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# Control Synthesis for Stochastic Switched Systems using the Tamed Euler Method

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## ABSTRACT

In this paper, we explain how, under the one-sided Lipschitz (OSL) hypothesis, one can find an error bound for a variant of the Euler-Maruyama approximation method for stochastic switched systems. We then explain how this bound can be used to control stochastic switched system in order to stabilize them in a given region. The method is illustrated on several examples of the literature.

## 1 INTRODUCTION

Control synthesis for stochastic switched systems has been recently explored using the construction of *approximately bisimilar symbolic* models. This approach relies on the hypothesis of *incremental stability* of the stochastic switched system (or existence of a common/multiple Lyapunov function) [8–10], which concerns only a small part of the real systems.

Here we address the problem of control synthesis in a more general setting. We do not use the construction of approximately bisimilar symbolic models, but the (tamed) Euler method [3] for stochastic switched systems. We thus follow the lines of previous work on control synthesis for deterministic switched systems [1]. Unlike [8–10], the Euler-based method requires neither state-space discretization nor input set discretization, thus avoiding a source of combinatorial explosion.

The correctness of these Euler-based methods does not rely on the hypothesis of incremental stability as in [8, 10], but on the hypothesis of ‘one-sided Lipschitz (OSL)’ condition with constant  $\lambda \in \mathbb{R}^d$  (also called ‘monotonicity’/‘dissipativity’, see [7]). It can be seen that if a stochastic switched system satisfies an OSL condition with  $\lambda < 0$ , then the function  $V(x, x') = \|x - x'\|^2$  is a *common incremental Lyapunov function* in the sense of [9], from which it follows that the switched system is incrementally stable, and can be treated by approximate bisimulation. However, Euler-based methods also apply when the system is not incrementally stable, in which case the constant  $\lambda$  is necessarily positive.

The plan of the paper is as follows: In Section 2, we give an explicit upper bound on the mean square error of the tamed Euler method for SDEs under OSL condition. We apply the result in order to ensure properties of stochastic switched systems, such as ‘(R, S)-stability’ (Section 3). We conclude in Section 4.

## 2 BOUNDING THE ERROR OF THE TAMED EULER METHOD

### 2.1 Assumptions

The symbol  $\|\cdot\|$  denotes the Euclidean norm on  $\mathbb{R}^d$ . The symbol  $\langle \cdot, \cdot \rangle$  denotes the scalar product of two vectors of  $\mathbb{R}^d$ . Given a point  $x \in \mathbb{R}^d$  and a positive real  $r > 0$ , the ball  $B(x, r)$  of centre  $x$  and radius  $r$  is the set  $\{y \in \mathbb{R}^d \mid \|x - y\| \leq r\}$ .

Let  $\tau \in (0, \infty)$  be a fixed real number, let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space with normal filtration  $(\mathcal{F}_t)_{t \in [0, \tau]}$ , let  $d, m \in \mathbb{N} := \{1, 2, \dots\}$  let  $W = (W^{(1)}, \dots, W^{(m)}) : [0, \tau] \times \Omega \rightarrow \mathbb{R}^m$  be an  $m$ -dimensional standard  $(W_t)_{t \in [0, \tau]}$ -Brownian motion and let  $x_0 : \Omega \rightarrow \mathbb{R}^d$  be an  $\mathcal{F}_0/\mathcal{B}(\mathbb{R}^d)$ -measurable mapping with  $\mathbb{E}[\|x_0\|^p] < \infty$  for all  $p \in [1, \infty)$ . Moreover, let  $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$  be a continuously differentiable and globally one-sided Lipschitz continuous function whose derivative grows at most polynomially and let  $g = (g_{i,j})_{i \in \{1, \dots, d\}, j \in \{1, \dots, m\}} : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times m}$  be a globally Lipschitz continuous function.

Then consider the Stochastic Differential Equations (SDE):

$$dX_t = f(X_t)dt + g(X_t)dW_t, \quad X_0 = x_0 \quad (1)$$

for  $t \in [0, \tau]$ . The drift coefficient  $f$  is the infinitesimal mean of the process  $X$  and the diffusion coefficient  $g$  is the infinitesimal standard deviation of the process  $X$ . Under the above assumptions, the SDE (1) is known to have a unique strong solution. More formally, there exists an adapted stochastic process  $X : [0, \tau] \times \Omega \rightarrow \mathbb{R}^d$  with continuous sample paths fulfilling

$$X_{t, x_0} = x_0 + \int_0^t f(X_s)ds + \int_0^t g(X_s)dW_s$$

for all  $t \in [0, \tau]$   $\mathbb{P}$ -a.s. (see, e.g., [6]).

We denote by  $X_{t, x_0}$  the solution of Equation (1) at time  $t$  from initial condition  $X_{0, x_0} = x_0$   $\mathbb{P}$ -a.s., in which  $x_0$  is a random variable that is measurable in  $\mathcal{F}_0$ .

We suppose that  $f$  behaves polynomially and  $g$  is Lipschitz, i.e.: there exist constants  $D \in \mathbb{R}_{\geq 0}$ ,  $q \in \mathbb{N}$  and  $L_g \in \mathbb{R}_{\geq 0}$  such that, for all  $x, y \in \mathbb{R}^d$

$$\|f(x) - f(y)\|^2 \leq D\|x - y\|^2(1 + \|x\|^q + \|y\|^q) \quad (H1)$$

$$\|g(x) - g(y)\| \leq L_g\|x - y\| \quad (H2)$$

We also assume that the SDE (1) satisfies the following one-sided Lipschitz (OSL) condition with constant  $\lambda \in \mathbb{R}$ :

$$\exists \lambda \in \mathbb{R} \forall x, y \in \mathbb{R}^d : \langle f(y) - f(x), y - x \rangle \leq \lambda \|y - x\|^2 \quad (H3)$$

REMARK 1. Constants  $\lambda$ ,  $L_g$  and  $D$  can be computed using (constrained) optimization algorithms (see [1]).

### 2.2 Tamed Euler approximation

The standard way to extend the classical Euler method for ordinary differential equations to the SDE (1) is the Euler-Maruyama scheme [4]. More precisely, given  $z : \Omega \rightarrow \mathbb{R}^d$  an  $\mathcal{F}_0/\mathcal{B}(\mathbb{R}^d)$ -measurable mapping with  $\mathbb{E}[\|z\|^p] < \infty$  for all  $p \in [1, \infty)$ , the explicit Euler-Maruyama (EM) method for the SDE (1) is given by the mappings  $Y_{n,z}^N : \Omega \rightarrow \mathbb{R}^d$ ,  $n \in \{0, 1, \dots, N\}$ , which satisfy  $Y_{0,z}^N = z$  and

$$Y_{n+1,z}^N = Y_{n,z}^N + \frac{\tau}{N} \cdot f(Y_{n,z}^N) + g(Y_{n,z}^N)(W_{(n+1)\tau/N} - W_{n\tau/N})$$

for all  $n \in \{0, 1, \dots, N-1\}$  and all  $N \in \mathbb{N}$ . See [4]. Unfortunately, the convergence results for the EM scheme does not hold when the drift function  $f$  of the SDE (1) behaves polynomially (and not linearly). For the sake of generality, we will now adopt a refined scheme, which has been proposed recently in order to overcome this difficulty [3]. Let  $\underline{X}_{n,z}^N : \Omega \rightarrow \mathbb{R}^d$ ,

$$\underline{X}_{n+1,z}^N = \underline{X}_{n,z}^N + \frac{\frac{\tau}{N} \cdot f(\underline{X}_{n,z}^N)}{1 + \frac{\tau}{N} \cdot \|f(\underline{X}_{n,z}^N)\|} + g(\underline{X}_{n,z}^N)(W_{\frac{(n+1)\tau}{N}} - W_{\frac{n\tau}{N}}) \quad (2)$$

for all  $n \in \{0, 1, \dots, N-1\}$  and all  $N \in \mathbb{N}$ . We refer to the numerical method (2) as a *tamed Euler scheme* [3]. In this method the drift term  $\frac{\tau}{N} \cdot f(\underline{X}_{n,z}^N)$  is “tamed” by the factor  $1/(1 + \frac{\tau}{N} \cdot \|f(\underline{X}_{n,z}^N)\|)$  for  $n \in \{0, 1, \dots, N-1\}$  and  $N \in \mathbb{N}$  in (2).

A time continuous interpolation of the time discrete numerical approximations (2) is also introduced in [3] as follows. Let  $\tilde{X}_z^N : [0, \tau] \times \Omega \rightarrow \mathbb{R}^d$ ,  $N \in \mathbb{N}$ , be a sequence of stochastic processes given by

$$\tilde{X}_{t,z}^N = \tilde{X}_{n,z}^N + \frac{(t - n\tau/N) \cdot f(\tilde{X}_{n,z}^N)}{1 + \tau/N \cdot \|f(\tilde{X}_{n,z}^N)\|} + g(\tilde{X}_{n,z}^N)(W_t - W_{\frac{n\tau}{N}})$$

for all  $t \in [\frac{n\tau}{N}, \frac{(n+1)\tau}{N}]$ ,  $n \in \{0, 1, \dots, N-1\}$  and all  $N \in \mathbb{N}$ . Note that  $\tilde{X}_{t,z}^N : [0, \tau] \times \Omega \rightarrow \mathbb{R}^d$  is an adapted stochastic process with continuous sample paths for every  $N \in \mathbb{N}$ .

Let us define  $\underline{X}_{t,z}^N$  by

$$\underline{X}_{t,z}^N := \underline{X}_{n,z}^N \quad \text{for } t \in [\frac{n\tau}{N}, \frac{(n+1)\tau}{N}].$$

Note that  $\tilde{X}_{t,z}^N = \underline{X}_{t,z}^N = \underline{X}_{n,z}^N$  at time  $t = \frac{n\tau}{N}$  for  $n \in \{0, 1, \dots, N\}$ .

The following theorem is proven in [3]:

**THEOREM 1.** [3] *Suppose (H1)(H2)(H3). Let the setting in this section be fulfilled, and  $z : \Omega \rightarrow \mathbb{R}^d$  be an  $\mathcal{F}_0/\mathcal{B}(\mathbb{R}^d)$ -measurable mapping with  $\mathbb{E}[\|z\|^p] < \infty$  for all  $p \in [1, \infty)$ . Then, for all  $p \in [1, \infty)$*

$$\sup_{N \in \mathbb{N}} \sup_{n \in \{0, 1, \dots, N\}} \mathbb{E}[\|\underline{X}_{n,z}^N\|^p] < \infty$$

For the sake of simplicity, the number  $N$  of subsampling steps is now left implicit. From Theorem 1, it follows (cf. Lemma 4.3, [2]):

**LEMMA 2.** *Suppose (H1)(H2)(H3). Let the setting in this section be fulfilled, and  $z : \Omega \rightarrow \mathbb{R}^d$  be an  $\mathcal{F}_0/\mathcal{B}(\mathbb{R}^d)$ -measurable mapping with  $\mathbb{E}[\|z\|^p] < \infty$  for all  $p \in [1, \infty)$ . Then, for any even integer  $r \geq 2$ , there exist two constants  $E_{r,z}$  and  $F_{r,z}$  such that*

$$\sup_{0 \leq t \leq \tau} \mathbb{E}\|\underline{X}_{t,z} - \tilde{X}_{t,z}\|^r \leq (\Delta_t)^{\frac{r}{2}} (E_{r,z}(\Delta_t)^{\frac{r}{2}} + F_{r,z}d).$$

with  $\Delta_t = \tau/N$  and:

$$E_{r,z} = 2^r (\|f(0)\|^r + D_2^{\frac{r+1}{2}} (1 + \mathbb{E} \sup_{0 \leq t \leq \tau} \|\underline{X}_{t,z}\|^{qr})^{\frac{1}{2}} (\mathbb{E} \sup_{0 \leq t \leq \tau} \|\underline{X}_{t,z}\|^{2r})^{\frac{1}{2}}),$$

$$F_{r,z} = 2^r (\|g(0)\|^{2r} + L_g^r \mathbb{E} \sup_{0 \leq t \leq \tau} \|\underline{X}_{t,z}\|^{\frac{r}{2}}).$$

**PROOF.** see Appendix.  $\square$

**REMARK 2.** *Constants  $E_{r,z}$  and  $F_{r,z}$  are computed using the constants  $\lambda$  and  $L_g$  (see Remark 1), and the expected values of  $\underline{X}_{t,z}$  at each time  $t = 0, \Delta t, 2\Delta t, \dots, N\Delta t$ . These expected values are computed using a Monte Carlo method (by averaging here the value of  $10^4$  samplings).*

## 2.3 Mean square error bounding

The following Theorem holds for SDE (1). This corresponds to a stochastic version of Theorem 1 of [1], showing that a similar result holds on average, using the tamed Euler method of [3]. It is an adaptation of Theorem 4.4 in [2].

**THEOREM 3.** *Given the SDE system (1) satisfying (H1)-(H2)-(H3). Let  $\delta_0 \in \mathbb{R}_{\geq 0}$ . Suppose that  $z$  is a random variable on  $\mathbb{R}^d$  such that*

$$\mathbb{E}[\|x_0 - z\|^2] \leq \delta_0^2.$$

Then, we have, for all  $\tau \geq 0$ :

$$\mathbb{E}[\sup_{0 \leq t \leq \tau} \|X_{t,x_0} - \tilde{X}_{t,z}\|^2] \leq \delta_{\tau,\delta_0}^2,$$

with  $\delta_{\tau,\delta_0}^2 := \beta(\tau)e^{\gamma\tau}$ , where:

$$\gamma = 2(\sqrt{\Delta_t} + 2\lambda + L_g^2 + 128L_g^4), \text{ and}$$

$$\beta(\tau) = 2\delta_0^2$$

$$+ 2\tau\Delta_t L_g^2 (1 + 128L_g^2)(F_{2,z}d + E_{2,z}\Delta_t)$$

$$+ 4\tau\sqrt{\Delta_t} D(F_{4,z}d + E_{4,z}\Delta_t^{\frac{1}{2}})$$

$$(1 + 4\mathbb{E} \sup_{0 \leq t \leq \tau} \|\underline{X}_{t,z}\|^{2q} + 4\mathbb{E} \sup_{0 \leq t \leq \tau} \|\tilde{X}_{t,z}\|^{2q})^{\frac{1}{2}}. \quad (3)$$

with  $\Delta_t = \tau/N$ .

**PROOF.** The proof closely follows the proof of Theorem 4.4 in [2]. Let  $e_t = X_{t,x_0} - \tilde{X}_{t,z}$ . We have, for all  $0 \leq t \leq \tau$ :

$$de_t = (f(X_{t,x_0}) - f(z))dt + (g(X_{t,x_0}) - g(z))dW_t. \quad (4)$$

Then, by using Equation (4) and the integral version of Itô formula applied to function  $x \mapsto \|x\|^2$  we obtain

$$\begin{aligned} \|e_t\|^2 &= \\ \|e_0\|^2 &+ \int_0^t 2\langle e_s, f(X_{s,x_0}) - f(\underline{X}_{s,z}) \rangle ds \\ &+ \int_0^t \|g(X_{s,x_0}) - g(\underline{X}_{s,z})\|^2 ds + M(t), \end{aligned} \quad (5)$$

where  $e_0 = x_0 - z$ , and

$$M(t) = \int_0^t 2\langle e_s, g(X_{s,x_0}) - g(\underline{X}_{s,z}) \rangle dW_s.$$

So we have using (H2):

$$\begin{aligned}
\|e_t\|^2 &\leq \\
&\|e_0\|^2 + \int_0^t 2\langle e_s, f(X_{s,x_0}) - f(\tilde{X}_{s,z}) \rangle ds \\
&+ L_g^2 \int_0^t \|X_{s,x_0} - \underline{X}_{s,z}\|^2 ds \\
&+ \int_0^t 2\langle e_s, f(\tilde{X}_{s,z}) - f(\underline{X}_{s,z}) \rangle ds + M(t).
\end{aligned} \tag{6}$$

So we have using (H3) and Young's inequality:

$$\begin{aligned}
\|e_t\|^2 &\leq \\
&\|e_0\|^2 + \int_0^t (2\lambda \|e_s\|^2 + L_g^2 \|e_s\|^2) ds \\
&+ L_g^2 \int_0^t \|\underline{X}_{s,z} - \tilde{X}_{s,z}\|^2 ds \\
&+ \int_0^t \left( \frac{1}{\sqrt{\Delta_t}} \|f(\tilde{X}_{s,z}) - f(\underline{X}_{s,z})\|^2 + \sqrt{\Delta_t} \|e_s\|^2 \right) ds \\
&+ M(t).
\end{aligned} \tag{7}$$

So we have using (H1), for all  $0 \leq t \leq \tau$ :

$$\begin{aligned}
\|e_t\|^2 &\leq \\
&\|e_0\|^2 + (\sqrt{\Delta_t} + 2\lambda + L_g^2) \int_0^t \|e_s\|^2 ds \\
&+ L_g^2 \int_0^t \|\underline{X}_{s,z} - \tilde{X}_{s,z}\|^2 ds \\
&+ \frac{D}{\sqrt{\Delta_t}} \\
&\quad \int_0^t (1 + \|\underline{X}_{s,z}\|^q + \|\tilde{X}_{s,z}\|^q) \|\underline{X}_{s,z} - \tilde{X}_{s,z}\|^2 ds \\
&+ M(t).
\end{aligned} \tag{8}$$

It follows using Lemma 2 for  $r = 2$ , and Cauchy-Schwarz inequality:

$$\begin{aligned}
\mathbb{E} \left[ \sup_{0 \leq s \leq t} \|e_s\|^2 \right] &\leq \\
&\mathbb{E} \|e_0\|^2 + (\sqrt{\Delta_t} + 2\lambda + L_g^2) \int_0^t \mathbb{E} \|e_s\|^2 ds \\
&+ L_g^2 \tau \Delta_t (E_{2,z} \Delta_t + F_{2,z} d) \\
&+ \frac{D}{\sqrt{\Delta_t}} \\
&\int_0^t (\mathbb{E} (1 + \|\underline{X}_{s,z}\|^q + \|\tilde{X}_{s,z}\|^q)^2)^{\frac{1}{2}} (\mathbb{E} \|\underline{X}_{s,z} - \tilde{X}_{s,z}\|^4)^{\frac{1}{2}} ds \\
&+ m(t),
\end{aligned} \tag{9}$$

where

$$m(t) = \mathbb{E} \left[ \sup_{0 \leq s \leq t} \|M(s)\| \right].$$

Hence, using using Lemma 2 for  $r = 4$ , and inequality  $(a + b)^r \leq 2^r (a^r + b^r)$ :

$$\begin{aligned}
\mathbb{E} \left[ \sup_{0 \leq s \leq t} \|e_s\|^2 \right] &\leq \\
&\mathbb{E} \|e_0\|^2 + (\sqrt{\Delta_t} + 2\lambda + L_g^2) \int_0^t \mathbb{E} \|e_s\|^2 ds \\
&+ L_g^2 \tau \Delta_t (E_{2,z} \Delta_t + F_{2,z} d) \\
&+ 2D\tau \sqrt{\Delta_t} (E_{4,z} \Delta_t^2 + F_{4,z} d)^{\frac{1}{2}} \\
&(1 + 4\mathbb{E} \sup_{0 \leq t \leq \tau} \|\underline{X}_{t,z}\|^{2q} + 4\mathbb{E} \sup_{0 \leq t \leq \tau} \|\tilde{X}_{t,z}\|^{2q})^{\frac{1}{2}} \\
&+ m(t).
\end{aligned} \tag{10}$$

On the other hand, from the Burkholder-Davis-Gundy inequality, we get:

$$m(t) \leq 16\mathbb{E} \left[ \int_0^t \|e_s\|^2 \|g(X_{s,x_0}) - g(\underline{X}_{s,z})\|^2 ds \right]^{\frac{1}{2}}$$

Hence, using (H2):

$$m(t) \leq 16L_g^2 \mathbb{E} \left[ \sup_{0 \leq s \leq t} \|e_s\|^2 \int_0^t \|X_{s,x_0} - \underline{X}_{s,z}\|^2 ds \right]^{\frac{1}{2}}$$

Then, using Young's inequality (for any  $\alpha > 0$ ):

$$m(t) \leq 8L_g^2 (\alpha \mathbb{E} \left[ \sup_{0 \leq s \leq t} \|e_s\|^2 \right] + \frac{1}{\alpha} \mathbb{E} \left[ \int_0^t \|X_{s,x_0} - \underline{X}_{s,z}\|^2 ds \right]).$$

Hence, by using Lemma 2 for  $r = 2$ :

$$\begin{aligned}
m(t) &\leq 8\alpha L_g^2 \mathbb{E} \left[ \sup_{0 \leq s \leq t} \|e_s\|^2 \right] \\
&\quad + \frac{8L_g^2}{\alpha} \int_0^t \mathbb{E} \left[ \sup_{0 \leq r \leq s} \|e_r\|^2 \right] ds \\
&\quad + \frac{8L_g^2}{\alpha} \tau \Delta_t (E_{2,z} \Delta_t + F_{2,z} d).
\end{aligned} \tag{11}$$

Hence, letting  $\alpha = \frac{1}{16L_g^2}$ , we have by replacing in (10):

$$\begin{aligned}
\frac{1}{2} \mathbb{E} \left[ \sup_{0 \leq s \leq t} \|e_s\|^2 \right] &\leq \\
&\delta_0^2 \\
&+ (\sqrt{\Delta_t} + 2\lambda + L_g^2 + 128L_g^4) \\
&\quad \int_0^t \mathbb{E} \left[ \sup_{0 \leq r \leq s} \|e_r\|^2 \right] ds \\
&+ \tau (L_g^2 + 128L_g^4) \Delta_t (E_{2,z} \Delta_t + F_{2,z} d) \\
&+ \tau 2D\sqrt{\Delta_t} (E_{4,z} \Delta_t^2 + F_{4,z} d)^{\frac{1}{2}} \\
&(1 + 4\mathbb{E} \sup_{0 \leq t \leq \tau} \|\underline{X}_{t,z}\|^{2q} + 4\mathbb{E} \sup_{0 \leq t \leq \tau} \|\tilde{X}_{t,z}\|^{2q})^{\frac{1}{2}}.
\end{aligned} \tag{12}$$

It results from Gronwall's inequality:

$$\mathbb{E} \left[ \sup_{0 \leq t \leq \tau} \|e_t\|^2 \right] = \beta(\tau) e^{Y\tau},$$

with

$$\begin{aligned}
\gamma &= 2(\sqrt{\Delta_t} + 2\lambda + L_g^2 + 128L_g^4), \text{ and} \\
\beta(\tau) &= 2\delta_0^2 \\
&\quad + 2\tau(\Delta_t L_g^2(1 + 128L_g^2)(F_{2,z}d + E_{2,z}\Delta_t) \\
&\quad + 4\tau\sqrt{\Delta_t}D(F_{4,z}d + E_{4,z}\Delta_t^2)^{\frac{1}{2}} \\
&\quad (1 + 4\mathbb{E} \sup_{0 \leq t \leq \tau} \|\tilde{X}_{t,z}\|^{2q} + 4\mathbb{E} \sup_{0 \leq t \leq \tau} \|\tilde{X}_{t,z}\|^{2q})^{\frac{1}{2}}.
\end{aligned} \tag{13}$$

□

It follows from Theorem 3 and Jensen's inequality:

**PROPOSITION 1.** Consider two points  $x_0$  and  $z$  of  $\mathbb{R}^d$ , and a positive real number  $\delta_0$ . Suppose that  $x_0 \in B(z, \delta_0)$  (i.e.  $\|x_0 - z\| \leq \delta_0$ ). Then  $\mathbb{E}X_{t,x_0} \in B(\tilde{X}_{t,z}, \delta_{t,\delta_0})$  for all  $t \in [0, \tau]$ .

It also follows from Theorem 3:

**PROPOSITION 2.** In the setting of Theorem 3, the expression  $\delta_{\tau,\delta_0}$  tends to

$$\delta_0 \sqrt{2e^{2\lambda\tau + L_g^2 + 128L_g^4}}$$

when  $\Delta_t$  tends to 0 (i.e., when  $N$  tends to  $\infty$ ).

## 2.4 Implementation

This method has been implemented in the interpreted language Octave, and the experiments performed on a 2.80 GHz Intel Core i7-4810MQ CPU with 8 GB of memory. The implementation is an adaptation of the program described in [1] for controlling deterministic switched systems, but makes use of the tamed Euler scheme for SDEs (with the error function  $\delta$  given in Theorem 3) instead of the classical Euler scheme.

**EXAMPLE 1.** Consider the following system, corresponding to the example in Section 6.2 of [8] (cf. [9]) for mode  $u = 1$ :

$$\begin{aligned}
dx_1 &= (-0.25x_1 + x_2 + 0.25)dt + 0.05x_1dW_t^1 \\
dx_2 &= (-2x_1 - 0.25x_2 - 2)dt + 0.05x_2dW_t^2
\end{aligned}$$

The program gives (for  $\tau = 1$ ,  $\Delta_t = \tau/10^4$ ):  $q = 0$ ,  $D = 1.36$ ,  $L_g = 0.05$ ,  $\lambda = 0.25$ ; and for  $z = (-4, -3.8)$ :  $E_{2,z} = 893.3$ ,  $E_{4,z} = 2.14 \cdot 10^5$ ,  $F_{2,z} = 0.002$ ,  $F_{4,z} = 4.9 \cdot 10^{-6}$ .

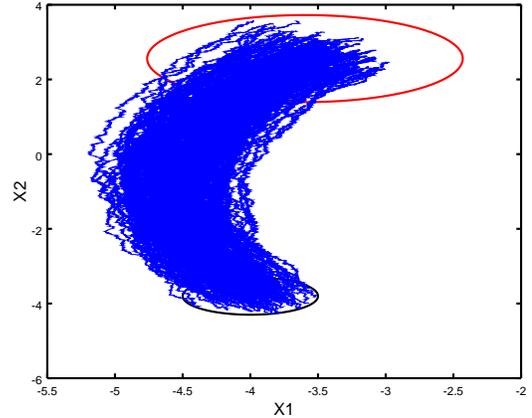
Consider now the system corresponding to the example of [8] for mode  $u = 2$ :

$$\begin{aligned}
dx_1 &= (-0.25x_1 + 2x_2 - 0.25)dt + 0.05x_1dW_t^1 \\
dx_2 &= (-x_1 - 0.25x_2 + 1)dt + 0.05x_2dW_t^2
\end{aligned}$$

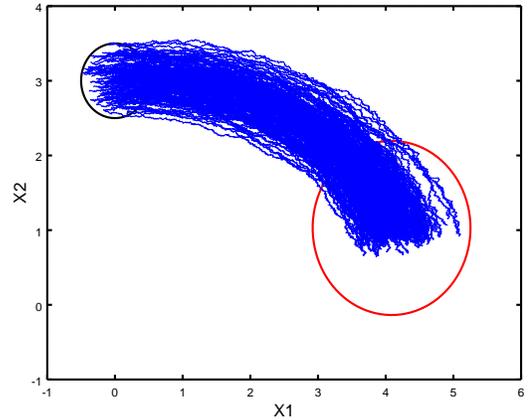
The program gives (for  $\tau = 1$ ,  $\Delta_t = \tau/10^4$ ):  $q = 0$ ,  $D = 1.36$ ,  $L_g = 0.05$ ,  $\lambda = 0.25$ , and, for  $z = (0, 3)$ :  $E_{2,z} = 543.2$ ,  $E_{4,z} = 7.94 \cdot 10^4$ ,  $F_{2,z} = 0.0442$ ,  $F_{4,z} = 0.00178$ .

Both computations take less than 10 s. of CPU time. Simulations of the two systems are given in Figure 1 for mode  $u = 1$  and starting point  $z = (-4, 3.8)$ , and in Figure 2 for mode  $u = 2$  and starting point  $z = (0, 3)$ . On each figure, the initial ball ( $t = 0$ ) is depicted in black, the final ball ( $t = \tau$ ) in red, and 200 random sampling trajectories in blue for  $t \in [0, \tau]$ .<sup>1</sup>

<sup>1</sup>Note that, in the figures, all the end points (at  $t = \tau$ ) of the sampling trajectories lie in the final ball, but this is not true in general; we only know by Proposition 1 that, for all starting point  $x_0$  of the initial ball, the expected value of the end point lie in the final ball.



**Figure 1:** Example 1 with mode  $u = 1$ ,  $\tau = 1$ ,  $\Delta_t = 10^{-4}$ , initial ball  $B(z, \delta_0)$  with  $z = (-4, 3.8)$  and  $\delta_0 = 0.5$ , final ball  $B(z_1, \delta_{\tau,\delta_0})$  with  $z_1 = (-3.6, 2.56)$  and  $\delta_{\tau,\delta_0} = 1.17$



**Figure 2:** Example 1 with mode  $u = 2$ ,  $\tau = 1$ ,  $\Delta_t = 10^{-4}$ , initial ball  $B(z, \delta_0)$  with  $z = (0, 3)$  and  $\delta_0 = 0.5$ , final ball  $B(z_1, \delta_{\tau,\delta_0})$  with  $z_1 = (0.79, -0.63)$  and  $\delta_{\tau,\delta_0} = 1.17$

### 3 SAMPLED STOCHASTIC SWITCHED SYSTEMS

#### 3.1 Stochastic switched system as a finite collection of SDEs

We now consider a finite number of SDEs. Each SDE is referred to as a *mode*  $j$ , and the set of modes is referred to as  $U = \{1, \dots, M\}$ . We will denote by  $X_{t,x_0}^j$  the solution at time  $t$  of the system:

$$\begin{aligned} dx(t) &= f_j(x(t)) + g_j(x(t))dW_t^j, \\ x(0) &= x_0. \end{aligned} \quad (14)$$

where  $x_0$  is a random variable that is measurable in  $\mathcal{F}_0$ . Hypotheses (H1-H2-H3), as defined in Section 2, are naturally extended to every mode  $j$  of  $U$ . Accordingly, constants  $L_g, \lambda, F$  associated to SDE (1) in Section 2, now become  $L_{g_j}, \lambda_j, F_j$  respectively, for each  $j \in U$ .

Likewise, for each  $j \in U$ , the nonnegative real  $(\delta_{t,\delta_0})^2$  becomes  $(\delta_{t,\delta_0}^j)^2$  for each mode  $j$ ; the approximate continuous-time solution of (14) starting from  $z$ , is denoted by  $\tilde{X}_{t,z}^j$ , and the approximate staircase solution by  $\underline{X}_{t,z}^j$ .

#### 3.2 Control patterns

The control laws that we now consider are “piecewise constant of duration  $\tau$ ” in the sense that, every  $\tau$  seconds, they select a given mode (see [8]). We call “(control) pattern of length  $k$ ” a sequence of  $k$  modes (i.e., an element of  $U^k$ ). Each pattern  $\pi$  of the form  $j_1 j_2 \dots j_k$  corresponds to the selection of mode  $j_1$  for time  $t \in [0, \tau)$ , then mode  $j_2$  for  $t \in [\tau, 2\tau)$ , and so on, until  $t = k\tau$ . We assume that the solution of the system is continuous at sampling instants  $t = \tau, 2\tau, \dots$  (which means that there is no “reset” of the system at sampling instants).

Given a stochastic switched system, a pattern  $\pi$  of length  $k$  and an initial random variable  $z$ , one constructs the “approximate solution controlled by  $\pi$ ” by composing together the approximations obtained by successive application of the modes of  $\pi$ . Formally, the “continuous” approximate solution  $\tilde{X}_{t,z}^\pi$  is defined at time  $t \in [0, k\tau]$  as follows:

- $\tilde{X}_{t,z}^\pi = \tilde{X}_{t,z}^j$  if  $\pi = j \in U, k = 1$  and  $t \in [0, \tau]$ , and
- $\tilde{X}_{(k-1)\tau+t',z}^\pi = \tilde{X}_{t',z'}^j$  with  $z' = \tilde{X}_{(k-1)\tau,z}^{\pi'}$  if  $k \geq 2, t' \in [0, \tau], \pi = \pi' * j$  for some  $j \in U$  and  $\pi' \in U^{k-1}$ .

The “staircase” approximate solution  $\underline{X}_{t,z}^\pi$  is defined analogously. Likewise, given an initial error radius  $\delta_0 > 0$  and a pattern  $\pi$  of length  $k \geq 1$ , one defines the error radius  $\delta_{t,\delta_0}^\pi$  as follows:<sup>2</sup>

- $\delta_{t,\delta_0}^\pi = \delta_{t,\delta_0}^j$  if  $\pi = j \in U, k = 1$  and  $t \in [0, \tau]$ , and
- $\delta_{(k-1)\tau+t',\delta_0}^\pi = \delta_{t',\delta'}^j$  with  $\delta' = \delta_{(k-1)\tau,\delta_0}^{\pi'}$ , if  $k \geq 2, t' \in [0, \tau], \pi = \pi' * j$  for some  $j \in U$  and  $\pi' \in U^{k-1}$ .

<sup>2</sup>For the sake of simplicity, we suppose that the number of steps of subsampling  $N$  is the same whatever the mode  $j$  of the pattern  $\pi$  is, hence the stepsize of the subsampling is always equal to  $\Delta_t = \tau/N$ ; in full generality, we should write  $N_j$  instead of  $N$  to express the dependence.

#### 3.3 Controlled $(R, S)$ -stability

Given a rectangle  $R \subset \mathbb{R}^d$  and a rectangle  $S \subset \mathbb{R}^d$  such that  $R \subseteq S$ , we now extend the problem of “controlled  $(R, S)$ -stability”, as defined in [1] for deterministic switched systems, to SDEs, as follows:

For all starting point  $x_0 \in R$ , find a pattern  $\pi$  of length  $k$  such that

- $\mathbb{E}X_{t,x_0}^\pi \in R$  for  $t = k\tau$
- $\mathbb{E}X_{t,x_0}^\pi \in S$  for all  $t = \tau, 2\tau, 3\tau, \dots$

It is easy to see that, in order to solve this problem, it suffices to exhibit a finite set of points  $z_1, \dots, z_p$  of  $S$ , and a positive real  $\delta_0 > 0$  such that:

- (1) all the balls  $B(z_i, \delta_0), i = 1, \dots, p$ , cover  $R$ , and are included into  $S$  (i.e.  $R \subseteq \bigcup_{i=1}^p B(z_i, \delta_0) \subseteq S$ );
- (2) for each  $i = 1, \dots, p$ , there is a pattern  $\pi$  of length  $k$  such that:
  - $B_{i,\pi,t} \subseteq S$  for  $t = \tau, 2\tau, \dots, (k-1)\tau$ , and
  - $B_{i,\pi,t} \subseteq R$  for  $t = k\tau$ .
where  $B_{i,\pi,t} := B(\mathbb{E}\tilde{X}_{t,z_i}^\pi, \delta_{t,\delta_0}^\pi)$ .

By repeated application of the patterns  $\pi_1, \dots, \pi_p$ , one defines a control that makes any trajectory starting from  $R$  return to  $R$  infinitely often while always belonging to  $S$  at sampling instants  $t = \tau, 2\tau, 3\tau, \dots$

The program mentioned in Section 2.4, has been extended in order to find, by exhaustive search, patterns that make the balls covering  $R$  return to  $R$ , and such that the intermediate balls (at  $t = \tau, 2\tau, \dots$ ) belong to  $S$ . We now give an application of this program.

EXAMPLE 2. Consider the system (see [8, 9]):

$$\begin{aligned} dx_1 &= (-0.25x_1 + ux_2 + (-1)^u 0.25)dt + 0.01x_1 dW_t^1 \\ dx_2 &= ((u-3)x_1 - 0.25x_2 + (-1)^u(3-u))dt + 0.01x_2 dW_t^2 \end{aligned}$$

where  $u = 1, 2$ .

For  $\tau = 0.5, \Delta_t = 10^{-4}$ , one finds (for all mode  $u = 1, 2$ ):

$q = 0, D = 1.36, L_g = 0.01, \lambda = 0.25$ ; for  $z = (-4, -3.8)$ :  $E_{2,z} = 893.31, E_{4,z} = 2.14 \cdot 10^5, F_{2,z} = 0.002, F_{4,z} = 4.9 \cdot 10^{-6}$ ; and for  $z = (0, 3)$ :  $E_{2,z} = 543.22, E_{4,z} = 7.94 \cdot 10^4, F_{2,z} = 0.0442, F_{4,z} = 0.00178$ .

Our program shows  $(R, S)$ -stability of the system for  $R = [-5, 5] \times [-4.4]$  and  $S = [-8, 8] \times [-7, 7]$ : given a covering of  $R$  with balls of radius  $\delta_0 = 0.1$ , the program finds, by exhaustive search, patterns of length  $\leq 5$  that make the balls return to  $R$ . It takes 6 hours of CPU time. Figures 3, 4, 5 and 6 depict in black the initial balls (at  $t = 0$ ) centered at the corners of  $R$ ; and for each initial ball, the pattern that sends the ball back to  $R$  (at time  $t = k\tau$ ); the intermediate balls (at  $t = \tau, 2\tau, \dots, (k-1)\tau$ ) are depicted in red, and 200 sampling trajectories drawn in blue.

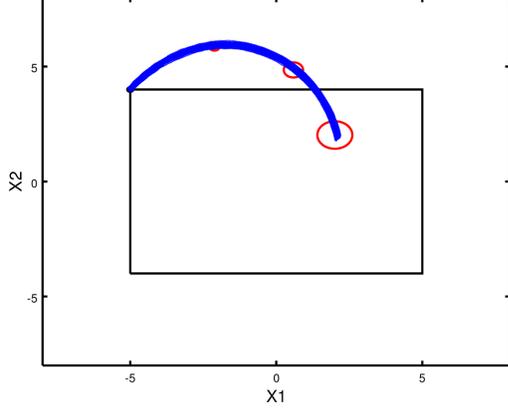
#### 3.4 Other applications

Our Euler-based method can also be used to control systems in order to achieve *reachability* properties. We sketched out this point in the following example.

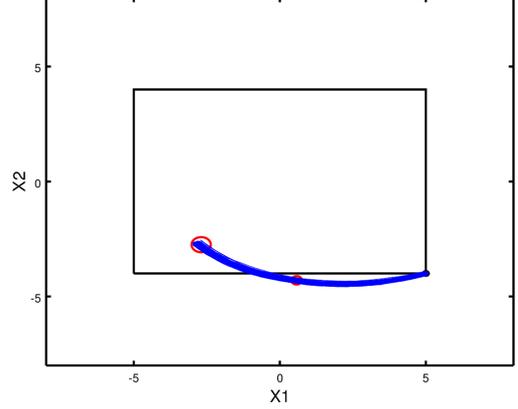
EXAMPLE 3. (the slit problem)

The problem is adapted from [5]. The controlled dynamics is:

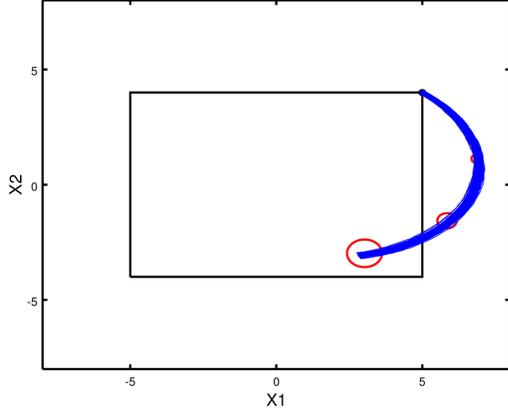
$$dX = udt + dW, \quad X_0 = 1$$



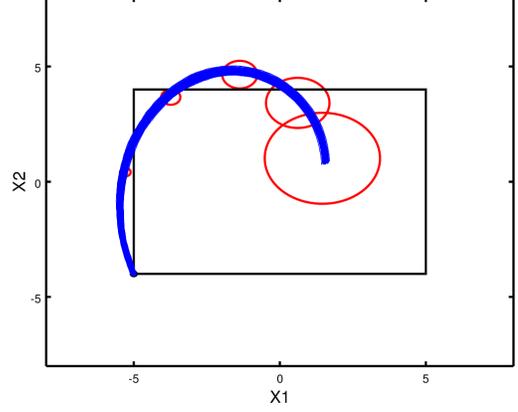
**Figure 3: initial ball  $B(z, \delta_0)$  with  $z = (-5, 4)$  and  $\delta_0 = 0.1$ ; pattern =  $(1 \cdot 1 \cdot 1)$ ;  $\tau = 0.5$**



**Figure 5: initial ball  $B(z, \delta_0)$  with  $z = (5, -4)$  and  $\delta_0 = 0.1$ ; pattern =  $(2 \cdot 2)$ ;  $\tau = 0.5$**



**Figure 4: initial ball  $B(z, \delta_0)$  with  $z = (5, 4)$  and  $\delta_0 = 0.1$ ; pattern =  $(2 \cdot 2 \cdot 2)$ ;  $\tau = 0.5$**



**Figure 6: initial ball  $B(z, \delta_0)$  with  $z = (-5, -4)$  and  $\delta_0 = 0.1$ ; pattern =  $(1 \cdot 1 \cdot 1 \cdot 1 \cdot 1)$ ;  $\tau = 0.5$**

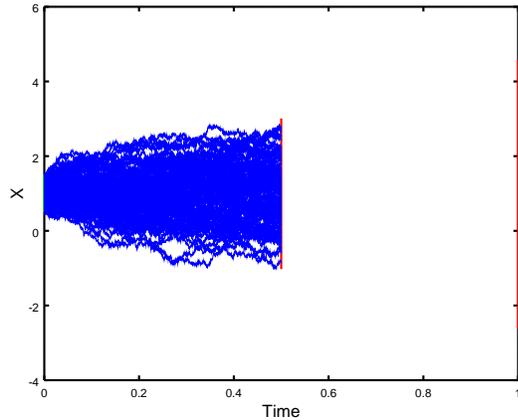
with mode  $u \in \{-6, -5, -4, -3, -2, 1, 0, 1, 2, 3, 4, 5, 6\}$ . We have (at  $t = 0.5$ ) a slit at  $x \in [-1, -4]$ . The objective is thus to control the system so that  $x(t) \in S = [-1, -4]$  at  $t = 0.5$ .

One has, for all mode:  $q = 0, D = 0, L_g = 0, \lambda = 0$ . For  $\delta_0 = 0.5$ , an initial point  $z = 1$  and a sampling time  $\tau = 0.5$  with subsampling  $\Delta_t = 10^{-3}$ , one has for mode  $u = -6$ :  $E_{2,z} = 144, E_{4,z} = 20736, F_{2,z} = 4, F_{4,z} = 16$ ; and for mode  $u = 0$ :  $E_{2,z} = 0, E_{4,z} = 0, F_{2,z} = 4, F_{4,z} = 16$ .

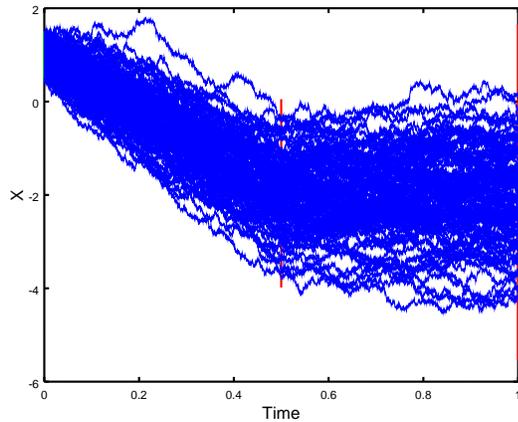
Suppose that all the trajectories start at  $x_0$  with  $x_0 \in B(z, \delta_0)$  (i.e.,  $|x_0 - z| \leq 0.5$ ), with  $z = 1$  and  $\delta_0 = 0.5$ . When there is no control ( $u = 0$ ), at time  $t = 0.5$ , the expected value of  $X_{t,x_0}$  is in  $B(z_1, \delta_{t,\delta_0})$  with  $z_1 = 1$  and  $\delta_{t,\delta_0} = 2$ . From Markov's inequality, it follows that the trajectories pass by  $S = [-1, -4]$  at  $t = 0.5$  with low probability: see Figure 7. On the other hand, with control  $u = -6$ , at time  $t = \tau = 0.5$ , the expected value of  $X_{t,x_0}$  is now in  $B(z_1, \delta_{t,\delta_0})$  with  $z_1 = -2$  and  $\delta_{\tau,\delta_0} = 2$ . This explains why the trajectories now pass by  $S = [-1, -4]$  at  $t = 0.5$  with high probability: see Figure 8.

#### 4 FINAL REMARKS AND FUTURE WORK

We have explained how to use an Euler-based method in order to control stochastic switched systems. We have focused our work on the property of  $(R, S)$ -stability, but it can also be used for achieving reachability properties. In the future, we plan to experiment the method with examples where the drift functions behave polynomially. We would like also to find bounds not only for the expected values of the solutions, but for their variance.



**Figure 7: Example 3 without control ( $u = 0$ ) for  $t \in [0, \tau]$ ; initial ball  $B(z, \delta_0)$  with  $z = 1$  and  $\delta_0 = 0.5$ ; final ball  $B(z_1, \delta_1)$  (at  $t = \tau = 0.5$ ) with  $z_1 = 1, \delta_1 = 2$**



**Figure 8: Example 3 with control pattern  $(-6 \cdot 0)$ ; initial ball  $B(z, \delta_0)$  with  $z = 1$  and  $\delta_0 = 0.5$ ; intermediate ball  $B(z_1, \delta_1)$  (at  $t = \tau = 0.5$ ) with  $z_1 = -2, \delta_1 = 2$ ; final ball  $B(z_2, \delta_2)$  (at  $t = 2\tau$ ) with  $z_2 = -2, \delta_2 = 3.6$**

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## APPENDIX: PROOF OF LEMMA 2

PROOF. Let  $t \in [k\Delta_t, (k+1)\Delta_t)$ . Then (using the inequality  $(a+b)^r \leq 2^r(a^r + b^r)$ ):

$$\begin{aligned}
& \|\underline{X}_t - \tilde{X}_t\|^r \\
&= \|(t - t_k)f(\underline{X}_k) + g(\underline{X}_k)(W_t - W_{t_k})\|^r \\
&\leq 2^r((\Delta_t)^r \|f(\underline{X}_k)\|^r \\
&+ \|g(\underline{X}_k)\|^r \|W_t - W_{t_k}\|^r) \|\underline{X}_t - \tilde{X}_t\|^r \\
&\leq 2^r((\Delta_t)^r (\|f(\underline{X}_k) - f(0)\|^r + \|f(0)\|^r) \\
&+ (\|g(\underline{X}_k) - g(0)\|^r + \|g(0)\|^r) \|W_t - W_{t_k}\|^r) \\
&\leq 2^r((\Delta_t)^r (D((1 + \|\underline{X}_k\|^q) \|\underline{X}_k\|^2)^{\frac{r}{2}} + \|f(0)\|^r) \\
&+ (L_g^r \|\underline{X}_k\|^r + \|g(0)\|^r) \|W_t - W_{t_k}\|^r) \\
&\leq 2^r((\Delta_t)^r (D((1 + \|\underline{X}_k\|^q)^{\frac{r}{2}} \|\underline{X}_k\|^r) + \|f(0)\|^r) \\
&+ (L_g^r \|\underline{X}_k\|^r + \|g(0)\|^r) \|W_t - W_{t_k}\|^r) \\
&\leq 2^r((\Delta_t)^r (D2^{\frac{r}{2}} ((1 + \|\underline{X}_k\|^{\frac{qr}{2}}) \|\underline{X}_k\|^r) + \|f(0)\|^r) \\
&+ (L_g^r \|\underline{X}_k\|^r + \|g(0)\|^r) \|W_t - W_{t_k}\|^r).
\end{aligned} \tag{15}$$

$$\begin{aligned}
& \mathbb{E} \|\underline{X}_t - \tilde{X}_t\|^r \\
&\leq 2^r((\Delta_t)^r (\|f(0)\|^r + D2^{\frac{r}{2}} \\
&(\mathbb{E}[(1 + \|\underline{X}_k\|^{\frac{qr}{2}})^2])^{\frac{1}{2}} (\mathbb{E}[\|\underline{X}_k\|^{2r}])^{\frac{1}{2}} \\
&+ (\|g(0)\|^{2r} + L_g^r \mathbb{E} \|\underline{X}_k\|^{\frac{r}{2}}) \\
&(\mathbb{E}[\|W_t - W_{t_k}\|^{2r}])^{\frac{1}{2}})) \\
&\leq 2^r((\Delta_t)^r (\|f(0)\|^r + D2^{\frac{r+1}{2}} \\
&(1 + \mathbb{E}[\|\underline{X}_k\|^{qr}])^{\frac{1}{2}} (\mathbb{E}[\|\underline{X}_k\|^{2r}])^{\frac{1}{2}} \\
&+ (\|g(0)\|^{2r} + L_g^r \mathbb{E} \|\underline{X}_k\|^{\frac{r}{2}}) \\
&(\mathbb{E}[\|W_t - W_{t_k}\|^{2r}])^{\frac{1}{2}})) \\
&\leq 2^r((\Delta_t)^r (\|f(0)\|^r + D2^{\frac{r+1}{2}} \\
&(1 + \mathbb{E}[\|\underline{X}_k\|^{qr}])^{\frac{1}{2}} (\mathbb{E}[\|\underline{X}_k\|^{2r}])^{\frac{1}{2}} \\
&+ (\|g(0)\|^{2r} + L_g^r \mathbb{E} \|\underline{X}_k\|^{\frac{r}{2}}) d(t - t_k)^{\frac{r}{2}})) \\
&\leq 2^r((\Delta_t)^r (\|f(0)\|^r + D2^{\frac{r+1}{2}} \\
&(1 + \sup_{0 \leq t \leq \tau} \mathbb{E} \|\underline{X}_t\|^{qr})^{\frac{1}{2}} (\mathbb{E} \sup_{0 \leq t \leq \tau} \|\underline{X}_t\|^{2r})^{\frac{1}{2}} \\
&+ (\|g(0)\|^{2r} + L_g^r \mathbb{E} \sup_{0 \leq t \leq \tau} \|\underline{X}_t\|^{\frac{r}{2}}) d(\Delta_t)^{\frac{r}{2}}) \\
&\leq 2^r(\Delta_t)^{\frac{r}{2}} ((\Delta_t)^{\frac{r}{2}} (\|f(0)\|^r + D2^{\frac{r+1}{2}} \\
&(1 + \mathbb{E} \sup_{0 \leq t \leq \tau} \|\underline{X}_t\|^{qr})^{\frac{1}{2}} (\mathbb{E} \sup_{0 \leq t \leq \tau} \|\underline{X}_t\|^{2r})^{\frac{1}{2}} \\
&+ d(\|g(0)\|^{2r} + L_g^r \mathbb{E} \sup_{0 \leq t \leq \tau} \|\underline{X}_t\|^{\frac{r}{2}})).
\end{aligned} \tag{16}$$

Hence:

$$\begin{aligned}
& \sup_{0 \leq t \leq \tau} \mathbb{E} \|\underline{X}_t - \tilde{X}_t\|^r \leq (\Delta_t)^{\frac{r}{2}} (E_{r,z}(\Delta_t)^{\frac{r}{2}} + F_{r,z}d) \\
& \text{with } E_{r,z} = 2^r (\|f(0)\|^r + D2^{\frac{r+1}{2}} (1 + \mathbb{E} \sup_{0 \leq t \leq \tau} \|\underline{X}_t\|^{qr})^{\frac{1}{2}} \\
& \quad (\mathbb{E} \sup_{0 \leq t \leq \tau} \|\underline{X}_t\|^{2r})^{\frac{1}{2}}), \\
& F_{r,z} = 2^r (\|g(0)\|^{2r} + L_g^r \mathbb{E} \sup_{0 \leq t \leq \tau} \|\underline{X}_t\|^{\frac{r}{2}}). \quad \square
\end{aligned}$$