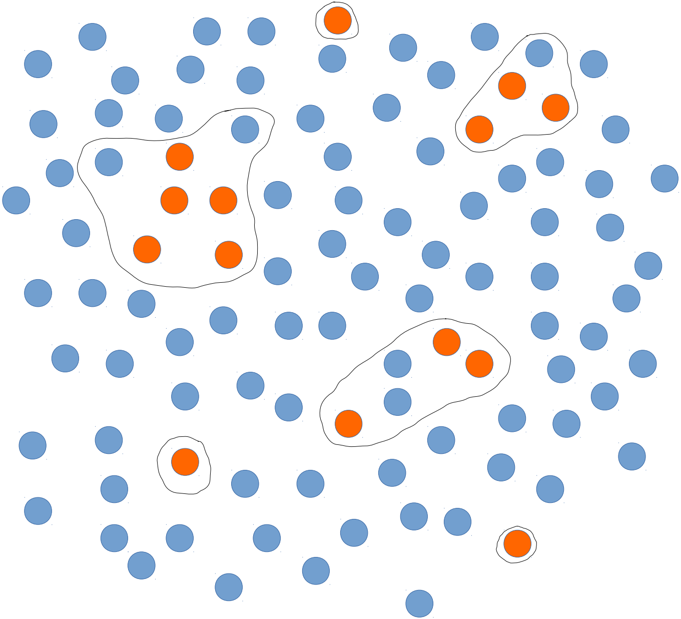
Complexity of the decoding algorithm

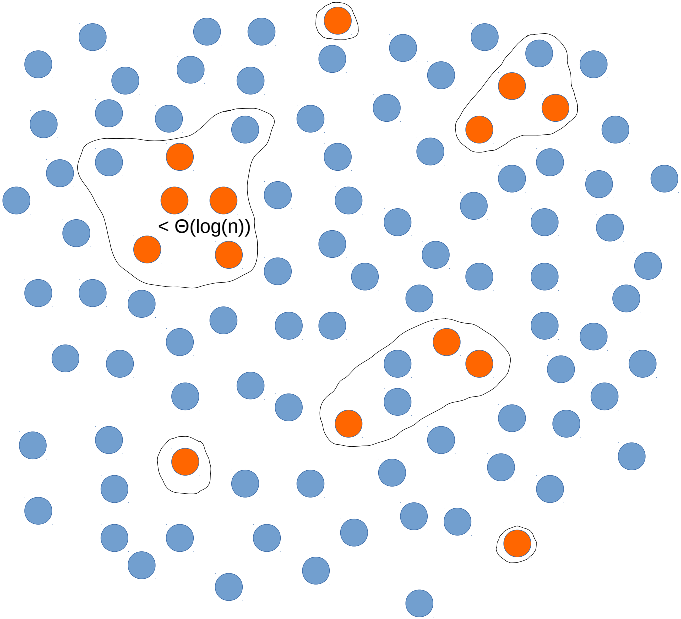
The number of flips is linear in the size of the initial error







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Ideas to improve this bound :

1. Improve the bound in the percolation theorem :
   What is the critical probability ?
2. Restrict the proof to interesting clusters :
   - The diameter of an interesting cluster is $\leq \Theta(ln(ln(n)))$
   - The number of edges inside an interesting cluster is large (ideas from bootstrap percolation)

Conclusion

The hypergraph product of an expander code :

- is an LDPC quantum code
- has a constant rate
- has a minimal distance : $d = \Theta(\sqrt{n})$

The decoding algorithm :

- has a capacity of correction : $\Theta(\sqrt{n})$
- corrects the error with high probability for the depolarizing channel

Futur work :

- improve our bound
- apply this result to fault tolerant quantum computation (Gottesman)

Thank you for your attention