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# On the Riesz means of $\delta_k(n)$

Saurabh Kumar Singh

**Abstract.** Let  $k \geq 1$  be an integer. Let  $\delta_k(n)$  denote the maximum divisor of  $n$  which is co-prime to  $k$ . We study the error term of the general  $m$ -th Riesz mean of the arithmetical function  $\delta_k(n)$  for any positive integer  $m \geq 1$ , namely the error term  $E_{m,k}(x)$  where

$$\frac{1}{m!} \sum_{n \leq x} \delta_k(n) \left(1 - \frac{n}{x}\right)^m = M_{m,k}(x) + E_{m,k}(x).$$

We establish a non-trivial upper bound for  $|E_{m,k}(x)|$ , for any integer  $m \geq 1$ .

**Keywords.** Euler-totient function, Generating functions, Riemann zeta-function.

**2010 Mathematics Subject Classification.** Primary 11A25; Secondary 11N37.

## 1. Introduction

For any fixed positive integer  $k$ , we define

$$\delta_k(n) = \max\{d : d \mid n, (d, k) = 1\}. \quad (1.1)$$

Joshi and Vaidya [JV] proved that

$$\sum_{n \leq x} \delta_k(n) = \frac{k}{2\sigma(k)} x^2 + E_k(x), \quad (1.2)$$

with  $E_k(x) = O(x)$  and  $\sigma(k) = \sum_{d|k} d$ , when  $k$  is a square free positive integer. They also proved that when  $k = p$ , a prime,

$$\underline{\lim}_{n \rightarrow \infty} \frac{E_p(x)}{x} = -\frac{p}{p+1}, \quad \text{and} \quad \overline{\lim}_{n \rightarrow \infty} \frac{E_p(x)}{x} = \frac{p}{p+1}.$$

It was proved by Maxsein and Herzog [MH] that for any square free positive integer  $k$ ,

$$\underline{\lim}_{n \rightarrow \infty} \frac{E_k(x)}{x} \leq -\frac{k}{\sigma(k)}, \quad \text{and} \quad \overline{\lim}_{n \rightarrow \infty} \frac{E_k(x)}{x} \geq \frac{k}{\sigma(k)}.$$

Around the same time, Adhikari, Balasubramanian and Sankaranarayanan [ABS] proved the above results by a different method. While a tauberian theorem of Hardy-Littlewood and Karamata was used in [MH] to get the asymptotic formula for  $\sum_{n \leq x} \gamma_k(n)$ , where  $\gamma_k(n)$  is defined by the relation  $\delta_k(n) = \gamma_k * I(n)$  where  $*$  is the Dirichlet convolution and  $I$  is the identity function, the method of [ABS] consists of averaging over arithmetical progressions.

For  $k \geq 1$  and square free, Harzog and Maxsein [MH] had also observed that

$$\limsup_{x \rightarrow \infty} \frac{E_k(x)}{x} \leq \frac{1}{2} d(k),$$

where  $d(k)$  denotes the number of divisors of  $k$ . Later Adhikari and Balasubramanian [AB] improved this result of Maxsein and Herzog by showing that

$$\overline{\lim}_{n \rightarrow \infty} \frac{|E_k(x)|}{x} \leq \frac{1}{2} \left( 1 - \frac{1}{p+1} \right) d(k),$$

where  $p$  denotes the smallest prime dividing  $k$ .

Writing

$$H_k(x) = \sum_{n \leq x} \frac{\delta_k(n)}{n} - \frac{kx}{\sigma(k)},$$

one observes (see [ABS]) that

$$\frac{E_k(x)}{x} = H_k(x) + O(1).$$

In [AS], more precise upper and lower bounds for the quantities  $\underline{\lim} H_k(x)$  and  $\overline{\lim} H_k(x)$  were established. The aim of this article is to study the error term of the general  $m$ -th Riesz mean related to the arithmetic function  $\delta_k(n)$  for any positive integer  $m \geq 1$  and  $k \geq 1$ . More precisely, we write

$$\frac{1}{m!} \sum_{n \leq x} \delta_k(n) \left( 1 - \frac{n}{x} \right)^m = M_{m,k}(x) + E_{m,k}(x) \quad (1.3)$$

where  $M_{m,k}(x)$  is the main term (exists) and  $E_{m,k}(x)$  is the error term of the sum under investigation. We prove the following.

**Theorem 1.1.** *Let  $x \geq x_0$  where  $x_0$  is a sufficiently large positive number and let  $c(\eta) = \frac{2}{1-2^{-\eta}}$  for any  $\eta > 0$ . For any integer  $m \geq 1$  and for any integer  $k \geq 1$ , we have*

$$\frac{1}{m!} \sum_{n \leq x} \delta_k(n) \left( 1 - \frac{n}{x} \right)^m = \frac{x^2}{(m+2)!} \prod_{p|k} \frac{p}{p+1} + E_{m,k}(x),$$

where

$$E_{1,k}(x) \ll kc(1/2)^{\omega(k)} x^{\frac{1}{2}} \log x,$$

and for  $m \geq 2$ , we have

$$E_{m,k}(x) \ll kc(\eta)^{\omega(k)} x^\eta$$

for any small fixed positive constant  $\eta$  and the implied constant is independent of  $m$ .

## 2. Notation

1. Throughout the paper,  $s = \sigma + it$ ; the parameters  $T$  and  $x$  are sufficiently large real numbers and  $m$  is an integer  $\geq 1$ .
2.  $\eta, \epsilon$  always denote sufficiently small positive constants.
3. As usual  $\zeta(s)$  denotes the Riemann zeta-function.
4.  $k$  is any square free positive integer.

### 3. Some Lemmas

Generating function for  $\delta_k(n)$  is given by:

**Lemma 3.1.** *We have*

$$\sum_{n=1}^{\infty} \frac{\delta_k(n)}{n^s} = \zeta(s-1)G(s),$$

where

$$G(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s} = \prod_{p|k} \left( \frac{1 - \frac{p}{p^s}}{1 - \frac{1}{p^s}} \right) \ll k c(\eta)^{\omega(k)},$$

for  $\sigma \geq \eta$  and

$$c(\eta) = \frac{2}{1 - 2^{-\eta}}.$$

*Proof.* We have (see [ABS, equation 2.2]),

$$\begin{aligned} \sum_{n=2}^{\infty} \frac{\delta_k(n)}{n^s} &= \prod_p \left( 1 + \frac{\delta_k(p)}{p^s} + \frac{\delta_k(p^2)}{p^{2s}} + \dots \right) \\ &= \prod_{p|k} \left( 1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \dots \right) \prod_{p \nmid k} \left( 1 + \frac{p}{p^s} + \frac{p^2}{p^{2s}} + \dots \right) \\ &= \zeta(s-1) \prod_{p|k} \frac{1 - \frac{1}{p^{s-1}}}{1 - \frac{1}{p^s}} := \zeta(s-1)G(s), \end{aligned}$$

since

$$\delta_k(p^m) = \begin{cases} 1 & \text{if } p \mid k \\ p^m & \text{if } p \nmid k. \end{cases}$$

And for  $\sigma \geq \eta$  ( $> 0$ ), we observe that

$$|G(s)| = \prod_{p|k} \left| \frac{1 - \frac{1}{p^{s-1}}}{1 - \frac{1}{p^s}} \right| \leq \prod_{p|k} \frac{1 + p^{1-\eta}}{1 - \frac{1}{p^\eta}} \leq \prod_{p|k} \frac{2p}{1 - \frac{1}{2^\eta}} \leq kc(\eta)^{\omega(k)}.$$

**Lemma 3.2.** *Let  $m$  be an integer  $\geq 1$ . Let  $c$  and  $y$  be any positive real numbers and  $T \geq T_0$  where  $T_0$  is sufficiently large. Then we have,*

$$\frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{y^s}{s(s+1)\cdots(s+m)} ds = \begin{cases} \frac{1}{m!} \left(1 - \frac{1}{y}\right)^m + O\left(\frac{4^m y^c}{T^m}\right) & \text{if } y \geq 1, \\ O\left(\frac{1}{T^m}\right) & \text{if } 0 < y \leq 1. \end{cases}$$

*Proof.* See [SS, Lemma 3.2] and also [In, p.31 Theorem B]).

**Lemma 3.3.** *The Riemann zeta-function  $\zeta(s)$  is extended as a meromorphic function in the whole complex plane  $\mathbb{C}$  with a simple pole at  $s = 1$  and it satisfies a functional equation  $\zeta(s) = \chi(s)\zeta(1-s)$  where*

$$\chi(s) = \frac{\pi^{-(1-s)/2} \Gamma\left(\frac{1-s}{2}\right)}{\pi^{-s/2} \Gamma\left(\frac{s}{2}\right)}.$$

Also, in any bounded vertical strip, using Stirling's formula, we have

$$\chi(s) = \left(\frac{2\pi}{t}\right)^{\sigma+it-1/2} e^{i(t+\frac{\pi}{4})} (1 + O(t^{-1}))$$

as  $|t| \rightarrow \infty$ . Thus, in any bounded vertical strip,

$$|\chi(s)| \asymp t^{1/2-\sigma} (1 + O(t^{-1}))$$

as  $|t| \rightarrow \infty$ .

*Proof.* See [T, p.116] or [Iv, p.8-12].

**Lemma 3.4.** *We have for  $t \geq t_0$  (sufficiently large),*

$$\zeta\left(\frac{1}{2} + it\right) \ll t^{1/6}(\log t)^{3/2}$$

and

$$\zeta(1 + it) \ll \log t.$$

*Proof.* See [T, page 99, Theorem 5.5] and [T, page 49, Theorem 3.5]

## 4. Proof of theorem 1.1

From Lemma 3.2, with  $c = 2 + \frac{1}{\log x}$  and writing  $F(s) := \zeta(s-1)G(s)$ , we have

$$\begin{aligned} S &:= \sum_{n \leq x} \delta_k(n) \left(1 - \frac{n}{x}\right)^m = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s) \frac{x^s}{s(s+1)\cdots(s+m)} ds \\ &= \frac{1}{2\pi i} \int_{c-iT}^{c+iT} F(s) \frac{x^s}{s(s+1)\cdots(s+m)} ds + O\left(\frac{4^m x^c \log x}{T^m}\right). \end{aligned} \quad (4.4)$$

Note that the tail portion error term in the above expression is actually

$$\ll \frac{4^m}{T^m} x^c \sum_{n \leq x} \frac{\delta_k(n)}{n^c} \ll \frac{4^m x^c \log x}{T^m},$$

since  $\delta_k(n) \leq n$ .

**Case 1:** Let  $m = 1$ . We move the line of integration in the above integral to  $\Re s = \frac{1}{2}$ . In the rectangular contour formed by the line segments joining the points  $c - iT$ ,  $c + iT$ ,  $\frac{1}{2} + iT$ ,  $\frac{1}{2} - iT$  and  $c - iT$  in the anticlockwise order, we observe that  $s = 2$  is a simple pole of the integrand. Thus we get the main term  $\frac{x^2}{(m+2)!} \prod_{p|k} \frac{p}{p+1}$  from the residue coming from the pole  $s = 2$ .

We note that

$$\begin{aligned} &\frac{1}{2\pi i} \int_{c-iT}^{c+iT} F(s) \frac{x^s}{s(s+1)} ds \\ &= \frac{1}{2\pi i} \left\{ \int_{\frac{1}{2}+iT}^{c+iT} \cdots + \int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \cdots + \int_{c-iT}^{\frac{1}{2}-iT} \cdots \right\} + \text{sum of the residues.} \end{aligned} \quad (4.5)$$

The left vertical line segment contributes the quantity:

$$\begin{aligned}
Q_1 &:= \frac{1}{2\pi} \int_{-T}^T F(1/2 + it) \frac{x^{1/2+it} dt}{(-1/2 + it)(1/2 + it)} dt \\
&= \frac{1}{2\pi} \left( \int_{|t| \leq t_0} + \int_{t_0 < |t| \leq T} \right) \frac{x^{\frac{1}{2}+it} \zeta\left(-\frac{1}{2} + it\right) G\left(\frac{1}{2} + it\right) dt}{\left(\frac{1}{2} + it\right) \left(\frac{1}{2} + it\right)} \\
&\ll k c(1/2)^{\omega(k)} x^{1/2} + k c(1/2)^{\omega(k)} x^{1/2} \int_{t_0 < |t| \leq T} t^{1/2 - (-1/2)} \left| \zeta(3/2 + it) G\left(\frac{1}{2} + it\right) \right| \frac{dt}{t^2} \\
&\ll k c(1/2)^{\omega(k)} x^{1/2} + k c(1/2)^{\omega(k)} x^{1/2} \int_{t_0 < |t| \leq T} \frac{dt}{t}. \\
&\ll k c(1/2)^{\omega(k)} x^{1/2} \log T. \tag{4.6}
\end{aligned}$$

Now we will estimate the contributions coming from the upper horizontal line (estimation for the lower horizontal line is similar).

The horizontal lines in total contribute a quantity which is in absolute value

$$\begin{aligned}
&\ll \int_{1/2}^c \left| \zeta(\sigma - 1 + iT) G(\sigma + iT) \frac{x^{\sigma+iT}}{(\sigma + iT)(\sigma + 1 + iT)} \right| d\sigma \\
&\ll \left( \int_{1/2}^1 + \int_1^{3/2} + \int_{3/2}^c \right) |\zeta(\sigma - 1 + iT) G(\sigma + iT)| \frac{x^\sigma}{T^2} d\sigma \\
&\ll k c(1/2)^{\omega(k)} \left\{ \left( \int_{1/2}^1 + \int_1^{3/2} \right) T^{1/2 - \sigma + 1} |\zeta(2 - \sigma + iT)| \frac{x^\sigma}{T^2} d\sigma \right. \\
&\quad \left. + \int_{3/2}^c |\zeta(\sigma - 1 + iT)| \frac{x^\sigma}{T^2} d\sigma \right\} \text{ (by Lemma 3.3)} \\
&\ll k c(1/2)^{\omega(k)} \left( \frac{x \log T}{T} + \frac{x^{3/2} \log T}{T^{3/2}} + \frac{x^2 \log T}{T^{11/6}} \right) \text{ (by Lemma 3.4)}.
\end{aligned}$$

Collecting all the estimates, and taking  $T = x^{10}$  we get:

$$\begin{aligned}
E_{1,k}(x) &\ll k c(1/2)^{\omega(k)} \left( x^{1/2} \log T + \frac{x^2}{T} + \frac{x \log T}{T} + \frac{x^{3/2} \log T}{T^{3/2}} + \frac{x^2 \log T}{T^{11/6}} \right) \\
&\ll k c(1/2)^{\omega(k)} x^{1/2} \log x. \tag{4.7}
\end{aligned}$$

**Case 2:** Let  $m \geq 2$ . We move the line of integration to  $\Re s = \eta (> 0)$ .

We note that

$$\begin{aligned}
&\frac{1}{2\pi i} \int_{c-iT}^{c+iT} F(s) \frac{x^s}{s(s+1)\cdots(s+m)} ds \\
&= \frac{1}{2\pi i} \left\{ \int_{\delta+iT}^{c+iT} \cdots + \int_{\delta-iT}^{\delta+iT} \cdots + \int_{c-iT}^{\delta-iT} \cdots \right\} + \text{sum of the residue.} \tag{4.8}
\end{aligned}$$

The left vertical line segment contributes the quantity:

$$\begin{aligned}
Q_m &:= \frac{1}{2\pi} \int_{-T}^T F(\eta + it) \frac{x^{\eta+it} dt}{(\eta + it)(\eta + 1 + it) \cdots (\eta + m + it)} \\
&= \frac{1}{2\pi} \left( \int_{|t| \leq t_0} + \int_{t_0 < |t| \leq T} \right) \frac{x^{\eta+it} \zeta(\eta - 1 + it) G(\eta + it) dt}{(\eta + it)(\eta + 1 + it) \cdots (\eta + m + it)} \\
&\ll k c(\eta)^{\omega(k)} x^\eta + k c(\eta)^{\omega(k)} x^\eta \int_{t_0 < |t| \leq T} t^{1/2-(\eta-1)} |\zeta(3/2 - \eta + it) G(\eta + it)| \frac{dt}{t^{m+1}} \\
&\ll k c(\eta)^{\omega(k)} x^\eta + k c(\eta)^{\omega(k)} x^\eta \int_{t_0 < |t| \leq T} \frac{t^{3/2-\eta}}{t^3} dt. \\
&\ll k c(\eta)^{\omega(k)} x^\eta. \tag{4.9}
\end{aligned}$$

Now we will estimate the contributions coming from the upper horizontal line (estimation for the lower horizontal line is similar).

The horizontal lines in total contribute a quantity which is in absolute value

$$\begin{aligned}
&\ll \int_\eta^c \left| \zeta(\sigma - 1 + iT) G(\sigma + iT) \frac{x^{\sigma+iT}}{(\sigma + iT)(\sigma + 1 + iT) \cdots (\sigma + m + iT)} \right| d\sigma \\
&\ll c(\eta)^{\omega(k)} k \left( \int_\eta^1 + \int_1^{3/2} + \int_{3/2}^c \right) |\zeta(\sigma - 1 + iT)| \frac{x^\sigma}{T^{k+1}} \\
&\ll k c(\eta)^{\omega(k)} \left\{ \left( \int_\eta^{1/2} + \int_{1/2}^1 + \int_1^{3/2} \right) T^{1/2-\sigma+1} |\zeta(2 - \sigma + iT)| \frac{x^\sigma}{T^{m+1}} d\sigma \right. \\
&\quad \left. + \int_{3/2}^c |\zeta(\sigma - 1 + iT)| \frac{x^\sigma}{T^{m+1}} d\sigma \right\} \\
&\ll k c(\eta)^{\omega(k)} \left( \frac{x^{1/2}}{T^{m-1/2+\eta}} + \frac{x \log T}{T^m} + \frac{x^{3/2} (\log T)^{3/2}}{T^{m+5/6}} + \frac{x^2 (\log T)^{3/2}}{T^{m+5/6}} \right)
\end{aligned}$$

Collecting all the estimates, and taking  $T = x^{10}$ , for  $m \geq 2$  we get:

$$E_{m,k}(x) \ll k c(\eta)^{\omega(k)} x^\eta. \tag{4.10}$$

This proves Theorem 1.1.

**Remark 4.1.** For  $m \geq 2$  we may try to move the line of integration slightly left of vertical line 0. On the line  $\Re s = 0$ , the function  $G(s)$  has simple poles at the points  $s(\ell, p) = \frac{2\pi i \ell}{\log p} \quad \forall \ell \in \mathbb{Z}$  and for each prime  $p \mid k$ . let  $p_1, p_2, \dots, p_{r_k}$  be the primes dividing  $k$ . The total contribution from the simple poles at the points  $s(\ell, p) = \frac{2\pi i \ell}{\log p_j}$  for  $1 \leq j \leq r_k$  is given by:

$$M = \sum_{j=1}^{r_k} \sum_{|\ell| \leq \frac{T \log p_j}{2\pi}} \zeta \left( \frac{2\pi i \ell}{\log p_j} - 1 \right) \prod_{p_i \neq p_j} \left( 1 - \frac{p_i}{p_j^{\frac{2\pi i \ell}{\log p_j}}} \right) \frac{x^{\frac{2\pi i \ell}{\log p_j}}}{\frac{2\pi i \ell}{\log p_j} \left( \frac{2\pi i \ell}{\log p_j} + 1 \right) \cdots \left( \frac{2\pi i \ell}{\log p_j} + m \right)}.$$

If one establishes that  $M = o(x^\eta)$ , then this will improve the error term. This seems to be really difficult.

**Remark 4.2.** From the Theorem 1.1 observe that

$$E_{1,k}(x) \ll_{\epsilon} x^{1/2+10\epsilon}$$

uniformly for  $3 \leq k \ll x^{\epsilon}$  since  $\omega(k) \ll \frac{\log}{\log \log k}$  for  $k \geq 3$ . Also  $E_{m,k}(x) \ll x^{c_1 \eta}$  uniformly for  $3 \leq k \ll x^{\epsilon}$ , where  $c_1$  is effective positive constant.

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