



**HAL**  
open science

## Quantum expander codes

Antoine Gropellier, Anthony Leverrier, Omar Fawzi

► **To cite this version:**

Antoine Gropellier, Anthony Leverrier, Omar Fawzi. Quantum expander codes. Journées codage et cryptographie 2017, Apr 2017, La Bresse, France. hal-01671485

**HAL Id: hal-01671485**

**<https://hal.science/hal-01671485v1>**

Submitted on 22 Dec 2017

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Quantum expander codes

**Antoine Gropellier** & Anthony Leverrier & Omar Fawzi

27 avril 2017



# Content of the talk

The hypergraph product of an expander code :

- is an LDPC quantum code
- has a constant rate
- has a minimal distance :  $d = \Theta(\sqrt{n})$

The decoding algorithm :

- has a capacity of correction :  $\Theta(\sqrt{n})$
- **corrects the error with high probability for the depolarizing channel**

# Content of the talk

- 1 Classical expander codes
- 2 Quantum expander codes
- 3 My contribution

- 1 Classical expander codes
- 2 Quantum expander codes
- 3 My contribution

## Definition : classical error correcting codes

$\mathcal{C}$  is a  $[n, k]$ -error correcting code if :

- $\mathcal{C}$  is a  $k$ -dimensional subspace of  $(\mathbb{Z}/2\mathbb{Z})^n$

### Definition : classical error correcting codes

$\mathcal{C}$  is a  $[n, k]$ -error correcting code if :

- $\mathcal{C}$  is a  $k$ -dimensional subspace of  $(\mathbb{Z}/2\mathbb{Z})^n$

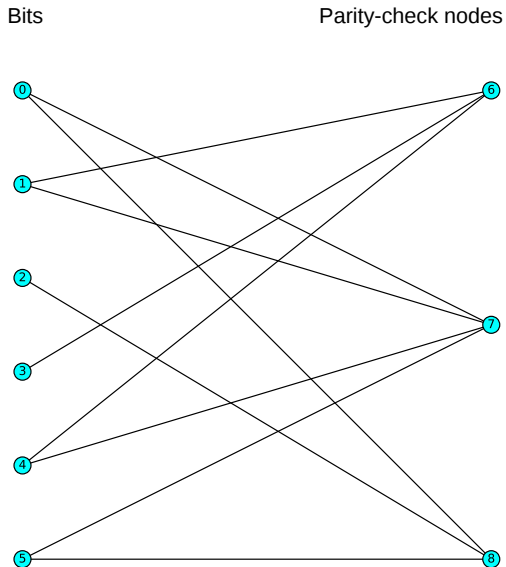
### Definition : parity check matrix of a code

$H \in \mathcal{M}_{n-k, n}$  is a parity check matrix for  $\mathcal{C}$  if :

$$c \in \mathcal{C} \Leftrightarrow H \cdot c = 0$$

# Tanner graph of a code

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$





## Theorem [Sipser & Spielman, '96]

We can construct a good family  $(\mathcal{C}_n)_{n \in \mathbb{N}}$  of  $[n, k, d]$ -error correcting codes. Here "Good" means :

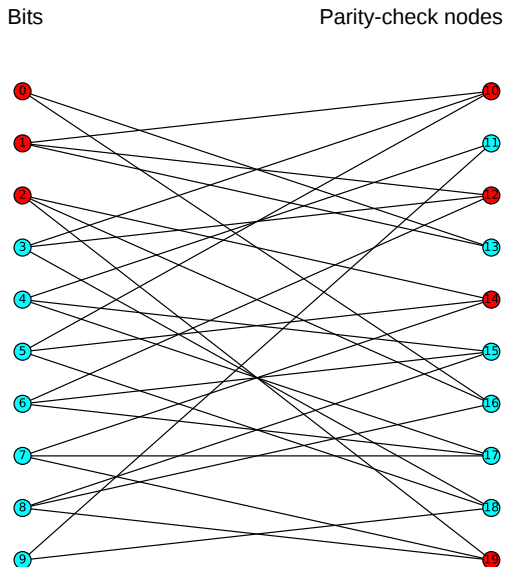
- This family is LDPC
- $k$  and  $d$  are linear in  $n$
- There exists an efficient correcting algorithm

## Remark

Good expander codes can be found efficiently by picking a random biregular graph

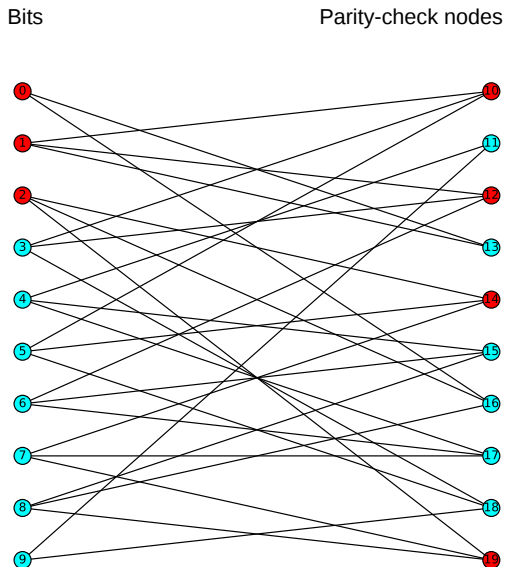
# Decoding algorithm

- Error :  
 $e_0 = \{0, 1, 2\}$
- Unsatisfied check-nodes (syndrome) :  
 $\{10, 12, 14, 19\}$
- Satisfied check-nodes :  
 $\{11, 13, 15, 16, 17, 18\}$

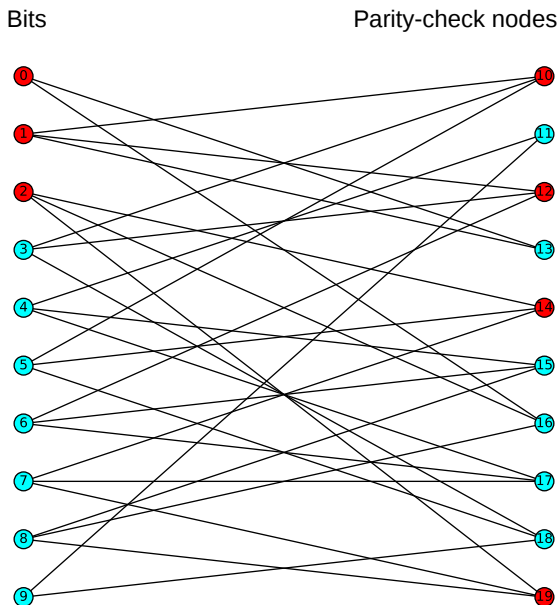


# Decoding algorithm

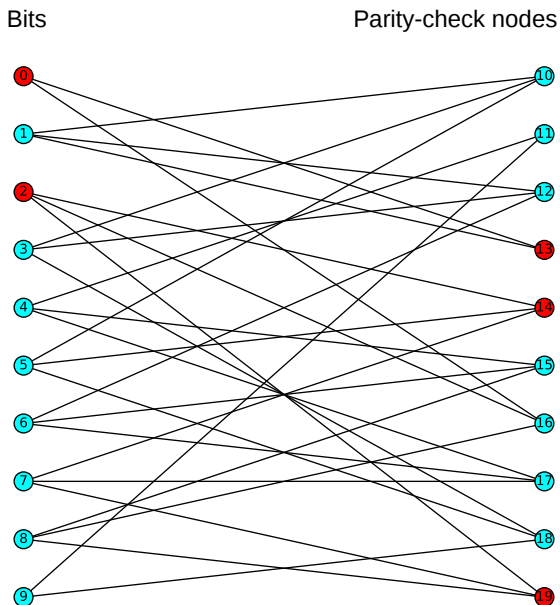
- INPUT : syndrome
- The error  $e_0$  is unknown
- OUTPUT :  $e$   
a set of bits
- GOAL :  $e = e_0$
- The algorithm flips a bit when it decreases the syndrome



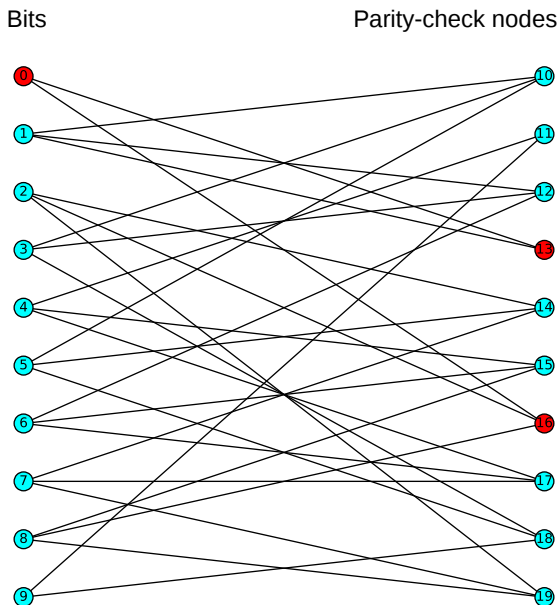
# Decoding algorithm : first example



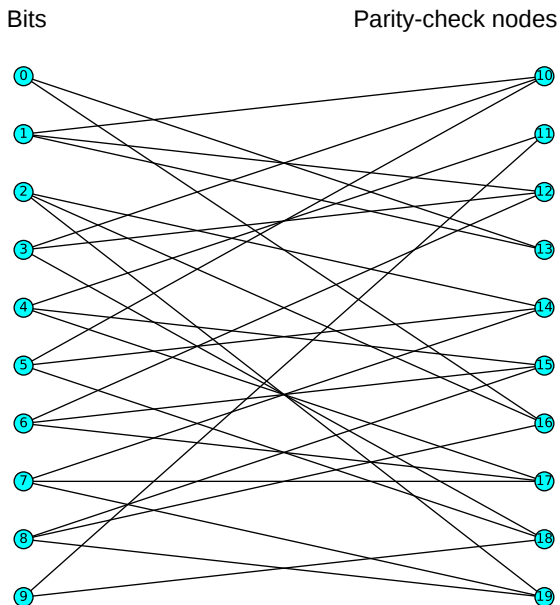
# Decoding algorithm : first example



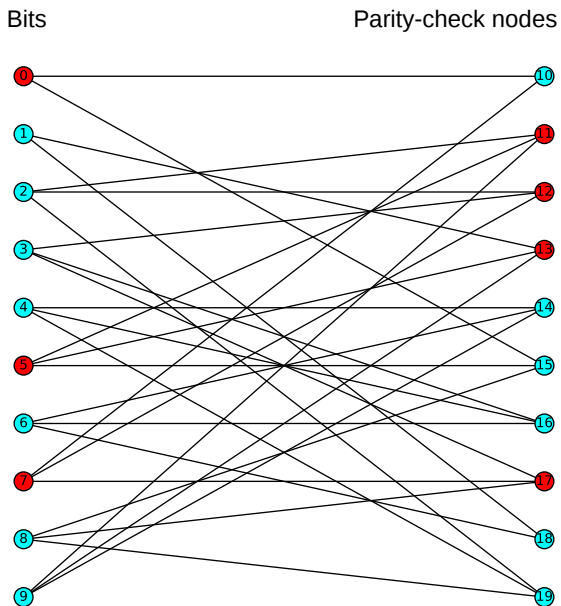
# Decoding algorithm : first example



# Decoding algorithm : first example

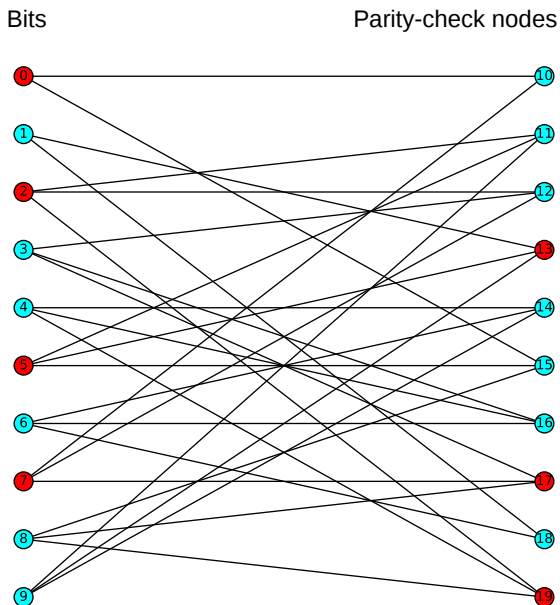


# Decoding algorithm : second example

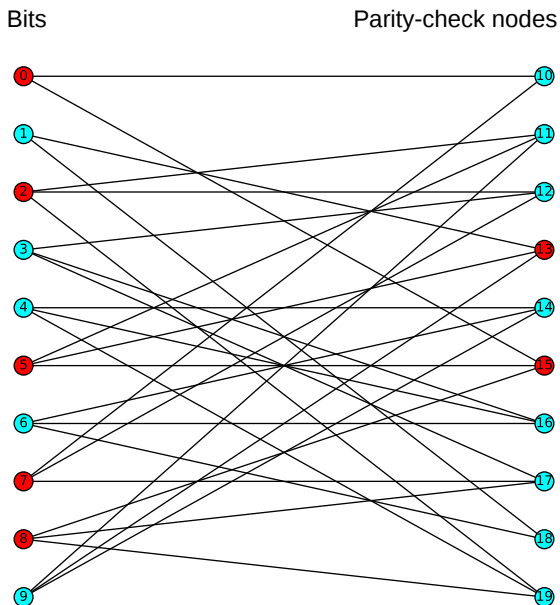




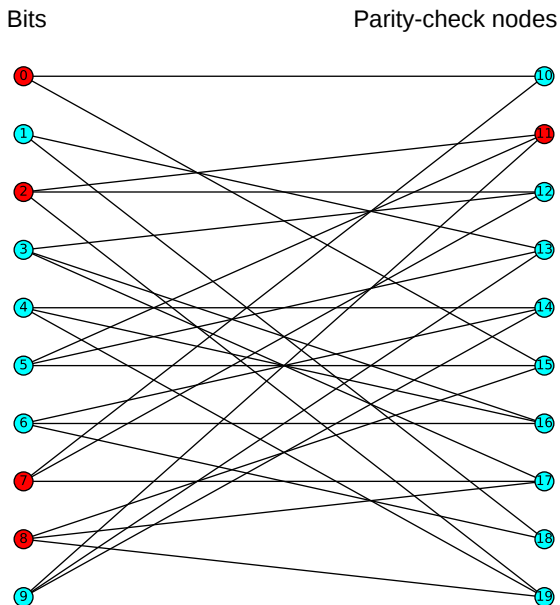
# Decoding algorithm : second example



# Decoding algorithm : second example



# Decoding algorithm : second example



# Plan

- 1 Classical expander codes
- 2 Quantum expander codes
- 3 My contribution

# From classical to quantum error correcting codes

- Bit :  $b \in \mathbb{Z}/2\mathbb{Z}$
- Q-bit :  $|\psi\rangle \in \mathbb{C}^2$ ,  
 $\| |\psi\rangle \|_2 = 1$

# From classical to quantum error correcting codes

- Bit :  $b \in \mathbb{Z}/2\mathbb{Z}$
- A  $[n, k]$ -code is a  $k$ -dimensional subspace of  $(\mathbb{Z}/2\mathbb{Z})^n$
- Q-bit :  $|\psi\rangle \in \mathbb{C}^2$ ,  
 $\| |\psi\rangle \|_2 = 1$
- A  $[[n, k]]$ -code is a  $2^k$ -dimensional subspace of  $\mathbb{C}^{2^n}$

# From classical to quantum error correcting codes

- Bit :  $b \in \mathbb{Z}/2\mathbb{Z}$
- A  $[n, k]$ -code is a  $k$ -dimensional subspace of  $(\mathbb{Z}/2\mathbb{Z})^n$
- Classical error : Flip
- Q-bit :  $|\psi\rangle \in \mathbb{C}^2$ ,  
 $\| |\psi\rangle \|_2 = 1$
- A  $[[n, k]]$ -code is a  $2^k$ -dimensional subspace of  $\mathbb{C}^{2^n}$
- Quantum error :  
X-Pauli error  
Z-Pauli error

$$\text{X-Pauli error : } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\text{Z-Pauli error : } \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

## CSS codes : Calderbank, Shor and Steane

We can construct a quantum error correcting code using  $\mathcal{C}_X$  and  $\mathcal{C}_Z$  two classical error correcting codes such that  $\mathcal{C}_X^\perp \subseteq \mathcal{C}_Z$



## Remark

The difficulty for constructing a quantum error correcting code is to find two classical codes which are orthogonal

## Remark

The difficulty for constructing a quantum error correcting code is to find two classical codes which are orthogonal

## Hypergraph product [Tillich & Zémor, '09]

Using a classical code  $\mathcal{C}$ , we can construct a  $[[n, k, d]]$ -CSS code with :

- $k = \Theta(n)$
- $d = \Theta(\sqrt{n})$  : we can correct any error on  $d$  Q-bits

## Remark

The difficulty for constructing a quantum error correcting code is to find two classical codes which are orthogonal

## Hypergraph product [Tillich & Zémor, '09]

Using a classical code  $\mathcal{C}$ , we can construct a  $[[n, k, d]]$ -CSS code with :

- $k = \Theta(n)$
- $d = \Theta(\sqrt{n})$  : we can correct any error on  $d$  Q-bits

## Theorem [Leverrier & Tillich & Zémor, '15]

For the hypergraph product of an expander code ( $\epsilon < 1/6$ ) :

There is an efficient decoding algorithm for this code.

This algorithm corrects any error of size  $\leq \Theta(\sqrt{n})$

This algorithm is very close to the algorithm of Sipser and Spielman

- 1 Classical expander codes
- 2 Quantum expander codes
- 3 My contribution**

# Our work

- **Question** : What happens for random errors of size  $\Theta(n)$ ?
- **Depolarizing channel** : each Q-bit has an X-type error (resp. Z-type error) with probability  $p$  independently

# Our work

- **Question** : What happens for random errors of size  $\Theta(n)$ ?
- **Depolarizing channel** : each Q-bit has an X-type error (resp. Z-type error) with probability  $p$  independently

## Theorem : what we proved

For a probability of error  $p < \frac{1}{(ed)^{2d}}$  :

$$\lim_{n \rightarrow +\infty} \mathbb{P}(\mathcal{A} \text{ corrects the error}) = 1$$

# Our work

- **Question** : What happens for random errors of size  $\Theta(n)$  ?
- **Depolarizing channel** : each Q-bit has an X-type error (resp. Z-type error) with probability  $p$  independently

## Theorem : what we proved

For a probability of error  $p < \frac{1}{(ed)^{2d}}$  :

$$\lim_{n \rightarrow +\infty} \mathbb{P}(\mathcal{A} \text{ corrects the error}) = 1$$

**Idea** : The algorithm is local :

- If two errors are far they don't interact
- The initial error can be decomposed in clusters of size  $O(\ln(n))$
- If there is no cluster of size  $\Theta(\sqrt{n})$  during the algorithm, the error will be corrected

# A motivation : fault-tolerant quantum computation

## Threshold Theorem [Ben-Or & Aharonov, '97]

We can simulate a quantum circuit with perfect gates by a circuit with noisy gates of size **quasi-linear**

## Theorem : what we proved

For a probability of error  $p < \frac{1}{(ed)^{2d}}$  :

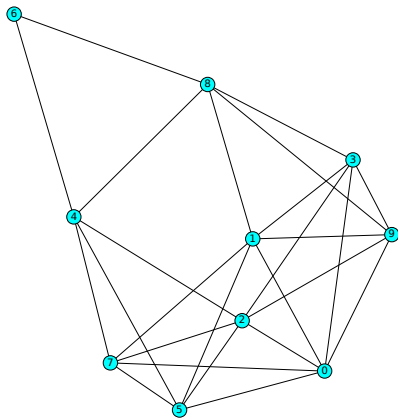
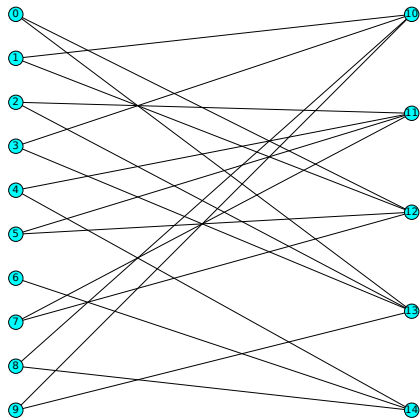
$$\lim_{n \rightarrow +\infty} \mathbb{P}(\mathcal{A} \text{ corrects the error}) = 1$$

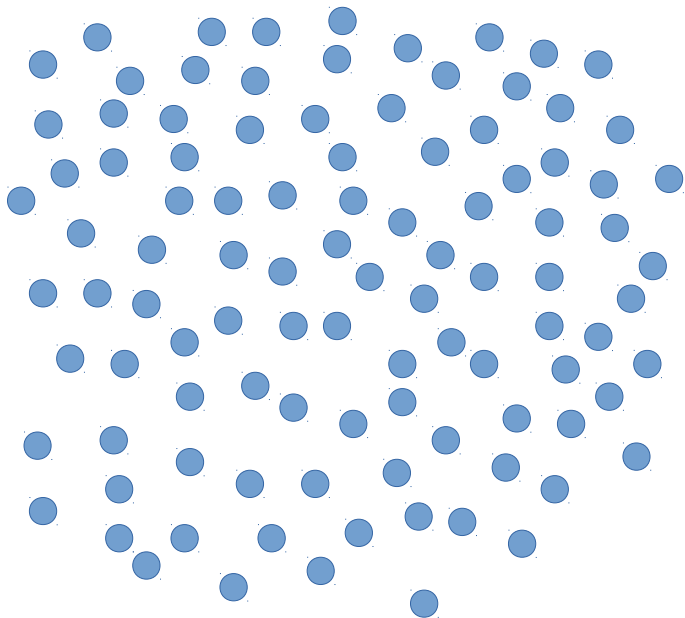
## What we hope to prove using [Gottesman, '13]

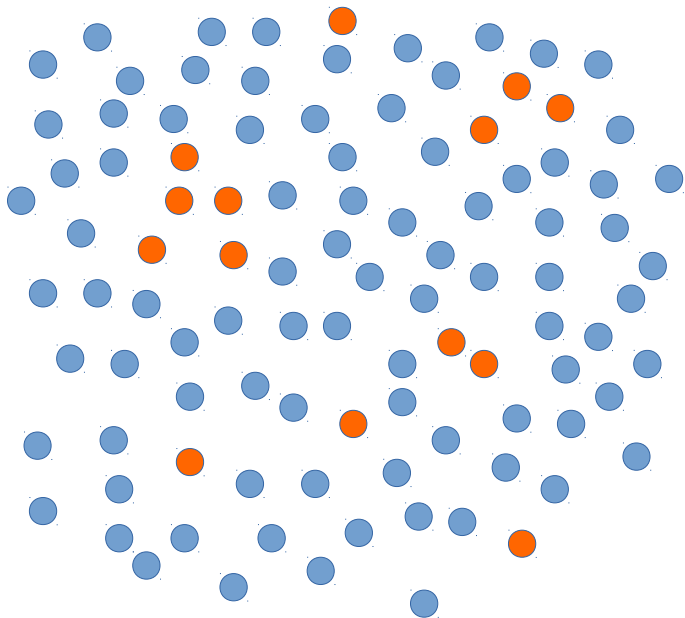
We can simulate a quantum circuit with perfect gates by a circuit with noisy gates of size **linear**

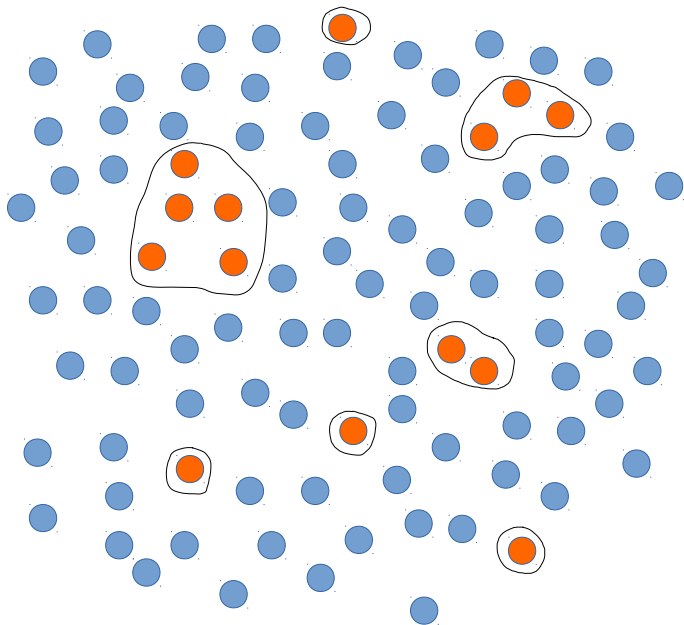


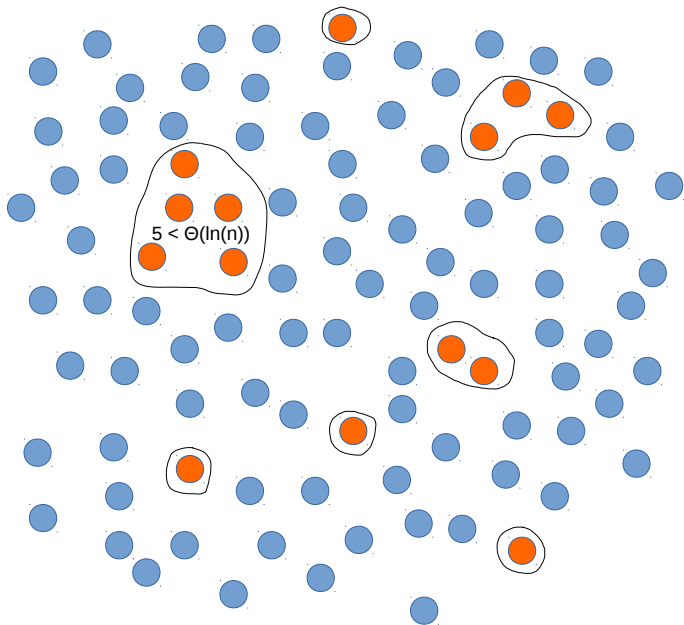
# Locality of the algorithm : the bit-graph











In the following, we will say whp  $P$  (with high probability the property  $P$  holds) if :  $\lim_{n \rightarrow +\infty} \mathbb{P}(P) = 1$

In the following, we will say whp  $P$  (with high probability the property  $P$  holds) if :  $\lim_{n \rightarrow +\infty} \mathbb{P}(P) = 1$

## Percolation Theorem

For a probability of error  $p < \frac{1}{1+d}$ , whp :

- The size of any connected components is  $\leq \Theta(\ln(n))$

For a probability of error  $p > \frac{1}{1+d}$ , whp :

- There is a connected component of size  $\Theta(n)$

In the following, we will say whp  $P$  (with high probability the property  $P$  holds) if :  $\lim_{n \rightarrow +\infty} \mathbb{P}(P) = 1$

## Percolation Theorem

For a probability of error  $p < \frac{1}{1+d}$ , whp :

- The size of any connected components is  $\leq \Theta(\ln(n))$

For a probability of error  $p > \frac{1}{1+d}$ , whp :

- There is a connected component of size  $\Theta(n)$

### Good news :

The algorithm corrects any error of size  $\leq \Theta(\sqrt{n})$

The algorithm corrects any error of size  $\leq \Theta(\ln(n))$



In the following, we will say whp  $P$  (with high probability the property  $P$  holds) if :  $\lim_{n \rightarrow +\infty} \mathbb{P}(P) = 1$

## Percolation Theorem

For a probability of error  $p < \frac{1}{1+d}$ , whp :

- The size of any connected components is  $\leq \Theta(\ln(n))$

For a probability of error  $p > \frac{1}{1+d}$ , whp :

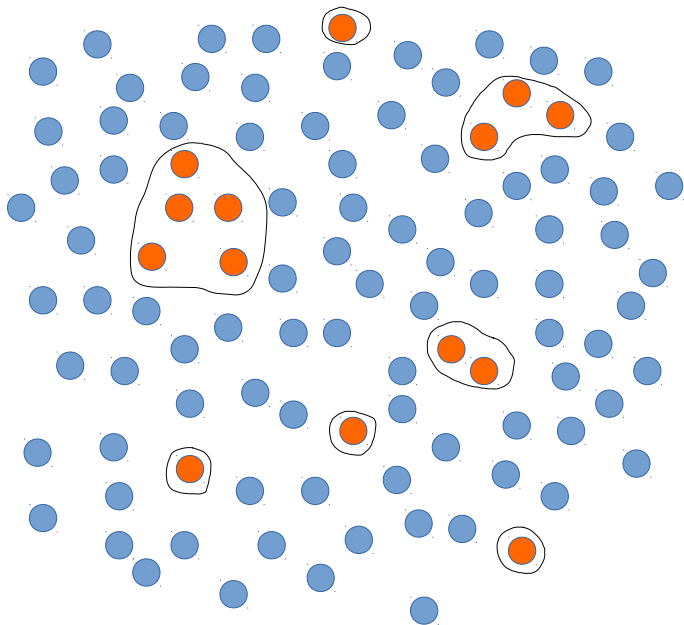
- There is a connected component of size  $\Theta(n)$

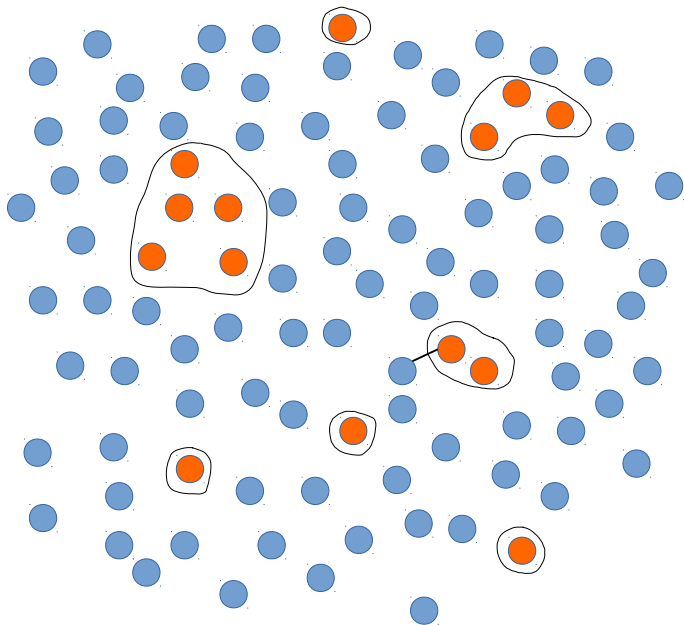
### Good news :

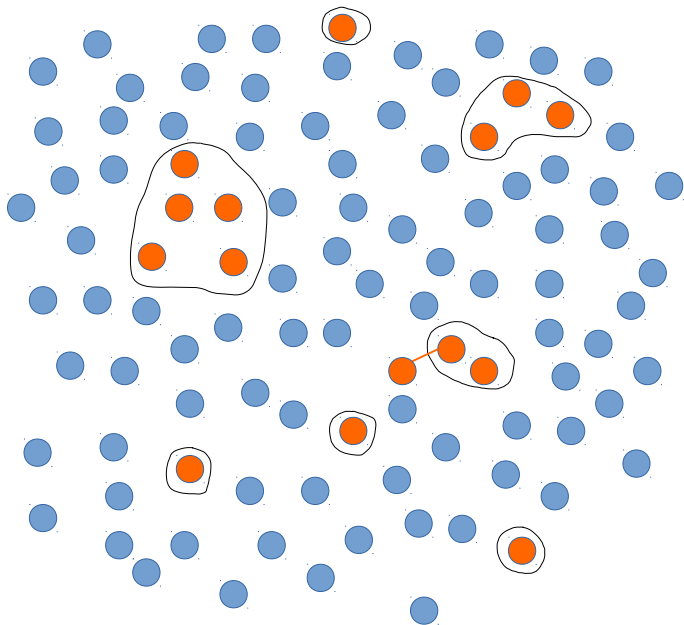
The algorithm corrects any error of size  $\leq \Theta(\sqrt{n})$

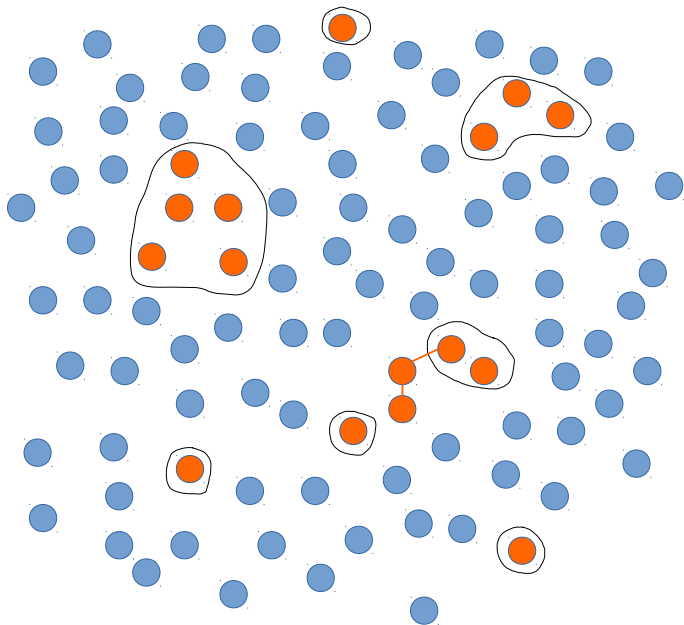
The algorithm corrects any error of size  $\leq \Theta(\ln(n))$

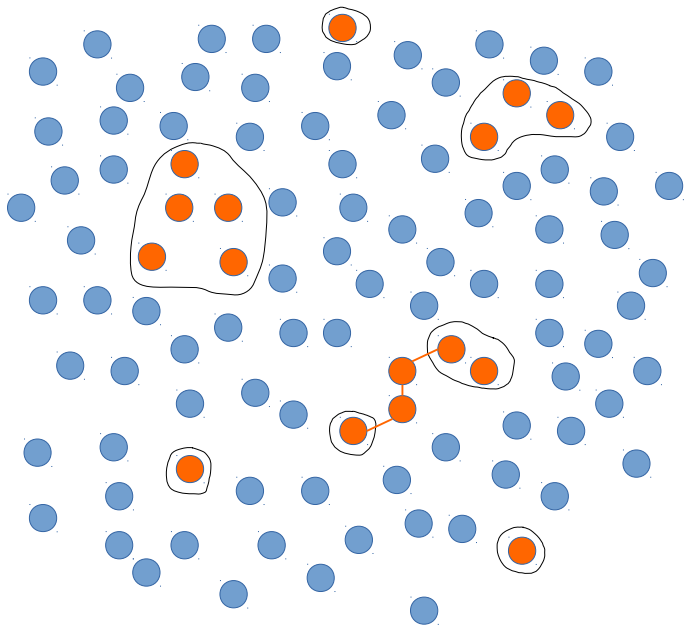
**Problem** : Some clusters can merge during the decoding











## Percolation Theorem

For a probability of error  $p < \frac{1}{1+d}$ , whp :

- The size of any connected components is  $\leq \Theta(\ln(n))$

## Percolation Theorem

For a probability of error  $p < \frac{1}{1+d}$ , whp :

- The size of any connected components is  $\leq \Theta(\ln(n))$

## Percolation theorem, reformulation

For a probability of error  $p < \frac{1}{1+d}$ , whp :

- if  $|X \cap E(p)| \geq 1 \times |X|$  then  $|X| < \Theta(\ln(n))$



## Percolation Theorem

For a probability of error  $p < \frac{1}{1+d}$ , whp :

- The size of any connected components is  $\leq \Theta(\ln(n))$

## Percolation theorem, reformulation

For a probability of error  $p < \frac{1}{1+d}$ , whp :

- if  $|X \cap E(p)| \geq 1 \times |X|$  then  $|X| < \Theta(\ln(n))$

## Percolation theorem, generalisation

$\forall \alpha > 0$ , if  $p < \text{cst}(\alpha, d)$ , whp :

- if  $|X \cap E(p)| \geq \alpha \times |X|$  then  $|X| < \Theta(\ln(n))$

## Percolation Theorem

For a probability of error  $p < \frac{1}{1+d}$ , whp :

- The size of any connected components is  $\leq \Theta(\ln(n))$

## Percolation theorem, reformulation

For a probability of error  $p < \frac{1}{1+d}$ , whp :

- if  $|X \cap E(p)| \geq 1 \times |X|$  then  $|X| < \Theta(\ln(n))$

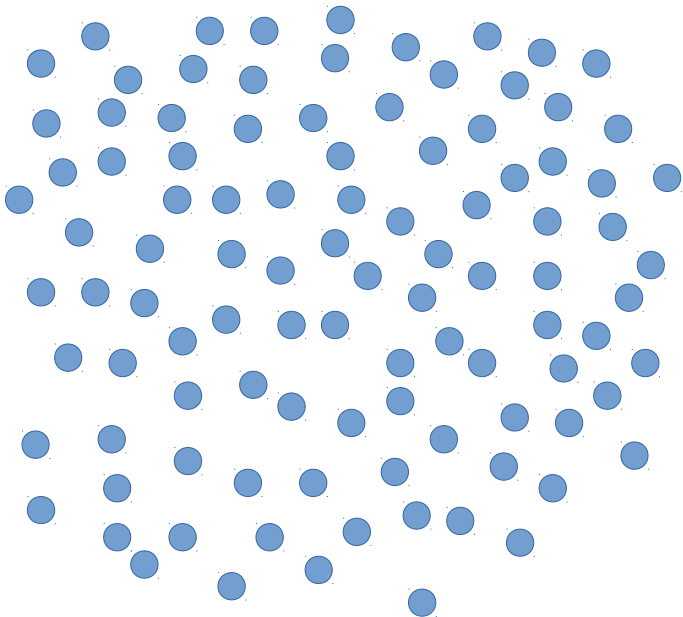
## Percolation theorem, generalisation

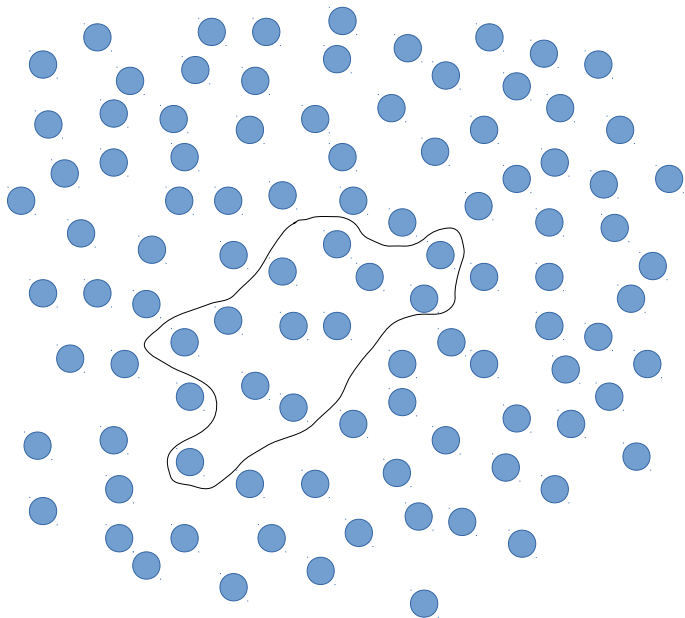
$\forall \alpha > 0$ , if  $p < cst(\alpha, d)$ , whp :

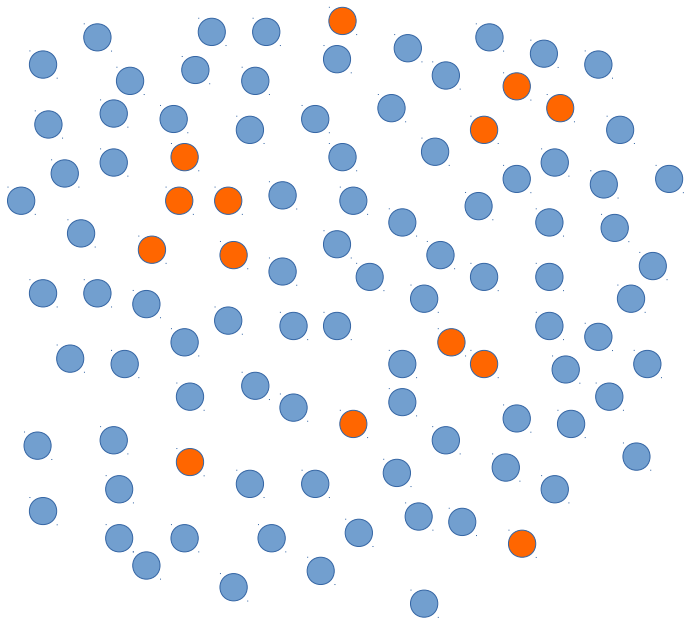
- if  $|X \cap E(p)| \geq \alpha \times |X|$  then  $|X| < \Theta(\ln(n))$

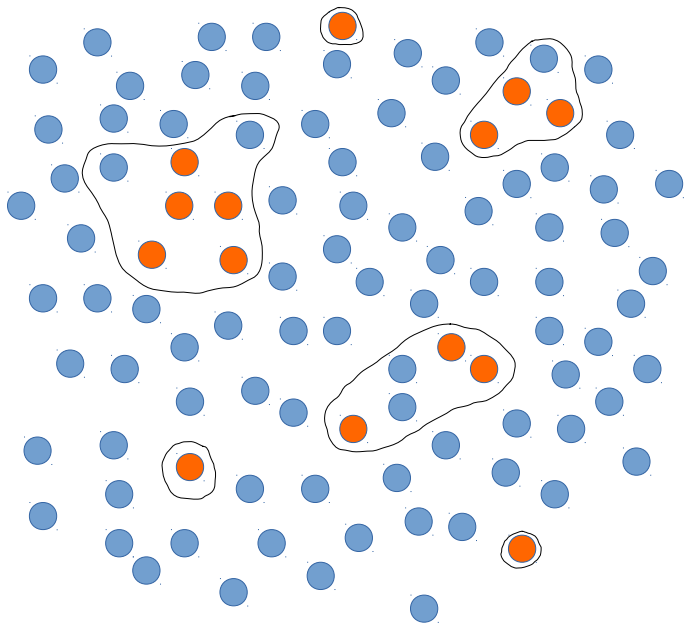
## Complexity of the decoding algorithm

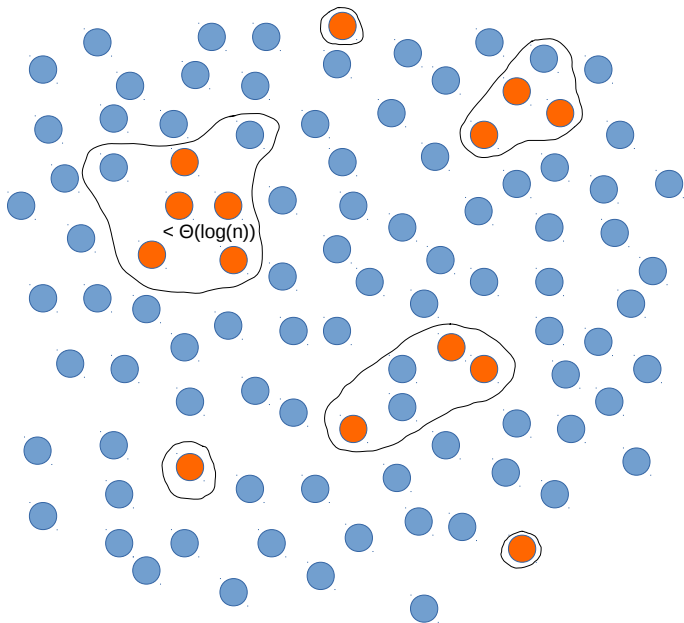
The number of flips is linear in the size of the initial error











## Theorem

For a probability of error  $p < \frac{1}{(ed)^{2d}}$  :

$$\lim_{n \rightarrow +\infty} \mathbb{P}(\mathcal{A} \text{ corrects the error}) = 1$$



## Theorem

For a probability of error  $p < \frac{1}{(ed)^{2d}}$  :

$$\lim_{n \rightarrow +\infty} \mathbb{P}(\mathcal{A} \text{ corrects the error}) = 1$$

### Ideas to improve this bound :

- 1 Improve the bound in the percolation theorem :  
What is the critical probability ?
- 2 Restrict the proof to interesting clusters :
  - \* The diameter of an interesting cluster is  $\leq \Theta(\ln(\ln(n)))$
  - \* The number of edges inside an interesting cluster is large  
(ideas from bootstrap percolation)

# Conclusion

The hypergraph product of an expander code :

- is an LDPC quantum code
- has a constant rate
- has a minimal distance :  $d = \Theta(\sqrt{n})$

The decoding algorithm :

- has a capacity of correction :  $\Theta(\sqrt{n})$
- corrects the error with high probability for the depolarizing channel

Futur work :

- improve our bound
- apply this result to fault tolerant quantum computation (Gottesman)

Thank you for your attention