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On the Riesz means of $\delta_k(n)$

Saurabh Kumar Singh

Abstract. Let $k \geq 1$ be an integer. Let $\delta_k(n)$ denote the maximum divisor of n which is co-prime to k . We study the error term of the general m -th Riesz mean of the arithmetical function $\delta_k(n)$ for any positive integer $m \geq 1$, namely the error term $E_{m,k}(x)$ where

$$\frac{1}{m!} \sum_{n \leq x} \delta_k(n) \left(1 - \frac{n}{x}\right)^m = M_{m,k}(x) + E_{m,k}(x).$$

We establish a non-trivial upper bound for $|E_{m,k}(x)|$, for any integer $m \geq 1$.

Keywords. Euler-totient function, Generating functions, Riemann zeta-function.

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1. Introduction

For any fixed positive integer k , we define

$$\delta_k(n) = \max\{d : d \mid n, (d, k) = 1\}. \quad (1.1)$$

Joshi and Vaidya [JV] proved that

$$\sum_{n \leq x} \delta_k(n) = \frac{k}{2\sigma(k)} x^2 + E_k(x), \quad (1.2)$$

with $E_k(x) = O(x)$ and $\sigma(k) = \sum_{d|k} d$, when k is a square free positive integer. They also proved that when $k = p$, a prime,

$$\underline{\lim}_{n \rightarrow \infty} \frac{E_p(x)}{x} = -\frac{p}{p+1}, \quad \text{and} \quad \overline{\lim}_{n \rightarrow \infty} \frac{E_p(x)}{x} = \frac{p}{p+1}.$$

It was proved by Maxsein and Herzog [MH] that for any square free positive integer k ,

$$\underline{\lim}_{n \rightarrow \infty} \frac{E_k(x)}{x} \leq -\frac{k}{\sigma(k)}, \quad \text{and} \quad \overline{\lim}_{n \rightarrow \infty} \frac{E_k(x)}{x} \geq \frac{k}{\sigma(k)}.$$

Around the same time, Adhikari, Balasubramanian and Sankaranarayanan [ABS] proved the above results by a different method. While a tauberian theorem of Hardy-Littlewood and Karamata was used in [MH] to get the asymptotic formula for $\sum_{n \leq x} \gamma_k(n)$, where $\gamma_k(n)$ is defined by the relation $\delta_k(n) = \gamma_k * I(n)$ where $*$ is the Dirichlet convolution and I is the identity function, the method of [ABS] consists of averaging over arithmetical progressions.

For $k \geq 1$ and square free, Harzog and Maxsein [MH] had also observed that

$$\limsup_{x \rightarrow \infty} \frac{E_k(x)}{x} \leq \frac{1}{2} d(k),$$

where $d(k)$ denotes the number of divisors of k . Later Adhikari and Balasubramanian [AB] improved this result of Maxsein and Herzog by showing that

$$\overline{\lim}_{n \rightarrow \infty} \frac{|E_k(x)|}{x} \leq \frac{1}{2} \left(1 - \frac{1}{p+1}\right) d(k),$$

where p denotes the smallest prime dividing k .

Writing

$$H_k(x) = \sum_{n \leq x} \frac{\delta_k(n)}{n} - \frac{kx}{\sigma(k)},$$

one observes (see [ABS]) that

$$\frac{E_k(x)}{x} = H_k(x) + O(1).$$

In [AS], more precise upper and lower bounds for the quantities $\underline{\lim} H_k(x)$ and $\overline{\lim} H_k(x)$ were established. The aim of this article is to study the error term of the general m -th Riesz mean related to the arithmetic function $\delta_k(n)$ for any positive integer $m \geq 1$ and $k \geq 1$. More precisely, we write

$$\frac{1}{m!} \sum_{n \leq x} \delta_k(n) \left(1 - \frac{n}{x}\right)^m = M_{m,k}(x) + E_{m,k}(x) \quad (1.3)$$

where $M_{m,k}(x)$ is the main term (exists) and $E_{m,k}(x)$ is the error term of the sum under investigation. We prove the following.

Theorem 1.1. *Let $x \geq x_0$ where x_0 is a sufficiently large positive number and let $c(\eta) = \frac{2}{1-2^{-\eta}}$ for any $\eta > 0$. For any integer $m \geq 1$ and for any integer $k \geq 1$, we have*

$$\frac{1}{m!} \sum_{n \leq x} \delta_k(n) \left(1 - \frac{n}{x}\right)^m = \frac{x^2}{(m+2)!} \prod_{p|k} \frac{p}{p+1} + E_{m,k}(x),$$

where

$$E_{1,k}(x) \ll kc(1/2)^{\omega(k)} x^{\frac{1}{2}} \log x,$$

and for $m \geq 2$, we have

$$E_{m,k}(x) \ll kc(\eta)^{\omega(k)} x^\eta$$

for any small fixed positive constant η and the implied constant is independent of m .

2. Notation

1. Throughout the paper, $s = \sigma + it$; the parameters T and x are sufficiently large real numbers and m is an integer ≥ 1 .
2. η, ϵ always denote sufficiently small positive constants.
3. As usual $\zeta(s)$ denotes the Riemann zeta-function.
4. k is any square free positive integer.

3. Some Lemmas

Generating function for $\delta_k(n)$ is given by:

Lemma 3.1. *We have*

$$\sum_{n=1}^{\infty} \frac{\delta_k(n)}{n^s} = \zeta(s-1)G(s),$$

where

$$G(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s} = \prod_{p|k} \left(\frac{1 - \frac{p}{p^s}}{1 - \frac{1}{p^s}} \right) \ll k c(\eta)^{\omega(k)},$$

for $\sigma \geq \eta$ and

$$c(\eta) = \frac{2}{1 - 2^{-\eta}}.$$

Proof. We have (see [ABS, equation 2.2]),

$$\begin{aligned} \sum_{n=2}^{\infty} \frac{\delta_k(n)}{n^s} &= \prod_p \left(1 + \frac{\delta_k(p)}{p^s} + \frac{\delta_k(p^2)}{p^{2s}} + \dots \right) \\ &= \prod_{p|k} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \dots \right) \prod_{p \nmid k} \left(1 + \frac{p}{p^s} + \frac{p^2}{p^{2s}} + \dots \right) \\ &= \zeta(s-1) \prod_{p|k} \frac{1 - \frac{1}{p^{s-1}}}{1 - \frac{1}{p^s}} := \zeta(s-1)G(s), \end{aligned}$$

since

$$\delta_k(p^m) = \begin{cases} 1 & \text{if } p | k \\ p^m & \text{if } p \nmid k. \end{cases}$$

And for $\sigma \geq \eta$ (> 0), we observe that

$$|G(s)| = \prod_{p|k} \left| \frac{1 - \frac{1}{p^{s-1}}}{1 - \frac{1}{p^s}} \right| \leq \prod_{p|k} \frac{1 + p^{1-\eta}}{1 - \frac{1}{p^\eta}} \leq \prod_{p|k} \frac{2p}{1 - \frac{1}{2^\eta}} \leq kc(\eta)^{\omega(k)}.$$

Lemma 3.2. *Let m be an integer ≥ 1 . Let c and y be any positive real numbers and $T \geq T_0$ where T_0 is sufficiently large. Then we have,*

$$\frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{y^s}{s(s+1)\cdots(s+m)} ds = \begin{cases} \frac{1}{m!} \left(1 - \frac{1}{y}\right)^m + O\left(\frac{4^m y^c}{T^m}\right) & \text{if } y \geq 1, \\ O\left(\frac{1}{T^m}\right) & \text{if } 0 < y \leq 1. \end{cases}$$

Proof. See [SS, Lemma 3.2] and also [In, p.31 Theorem B]).

Lemma 3.3. *The Riemann zeta-function $\zeta(s)$ is extended as a meromorphic function in the whole complex plane \mathbb{C} with a simple pole at $s = 1$ and it satisfies a functional equation $\zeta(s) = \chi(s)\zeta(1-s)$ where*

$$\chi(s) = \frac{\pi^{-(1-s)/2} \Gamma\left(\frac{1-s}{2}\right)}{\pi^{-s/2} \Gamma\left(\frac{s}{2}\right)}.$$

Also, in any bounded vertical strip, using Stirling's formula, we have

$$\chi(s) = \left(\frac{2\pi}{t}\right)^{\sigma+it-1/2} e^{i(t+\frac{\pi}{4})} (1 + O(t^{-1}))$$

as $|t| \rightarrow \infty$. Thus, in any bounded vertical strip,

$$|\chi(s)| \asymp t^{1/2-\sigma} (1 + O(t^{-1}))$$

as $|t| \rightarrow \infty$.

Proof. See [T, p.116] or [Iv, p.8-12].

Lemma 3.4. We have for $t \geq t_0$ (sufficiently large),

$$\zeta\left(\frac{1}{2} + it\right) \ll t^{1/6}(\log t)^{3/2}$$

and

$$\zeta(1 + it) \ll \log t.$$

Proof. See [T, page 99, Theorem 5.5] and [T, page 49, Theorem 3.5]

4. Proof of theorem 1.1

From Lemma 3.2, with $c = 2 + \frac{1}{\log x}$ and writing $F(s) := \zeta(s-1)G(s)$, we have

$$\begin{aligned} S &:= \sum_{n \leq x} \delta_k(n) \left(1 - \frac{n}{x}\right)^m = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s) \frac{x^s}{s(s+1)\cdots(s+m)} ds \\ &= \frac{1}{2\pi i} \int_{c-iT}^{c+iT} F(s) \frac{x^s}{s(s+1)\cdots(s+m)} ds + O\left(\frac{4^m x^c \log x}{T^m}\right). \end{aligned} \quad (4.4)$$

Note that the tail portion error term in the above expression is actually

$$\ll \frac{4^m}{T^m} x^c \sum_{n \leq x} \frac{\delta_k(n)}{n^c} \ll \frac{4^m x^c \log x}{T^m},$$

since $\delta_k(n) \leq n$.

Case 1: Let $m = 1$. We move the line of integration in the above integral to $\Re s = \frac{1}{2}$. In the rectangular contour formed by the line segments joining the points $c - iT$, $c + iT$, $\frac{1}{2} + iT$, $\frac{1}{2} - iT$ and $c - iT$ in the anticlockwise order, we observe that $s = 2$ is a simple pole of the integrand. Thus we get the main term $\frac{x^2}{(m+2)!} \prod_{p|k} \frac{p}{p+1}$ from the residue coming from the pole $s = 2$.

We note that

$$\begin{aligned} &\frac{1}{2\pi i} \int_{c-iT}^{c+iT} F(s) \frac{x^s}{s(s+1)} ds \\ &= \frac{1}{2\pi i} \left\{ \int_{\frac{1}{2}+iT}^{c+iT} \cdots + \int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \cdots + \int_{c-iT}^{\frac{1}{2}-iT} \cdots \right\} + \text{sum of the residues.} \end{aligned} \quad (4.5)$$

The left vertical line segment contributes the quantity:

$$\begin{aligned}
Q_1 &:= \frac{1}{2\pi} \int_{-T}^T F(1/2 + it) \frac{x^{1/2+it} dt}{(-1/2 + it)(1/2 + it)} dt \\
&= \frac{1}{2\pi} \left(\int_{|t| \leq t_0} + \int_{t_0 < |t| \leq T} \right) \frac{x^{\frac{1}{2}+it} \zeta\left(-\frac{1}{2} + it\right) G\left(\frac{1}{2} + it\right) dt}{\left(\frac{1}{2} + it\right) \left(\frac{1}{2} + it\right)} \\
&\ll k c(1/2)^{\omega(k)} x^{1/2} + k c(1/2)^{\omega(k)} x^{1/2} \int_{t_0 < |t| \leq T} t^{1/2 - (-1/2)} \left| \zeta(3/2 + it) G\left(\frac{1}{2} + it\right) \right| \frac{dt}{t^2} \\
&\ll k c(1/2)^{\omega(k)} x^{1/2} + k c(1/2)^{\omega(k)} x^{1/2} \int_{t_0 < |t| \leq T} \frac{dt}{t}. \\
&\ll k c(1/2)^{\omega(k)} x^{1/2} \log T. \tag{4.6}
\end{aligned}$$

Now we will estimate the contributions coming from the upper horizontal line (estimation for the lower horizontal line is similar).

The horizontal lines in total contribute a quantity which is in absolute value

$$\begin{aligned}
&\ll \int_{1/2}^c \left| \zeta(\sigma - 1 + iT) G(\sigma + iT) \frac{x^{\sigma+iT}}{(\sigma + iT)(\sigma + 1 + iT)} \right| d\sigma \\
&\ll \left(\int_{1/2}^1 + \int_1^{3/2} + \int_{3/2}^c \right) |\zeta(\sigma - 1 + iT) G(\sigma + iT)| \frac{x^\sigma}{T^2} d\sigma \\
&\ll k c(1/2)^{\omega(k)} \left\{ \left(\int_{1/2}^1 + \int_1^{3/2} \right) T^{1/2-\sigma+1} |\zeta(2 - \sigma + iT)| \frac{x^\sigma}{T^2} d\sigma \right. \\
&\quad \left. + \int_{3/2}^c |\zeta(\sigma - 1 + iT)| \frac{x^\sigma}{T^2} d\sigma \right\} \text{ (by Lemma 3.3)} \\
&\ll k c(1/2)^{\omega(k)} \left(\frac{x \log T}{T} + \frac{x^{3/2} \log T}{T^{3/2}} + \frac{x^2 \log T}{T^{11/6}} \right) \text{ (by Lemma 3.4)}.
\end{aligned}$$

Collecting all the estimates, and taking $T = x^{10}$ we get:

$$\begin{aligned}
E_{1,k}(x) &\ll k c(1/2)^{\omega(k)} \left(x^{1/2} \log T + \frac{x^2}{T} + \frac{x \log T}{T} + \frac{x^{3/2} \log T}{T^{3/2}} + \frac{x^2 \log T}{T^{11/6}} \right) \\
&\ll k c(1/2)^{\omega(k)} x^{1/2} \log x. \tag{4.7}
\end{aligned}$$

Case 2: Let $m \geq 2$. We move the line of integration to $\Re s = \eta (> 0)$.

We note that

$$\begin{aligned}
&\frac{1}{2\pi i} \int_{c-iT}^{c+iT} F(s) \frac{x^s}{s(s+1)\cdots(s+m)} ds \\
&= \frac{1}{2\pi i} \left\{ \int_{\delta+iT}^{c+iT} \cdots + \int_{\delta-iT}^{\delta+iT} \cdots + \int_{c-iT}^{\delta-iT} \cdots \right\} + \text{sum of the residue.} \tag{4.8}
\end{aligned}$$

The left vertical line segment contributes the quantity:

$$\begin{aligned}
Q_m &:= \frac{1}{2\pi} \int_{-T}^T F(\eta + it) \frac{x^{\eta+it} dt}{(\eta + it)(\eta + 1 + it) \cdots (\eta + m + it)} \\
&= \frac{1}{2\pi} \left(\int_{|t| \leq t_0} + \int_{t_0 < |t| \leq T} \right) \frac{x^{\eta+it} \zeta(\eta - 1 + it) G(\eta + it) dt}{(\eta + it)(\eta + 1 + it) \cdots (\eta + m + it)} \\
&\ll k c(\eta)^{\omega(k)} x^\eta + k c(\eta)^{\omega(k)} x^\eta \int_{t_0 < |t| \leq T} t^{1/2-(\eta-1)} |\zeta(3/2 - \eta + it) G(\eta + it)| \frac{dt}{t^{m+1}} \\
&\ll k c(\eta)^{\omega(k)} x^\eta + k c(\eta)^{\omega(k)} x^\eta \int_{t_0 < |t| \leq T} \frac{t^{3/2-\eta}}{t^3} dt. \\
&\ll k c(\eta)^{\omega(k)} x^\eta. \tag{4.9}
\end{aligned}$$

Now we will estimate the contributions coming from the upper horizontal line (estimation for the lower horizontal line is similar).

The horizontal lines in total contribute a quantity which is in absolute value

$$\begin{aligned}
&\ll \int_\eta^c \left| \zeta(\sigma - 1 + iT) G(\sigma + iT) \frac{x^{\sigma+iT}}{(\sigma + iT)(\sigma + 1 + iT) \cdots (\sigma + m + iT)} \right| d\sigma \\
&\ll c(\eta)^{\omega(k)} k \left(\int_\eta^1 + \int_1^{3/2} + \int_{3/2}^c \right) |\zeta(\sigma - 1 + iT)| \frac{x^\sigma}{T^{k+1}} \\
&\ll k c(\eta)^{\omega(k)} \left\{ \left(\int_\eta^{1/2} + \int_{1/2}^1 + \int_1^{3/2} \right) T^{1/2-\sigma+1} |\zeta(2 - \sigma + iT)| \frac{x^\sigma}{T^{m+1}} d\sigma \right. \\
&\quad \left. + \int_{3/2}^c |\zeta(\sigma - 1 + iT)| \frac{x^\sigma}{T^{m+1}} d\sigma \right\} \\
&\ll k c(\eta)^{\omega(k)} \left(\frac{x^{1/2}}{T^{m-1/2+\eta}} + \frac{x \log T}{T^m} + \frac{x^{3/2} (\log T)^{3/2}}{T^{m+5/6}} + \frac{x^2 (\log T)^{3/2}}{T^{m+5/6}} \right)
\end{aligned}$$

Collecting all the estimates, and taking $T = x^{10}$, for $m \geq 2$ we get:

$$E_{m,k}(x) \ll k c(\eta)^{\omega(k)} x^\eta. \tag{4.10}$$

This proves Theorem 1.1.

Remark 4.1. For $m \geq 2$ we may try to move the line of integration slightly left of vertical line 0. On the line $\Re s = 0$, the function $G(s)$ has simple poles at the points $s(\ell, p) = \frac{2\pi i \ell}{\log p} \quad \forall \ell \in \mathbb{Z}$ and for each prime $p \mid k$. let p_1, p_2, \dots, p_{r_k} be the primes dividing k . The total contribution from the simple poles at the points $s(\ell, p) = \frac{2\pi i \ell}{\log p_j}$ for $1 \leq j \leq r_k$ is given by:

$$M = \sum_{j=1}^{r_k} \sum_{|\ell| \leq \frac{T \log p_j}{2\pi}} \zeta \left(\frac{2\pi i \ell}{\log p_j} - 1 \right) \prod_{p_i \neq p_j} \left(1 - \frac{p_i}{p_j^{\frac{2\pi i \ell}{\log p_j}}} \right) \frac{x^{\frac{2\pi i \ell}{\log p_j}}}{\frac{2\pi i \ell}{\log p_j} \left(\frac{2\pi i \ell}{\log p_j} + 1 \right) \cdots \left(\frac{2\pi i \ell}{\log p_j} + m \right)}.$$

If one establishes that $M = o(x^\eta)$, then this will improve the error term. This seems to be really difficult.

Remark 4.2. From the Theorem 1.1 observe that

$$E_{1,k}(x) \ll_{\epsilon} x^{1/2+10\epsilon}$$

uniformly for $3 \leq k \ll x^{\epsilon}$ since $\omega(k) \ll \frac{\log}{\log \log k}$ for $k \geq 3$. Also $E_{m,k}(x) \ll x^{c_1 \eta}$ uniformly for $3 \leq k \ll x^{\epsilon}$, where c_1 is effective positive constant.

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