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# Dual Ramanujan-Fourier series

Noboru Ushiroya

**Abstract.** Let  $c_q(n)$  be the Ramanujan sums. Many results concerning Ramanujan-Fourier series  $f(n) = \sum_{q=1}^{\infty} a_q c_q(n)$  are obtained by many mathematicians. In this paper we study series of the form  $f(q) = \sum_{n=1}^{\infty} a_n c_q(n)$ , which we call dual Ramanujan-Fourier series. We extend Lucht's theorem and Delange's theorem to this case and obtain some results.

**Keywords.** Ramanujan-Fourier series, Ramanujan sums, arithmetic functions, multiplicative functions.

**2010 Mathematics Subject Classification.** 11A25, 11N37.

## 1. Introduction

For  $q, n \in \mathbb{N} = \{1, 2, \dots\}$ , the Ramanujan sums  $c_q(n)$  are defined in [Ra] by

$$c_q(n) = \sum_{\substack{k=1 \\ (k,q)=1}}^q \exp\left(\frac{2\pi i k n}{q}\right),$$

where  $(k, q)$  is the greatest common divisor of  $k$  and  $q$ . Let  $f : \mathbb{N} \mapsto \mathbb{C}$  be an arithmetic function. Ramanujan [Ra] investigated its Ramanujan-Fourier series which is an infinite series of the form

$$f(n) = \sum_{q=1}^{\infty} a_q c_q(n), \quad (1.1)$$

where  $a_q$  are called the Ramanujan-Fourier coefficients of  $f$ , and he obtained the following results.

$$\frac{\sigma_s(n)}{n^s} = \zeta(s+1) \sum_{q=1}^{\infty} \frac{c_q(n)}{q^{s+1}}, \quad (1.2)$$

$$\frac{\varphi_s(n)}{n^s} = \frac{1}{\zeta(s+1)} \sum_{q=1}^{\infty} \frac{\mu(q)}{\varphi_{s+1}(q)} c_q(n), \quad (1.3)$$

$$\tau(n) = - \sum_{q=1}^{\infty} \frac{\log q}{q} c_q(n), \quad (1.4)$$

$$r(n) = \pi \sum_{q=1}^{\infty} \frac{(-1)^{q-1}}{2q-1} c_{2q-1}(n), \quad (1.5)$$

where  $\sigma_s(n) = \sum_{d|n} d^s$  with  $s > 0$ ,  $\zeta(s)$  is the Riemann zeta function,  $\varphi_s(n) = n^s \prod_{p|n} (1 - 1/p^s)$ ,  $\tau(n) = \sum_{d|n} 1$ ,  $\mu$  is the Möbius function and  $r(n)$  is the number of representations of  $n$  as the sum of two squares.

Ramanujan [Ra] also investigated dual Ramanujan-Fourier series of the form

$$f(q) = \sum_{n=1}^{\infty} a_n c_q(n),$$

and he obtained the following results.

$$(\text{id}^{1-s} * \mu)(q) = \varphi_{1-s}(q) = \frac{1}{\zeta(s)} \sum_{n=1}^{\infty} \frac{c_q(n)}{n^s} \quad \text{if } s > 1, \quad (1.6)$$

$$\Lambda(q) = - \sum_{n=1}^{\infty} \frac{c_q(n)}{n} \quad \text{if } q \geq 2, \quad (1.7)$$

where  $\text{id}$  is the function  $\text{id}(n) = n$ ,  $f * g$  denotes the Dirichlet convolution of  $f$  and  $g$ , and  $\Lambda(q)$  denotes the von Mangoldt function.

We investigate dual Ramanujan-Fourier series and obtain theorems which are extensions of the results due to Delange and Lucht. Several examples are given. The method used in this paper is quite elementary.

## 2. Preliminaries

$$\text{Let } \delta(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases} \quad \text{and let } \delta(m, n) = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n. \end{cases}$$

We set  $D(m, n) = m\delta(m, n)$ . Obviously,  $D(m, n) = D(n, m)$  holds.

Let  $f, g : \mathbb{N} \mapsto \mathbb{C}$  be arithmetic functions. The Dirichlet convolution of  $f$  and  $g$  is defined by

$$(f * g)(n) = \sum_{d|n} f(d)g(n/d).$$

For two arithmetic functions, one of which is a function of one variable, the other a function of two variables, we define similar types of convolutions as follows.

**Definition 2.1.** *Let  $f : \mathbb{N} \mapsto \mathbb{C}$  be an arithmetic function and  $g : \mathbb{N} \times \mathbb{N} \mapsto \mathbb{C}$  be an arithmetic function of two variables. We define  $f *_\ell g : \mathbb{N} \times \mathbb{N} \mapsto \mathbb{C}$  and  $g *_r f : \mathbb{N} \times \mathbb{N} \mapsto \mathbb{C}$  as follows.*

$$(f *_\ell g)(q, n) = (f(\cdot) * g(\cdot, n))(q) = \sum_{d|q} f\left(\frac{q}{d}\right)g(d, n),$$

$$(g *_r f)(q, n) = (g(q, \cdot) * f(\cdot))(n) = \sum_{d|n} g(q, d)f\left(\frac{n}{d}\right).$$

It is clear that the following lemma holds.

**Lemma 2.1.** *Let  $f, h : \mathbb{N} \mapsto \mathbb{C}$  be arithmetic functions and let  $g : \mathbb{N} \times \mathbb{N} \mapsto \mathbb{C}$  be an arithmetic function of two variables. Then we have*

$$(f *_\ell g) *_r h = f *_\ell (g *_r h),$$

$$f *_\ell (h *_\ell g) = (f *_\ell h) *_\ell g,$$

$$(g *_r f) *_r h = g *_r (f *_r h).$$

We note that  $((f *_\ell g) *_r h)(q, n)$  can also be written as  $\sum_{d_1|q, d_2|n} f(q/d_1)g(d_1, d_2)h(n/d_2)$ . We simply write  $f *_\ell g *_r h$  instead of  $(f *_\ell g) *_r h$  or  $f *_\ell (g *_r h)$ .

It is easy to see that the following lemma holds.

**Lemma 2.2.** *Let  $f : \mathbb{N} \mapsto \mathbb{C}$  be an arithmetic function. Then we have*

$$(f *_{\ell} D)(q, n) = I_{n|q} f\left(\frac{q}{n}\right)n = \begin{cases} f\left(\frac{q}{n}\right)n & \text{if } n \mid q \\ 0 & \text{if } n \nmid q, \end{cases}$$

$$(D *_{r} f)(q, n) = I_{q|n} f\left(\frac{n}{q}\right)q = \begin{cases} f\left(\frac{n}{q}\right)q & \text{if } q \mid n \\ 0 & \text{if } q \nmid n, \end{cases}$$

where  $I_{n|q} = \begin{cases} 1 & \text{if } n \mid q \\ 0 & \text{if } n \nmid q. \end{cases}$

*Proof.* By definiton, we have

$$(f *_{\ell} D)(q, n) = \sum_{d|q} f\left(\frac{q}{d}\right)D(d, n) = \sum_{d|q} f\left(\frac{q}{d}\right)n\delta(d, n) = I_{n|q} f\left(\frac{q}{n}\right)n.$$

The proof of the second assertion is similar. □

Let  $f, g : \mathbb{N} \times \mathbb{N} \mapsto \mathbb{C}$  be arithmetic functions of two variables. The Dirichlet convolution of  $f$  and  $g$  is defined by

$$(f * g)(q, n) = \sum_{d_1|q, d_2|n} f(d_1, d_2)g(q/d_1, n/d_2).$$

Let  $f : \mathbb{N} \mapsto \mathbb{C}$  be an arithmetic function. We note that, if we define  $f \otimes \delta, \delta \otimes f : \mathbb{N} \times \mathbb{N} \mapsto \mathbb{C}$  by

$$(f \otimes \delta)(q, n) = f(q)\delta(n),$$

$$(\delta \otimes f)(q, n) = \delta(q)f(n),$$

then we have for  $g : \mathbb{N} \times \mathbb{N} \mapsto \mathbb{C}$

$$(f *_{\ell} g)(q, n) = ((f \otimes \delta) * g)(q, n),$$

$$(g *_{r} f)(q, n) = (g * (\delta \otimes f))(q, n).$$

We say that  $f : \mathbb{N} \mapsto \mathbb{C}$  is a multiplicative function if  $f$  satisfies

$$f(n_1 n_2) = f(n_1)f(n_2)$$

for any  $n_1, n_2 \in \mathbb{N}$  satisfying  $(n_1, n_2) = 1$ . It is well known that if  $f$  and  $g$  are multiplicative functions, then  $f * g$  also becomes a multiplicative function. We say that  $f : \mathbb{N} \times \mathbb{N} \mapsto \mathbb{C}$  is a multiplicative function of two variables if  $f$  satisfies

$$f(q_1 q_2, n_1 n_2) = f(q_1, n_1) f(q_2, n_2)$$

for any  $q_1, q_2, n_1, n_2 \in \mathbb{N}$  satisfying  $(q_1 n_1, q_2 n_2) = 1$ . It is well known that if  $f$  and  $g$  are multiplicative functions of two variables, then  $f * g$  also becomes a multiplicative function of two variables.

It is easy to see that the following lemma holds.

**Lemma 2.3.** *Let  $f, h : \mathbb{N} \mapsto \mathbb{C}$  be multiplicative functions and let  $g : \mathbb{N} \times \mathbb{N} \mapsto \mathbb{C}$  be a multiplicative function of two variables. Then  $f *_{\ell} g, g *_{r} h$  and  $f *_{\ell} g *_{r} h$  are all multiplicative functions of two variables.*

*Proof.* If  $f$  is multiplicative, then  $f \otimes \delta$  is also multiplicative as a function of two variables. Therefore  $f *_{\ell} g = (f \otimes \delta) * g$  is also multiplicative as a function of two variables. Similarly,  $g *_{r} h$  and  $f *_{\ell} g *_{r} h$  are multiplicative functions of two variables. □

Ramanujan [Ra] proved that  $c_q(n)$  can be written as

$$c_q(n) = \sum_{d|(q,n)} \mu(q/d)d.$$

We show that  $c_q(n)$  can also be written as follows.

**Lemma 2.4.**

$$c_q(n) = (\mu *_{\ell} D *_{r} \mathbf{1})(q, n),$$

where  $\mathbf{1}(n) = 1$  for every  $n \in \mathbb{N}$ .

*Proof.* By definition, we have

$$(\mu *_{\ell} D *_{r} \mathbf{1})(q, n) = \sum_{d_1|q, d_2|n} \mu(q/d_1)d_1\delta(d_1, d_2)\mathbf{1}(n/d_2) = \sum_{d|(q,n)} \mu(q/d)d.$$

□

Hardy [Ha] proved that, for fixed  $n$ ,  $q \mapsto c_q(n)$  is a multiplicative function. Johnson [Jo] proved that  $(q, n) \mapsto c_q(n)$  is a multiplicative function of two variables. We remark that the multiplicativity of  $(q, n) \mapsto c_q(n)$  is trivial from Lemma 2.4 and Lemma 2.3 since  $D : \mathbb{N} \times \mathbb{N} \mapsto \mathbb{C}$  is multiplicative as a function of two variables.

It is well known that the following holds ([Si]). For a fixed integer  $k$ ,

$$\sum_{q|k} c_q(n) = I_{k|n}k = \begin{cases} k & \text{if } k | n \\ 0 & \text{if } k \nmid n. \end{cases}$$

We give another expression of the above in the following lemma. We simply write  $c$  instead of  $c(\cdot)$ . We note that  $\sum_{q|k} c_q(n)$  can be written as  $(\mathbf{1} *_{\ell} c)(k, n)$ .

**Lemma 2.5.** *We have*

$$\begin{aligned} (\mathbf{1} *_{\ell} c)(q, n) &= qI_{q|n} = \begin{cases} q & \text{if } q | n \\ 0 & \text{if } q \nmid n, \end{cases} \quad \text{and} \\ (c *_{r} \mu)(q, n) &= nI_{n|q}\mu\left(\frac{q}{n}\right) = \begin{cases} n\mu\left(\frac{q}{n}\right) & \text{if } n | q \\ 0 & \text{if } n \nmid q. \end{cases} \end{aligned}$$

*Proof.* Since  $c = \mu *_{\ell} D *_{r} \mathbf{1}$  and  $\mathbf{1} * \mu = \delta$ , we have by Lemma 2.2

$$\begin{aligned} (\mathbf{1} *_{\ell} c)(q, n) &= (\mathbf{1} *_{\ell} (\mu *_{\ell} D *_{r} \mathbf{1}))(q, n) = ((\mathbf{1} * \mu) *_{\ell} (D *_{r} \mathbf{1}))(q, n) = (D *_{r} \mathbf{1})(q, n) = qI_{q|n}, \quad \text{and} \\ (c *_{r} \mu)(q, n) &= ((\mu *_{\ell} D *_{r} \mathbf{1}) *_{r} \mu)(q, n) = ((\mu *_{\ell} D) *_{r} (\mathbf{1} * \mu))(q, n) = (\mu *_{\ell} D)(q, n) = nI_{n|q}\mu\left(\frac{q}{n}\right). \end{aligned}$$

□

### 3. Some Results

In this section we show some results concerning dual Ramanujan-Fourier series. First, we introduce the following Lucht's theorem concerning Ramanujan-Fourier series.

**Theorem 3.1. (Lucht [Lu])** *Let  $a : \mathbb{N} \mapsto \mathbb{C}$  be an arithmetic function. If the series*

$$A(n) := n \sum_{k=1}^{\infty} \mu(k)a(kn)$$

*converges for every  $n \in \mathbb{N}$ , then for  $f(n) = (A * \mathbf{1})(n)$ , we have*

$$f(n) = \sum_{q=1}^{\infty} a(q)c_q(n).$$

Lucht obtained (1.4) by taking  $a(n) = -\frac{\log n}{n}$ . In this case, we see that

$$A(n) = n \sum_{k=1}^{\infty} \mu(k)a(kn) = -n \sum_{k=1}^{\infty} \mu(k) \frac{\log k + \log n}{kn} = \mathbf{1}(n), \tag{3.8}$$

since  $\sum_{k=1}^{\infty} \frac{\mu(k) \log k}{k} = -1$  and  $\sum_{k=1}^{\infty} \frac{\mu(k)}{k} = 0$ . Therefore  $f = A * \mathbf{1} = \mathbf{1} * \mathbf{1} = \tau$  satisfies (1.4). We would like to extend Lucht's theorem to the case of dual Ramanujan-Fourier series. We show the following theorem which is "dual" to Lucht's theorem.

**Theorem 3.2.** *Let  $a : \mathbb{N} \mapsto \mathbb{C}$  be an arithmetic function. If the series*

$$A(q) := q \sum_{k=1}^{\infty} a(kq)$$

*converges for every  $q \in \mathbb{N}$ , then for  $f(q) = (A * \mu)(q)$ , we have*

$$f(q) = \sum_{n=1}^{\infty} a(n)c_q(n). \tag{3.9}$$

*Proof.* Since  $c_q(n) = (\mu *_{\ell} D *_{\ell} \mathbf{1})(q, n)$ , we have by Lemma 2.2

$$\begin{aligned} \sum_{n \leq x} a(n)c_q(n) &= \sum_{n \leq x} a(n)((\mu *_{\ell} D) *_{\ell} \mathbf{1})(q, n) = \sum_{n \leq x} a(n) \sum_{d|n} (\mu *_{\ell} D)(q, d) \mathbf{1}\left(\frac{n}{d}\right) \\ &= \sum_{dk \leq x} a(dk)(\mu *_{\ell} D)(q, d) = \sum_{dk \leq x} a(dk) I_{d|q} \mu\left(\frac{q}{d}\right) d \\ &= \sum_{d \leq x} I_{d|q} \mu\left(\frac{q}{d}\right) d \sum_{k \leq x/d} a(dk) = \sum_{d|q} \mu\left(\frac{q}{d}\right) (d \sum_{k \leq x/d} a(dk)), \end{aligned}$$

where  $x$  is a sufficiently large real number. Letting  $x \rightarrow \infty$ , we have

$$\sum_{n=1}^{\infty} a(n)c_q(n) = \sum_{d|q} \mu\left(\frac{q}{d}\right) A(d) = (\mu * A)(q) = f(q),$$

which proves Theorem 3.2. □

**Remark 3.1.** *It is easy to see that, if we set  $a(n) = 1/n^s$  ( $s > 1$ ), then we obtain (1.6) since*

$$A(q) = q \sum_{k=1}^{\infty} a(kq) = q \sum_{k=1}^{\infty} \frac{1}{(kq)^s} = \zeta(s) \frac{1}{q^{s-1}} = \zeta(s) \text{id}^{1-s}(q).$$

We show other examples of Theorem 3.2 below.

**Example 3.1.** Let  $\omega(q)$  denote the number of distinct prime divisors of  $q$  and let  $\lambda(q) = (-1)^{\Omega(q)}$  be the Liouville function where  $\Omega(q)$  is the number of prime factors of  $q$ , counted with multiplicity. Then we have

$$2^{\omega(q)}\lambda(q) = -\frac{1}{\zeta(2)} \sum_{n=1}^{\infty} \frac{\lambda(n) \log n}{n} c_q(n).$$

*Proof.* Let  $a(n) = \lambda(n) \log n/n$ . Then, noting that  $\lambda$  is completely multiplicative, we have

$$\begin{aligned} A(q) &= q \sum_{k=1}^{\infty} a(kq) = q \sum_{k=1}^{\infty} \frac{\lambda(kq)(\log k + \log q)}{kq} \\ &= \lambda(q) \sum_{k=1}^{\infty} \frac{\lambda(k)(\log k + \log q)}{k} = -\zeta(2)\lambda(q), \end{aligned}$$

since  $\sum_{k=1}^{\infty} \frac{\lambda(k) \log k}{k} = -\zeta(2)$  and  $\sum_{k=1}^{\infty} \frac{\lambda(k)}{k} = 0$ . Therefore, if we set

$$f(q) = (A * \mu)(q) = -\zeta(2)(\lambda * \mu)(q) = -\zeta(2)2^{\omega(q)}\lambda(q),$$

then Theorem 3.2 gives the desired result.  $\square$

**Example 3.2.** Let  $s > 1$ . Then we have

$$\frac{1}{\zeta(s)} \left( \frac{(\text{id})\mu}{\varphi_s} * \mu \right)(q) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} c_q(n). \quad (3.10)$$

*Proof.* Setting  $a(n) = \mu(n)/n^s$ , we have

$$\begin{aligned} A(q) &= q \sum_{k=1}^{\infty} a(kq) = q \sum_{k=1}^{\infty} \frac{\mu(kq)}{(kq)^s} = q \sum_{\substack{k \geq 1 \\ (k,q)=1}} \frac{\mu(k)\mu(q)}{k^s q^s} \\ &= \frac{q\mu(q)}{q^s} \prod_{p|q} \left(1 - \frac{1}{p^s}\right) = \frac{q\mu(q)}{q^s} \frac{1}{\zeta(s) \prod_{p|q} (1 - 1/p^s)} \\ &= \frac{1}{\zeta(s)} \frac{\text{id}(q)\mu(q)}{\varphi_s(q)}. \end{aligned}$$

Therefore if we set  $f = A * \mu = \frac{1}{\zeta(s)} \frac{(\text{id})\mu}{\varphi_s} * \mu$ , then we see that (3.10) holds by Theorem 3.2.  $\square$

Ramanujan [Ra] proved

$$\sum_{q=1}^{\infty} \frac{c_q(n)}{q} = 0, \quad (3.11)$$

which is well known to be equivalent to the prime number theorem. We remark that, letting  $s \rightarrow 1$  in (3.10), we obtain

$$\sum_{n=1}^{\infty} \frac{\mu(n)c_q(n)}{n} = 0, \quad (3.12)$$

which is "dual" to (3.11).

We also have another example which is "dual" to (3.11).

**Example 3.3.** *We have*

$$\sum_{n=1}^{\infty} \frac{\lambda(n)c_q(n)}{n} = 0. \tag{3.13}$$

*Proof.* Taking  $a(n) = \lambda(n)/n$  in Theorem 3.2, we have

$$A(q) = q \sum_{k=1}^{\infty} a(kq) = q \sum_{k=1}^{\infty} \frac{\lambda(kq)}{kq} = \lambda(q) \sum_{k=1}^{\infty} \frac{\lambda(k)}{k} = 0,$$

from which  $f = A * \mu = 0$  follows. □

Next we introduce Delange’s theorem concerning Ramanujan-Fourier series. Given an arithmetic function  $a : \mathbb{N} \mapsto \mathbb{C}$ , it is convenient to use Theorem 3.2 in order to find  $f$  satisfying (3.9). However, given  $f$ , it is not convenient to use Theorem 3.2 in order to find  $a$  satisfying (3.9). In the case of Ramanujan-Fourier series, it is sometimes useful to use the following Delange’s theorem in order to find  $a$  satisfying (1.1) for given  $f$ . We will extend Delange’s theorem to the case of dual Ramanujan-Fourier series later.

**Theorem 3.3. (Delange [De])** *Let  $f(n)$  be an arithmetic function satisfying*

$$\sum_{n=1}^{\infty} 2^{\omega(n)} \frac{|(f * \mu)(n)|}{n} < \infty. \tag{3.14}$$

*Then its Ramanujan-Fourier series is pointwise convergent and*

$$f(n) = \sum_{q=1}^{\infty} a(q)c_q(n)$$

*holds where*

$$a(q) = \sum_{m=1}^{\infty} \frac{(f * \mu)(qm)}{qm}.$$

*Moreover, if  $f$  is a multiplicative function, then  $a(q)$  can be rewritten as*

$$a(q) = \prod_{p \in \mathcal{P}} \left( \sum_{e=\nu_p(q)}^{\infty} \frac{(f * \mu)(p^e)}{p^e} \right), \tag{3.15}$$

*where  $\mathcal{P}$  denotes the set of prime numbers and  $\nu_p(q) = \begin{cases} \alpha & \text{if } p^\alpha || q \\ 0 & \text{if } p \nmid q. \end{cases}$*

Lucht [Lu] showed that Theorem 3.3 can easily be obtained from Theorem 3.1. We would like to extend Theorem 3.3 to the case of dual Ramanujan-Fourier series by using Theorem 3.2.

**Theorem 3.4.** *Let  $f$  be an arithmetic function satisfying*

$$\sum_{q=1}^{\infty} \frac{|(f * \mathbf{1})(q)|}{q} \tau(q) < \infty. \tag{3.16}$$

*Then its dual Ramanujan-Fourier series is pointwise convergent and*

$$f(q) = \sum_{n=1}^{\infty} a(n)c_q(n)$$

*holds where*

$$a(n) = \sum_{m=1}^{\infty} \frac{(f * \mathbf{1})(nm)}{nm} \mu(m). \tag{3.17}$$



**Remark 3.2.** If  $f$  is a multiplicative function satisfying

$$\sum_{p \in \mathcal{P}} \sum_{e=1}^{\infty} \frac{|(f * \mathbf{1})(p^e)|}{p^e} (e+1) < \infty, \quad (3.18)$$

then its dual Ramanujan-Fourier series is pointwise convergent and

$$f(q) = \sum_{n=1}^{\infty} a(n) c_q(n)$$

holds where

$$a(n) = \sum_{m=1}^{\infty} \frac{(f * \mathbf{1})(nm)}{nm} \mu(m).$$

Moreover,  $a(n)$  can be rewritten as

$$a(n) = \prod_{p \in \mathcal{P}} \left( \frac{(f * \mathbf{1})(p^{\nu_p(n)})}{p^{\nu_p(n)}} - \frac{(f * \mathbf{1})(p^{\nu_p(n)+1})}{p^{\nu_p(n)+1}} \right). \quad (3.19)$$

*Proof of Theorem 3.4.* We first see that  $A(q) = q \sum_{k=1}^{\infty} a(kq)$  converges since

$$\begin{aligned} \sum_{k \leq x} |a(kq)| &\leq \sum_{k \leq x} \sum_{m=1}^{\infty} \frac{|(f * \mathbf{1})(kqm)|}{kqm} |\mu(m)| \\ &\leq \sum_{\ell=1}^{\infty} \frac{|(f * \mathbf{1})(\ell)|}{\ell} \sum_{k \leq x, k|\ell} 1 \leq \sum_{\ell=1}^{\infty} \frac{|(f * \mathbf{1})(\ell)|}{\ell} \tau(\ell) < \infty \end{aligned}$$

holds for every  $x > 1$ . Using Lemma 2.5 we can rewrite  $A(q)$  as follows.

$$A(q) = q \sum_{k=1}^{\infty} a(kq) = \sum_{m=1}^{\infty} a(m) q I_{q|m} = \sum_{m=1}^{\infty} a(m) (\mathbf{1} *_{\ell} c)(q, m).$$

From this we have

$$\begin{aligned} f(q) &= (\mu * A)(q) = \sum_{m=1}^{\infty} a(m) (\mu *_{\ell} (\mathbf{1} *_{\ell} c))(q, m) = \sum_{m=1}^{\infty} a(m) ((\mu * \mathbf{1}) *_{\ell} c)(q, m) \\ &= \sum_{m=1}^{\infty} a(m) (\delta *_{\ell} c)(q, m) = \sum_{m=1}^{\infty} a(m) c_q(m). \end{aligned}$$

which completes the proof of Theorem 3.4.  $\square$

*Proof of Remark 3.2.* If  $f$  is a multiplicative function, then  $q \mapsto \frac{(f * \mathbf{1})(q)}{q} \tau(q)$  is also a multiplicative function. Using  $1 + x \leq \exp(x)$ , we see that (3.16) follows from (3.18) since

$$\begin{aligned} \sum_{q \leq Q} \frac{|(f * \mathbf{1})(q)|}{q} \tau(q) &\leq \prod_{p \in \mathcal{P}} \left( 1 + \sum_{e=1}^{\infty} \frac{|(f * \mathbf{1})(p^e)|}{p^e} \tau(p^e) \right) \\ &\leq \prod_{p \in \mathcal{P}} \exp \left( \sum_{e=1}^{\infty} \frac{|(f * \mathbf{1})(p^e)|}{p^e} \tau(p^e) \right) = \exp \left( \sum_{p \in \mathcal{P}} \sum_{e=1}^{\infty} \frac{|(f * \mathbf{1})(p^e)|}{p^e} (e+1) \right) < \infty \end{aligned}$$

holds for every  $Q > 1$ . Therefore (3.17) holds by Theorem 3.4. In the expression of (3.17), we set  $n = \prod p_j^{e_j}$ ,  $m = r \prod p_j^{d_j}$  where  $(r, n) = 1$ ,  $e_j \geq 1$ , and  $d_j \geq 0$ . Then we have

$$a(n) = \sum_{d_j \geq 0, r \geq 1, (r, n) = 1} \frac{(f * \mathbf{1})(r \prod p_j^{e_j + d_j})}{r \prod p_j^{e_j + d_j}} \mu(r \prod p_j^{d_j}).$$

Since  $f * \mathbf{1}$  is multiplicative and since  $\mu(p_j^{d_j}) = 0$  if  $d_j \geq 2$  for some  $j$ , we obtain

$$\begin{aligned} a(n) &= \prod_j \left( \sum_{0 \leq d_j \leq 1} \frac{(f * \mathbf{1})(p_j^{e_j + d_j})}{p_j^{e_j + d_j}} \mu(p_j^{d_j}) \right) \times \sum_{r \geq 1, (r, n) = 1} \frac{(f * \mathbf{1})(r)}{r} \mu(r) \\ &= \prod_{p|n} \left( \frac{(f * \mathbf{1})(p^{\nu_p(n)})}{p^{\nu_p(n)}} - \frac{(f * \mathbf{1})(p^{\nu_p(n)+1})}{p^{\nu_p(n)+1}} \right) \times \prod_{p \nmid n} \left( 1 - \frac{(f * \mathbf{1})(p)}{p} \right) \\ &= \prod_{p \in \mathcal{P}} \left( \frac{(f * \mathbf{1})(p^{\nu_p(n)})}{p^{\nu_p(n)}} - \frac{(f * \mathbf{1})(p^{\nu_p(n)+1})}{p^{\nu_p(n)+1}} \right), \end{aligned}$$

which completes the proof of Remark 3.2. □

Several examples are shown below.

**Example 3.4.** *Let  $s > 1$ . Then we have*

$$\frac{\varphi_s(q)}{q^s} \mu(q) = \frac{1}{\zeta(s+1)} \sum_{n=1}^{\infty} \frac{\varphi(K(n))}{n \psi_{s+1}(K(n))} c_q(n),$$

where  $K(n) = \prod_{p|n} p$  and  $\psi_s(n) = n^s \prod_{p|n} (1 + 1/p^s)$ .

*Proof.* Let  $f(q) = \frac{\varphi_s(q)}{q^s} \mu(q)$ . Then it is easy to see that

$$f(p^e) = \begin{cases} -(1 - 1/p^s) & \text{if } e = 1 \\ 0 & \text{if } e \geq 2 \end{cases}$$

and

$$(f * \mathbf{1})(p^e) = \begin{cases} 1 & \text{if } e = 0 \\ 1/p^s & \text{if } e \geq 1, \end{cases}$$

from which we see that (3.18) holds. It is also easy to see that

$$\frac{(f * \mathbf{1})(p^e)}{p^e} - \frac{(f * \mathbf{1})(p^{e+1})}{p^{e+1}} = \begin{cases} 1 - 1/p^{s+1} & \text{if } e = 0 \\ \frac{1}{p^{e+s}}(1 - 1/p) & \text{if } e \geq 1. \end{cases}$$

From this and (3.19) we have

$$\begin{aligned} a(n) &= \prod_{p|n} \left( \frac{(f * \mathbf{1})(p^{\nu_p(n)})}{p^{\nu_p(n)}} - \frac{(f * \mathbf{1})(p^{\nu_p(n)+1})}{p^{\nu_p(n)+1}} \right) \prod_{p \nmid n} \left( \frac{(f * \mathbf{1})(p^{\nu_p(n)})}{p^{\nu_p(n)}} - \frac{(f * \mathbf{1})(p^{\nu_p(n)+1})}{p^{\nu_p(n)+1}} \right) \\ &= \prod_{p|n} \frac{1}{p^{\nu_p(n)+s}} (1 - 1/p) \prod_{p \nmid n} (1 - 1/p^{s+1}) \\ &= \prod_{p|n} \frac{1}{p^{\nu_p(n)+s}} \frac{1 - 1/p}{1 - 1/p^{s+1}} \prod_{p \in \mathcal{P}} (1 - 1/p^{s+1}) \\ &= \frac{1}{\zeta(s+1)} \prod_{p|n} \frac{1}{p^{\nu_p(n)}} \frac{p(1 - 1/p)}{p^{s+1}(1 - 1/p^{s+1})} = \frac{1}{\zeta(s+1)} \frac{\varphi(K(n))}{n \psi_{s+1}(K(n))}. \end{aligned}$$

□

**Example 3.5.** Let  $s > 1$ . Then we have

$$\frac{\sigma_s(q)}{q^s} \mu(q) = \frac{\zeta(s+1)}{\zeta(2s+2)} \sum_{n=1}^{\infty} \frac{(-1)^{\omega(n)} \varphi(K(n))}{n \psi_{s+1}(K(n))} c_q(n).$$

*Proof.* Let  $f(q) = \frac{\sigma_s(q)}{q^s} \mu(q)$ . Then it is easy to see that

$$f(p^e) = \begin{cases} -1 - 1/p^s & \text{if } e = 1 \\ 0 & \text{if } e \geq 2 \end{cases}$$

and

$$(f * \mathbf{1})(p^e) = \begin{cases} 1 & \text{if } e = 0 \\ -1/p^s & \text{if } e \geq 1, \end{cases}$$

from which we see that (3.18) holds. It is also easy to see that

$$\frac{(f * \mathbf{1})(p^e)}{p^e} - \frac{(f * \mathbf{1})(p^{e+1})}{p^{e+1}} = \begin{cases} 1 + 1/p^{s+1} & \text{if } e = 0 \\ \frac{-1}{p^{e+s}}(1 - 1/p) & \text{if } e \geq 1. \end{cases}$$

Therefore by (3.19) we have

$$\begin{aligned} a(n) &= \prod_{p|n} \left( \frac{(f * \mathbf{1})(p^{\nu_p(n)})}{p^{\nu_p(n)}} - \frac{(f * \mathbf{1})(p^{\nu_p(n)+1})}{p^{\nu_p(n)+1}} \right) \prod_{p|n} \left( \frac{(f * \mathbf{1})(p^{\nu_p(n)})}{p^{\nu_p(n)}} - \frac{(f * \mathbf{1})(p^{\nu_p(n)+1})}{p^{\nu_p(n)+1}} \right) \\ &= \prod_{p|n} \frac{-1}{p^{\nu_p(n)+s}} (1 - 1/p) \prod_{p|n} (1 + 1/p^{s+1}) \\ &= \prod_{p|n} \frac{-1}{p^{\nu_p(n)+s}} \frac{1 - 1/p}{1 + 1/p^{s+1}} \prod_{p \in \mathcal{P}} (1 + 1/p^{s+1}) \\ &= \frac{\zeta(s+1)}{\zeta(2s+2)} \prod_{p|n} \frac{-1}{p^{\nu_p(n)}} \frac{p(1 - 1/p)}{p^{s+1}(1 + 1/p^{s+1})} = \frac{\zeta(s+1)}{\zeta(2s+2)} \frac{(-1)^{\omega(n)} \varphi(K(n))}{n \psi_{s+1}(K(n))}. \end{aligned}$$

□

**Example 3.6.** We have

$$\lambda(q) = \sum_{n=1}^{\infty} a(n) c_q(n),$$

$$\text{where } a(n) = \frac{1}{n} \prod_{\substack{p|n \\ \nu_p(n): \text{odd}}} \frac{-1}{p}.$$

*Proof.* Let  $f(q) = \lambda(q)$ . Then it is easy to see that

$$(f * \mathbf{1})(p^e) = \begin{cases} 0 & \text{if } e \text{ is odd} \\ 1 & \text{if } e \text{ is even,} \end{cases}$$

from which we see that (3.18) holds. It is also easy to see that

$$\frac{(f * \mathbf{1})(p^e)}{p^e} - \frac{(f * \mathbf{1})(p^{e+1})}{p^{e+1}} = \begin{cases} -1/p^{e+1} & \text{if } e \text{ is odd} \\ 1/p^e & \text{if } e \text{ is even.} \end{cases}$$

Therefore by (3.19) we have

$$\begin{aligned} a(n) &= \prod_{p|n} \left( \frac{(f * \mathbf{1})(p^{\nu_p(n)})}{p^{\nu_p(n)}} - \frac{(f * \mathbf{1})(p^{\nu_p(n)+1})}{p^{\nu_p(n)+1}} \right) \prod_{p \nmid n} \left( \frac{(f * \mathbf{1})(p^{\nu_p(n)})}{p^{\nu_p(n)}} - \frac{(f * \mathbf{1})(p^{\nu_p(n)+1})}{p^{\nu_p(n)+1}} \right) \\ &= \prod_{\substack{p|n \\ \nu_p(n):\text{odd}}} \frac{-1}{p^{\nu_p(n)+1}} \prod_{\substack{p|n \\ \nu_p(n):\text{even}}} \frac{1}{p^{\nu_p(n)}} \prod_{p \nmid n} 1 \\ &= \prod_{p|n} \frac{1}{p^{\nu_p(n)}} \prod_{\substack{p|n \\ \nu_p(n):\text{odd}}} \frac{-1}{p} = \frac{1}{n} \prod_{\substack{p|n \\ \nu_p(n):\text{odd}}} \frac{-1}{p}. \end{aligned}$$

□

**Example 3.7.**

$$\frac{\varphi(q)}{q} \lambda(q) = \frac{1}{\zeta(2)} \sum_{n=1}^{\infty} \frac{I_{\text{square}}(n)}{n} c_q(n),$$

$$\text{where } I_{\text{square}}(n) = \begin{cases} 1 & \text{if } n \text{ is a perfect square} \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* Let  $f(q) = \frac{\varphi(q)}{q} \lambda(q)$ . Then it is easy to see that

$$(f * \mathbf{1})(p^e) = \begin{cases} 1/p & \text{if } e \text{ is odd} \\ 1 & \text{if } e \text{ is even.} \end{cases}$$

from which we see that (3.18) holds. It is also easy to see that

$$\frac{(f * \mathbf{1})(p^e)}{p^e} - \frac{(f * \mathbf{1})(p^{e+1})}{p^{e+1}} = \begin{cases} 0 & \text{if } e \text{ is odd} \\ \frac{1}{p^e} (1 - 1/p^2) & \text{if } e \text{ is even.} \end{cases}$$

Therefore, by (3.19),  $a(n) = 0$  if  $n$  is not a perfect square. If  $n$  is a perfect square, then we have

$$a(n) = \prod_{p|n} \frac{1}{p^{\nu_p(n)}} (1 - 1/p^2) \prod_{p \nmid n} (1 - 1/p^2) = \prod_{p \in \mathcal{P}} (1 - 1/p^2) \prod_{p|n} \frac{1}{p^{\nu_p(n)}} = \frac{1}{\zeta(2)} \frac{1}{n}.$$

Thus we can express  $a(n)$  as

$$a(n) = \frac{1}{\zeta(2)} \frac{I_{\text{square}}(n)}{n}$$

whether  $n$  is a perfect square or not. This completes the proof of Example 3.7. □

Let  $\mathcal{F}$  be the set of real valued arithmetic functions and let  $\mathcal{A} = \{a \in \mathcal{F} : \sum_q a(q)c_q(n) \text{ converges.}\}$ ,  $\mathcal{B} = \{b \in \mathcal{F} : \sum_n b(n)c_q(n) \text{ converges.}\}$ . If we define  $T : \mathcal{A} \mapsto \mathcal{F}$  and  $T^* : \mathcal{B} \mapsto \mathcal{F}$  by

$$\begin{aligned} (Ta)(n) &= \sum_q a(q)c_q(n), \\ (T^*b)(q) &= \sum_n b(n)c_q(n), \end{aligned}$$

respectively, then we have "formally"

$$\langle b, Ta \rangle = \langle T^*b, a \rangle,$$

where  $\langle b, a \rangle := \sum_n b(n)a(n)$  is an inner product of  $a$  and  $b$ . More precisely, we have the following trivial proposition.

**Proposition 3.1.** *If  $f(n) = \sum_q a(q)c_q(n)$  and  $g(q) = \sum_n b(n)c_q(n)$ . If*

$$\sum_{q,n} |a(q)c_q(n)b(n)| < \infty, \quad (3.20)$$

then

$$\sum_n f(n)b(n) = \sum_q a(q)g(q).$$

*Proof.*

$$\sum_n f(n)b(n) = \sum_{q,n} a(q)c_q(n)b(n) = \sum_q a(q)g(q).$$

□

As an example of the above proposition, we show the following example.

**Example 3.8.**

$$\sum_{n=1}^{\infty} \frac{\varphi(n)I_{n:square}}{n^2} = \sum_{q=1}^{\infty} \frac{\mu(q)^2}{q\psi(q)}.$$

*Proof.* By (1.3) with  $s = 1$  and Example 3.7, we have

$$\begin{aligned} \frac{\varphi(n)}{n} &= \frac{1}{\zeta(2)} \sum_{q=1}^{\infty} \frac{\mu(q)}{\varphi_2(q)} c_q(n), \\ \frac{\varphi(q)}{q} \lambda(q) &= \frac{1}{\zeta(2)} \sum_{n=1}^{\infty} \frac{I_{square}(n)}{n} c_q(n). \end{aligned}$$

We note that the right hand of (1.3) is absolutely convergent. Hence (3.20) holds. By Proposition 3.20 we have

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\varphi(n)}{n} \frac{I_{n:square}}{n} &= \sum_{q=1}^{\infty} \frac{\mu(q)}{\varphi_2(q)} \frac{\varphi(q)}{q} \lambda(q) = \sum_{q=1}^{\infty} \frac{\mu(q)\lambda(q) \prod_{p|q}(1-1/p)}{q^2 \prod_{p|q}(1-1/p^2)} \\ &= \sum_{q=1}^{\infty} \frac{\mu(q)^2}{q^2 \prod_{p|q}(1+1/p)} = \sum_{q=1}^{\infty} \frac{\mu(q)^2}{q\psi(q)}, \end{aligned}$$

which completes the proof of Example 3.8. □

Of course, Example 3.8 can also be obtained by expressing both sides as infinite products by prime numbers.

**Remark 3.3.** *We do not know whether we can loosen the condition (3.20) or not. If we can, then, for every  $f \in T\mathcal{A}$  such that  $f = Ta$  and for every  $g \in T^*\mathcal{B}$  such that  $g = T^*b$ , we have "formally"*

$$\begin{aligned} \sum_n \frac{f(n)\mu(n)}{n} &= \langle Ta, \frac{\mu}{\text{id}} \rangle = \langle a, T^* \frac{\mu}{\text{id}} \rangle = \langle a, 0 \rangle = 0, \\ \sum_q \frac{g(q)}{q} &= \langle T^*b, \frac{1}{\text{id}} \rangle = \langle b, T \frac{1}{\text{id}} \rangle = \langle b, 0 \rangle = 0, \end{aligned}$$

namely,  $\text{Im}T \perp \text{Ker}T^*$  and  $\text{Im}T^* \perp \text{Ker}T$ . However, we can't prove the above rigorously.

Next we consider Dirichlet series of a function expressed as Ramanujan-Fourier series or dual Ramanujan-Fourier series. We show the following theorem.

**Theorem 3.5.** *Suppose  $s > 1$ . Let  $f$  be an arithmetic function such that the Dirichlet series  $\sum_{n=1}^{\infty} \frac{f(n)}{n^s}$  converges absolutely and let  $a(\cdot)$  be a multiplicative function such that  $\sum_{k,n \geq 1} \frac{|a(kn)|}{n^{s-1}} < \infty$ .*

(i) *If  $f(n) = \sum_{q=1}^{\infty} a(q)c_q(n)$  converges absolutely, then the Dirichlet series of  $f$  is expressed as*

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \zeta(s) \prod_{p \in \mathcal{P}} \left( \sum_{e \geq 0} \frac{a(p^e) - a(p^{e+1})}{p^{e(s-1)}} \right).$$

(ii) *If  $f(q) = \sum_{n=1}^{\infty} a(n)c_q(n)$  converges absolutely, then the Dirichlet series of  $f$  is expressed as*

$$\sum_{q=1}^{\infty} \frac{f(q)}{q^s} = \frac{1}{\zeta(s)} \prod_{p \in \mathcal{P}} \left( \sum_{e_1 \geq 0} \frac{\sum_{e_2 \geq e_1} a(p^{e_2})}{p^{e_1(s-1)}} \right).$$

*Proof.* (i) Since  $f(n) = \sum_{q=1}^{\infty} a(q)c_q(n) = \sum_{q=1}^{\infty} a(q)(\mu_{\ell} * D_r * \mathbf{1})(q, n)$ , we have by Lemma 2.2

$$\begin{aligned} (f * \mu)(n) &= \sum_{q=1}^{\infty} a(q)((\mu_{\ell} * D_r * \mathbf{1}) * \mu)(q, n) = \sum_{q=1}^{\infty} a(q)(\mu_{\ell} * D)(q, n) \\ &= \sum_{q=1}^{\infty} a(q)I_{n|q} \mu\left(\frac{q}{n}\right)n = \sum_{\substack{q \geq 1 \\ n|q}} a(q)\mu\left(\frac{q}{n}\right)n. \end{aligned}$$

From this we have

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(f * \mu)(n)}{n^s} &= \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{\substack{q \geq 1 \\ n|q}} a(q)\mu\left(\frac{q}{n}\right)n = \sum_{q \geq 1} a(q) \sum_{n|q} \frac{\mu\left(\frac{q}{n}\right)}{n^{s-1}} = \sum_{q=1}^{\infty} a(q) \left( \frac{1}{\text{id}^{s-1}} * \mu \right)(q) \\ &= \prod_{p \in \mathcal{P}} \left( \sum_{e \geq 0} a(p^e) \left( \frac{1}{\text{id}^{s-1}} * \mu \right)(p^e) \right) = \prod_{p \in \mathcal{P}} \left( 1 + \sum_{e \geq 1} a(p^e) \left( \frac{1}{p^{e(s-1)}} - \frac{1}{p^{(e-1)(s-1)}} \right) \right) \\ &= \prod_{p \in \mathcal{P}} \left( 1 - a(p) + \sum_{e \geq 1} \frac{a(p^e) - a(p^{e+1})}{p^{e(s-1)}} \right) = \prod_{p \in \mathcal{P}} \left( \sum_{e \geq 0} \frac{a(p^e) - a(p^{e+1})}{p^{e(s-1)}} \right). \end{aligned}$$

Therefore we have

$$\frac{1}{\zeta(s)} \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_{p \in \mathcal{P}} \left( \sum_{e \geq 0} \frac{a(p^e) - a(p^{e+1})}{p^{e(s-1)}} \right).$$

(ii) We proceed in a similar manner. Since  $f(q) = \sum_{n=1}^{\infty} a(n)c_q(n) = \sum_{n=1}^{\infty} a(n)(\mu_{\ell} * D_r * \mathbf{1})(q, n)$ , we have by Lemma 2.2

$$\begin{aligned} (\mathbf{1} * f)(q) &= \sum_{n=1}^{\infty} a(n)(\mathbf{1} * (\mu_{\ell} * D_r * \mathbf{1}))(q, n) = \sum_{n=1}^{\infty} a(n)(D_r * \mathbf{1})(q, n) \\ &= \sum_{n=1}^{\infty} a(n)I_{q|n} q = \sum_{\substack{n \geq 1 \\ q|n}} a(n)q. \end{aligned}$$

From this we have

$$\begin{aligned}
\sum_{q=1}^{\infty} \frac{(\mathbf{1} * f)(q)}{q^s} &= \sum_{q=1}^{\infty} \frac{1}{q^s} \sum_{\substack{n \geq 1 \\ q|n}} a(n)q = \sum_{n \geq 1} a(n) \sum_{q|n} \frac{1}{q^{s-1}} \\
&= \sum_{n=1}^{\infty} a(n) \left( \frac{1}{\text{id}^{s-1}} * \mathbf{1} \right)(n) = \prod_{p \in \mathcal{P}} \left( \sum_{e \geq 0} a(p^e) \left( \frac{1}{\text{id}^{s-1}} * \mathbf{1} \right)(p^e) \right) \\
&= \prod_{p \in \mathcal{P}} \left( \sum_{e \geq 0} a(p^e) (\text{id}^{1-s} * \mathbf{1})(p^e) \right) = \prod_{p \in \mathcal{P}} \left( \sum_{e_2 \geq 0} a(p^{e_2}) \sum_{0 \leq e_1 \leq e_2} \frac{1}{p^{e_1(s-1)}} \right) \\
&= \prod_{p \in \mathcal{P}} \left( \sum_{e_1 \geq 0} \frac{\sum_{e_2 \geq e_1} a(p^{e_2})}{p^{e_1(s-1)}} \right).
\end{aligned}$$

Therefore we have

$$\zeta(s) \sum_{q=1}^{\infty} \frac{f(q)}{q^s} = \prod_{p \in \mathcal{P}} \left( \sum_{e_1 \geq 0} \frac{\sum_{e_2 \geq e_1} a(p^{e_2})}{p^{e_1(s-1)}} \right),$$

which completes the proof of Theorem 3.5.  $\square$

As an example of Theorem 3.5, we show the following example.

**Example 3.9.**

$$\frac{\lambda(q)K(q)\psi(q)}{q^2} = \frac{\zeta(2)}{\zeta(4)} \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^2} c_q(n).$$

*Proof.* Let  $a(n) = \lambda(n)/n^2$ . Then we have

$$\begin{aligned}
\sum_{e_2 \geq e_1} a(p^{e_2}) &= \frac{(-1)^{e_1}}{p^{2e_1}} + \frac{(-1)^{e_1+1}}{p^{2(e_1+1)}} + \cdots = \frac{(-1)^{e_1}}{p^{2e_1}(1+1/p^2)}, \quad \text{and} \\
\sum_{e_1 \geq 0} \frac{\sum_{e_2 \geq e_1} a(p^{e_2})}{p^{e_1(s-1)}} &= \frac{1}{1+1/p^2} \sum_{e \geq 0} \frac{(-1)^e}{p^{e(s+1)}} = \frac{1}{1+1/p^2} \frac{1}{1+1/p^{s+1}}.
\end{aligned}$$

From this we have

$$\frac{1}{\zeta(s)} \prod_{p \in \mathcal{P}} \left( \sum_{e_1 \geq 0} \frac{\sum_{e_2 \geq e_1} a(p^{e_2})}{p^{e_1(s-1)}} \right) = \frac{1}{\zeta(s)} \prod_{p \in \mathcal{P}} \frac{1}{1+1/p^2} \frac{1}{1+1/p^{s+1}} = \frac{1}{\zeta(s)} \frac{\zeta(4)}{\zeta(2)} \frac{\zeta(2s+2)}{\zeta(s+1)}.$$

Therefore, if we set  $f(q) = \sum_{n=1}^{\infty} a(n)c_q(n)$ , then  $f$  satisfies

$$\sum_{q=1}^{\infty} \frac{f(q)}{q^s} = \frac{1}{\zeta(s)} \frac{\zeta(4)}{\zeta(2)} \frac{\zeta(2s+2)}{\zeta(s+1)}.$$

On the other hand, if we set  $\tilde{f}(q) = \frac{\lambda(q)K(q)\psi(q)}{q^2}$ , then  $\tilde{f}$  satisfies

$$\begin{aligned}
\sum_{q=1}^{\infty} \frac{\tilde{f}(q)}{q^s} &= \prod_{p \in \mathcal{P}} \left( 1 + \sum_{e \geq 1} \frac{1}{p^{es}} \frac{\lambda(p^e)K(p^e)\psi(p^e)}{p^{2e}} \right) = \prod_{p \in \mathcal{P}} \left( 1 + \sum_{e \geq 1} \frac{1}{p^{es}} \frac{(-1)^e p \cdot p^e (1+1/p)}{p^{2e}} \right) \\
&= \prod_{p \in \mathcal{P}} \left( 1 + (p+1) \sum_{e \geq 1} \left( \frac{-1}{p^{e(s+1)}} \right) \right) = \prod_{p \in \mathcal{P}} \left( 1 - \frac{p+1}{p^{s+1}+1} \right) \\
&= \prod_{p \in \mathcal{P}} \frac{1-1/p^s}{1+1/p^{s+1}} = \frac{\zeta(2s+2)}{\zeta(s)\zeta(s+1)}.
\end{aligned}$$

By the uniqueness of the Dirichlet series, we have  $\tilde{f}(q) = \frac{\zeta(2)}{\zeta(4)}f(q)$ , namely

$$\frac{\lambda(q)K(q)\psi(q)}{q^2} = \frac{\zeta(2)}{\zeta(4)} \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^2} c_q(n).$$

□

## 4. The Case of Arithmetic Functions of Two Variables

In this section, we consider the case of arithmetic functions of two variables. We would like to extend theorems in section 3 to this case. In more detail, we consider Ramanujan-Fourier series

$$f(n_1, n_2) = \sum_{q_1, q_2=1}^{\infty} a(q_1, q_2) c_{q_1}(n_1) c_{q_2}(n_2),$$

and dual Ramanujan-Fourier series

$$f(q_1, q_2) = \sum_{n_1, n_2=1}^{\infty} a(n_1, n_2) c_{q_1}(n_1) c_{q_2}(n_2),$$

where  $f, a$  are arithmetic functions of two variables.

We use the same notations  $\mathbf{1}$  and  $\mu$  for the functions

$$\begin{aligned} \mathbf{1}(n_1, n_2) &= \mathbf{1}(n_1)\mathbf{1}(n_2), \\ \mu(n_1, n_2) &= \mu(n_1)\mu(n_2), \end{aligned}$$

respectively. Clearly,  $(\mu * \mathbf{1})(n_1, n_2) = \delta(n_1)\delta(n_2)$  holds.

We begin with the following theorem which is an extension of Theorem 3.1.

**Theorem 4.1.** *Let  $a : \mathbb{N} \times \mathbb{N} \mapsto \mathbb{C}$  be an arithmetic function of two variables. If the series*

$$A(n_1, n_2) := n_1 n_2 \sum_{k_1, k_2=1}^{\infty} \mu(k_1, k_2) a(k_1 n_1, k_2 n_2)$$

*converges for every  $n_1, n_2 \in \mathbb{N}$ , then for  $f(n_1, n_2) = (A * \mathbf{1})(n_1, n_2)$ , we have*

$$f(n_1, n_2) = \sum_{q_1, q_2=1}^{\infty} a(q_1, q_2) c_{q_1}(n_1) c_{q_2}(n_2).$$

*Proof.* Since  $c_q(n) = (\mu *_{\ell} D *_{r} \mathbf{1})(q, n)$ , we have

$$\begin{aligned} \sum_{\substack{q_1 \leq x \\ q_2 \leq y}} a(q_1, q_2) c_{q_1}(n_1) c_{q_2}(n_2) &= \sum_{\substack{q_1 \leq x \\ q_2 \leq y}} a(q_1, q_2) (\mu *_{\ell} D *_{r} \mathbf{1})(q_1, n_1) (\mu *_{\ell} D *_{r} \mathbf{1})(q_2, n_2) \\ &= \sum_{\substack{q_1 \leq x \\ q_2 \leq y}} a(q_1, q_2) \left( \sum_{d_1 | q_1} \mu\left(\frac{q_1}{d_1}\right) (D *_{r} \mathbf{1})(d_1, n_1) \right) \left( \sum_{d_2 | q_2} \mu\left(\frac{q_2}{d_2}\right) (D *_{r} \mathbf{1})(d_2, n_2) \right). \end{aligned}$$



Setting  $q_1 = d_1 k_1$ ,  $q_2 = d_2 k_2$  and using Lemma 2.2, we see that the above is equal to

$$\begin{aligned} & \sum_{\substack{d_1 k_1 \leq x \\ d_2 k_2 \leq y}} a(d_1 k_1, d_2 k_2) \mu(k_1) \mu(k_2) (D_\ell * \mathbf{1})(d_1, n_1) (D_\ell * \mathbf{1})(d_2, n_2) \\ &= \sum_{\substack{d_1 k_1 \leq x \\ d_2 k_2 \leq y}} a(d_1 k_1, d_2 k_2) \mu(k_1) \mu(k_2) I_{d_1 | n_1} d_1 I_{d_2 | n_2} d_2 \\ &= \sum_{\substack{d_1 \leq x \\ d_2 \leq y}} I_{d_1 | n_1} I_{d_2 | n_2} d_1 d_2 \sum_{\substack{k_1 \leq x/d_1 \\ k_2 \leq y/d_2}} \mu(k_1, k_2) a(d_1 k_1, d_2 k_2), \end{aligned}$$

where  $x, y$  are sufficiently large real numbers. Letting  $x, y \rightarrow \infty$ , we have

$$\begin{aligned} \sum_{q_1, q_2=1}^{\infty} a(q_1, q_2) c_{q_1}(n_1) c_{q_2}(n_2) &= \sum_{d_1, d_2=1}^{\infty} I_{d_1 | n_1} I_{d_2 | n_2} (d_1 d_2 \sum_{k_1, k_2=1}^{\infty} \mu(k_1, k_2) a(d_1 k_1, d_2 k_2)) \\ &= \sum_{\substack{d_1 | n_1 \\ d_2 | n_2}} A(d_1, d_2) = (A * \mathbf{1})(n_1, n_2) = f(n_1, n_2), \end{aligned}$$

which proves Theorem 4.1. □

The following theorem is an extension of Theorem 3.2.

**Theorem 4.2.** *Let  $a : \mathbb{N} \times \mathbb{N} \mapsto \mathbb{C}$  be an arithmetic function of two variables. If the series*

$$A(q_1, q_2) := q_1 q_2 \sum_{k_1, k_2=1}^{\infty} a(k_1 q_1, k_2 q_2)$$

*converges for every  $q_1, q_2 \in \mathbb{N}$ , then for  $f(q_1, q_2) = (A * \mu)(q_1, q_2)$ , we have*

$$f(q_1, q_2) = \sum_{n_1, n_2=1}^{\infty} a(n_1, n_2) c_{q_1}(n_1) c_{q_2}(n_2). \quad (4.21)$$

*Proof.* The proof proceeds along the same lines as the proof of Theorem 3.2. We have

$$\begin{aligned} \sum_{\substack{n_1 \leq x \\ n_2 \leq y}} a(n_1, n_2) c_{q_1}(n_1) c_{q_2}(n_2) &= \sum_{\substack{n_1 \leq x \\ n_2 \leq y}} a(n_1, n_2) (\mu_\ell * D_r * \mathbf{1})(q_1, n_1) (\mu_\ell * D_r * \mathbf{1})(q_2, n_2) \\ &= \sum_{\substack{n_1 \leq x \\ n_2 \leq y}} a(n_1, n_2) \left( \sum_{d_1 | n_1} (\mu_\ell * D)(q_1, d_1) \mathbf{1}\left(\frac{n_1}{d_1}\right) \right) \left( \sum_{d_2 | n_2} (\mu_\ell * D)(q_2, d_2) \mathbf{1}\left(\frac{n_2}{d_2}\right) \right). \end{aligned}$$

Setting  $n_1 = d_1 k_1$ ,  $n_2 = d_2 k_2$  and using Lemma 2.2, we see that the above is equal to

$$\begin{aligned} & \sum_{\substack{d_1 k_1 \leq x \\ d_2 k_2 \leq y}} a(d_1 k_1, d_2 k_2) (\mu_\ell * D)(q_1, d_1) (\mu_\ell * D)(q_2, d_2) \\ &= \sum_{\substack{d_1 k_1 \leq x \\ d_2 k_2 \leq y}} a(d_1 k_1, d_2 k_2) I_{d_1 | q_1} \mu\left(\frac{q_1}{d_1}\right) d_1 I_{d_2 | q_2} \mu\left(\frac{q_2}{d_2}\right) d_2 \\ &= \sum_{\substack{d_1 \leq x \\ d_2 \leq y}} I_{d_1 | q_1} I_{d_2 | q_2} \mu\left(\frac{q_1}{d_1}\right) \mu\left(\frac{q_2}{d_2}\right) (d_1 d_2 \sum_{\substack{k_1 \leq x/d_1 \\ k_2 \leq y/d_2}} a(d_1 k_1, d_2 k_2)), \end{aligned}$$

where  $x, y$  are sufficiently large real numbers. Letting  $x, y \rightarrow \infty$ , we have

$$\begin{aligned} \sum_{n_1, n_2=1}^{\infty} a(n_1, n_2) c_{q_1}(n_1) c_{q_2}(n_2) &= \sum_{d_1, d_2=1}^{\infty} I_{d_1|q_1} I_{d_2|q_2} \mu\left(\frac{q_1}{d_1}\right) \mu\left(\frac{q_2}{d_2}\right) A(d_1, d_2) \\ &= \sum_{\substack{d_1|q_1 \\ d_2|q_2}} \mu\left(\frac{q_1}{d_1}\right) \mu\left(\frac{q_2}{d_2}\right) A(d_1, d_2) = (\mu * A)(q_1, q_2) = f(q_1, q_2), \end{aligned}$$

which proves Theorem 4.2. □

The following example is an extension of Example 3.2.

**Example 4.1.** *Let  $s > 1$ . Then we have*

$$\left( \prod_{p \in \mathcal{P}} \left(1 - \frac{2}{p^s}\right) \right) \left( \frac{q_1 q_2 \mu(q_1 q_2)}{\widetilde{\varphi}_s(q_1) \widetilde{\varphi}_s(q_2)} * \mu \right)(q_1, q_2) = \sum_{n_1, n_2=1}^{\infty} \frac{\mu(n_1 n_2)}{(n_1 n_2)^s} c_{q_1}(n_1) c_{q_2}(n_2), \quad (4.22)$$

where  $\widetilde{\varphi}_s(q) = q^s \prod_{p|q} (1 - 2/p^s)$ .

*Proof.* Setting  $a(n_1, n_2) = \frac{\mu(n_1 n_2)}{(n_1 n_2)^s}$  we have

$$\begin{aligned} A(q_1, q_2) &= q_1 q_2 \sum_{k_1, k_2=1}^{\infty} a(k_1 q_1, k_2 q_2) = q_1 q_2 \sum_{k_1, k_2=1}^{\infty} \frac{\mu(k_1 q_1 k_2 q_2)}{(k_1 q_1 k_2 q_2)^s} \\ &= q_1 q_2 \sum_{\substack{k_1, k_2 \geq 1 \\ (k_1 k_2, q_1 q_2)=1}} \frac{\mu(k_1 k_2) \mu(q_1 q_2)}{(k_1 k_2)^s (q_1 q_2)^s} = \frac{\mu(q_1 q_2)}{(q_1 q_2)^{s-1}} \sum_{\substack{k \geq 1 \\ (k, q_1 q_2)=1}} \frac{\mu(k)}{k^s} \sum_{k_1|k} 1 \\ &= \frac{\mu(q_1 q_2)}{(q_1 q_2)^{s-1}} \sum_{\substack{k \geq 1 \\ (k, q_1 q_2)=1}} \frac{\mu(k) \tau(k)}{k^s} = \frac{\mu(q_1 q_2)}{(q_1 q_2)^{s-1}} \prod_{p|q_1 q_2} \left(1 + \frac{\mu(p) \tau(p)}{p^s}\right) \\ &= \frac{\mu(q_1 q_2)}{(q_1 q_2)^{s-1}} \prod_{p|q_1 q_2} \left(1 - \frac{2}{p^s}\right) = \frac{\mu(q_1 q_2)}{(q_1 q_2)^{s-1}} \frac{\prod_{p \in \mathcal{P}} (1 - 2/p^s)}{\prod_{p|q_1 q_2} (1 - 2/p^s)}. \end{aligned}$$

If  $(q_1, q_2) > 1$ , then  $A(q_1, q_2) = 0$  since  $\mu(q_1 q_2) = 0$ . If  $(q_1, q_2) = 1$ , then we have

$$\begin{aligned} A(q_1, q_2) &= \frac{\mu(q_1 q_2)}{(q_1 q_2)^{s-1}} \frac{\prod_{p \in \mathcal{P}} (1 - 2/p^s)}{\prod_{p|q_1} (1 - 2/p^s) \prod_{p|q_2} (1 - 2/p^s)} \\ &= \frac{q_1 q_2 \mu(q_1 q_2)}{(q_1^s \prod_{p|q_1} (1 - 2/p^s)) (q_2^s \prod_{p|q_2} (1 - 2/p^s))} \prod_{p \in \mathcal{P}} (1 - 2/p^s) \\ &= \frac{q_1 q_2 \mu(q_1 q_2)}{\widetilde{\varphi}_s(q_1) \widetilde{\varphi}_s(q_2)} \prod_{p \in \mathcal{P}} (1 - 2/p^s), \end{aligned}$$

which clearly holds also in the case  $(q_1, q_2) > 1$ . If we set  $f = A * \mu$ , then Theorem 4.2 gives the desired result. □

**Remark 4.1.** *We consider the case  $s \downarrow 1$  in (4.22), where the notation  $s \downarrow 1$  means that  $s$  approaches*

1 from above. Since

$$\begin{aligned}
\prod_{p \in \mathcal{P}} \left(1 - \frac{2}{p^s}\right) &= \left(1 - \frac{2}{2^s}\right) \prod_{\substack{p \in \mathcal{P} \\ p \geq 3}} \left(1 - \frac{2}{p^s}\right) = \left(1 - \frac{2}{2^s}\right) \prod_{\substack{p \in \mathcal{P} \\ p \geq 3}} \left(1 - 2/p^s\right) \frac{(1 - 1/p^s)^2}{1 - 2/p^s + 1/p^{2s}} \\
&= \left(1 - \frac{2}{2^s}\right) \prod_{\substack{p \in \mathcal{P} \\ p \geq 3}} \frac{(1 - 1/p^s)^2}{\frac{1 - 2/p^s + 1/p^{2s}}{(1 - 2/p^s)}} = \left(1 - \frac{2}{2^s}\right) \prod_{\substack{p \in \mathcal{P} \\ p \geq 3}} \frac{(1 - 1/p^s)^2}{1 + \frac{1}{p^{2s}(1 - 2/p^s)}} \\
&= \left(1 - \frac{2}{2^s}\right) \prod_{\substack{p \in \mathcal{P} \\ p \geq 3}} \left(1 - \frac{1}{p^s}\right)^2 \prod_{\substack{p \in \mathcal{P} \\ p \geq 3}} \frac{1}{1 + \frac{1}{p^s(p^s - 2)}} \\
&= \left(1 - \frac{2}{2^s}\right) \frac{1}{\left(1 - \frac{1}{2^s}\right)^2 \zeta^2(s)} \prod_{\substack{p \in \mathcal{P} \\ p \geq 3}} \frac{1}{1 + \frac{1}{p^s(p^s - 2)}},
\end{aligned}$$

we have

$$\begin{aligned}
&\lim_{s \downarrow 1} \left( \prod_{p \in \mathcal{P}} \left(1 - \frac{2}{p^s}\right) \right) \frac{q_1 q_2 \mu(q_1 q_2)}{\widetilde{\varphi}_s(q_1) \widetilde{\varphi}_s(q_2)} \\
&= \lim_{s \downarrow 1} \left(1 - \frac{2}{2^s}\right) \frac{1}{\left(1 - \frac{1}{2^s}\right)^2 \zeta^2(s)} \left( \prod_{\substack{p \in \mathcal{P} \\ p \geq 3}} \frac{1}{1 + \frac{1}{p^s(p^s - 2)}} \right) \frac{q_1 q_2 \mu(q_1 q_2)}{\widetilde{\varphi}_s(q_1) \widetilde{\varphi}_s(q_2)} \\
&= \lim_{s \downarrow 1} \left(1 - \frac{2}{2^s}\right) \frac{1}{\zeta^2(s)} \frac{q_1 q_2 \mu(q_1 q_2)}{q_1^s \prod_{p|q_1} (1 - 2/p^s) q_2^s \prod_{p|q_2} (1 - 2/p^s)} \frac{1}{\left(1 - \frac{1}{2}\right)^2} \prod_{\substack{p \in \mathcal{P} \\ p \geq 3}} \frac{1}{1 + \frac{1}{p(p-2)}} \\
&= \lim_{s \downarrow 1} \frac{1}{\zeta^2(s)} \frac{1 - 2/2^s}{\prod_{p|q_1} (1 - 2/p^s) \prod_{p|q_2} (1 - 2/p^s)} \mu(q_1 q_2) \frac{1}{\left(1 - \frac{1}{2}\right)^2} \prod_{\substack{p \in \mathcal{P} \\ p \geq 3}} \frac{1}{1 + \frac{1}{p(p-2)}} = 0,
\end{aligned}$$

where we note that, since  $\mu(q_1 q_2) = 0$  if  $q_1$  and  $q_2$  are even, we may assume  $q_1$  or  $q_2$  is odd. Therefore by letting  $s \downarrow 1$  in (4.22), we obtain

$$\sum_{n_1, n_2=1}^{\infty} \frac{\mu(n_1 n_2)}{n_1 n_2} c_{q_1}(n_1) c_{q_2}(n_2) = 0,$$

which is an extension of (3.12) to the case of two variables. Of course an extension of (3.13)

$$\sum_{n_1, n_2=1}^{\infty} \frac{\lambda(n_1 n_2)}{n_1 n_2} c_{q_1}(n_1) c_{q_2}(n_2) = 0$$

clearly holds since  $\lambda$  is completely multiplicative.

Next we consider extensions of Theorem 3.3 and Theorem 3.4. Ushiroya [Us] proved the following theorem which is an extension of Theorem 3.3.

**Theorem 4.3.** ([Us]) (i) Let  $f(n_1, n_2)$  be an arithmetic function of two variables satisfying

$$\sum_{n_1, n_2=1}^{\infty} 2^{\omega(n_1)} 2^{\omega(n_2)} \frac{|(f * \mu)(n_1, n_2)|}{n_1 n_2} < \infty.$$

Then its Ramanujan-Fourier series is pointwise convergent and

$$f(n_1, n_2) = \sum_{q_1, q_2=1}^{\infty} a(q_1, q_2) c_{q_1}(n_1) c_{q_2}(n_2)$$

holds where

$$a(q_1, q_2) = \sum_{m_1, m_2=1}^{\infty} \frac{(f * \mu)(m_1 q_1, m_2 q_2)}{m_1 q_1 m_2 q_2}.$$

(ii) Let  $f$  be a multiplicative function of two variables satisfying

$$\sum_{p \in \mathcal{P}} \sum_{\substack{e_1, e_2 \geq 0 \\ e_1 + e_2 \geq 1}} \frac{|(f * \mu)(p^{e_1}, p^{e_2})|}{p^{e_1 + e_2}} < \infty.$$

Then its Ramanujan-Fourier series is pointwise convergent and

$$f(n_1, n_2) = \sum_{q_1, q_2=1}^{\infty} a(q_1, q_2) c_{q_1}(n_1) c_{q_2}(n_2)$$

holds where

$$a(q_1, q_2) = \sum_{m_1, m_2=1}^{\infty} \frac{(f * \mu)(m_1 q_1, m_2 q_2)}{m_1 q_1 m_2 q_2}.$$

Moreover, if the mean value  $M(f) = \lim_{x \rightarrow \infty} \sum_{n \leq x} f(n)$  is not zero and if  $\{q_1, q_2\} > 1$ , where  $\{q_1, q_2\}$  denotes the least common multiple of  $q_1$  and  $q_2$ , then  $a(q_1, q_2)$  can be rewritten as

$$\begin{aligned} a(q_1, q_2) &= \prod_{p \in \mathcal{P}} \left( \sum_{e_1 = \nu_p(q_1)} \sum_{e_2 = \nu_p(q_2)} \frac{(f * \mu)(p^{e_1}, p^{e_2})}{p^{e_1 + e_2}} \right) \\ &= M(f) \prod_{p | \{q_1, q_2\}} \left\{ \left( \sum_{e_1 = \nu_p(q_1)} \sum_{e_2 = \nu_p(q_2)} \frac{(f * \mu)(p^{e_1}, p^{e_2})}{p^{e_1 + e_2}} \right) / \left( \sum_{e_1=0} \sum_{e_2=0} \frac{(f * \mu)(p^{e_1}, p^{e_2})}{p^{e_1 + e_2}} \right) \right\}. \end{aligned} \quad (4.23)$$

We remark that many examples of the form

$$f(n_1, n_2) = \sum_{q_1, q_2=1}^{\infty} a(q_1, q_2) c_{q_1}(n_1) c_{q_2}(n_2)$$

are obtained in [Us].

Next we extend Theorem 3.4 to dual Ramanujan-Fourier series.

**Theorem 4.4.** *Let  $f$  be an arithmetic function of two variables satisfying*

$$\sum_{q_1, q_2=1}^{\infty} \frac{|(f * \mathbf{1})(q_1, q_2)|}{q_1 q_2} \tau(q_1) \tau(q_2) < \infty. \quad (4.24)$$

Then its dual Ramanujan-Fourier series is pointwise convergent and

$$f(q_1, q_2) = \sum_{n_1, n_2=1}^{\infty} a(n_1, n_2) c_{q_1}(n_1) c_{q_2}(n_2)$$

holds where

$$a(n_1, n_2) = \sum_{m_1, m_2=1}^{\infty} \frac{(f * \mathbf{1})(n_1 m_1, n_2 m_2)}{n_1 m_1 n_2 m_2} \mu(m_1, m_2). \quad (4.25)$$

**Remark 4.2.** Let  $f$  be a multiplicative function of two variables satisfying

$$\sum_{p \in \mathcal{P}} \sum_{\substack{e_1, e_2 \geq 0 \\ e_1 + e_2 \geq 1}} \frac{|(f * \mathbf{1})(p^{e_1}, p^{e_2})|}{p^{e_1 + e_2}} (e_1 + 1)(e_2 + 1) < \infty. \quad (4.26)$$

Then its dual Ramanujan-Fourier series is pointwise convergent and

$$f(q_1, q_2) = \sum_{n_1, n_2=1}^{\infty} a(n_1, n_2) c_{q_1}(n_1) c_{q_2}(n_2)$$

holds where

$$a(n_1, n_2) = \sum_{m_1, m_2=1}^{\infty} \frac{(f * \mathbf{1})(n_1 m_1, n_2 m_2)}{n_1 m_1 n_2 m_2} \mu(m_1, m_2).$$

Moreover,  $a(n_1, n_2)$  can be rewritten as

$$a(n_1, n_2) = \prod_{p \in \mathcal{P}} \left( \frac{(f * \mathbf{1})(p^{\nu_p(n_1)}, p^{\nu_p(n_2)})}{p^{\nu_p(n_1) + \nu_p(n_2)}} - \frac{(f * \mathbf{1})(p^{\nu_p(n_1)+1}, p^{\nu_p(n_2)})}{p^{\nu_p(n_1) + \nu_p(n_2) + 1}} \right. \\ \left. - \frac{(f * \mathbf{1})(p^{\nu_p(n_1)}, p^{\nu_p(n_2)+1})}{p^{\nu_p(n_1) + \nu_p(n_2) + 1}} + \frac{(f * \mathbf{1})(p^{\nu_p(n_1)+1}, p^{\nu_p(n_2)+2})}{p^{\nu_p(n_1) + \nu_p(n_2) + 2}} \right).$$

*Proof of Theorem 4.4.* We proceed along the same lines as the proof of Theorem 3.4. We first see that  $A(q_1, q_2) = q_1 q_2 \sum_{k_1, k_2=1}^{\infty} a(k_1 q_1, k_2 q_2)$  converges since

$$\sum_{\substack{k_1 \leq x \\ k_2 \leq y}} |a(k_1 q_1, k_2 q_2)| \leq \sum_{\substack{k_1 \leq x \\ k_2 \leq y}} \sum_{m_1, m_2=1}^{\infty} \frac{|(f * \mathbf{1})(k_1 q_1 m_1, k_2 q_2 m_2)|}{k_1 q_1 m_1 k_2 q_2 m_2} |\mu(m_1, m_2)| \\ \leq \sum_{\ell_1, \ell_2=1}^{\infty} \frac{|(f * \mathbf{1})(\ell_1, \ell_2)|}{\ell_1 \ell_2} \sum_{\substack{k_1 \leq x, k_1 | \ell_1 \\ k_2 \leq y, k_2 | \ell_2}} 1 \\ \leq \sum_{\ell_1, \ell_2=1}^{\infty} \frac{|(f * \mathbf{1})(\ell_1, \ell_2)|}{\ell_1 \ell_2} \tau(\ell_1) \tau(\ell_2) < \infty.$$

Using Lemma 2.5 we can rewrite  $A(q_1, q_2)$  as

$$A(q_1, q_2) = q_1 q_2 \sum_{k_1, k_2=1}^{\infty} a(k_1 q_1, k_2 q_2) = \sum_{m_1, m_2=1}^{\infty} a(m_1, m_2) q_1 I_{q_1 | m_1} q_2 I_{q_2 | m_2} \\ = \sum_{m_1, m_2=1}^{\infty} a(m_1, m_2) (\mathbf{1} *_{\ell} c)(q_1, m_1) (\mathbf{1} *_{\ell} c)(q_2, m_2).$$

From this we have

$$f(q_1, q_2) = (\mu * A)(q_1, q_2) = \sum_{m_1, m_2=1}^{\infty} a(m_1, m_2) (\mu *_{\ell} (\mathbf{1} *_{\ell} c))(q_1, m_1) (\mu *_{\ell} (\mathbf{1} *_{\ell} c))(q_2, m_2) \\ = \sum_{m_1, m_2=1}^{\infty} a(m_1, m_2) (\delta *_{\ell} c)(q_1, m_1) (\delta *_{\ell} c)(q_2, m_2) \\ = \sum_{m_1, m_2=1}^{\infty} a(m_1, m_2) c_{q_1}(m_1) c_{q_2}(m_2).$$

This completes the proof of Theorem 4.4.  $\square$

*Proof of Remark 4.2.* We first note that, if  $f$  is a multiplicative function of two variables, then  $(q_1, q_2) \mapsto \frac{(f * \mathbf{1})(q_1, q_2)}{q_1 q_2} \tau(q_1) \tau(q_2)$  is also a multiplicative function of two variables. Using  $1 + x \leq \exp(x)$ , we see that (4.24) follows from (4.26) since

$$\begin{aligned} \sum_{\substack{q_1 \leq Q_1 \\ q_2 \leq Q_2}} \frac{|(f * \mathbf{1})(q_1, q_2)|}{q_1 q_2} \tau(q_1) \tau(q_2) &\leq \prod_{p \in \mathcal{P}} \left( 1 + \sum_{\substack{e_1, e_2 \geq 0 \\ e_1 + e_2 \geq 1}} \frac{|(f * \mathbf{1})(p^{e_1}, p^{e_2})|}{p^{e_1} p^{e_2}} \tau(p^{e_1}) \tau(p^{e_2}) \right) \\ &\leq \prod_{p \in \mathcal{P}} \exp \left( \sum_{\substack{e_1, e_2 \geq 0 \\ e_1 + e_2 \geq 1}} \frac{|(f * \mathbf{1})(p^{e_1}, p^{e_2})|}{p^{e_1} p^{e_2}} \tau(p^{e_1}) \tau(p^{e_2}) \right) \\ &= \exp \left( \sum_{p \in \mathcal{P}} \sum_{\substack{e_1, e_2 \geq 0 \\ e_1 + e_2 \geq 1}} \frac{|(f * \mathbf{1})(p^{e_1}, p^{e_2})|}{p^{e_1 + e_2}} (e_1 + 1)(e_2 + 1) \right) < \infty \end{aligned}$$

holds for any  $Q_1, Q_2 > 1$ . Therefore (4.25) holds by Theorem 4.4. In the expression of (4.25), we set, for  $i = 1, 2$ ,  $n_i = \prod p_j^{e_{ij}}$ ,  $m_i = r_i \prod p_j^{d_{ij}}$  where  $(r_i, n_1 n_2) = 1$ ,  $e_{ij} \geq 1$ , and  $d_{ij} \geq 0$ . Then we have

$$a(n_1, n_2) = \sum_{\substack{d_{ij} \geq 0, r_i \geq 1 \\ (r_i, n_1 n_2) = 1}} \frac{(f * \mathbf{1})(r_1 \prod p_j^{d_{1j} + e_{1j}}, r_2 \prod p_j^{d_{2j} + e_{2j}})}{r_1 r_2 \prod p_j^{d_{1j} + e_{1j} + d_{2j} + e_{2j}}} \mu(r_1 \prod p_j^{d_{1j}}, r_2 \prod p_j^{d_{2j}}).$$

Since  $f * \mathbf{1}$  is multiplicative and since  $\mu(p_j^{d_{1j}}, p_j^{d_{2j}}) = 0$  if  $d_{1j} \geq 2$  or  $d_{2j} \geq 2$  for some  $j$ , we obtain

$$\begin{aligned} a(n_1, n_2) &= \prod_j \left( \sum_{0 \leq d_{ij} \leq 1} \frac{(f * \mathbf{1})(p_j^{d_{1j} + e_{1j}}, p_j^{d_{2j} + e_{2j}})}{p_j^{d_{1j} + e_{1j} + d_{2j} + e_{2j}}} \mu(p_j^{d_{1j}}, p_j^{d_{2j}}) \right) \times \sum_{\substack{r_i \geq 1 \\ (r_i, n_1 n_2) = 1}} \frac{(f * \mathbf{1})(r_1, r_2)}{r_1 r_2} \mu(r_1, r_2) \\ &= \prod_{p | n_1 n_2} \left( \frac{(f * \mathbf{1})(p^{\nu_p(n_1)}, p^{\nu_p(n_2)})}{p^{\nu_p(n_1) + \nu_p(n_2)}} - \frac{(f * \mathbf{1})(p^{\nu_p(n_1) + 1}, p^{\nu_p(n_2)})}{p^{\nu_p(n_1) + \nu_p(n_2) + 1}} \right. \\ &\quad \left. - \frac{(f * \mathbf{1})(p^{\nu_p(n_1)}, p^{\nu_p(n_2) + 1})}{p^{\nu_p(n_1) + \nu_p(n_2) + 1}} + \frac{(f * \mathbf{1})(p^{\nu_p(n_1) + 1}, p^{\nu_p(n_2) + 2})}{p^{\nu_p(n_1) + \nu_p(n_2) + 2}} \right) \\ &\quad \times \prod_{p | n_1 n_2} \left( 1 - \frac{f(p, 1)}{p} - \frac{f(1, p)}{p} + \frac{f(p, p)}{p^2} \right) \\ &= \prod_{p \in \mathcal{P}} \left( \frac{(f * \mathbf{1})(p^{\nu_p(n_1)}, p^{\nu_p(n_2)})}{p^{\nu_p(n_1) + \nu_p(n_2)}} - \frac{(f * \mathbf{1})(p^{\nu_p(n_1) + 1}, p^{\nu_p(n_2)})}{p^{\nu_p(n_1) + \nu_p(n_2) + 1}} \right. \\ &\quad \left. - \frac{(f * \mathbf{1})(p^{\nu_p(n_1)}, p^{\nu_p(n_2) + 1})}{p^{\nu_p(n_1) + \nu_p(n_2) + 1}} + \frac{(f * \mathbf{1})(p^{\nu_p(n_1) + 1}, p^{\nu_p(n_2) + 2})}{p^{\nu_p(n_1) + \nu_p(n_2) + 2}} \right), \end{aligned}$$

which completes the proof of Remark 4.2.  $\square$

If we take  $f = \mu$  in Theorem 3.4, then it is obvious that

$$\mu(q) = \sum_{n=1}^{\infty} a(n) c_q(n)$$

holds where  $a(n) = \delta(n)$ . The following example is an extension of the above trivial example.

**Example 4.2.** *We have*

$$\mu(q_1q_2) = \frac{1}{\zeta(2)} \sum_{n_1, n_2=1}^{\infty} \frac{\mu(K((n_1, n_2)))\varphi(K((n_1, n_2)))}{n_1n_2\psi(K(n_1n_2))} c_{q_1}(n_1)c_{q_2}(n_2).$$

*Proof.* Let  $f(q_1, q_2) = \mu(q_1q_2)$ . Then it is easy to see that

$$\begin{aligned} (f * \mathbf{1})(p^k, 1) &= (f * \mathbf{1})(1, p^k) = 0 \quad \text{if } k \geq 1, \\ (f * \mathbf{1})(p^k, p^\ell) &= -1 \quad \text{if } k, \ell \geq 1. \end{aligned}$$

From this we see that (4.26) holds. We have

$$\begin{aligned} & \frac{(f * \mathbf{1})(p^k, p^\ell)}{p^{k+\ell}} - \frac{(f * \mathbf{1})(p^{k+1}, p^\ell)}{p^{k+\ell+1}} - \frac{(f * \mathbf{1})(p^k, p^{\ell+1})}{p^{k+\ell+1}} + \frac{(f * \mathbf{1})(p^{k+1}, p^{\ell+1})}{p^{k+\ell+2}} \\ &= \begin{cases} 1 - 1/p^2 & \text{if } k = \ell = 0 \\ 1/p^{\ell+1} - 1/p^{\ell+2} & \text{if } k = 0, \ell \geq 1 \\ 1/p^{k+1} - 1/p^{k+2} & \text{if } k \geq 1, \ell = 0 \\ -1/p^{k+\ell} + 2/p^{k+\ell+1} - 1/p^{k+\ell+2} & \text{if } k, \ell \geq 1. \end{cases} \end{aligned}$$

Therefore we have by Remark 4.2

$$\begin{aligned} a(n_1, n_2) &= \prod_{p|n_1n_2} \left(1 - \frac{1}{p^2}\right) \prod_{\substack{p|n_1 \\ p|n_2}} \left(\frac{1}{p^{\nu_p(n_2)+1}} - \frac{1}{p^{\nu_p(n_2)+2}}\right) \prod_{\substack{p|n_1 \\ p|n_2}} \left(\frac{1}{p^{\nu_p(n_1)+1}} - \frac{1}{p^{\nu_p(n_1)+2}}\right) \\ &\quad \times \prod_{\substack{p|n_1 \\ p|n_2}} \left(-\frac{1}{p^{\nu_p(n_1)+\nu_p(n_2)}} + \frac{2}{p^{\nu_p(n_1)+\nu_p(n_2)+1}} - \frac{1}{p^{\nu_p(n_1)+\nu_p(n_2)+2}}\right) \\ &= \prod_{p|n_1n_2} \left(1 - \frac{1}{p^2}\right) \prod_{\substack{p|n_1 \\ p|n_2}} \frac{1}{p^{\nu_p(n_2)+1}} \left(1 - \frac{1}{p}\right) \prod_{\substack{p|n_1 \\ p|n_2}} \frac{1}{p^{\nu_p(n_1)+1}} \left(1 - \frac{1}{p}\right) \\ &\quad \times \prod_{\substack{p|n_1 \\ p|n_2}} \frac{-1}{p^{\nu_p(n_1)+\nu_p(n_2)}} \left(1 - \frac{2}{p} + \frac{1}{p^2}\right) \\ &= \prod_{p \in \mathcal{P}} \left(1 - \frac{1}{p^2}\right) \prod_{\substack{p|n_1 \\ p|n_2}} \frac{1}{p^{\nu_p(n_2)+1}} \frac{1-1/p}{1-1/p^2} \prod_{\substack{p|n_1 \\ p|n_2}} \frac{1}{p^{\nu_p(n_1)+1}} \frac{1-1/p}{1-1/p^2} \prod_{\substack{p|n_1 \\ p|n_2}} \frac{-1}{p^{\nu_p(n_1)+\nu_p(n_2)}} \frac{(1-1/p)^2}{1-1/p^2} \\ &= \frac{1}{\zeta(2)} \prod_{\substack{p|n_1 \\ p|n_2}} \frac{1}{p^{\nu_p(n_2)+1}} \frac{1}{1+1/p} \prod_{\substack{p|n_1 \\ p|n_2}} \frac{1}{p^{\nu_p(n_1)+1}} \frac{1}{1+1/p} \prod_{p|(n_1, n_2)} \frac{-1}{p^{\nu_p(n_1)+\nu_p(n_2)}} \frac{1-1/p}{1+1/p} \\ &= \frac{1}{\zeta(2)} \prod_{p|n_1n_2} \frac{1}{p^{\nu_p(n_1)+\nu_p(n_2)}} \frac{1}{p(1+1/p)} \prod_{p|(n_1, n_2)} (-1)p(1-1/p) \\ &= \frac{1}{\zeta(2)} \frac{\mu(K((n_1, n_2)))\varphi(K((n_1, n_2)))}{n_1n_2\psi(K(n_1n_2))}, \end{aligned}$$

which completes the proof of Example 4.2.  $\square$

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