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Spectral analysis for non-stationary audio

Adrien Meynard and Bruno Torr sani

Abstract

A new approach for the analysis of non-stationary signals is proposed, with a focus on audio applications. Following earlier contributions, non-stationarity is modeled via stationarity-breaking operators acting on Gaussian stationary random signals. The focus is here on time warping and amplitude modulation, and an approximate maximum-likelihood approach based on suitable approximations in the wavelet transform domain is developed. This paper provides theoretical analysis of the approximations, and describes and analyses a corresponding estimation algorithm. The latter is tested and validated on synthetic as well as real audio signal.

Index Terms

Non-stationary signals, deformation, wavelet analysis, local spectrum, Doppler effect

I. INTRODUCTION

Non-stationarity is a key feature of acoustic signals, in particular audio signals. To mention a few examples, a large part of information carried by musical and speech signals is encoded by their non-stationary nature, as is the case for environment sounds (think of car noises for example, where non-stationarity informs about speed variations), and many animals (bats, dolphins,...) use non-stationary signals for localization and communication. Beyond acoustics, amplitude and frequency modulation are of prime importance in telecommunication.

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While stationarity can be given rigorous definitions, non-stationarity is a very wide concept, as there are infinitely many ways to depart from stationarity. The theory of random signals and processes (see [1], [2] and references therein) gives a clear meaning to the notion of stationarity. In the context of time series analysis, Priestley [2], [3] was one of the first to develop a systematic theory of non-stationary processes, introducing the class of locally stationary processes and the notion of evolutionary spectrum. A similar approach was followed in [4], who proposed a wavelet-based approach to covariance estimation for locally stationary processes (see also [5]). An alternate theory of locally stationary time series was developed by Dahlhaus [6] (see also [7] for a corresponding stationarity test). In a different context, frequency modulated stationary signal were considered in [8], [9], and time warping models were analyzed in [10]. In several of these approaches, wavelet, time-frequency and similar representations happen to play a key role for the characterization of non-stationarity.

In a deterministic setting, a popular non-stationarity model expresses the signal as a sum of K sinusoidal components $y(t) = \sum_{k=1}^K A_k(t) \cos(2\pi\phi_k(t))$. This model has been largely used in speech processing since early works by McAulay and Quatieri [11] (see also [12] and references therein for more recent developments, and [13], [14] for probabilistic approaches). The instantaneous frequencies ϕ'_k of each mode give important information about the physical phenomenon. Under smoothness assumptions on functions A_k and ϕ'_k , techniques such as ridge/multiridge detection (see [15] and references therein), synchrosqueezing or reassignment have been developed to extract these quantities from a single signal observation (see [16], [17] for recent accounts).

In sound processing, signals often possess an harmonic structure, which corresponds to a special case of the above model where each instantaneous frequency ϕ'_k is multiple of a fundamental frequency ϕ'_0 : $\phi'_k(t) = \lambda_k \phi'_0(t)$, $\lambda_k \in \mathbb{R}$. If the amplitudes A_k are such that $A_k(t) = \alpha_k A_0(t)$, we can describe such signals as a stationary signal $x(t) = \sum_{k=1}^K \alpha_k \cos(2\pi\lambda_k t + \varphi_k)$ modified by time warping and amplitude modulation: $y(t) = A_0(t)x(\phi_0(t))$. A major limit of this model is that each component is purely sinusoidal while audio signals often contain broadband information. However sounds originating from physical phenomena can often be modeled as stationary signals which have been deformed by a stationarity breaking operator (time warping, amplitude modulation,...). For example sounds generated by a variable speed engine or any stationary sound

deformed by Doppler effect can be described as such. A stochastic time warping model has been introduced in [18], [19], where wavelet-based approximation and estimation techniques were developed. In [9], [20], an approximate maximum-likelihood approach was proposed for the joint estimation of the time warping and power spectrum of the underlying Gaussian stationary signal, exploiting similar approximations.

In this paper, we build on results of [9], [20] which we extend and improve in several ways. We develop an approximate maximum likelihood method for estimating jointly time warping and amplitude modulation from a single realization. While the overall structure of the algorithm is similar, we formulate the problem as a continuous parameter estimation problem, which avoids quantization effects present in [9], [20], and allows computing a Cramér-Rao bound for assessing the precision of the estimate. After completing the estimation, the inverse deformation can be applied to the input signal, which yields an estimate for the power spectrum.

The plan of the paper is as follows. After giving some definitions and notations in Section II, we detail in Section III the non-stationary signal models we consider, and specify the assumptions made on the underlying stationary signal. We also analyze the effect of time warping and amplitude modulation in the wavelet domain, which we exploit in designing the estimation procedure. We finally propose an alternate estimation algorithm, and analyze the expected performances of the corresponding estimator. Section IV is devoted to numerical results, on both synthetic signals and real sounds. We also shortly describe in this section an extension published in [21] involving simultaneously time warping and frequency modulation. More mathematical developments are postponed to the Appendix.

II. NOTATIONS AND BACKGROUND

A. Random signals, stationarity

Throughout this paper, we will work in the framework of the theory of random signals. Signals of interest will be modeled as realizations of random processes¹ $t \in \mathbb{R} \rightarrow X_t \in \mathbb{C}$. In this paper, the random processes will be denoted by upper-case letters

¹Signals of interest are real-valued, however we will also use complex-valued functions since we will use complex-valued wavelet transforms later on.

while their realizations will be denoted by lower-case letters. The random processes will be assumed to zero mean ($\mathbb{E}\{X_t = 0\}$ for all t) and be second order, i.e. they have well defined covariance kernel $\mathbb{E}\{X_t\bar{X}_s\}$. A particularly interesting class of such stochastic processes is the class of second order (or weakly) stationary processes, for which $C_X(t-s) \triangleq \mathbb{E}\{X_t\bar{X}_s\}$ is a function of $t-s$ only. Under these assumptions, the Wiener-Khinchin theorem states that the covariance kernel may be expressed as the inverse Fourier transform of a non-negative measure $d\eta_X$, which we will assume to be continuous with respect to the Lebesgue measure: $d\eta_X(\nu) = \mathcal{S}_X(\nu)d\nu$, for some non-negative L^1 function \mathcal{S}_X called the power spectrum. We then write

$$C_X(t) = \int \mathcal{S}_X(\nu)e^{2i\pi\nu t}d\nu .$$

We refer to textbooks such as [1], [2] for a more complete mathematical account of the theory, and to [20] for an extension to distribution theory setting.

B. Elementary operators

Our approach rests on non-stationary models obtained by deformations of stationary random signals. We will mainly use as elementary operators the amplitude modulation A_α , translation T_τ , dilation D_s , and frequency modulation M_ν defined as follows:

$$\begin{aligned} A_\alpha x(t) &= \alpha x(t), & T_\tau x(t) &= x(t - \tau), \\ D_s x(t) &= q^{-s/2}x(q^{-s}t), & M_\nu x(t) &= e^{2i\pi\nu t}x(t). \end{aligned}$$

where $\alpha, \tau, s, \nu \in \mathbb{R}$ and $q > 0$ is a fixed number. The amplitude modulation commutes with the other three operators, which satisfy the commutation rules

$$T_\tau D_s = D_s T_{q^{-s}\tau}, \quad T_\tau M_\nu = e^{-2i\pi\nu\tau} M_\nu T_\tau, \quad M_\nu D_s = D_s M_{\nu q^s}.$$

C. Wavelet transform

Our analysis relies heavily on transforms such as the continuous wavelet transform (and discretized versions). In particular, the wavelet transform of a signal $X : t \in \mathbb{R} \rightarrow X_t$ is defined as:

$$\mathcal{W}_X(s, \tau) = \langle X, \psi_{s\tau} \rangle, \quad \text{with } \psi_{s\tau} = T_\tau D_s \psi. \quad (1)$$

where ψ is the analysis wavelet, i.e. a smooth function, with fast decay away from the origin. It may be shown that for suitable choices of ψ the wavelet transform is invertible

(see [15]), we will not use that property here. Notice that when X is a realization of a continuous time random process, it does not need to decay at infinity. However, for a suitably smooth and localized wavelet ψ , the wavelet transform can still be well defined (see [15], [20] for more details). In such a situation the wavelet transform of X is a two-dimensional random field, which we analyze in the next section. Besides, in this paper the analysis wavelet ψ is complex valued and belongs to the set $H^2(\mathbb{R}) = \{\psi \in L^2(\mathbb{R}) : \text{supp}(\hat{\psi}) \subset \mathbb{R}^+\}$. In that framework, a useful property is that if X is a real zero mean Gaussian random process then, \mathcal{W}_X is a complex zero mean circular Gaussian random field.

Classical choices of wavelets in $H^2(\mathbb{R})$ are (analytic) derivative of Gaussian ψ_k (which has k vanishing moments), and the sharp wavelet $\psi_{\#}$ (with infinitely many vanishing moments) introduced in [21]. These can be defined in the positive Fourier domain by

$$\hat{\psi}_k(\nu) = \nu^k e^{-k\nu^2/2\nu_0^2}, \quad \hat{\psi}_{\#}(\nu) = \epsilon \frac{\delta(\nu, \nu_0)}{\delta(\nu_1, \nu_0)}, \quad \nu > 0 \quad (2)$$

and vanish on the negative Fourier half axis. Here ν_0 is the mode of $\hat{\psi}$. In the expression of $\hat{\psi}_{\#}$, ν_1 is chosen so that $\hat{\psi}_{\#}(\nu_1) = \epsilon$ (a prescribed numerical tolerance at cutoff frequency ν_1), and the divergence δ is defined by $\delta(a, b) = \frac{1}{2} \left(\frac{a}{b} + \frac{b}{a} \right) - 1$.

III. JOINT ESTIMATION OF TIME WARPING AND AMPLITUDE MODULATION

A. Model and approximations

Let us first describe the deformation model we will mainly be using in the following. As said above, the non-stationary signals of interest are obtained as linear deformations of stationary random signals. The deformations of interest here are amplitude modulations and time warpings. Amplitude modulations are pointwise multiplications by smooth functions, defined as

$$\mathcal{A}_a : \quad \mathcal{A}_a x(t) = a(t)x(t), \quad (3)$$

where $a \in C^1$ is a real valued function, such that

$$0 < c_a \leq a(t) \leq C_a < \infty, \quad \forall t, \quad (4)$$

for some constants $c_a, C_a \in \mathbb{R}_+^*$. Time warpings are compositions with smooth and monotonic functions,

$$\mathcal{D}_\gamma : \quad \mathcal{D}_\gamma x(t) = \sqrt{\gamma'(t)}x(\gamma(t)) \quad (5)$$

where $\gamma \in C^2$ is a strictly increasing smooth function, satisfying the control condition [20]

$$0 < c_\gamma \leq \gamma'(t) \leq C_\gamma < \infty, \quad \forall t, \quad (6)$$

for some constants $c_\gamma, C_\gamma \in \mathbb{R}_+^*$.

Assume one is given a (unique) realization of a random signal of the form

$$Y = \mathcal{A}_a \mathcal{D}_\gamma X \quad (7)$$

where X is a stationary zero mean real random process with (unknown) power spectrum \mathcal{S}_X . The goal is to estimate the deformation functions a and γ from this realization of Y , exploiting the assumed stationarity of X .

Remark 1: Clearly enough, the stationarity assumption is not sufficient to yield unambiguous estimates, since affine functions $\gamma(t) = \lambda t + \mu$ do not break stationarity: for any stationary X , $\mathcal{D}_\gamma X$ is stationary too. Therefore, the warping function γ can only be estimated up to an affine function, as analyzed in [19] and [20]. Similarly, the amplitude function a can only be estimated up to a constant factor.

A key ingredient of our approach is the smoothness of the deformation functions a and γ , and their slow variations. This allows us to perform a local analysis using smooth and localized test functions, on which the action of \mathcal{A}_a and \mathcal{D}_γ can be approximated by their so-called *tangent operators* $\widetilde{\mathcal{A}}_a^\tau$ and $\widetilde{\mathcal{D}}_\gamma^\tau$ (see [19], [9], [20], [22]). More precisely, given a test function g located near $t = \tau$ (i.e. decaying fast enough as a function of $|t - \tau|$), Taylor expansions near $t = \tau$ yield

$$\mathcal{A}_a g(t) \approx \widetilde{\mathcal{A}}_a^\tau g(t), \quad \text{with } \widetilde{\mathcal{A}}_a^\tau \triangleq A_{a(\tau)}, \quad (8)$$

$$\mathcal{D}_\gamma g(t) \approx \widetilde{\mathcal{D}}_\gamma^\tau g(t), \quad \text{with } \widetilde{\mathcal{D}}_\gamma^\tau \triangleq T_\tau D_{-\log_q(\gamma'(\tau))} T_{-\gamma(\tau)}. \quad (9)$$

Therefore, the wavelet transform of Y will be approximated by $\mathcal{W}_Y(s, \tau) \approx \widetilde{\mathcal{W}}_Y(s, \tau) \triangleq \langle \widetilde{\mathcal{A}}_a^\tau \widetilde{\mathcal{D}}_\gamma^\tau X, T_\tau D_s \psi \rangle$, i.e.

$$\widetilde{\mathcal{W}}_Y(s, \tau) = a(\tau) \mathcal{W}_X \left(s + \log_q(\gamma'(\tau)), \gamma(\tau) \right). \quad (10)$$

Here we have used the standard commutation rules of translation and dilation operators given in Section II-B.

The result below provides a quantitative assessment of the quality of the approximation. There, we denote by $\|f\|_\infty = \text{ess sup}_t |f(t)|$ the essential absolute supremum of a function f .

Theorem 1: Let X be a second order zero mean stationary random process, let Y be the non-stationary process defined in (7). Let ψ be a smooth test function, localized in such a way that $|\psi(t)| \leq 1/(1+|t|^\beta)$ for some $\beta > 2$. Let \mathcal{W}_Y denote the wavelet transform of Y , $\widetilde{\mathcal{W}}_Y$ its approximation given in (10), and let $\varepsilon = \mathcal{W}_Y - \widetilde{\mathcal{W}}_Y$ denote the approximation error. Assume ψ and \mathcal{S}_X are such that

$$I_X^{(\rho)} \triangleq \sqrt{\int_0^\infty \xi^{2\rho} \mathcal{S}_X(\xi) d\xi} < \infty, \text{ where } \rho = \frac{\beta - 1}{\beta + 2}.$$

Then the approximation error ε is a second order, two-dimensional complex random field, and

$$\mathbb{E} \left\{ |\varepsilon(s, \tau)|^2 \right\} \leq C_a^2 q^{3s} (K_1 \|\gamma''\|_\infty + K_2 q^{\mu s} \|\gamma''\|_\infty^\rho + K_3 \|a'\|_\infty)^2$$

where

$$\begin{aligned} K_1 &= \frac{\beta \sigma_X}{2(\beta - 2) \sqrt{c_\gamma}}, & K_2 &= I_X^{(\rho)} \left(\frac{\pi}{2}\right)^\rho \sqrt{C_\gamma} \frac{4}{3\rho}, \\ K_3 &= \frac{\sqrt{C_\gamma} \beta \sigma_X}{(\beta - 2) c_a}, & \mu &= \frac{\beta - 4}{\beta + 2}, \end{aligned}$$

σ_X^2 being the variance of X .

The proof, which is an extension of the one given in [20], is given in appendix A.

Remark 2: The assumption on β ensures that the parameters belong to the following intervals: $1/4 < \rho < 1$ and $-1/2 < \mu < 1$. Therefore, the variance of the approximation error tends to zero when the scales are small (i.e. $s \rightarrow -\infty$). Besides, the error is inversely proportional to the speed of variations of γ' and a . This is consistent with the approximations of the deformation operators by their tangent operators made in equations (8) and (9).

From now on, we will assume the above approximations are valid, and work on the approximate random fields. The problem is then to estimate jointly a , γ from $\widetilde{\mathcal{W}}_Y$, which is a zero mean random field with covariance

$$\mathbb{E} \left\{ \widetilde{\mathcal{W}}_Y(s, \tau) \overline{\widetilde{\mathcal{W}}_Y(s', \tau')} \right\} = C(s, s', \tau, \tau') \quad (11)$$

where

$$\begin{aligned} C(s, s', \tau, \tau') &= a(\tau) a(\tau') q^{\frac{s+s'}{2}} \sqrt{\gamma'(\tau) \gamma'(\tau')} \int_0^\infty \mathcal{S}_X(\xi) \\ &\quad \times \overline{\hat{\psi}(q^s \gamma'(\tau) \xi)} \hat{\psi}(q^{s'} \gamma'(\tau') \xi) e^{2i\pi \xi (\gamma(\tau) - \gamma(\tau'))} d\xi. \end{aligned} \quad (12)$$

B. Estimation

1) *Estimation procedure:* Our goal is to estimate both deformation functions γ and a from the approximated wavelet transform $\widetilde{\mathcal{W}}_y$ of a realization y of Y , assuming the latter is a reliable approximation of the true wavelet transform. From now on, we additionally assume that X is a Gaussian random process. Therefore, $\widetilde{\mathcal{W}}_Y$ is a zero mean circular Gaussian random field and its probability density function is characterized by the covariance matrix. However, equation (12) shows that besides deformation functions the covariance also depends on the power spectrum \mathcal{S}_X of the underlying stationary signal X , which is unknown too. Therefore, the evaluation of the maximum likelihood estimate for a and γ requires a guess for \mathcal{S}_X . This constraint naturally brings the estimation strategy to an alternate algorithm. In [21], an estimate for the power spectrum was obtained at each iteration by computing a Welch periodogram on a “stationarized” signal $\mathcal{A}_{\tilde{a}}^{-1} \mathcal{D}_{\tilde{\gamma}}^{-1} Y$, \tilde{a} and $\tilde{\gamma}$ being the current estimates for the deformation functions a and γ . We use here a simpler estimate, computed directly from the wavelet coefficients. The two steps of the estimation algorithm are detailed below.

Remark 3: The alternate likelihood maximization strategy is reminiscent of the Expectation-Maximization (EM) algorithm, the power spectrum being the nuisance parameter. However, while it would be nice to apply directly the EM paradigm (whose convergence is proven) to our problem, the dimensionality of the latter (and the corresponding size of covariance matrices) forces us to make additional simplifications that depart from the EM scheme. Therefore we turn to a simpler approach with several dimension reduction steps.

(a) *Deformation estimation.* Assume that the power spectrum \mathcal{S}_X is known (in fact, only an estimate $\tilde{\mathcal{S}}_X$ is known). Thus, we are able to write the likelihood corresponding to the observations of the wavelet coefficients. Then the maximum likelihood estimator is implemented to determine the unknown functions γ and a .

The wavelet transform (1) is computed on a regular time-scale grid $\mathbf{\Lambda} = \mathbf{s} \times \boldsymbol{\tau}$, δ_s being the scale sampling step and F_s the time sampling frequency. The sizes of \mathbf{s} and $\boldsymbol{\tau}$ are respectively denoted by M_s and N_τ .

Considering the covariance expression (12) we want to estimate the vector of parameters $\boldsymbol{\Theta} = (\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_3) \triangleq (a(\boldsymbol{\tau})^2, \log_q(\gamma'(\boldsymbol{\tau})), \gamma(\boldsymbol{\tau}))$. Let $\mathbf{W}_y = \widetilde{\mathcal{W}}_y(\mathbf{\Lambda})$ denote the

discretized transform and let $\mathbf{C}_W(\Theta)$ be the corresponding covariance matrix. The related log-likelihood is

$$\mathcal{L}(\Theta) = -\frac{1}{2} \ln |\det(\mathbf{C}_W(\Theta))| - \frac{1}{2} \mathbf{C}_W(\Theta)^{-1} \mathbf{W}_y \cdot \mathbf{W}_y . \quad (13)$$

The matrix $\mathbf{C}_W(\Theta)$ is a matrix of size $M_s N_\tau \times M_s N_\tau$, which is generally huge. For instance, when the signal is a sound of 5 seconds sampled at frequency $F_s = 44.1$ kHz and the wavelet transform is computed on 8 scales, the matrix $\mathbf{C}_W(\Theta)$ has about 3.1 trillion elements which makes it numerically intractable. In addition, due to the redundancy of the wavelet transform, $\mathbf{C}_W(\Theta)$ turns out to be singular, which makes the evaluation of the likelihood impossible.

To overcome these issues, we use a block-diagonal regularization of the covariance matrix, obtained by forcing to zeros entries corresponding to different time indices. In other words, we disregard time correlations in the wavelet domain, which amounts to consider fixed time vector $\mathbf{w}_{y,\tau_n} = \widetilde{\mathcal{W}}_y(\mathbf{s}, \tau_n)$ as independent circular Gaussian vectors with zero mean and covariance matrix

$$\mathbf{C}(\Theta_n)_{ij} = \theta_{n,1} \mathbf{C}_0(\theta_{n,2})_{ij}, \quad 1 \leq i, j \leq M_s, \quad (14)$$

where

$$\mathbf{C}_0(\theta_{n,2})_{ij} = q^{(s_i+s_j)/2} \int_0^\infty \mathcal{S}_X(q^{-\theta_{n,2}\xi}) \overline{\hat{\psi}}(q^{s_i}\xi) \hat{\psi}(q^{s_j}\xi) d\xi . \quad (15)$$

In this situation, the regularized log likelihood \mathcal{L}^r splits into a sum of independent terms

$$\mathcal{L}^r(\Theta) = \sum_n \mathcal{L}(\Theta_n) ,$$

where $\Theta_n = (\theta_{n,1}, \theta_{n,2}) \triangleq (\theta_1(n), \theta_2(n))$ corresponds to the amplitude and warping parameters at fixed time $\tau_n = \tau(n)$. Notice that in such a formalism, $\theta_{n,3} = \gamma(\tau_n)$ does not appear any more in the covariance expression. Thus we are led to maximize independently for each n

$$\mathcal{L}(\Theta_n) = -\frac{1}{2} \ln |\det(\mathbf{C}(\Theta_n))| - \frac{1}{2} \mathbf{C}(\Theta_n)^{-1} \mathbf{w}_{y,\tau_n} \cdot \mathbf{w}_{y,\tau_n} . \quad (16)$$

For simplicity, the estimation procedure is done by an iterative algorithm (given in more details in part III-B2), which rests on two main steps. On the one hand, the log-likelihood is maximized with respect to $\theta_{n,2}$ using a gradient ascent method, for a fixed

value of $\theta_{n,1}$. On the other hand, for a fixed $\theta_{n,2}$, an estimate for $\theta_{n,1}$ is directly obtained which reads

$$\tilde{\theta}_{n,1} = \frac{1}{M_s} \mathbf{C}_0^{-1}(\theta_{n,2}) \mathbf{w}_{y,\tau_n} \cdot \mathbf{w}_{y,\tau_n}, \quad (17)$$

(b) *Spectrum estimation.* Assume the amplitude modulation and time-warping parameters θ_1 and θ_2 are known (in fact, only estimates $\tilde{\theta}_1$ and $\tilde{\theta}_2$ are known). For any n we can compute the wavelet transform

$$\frac{1}{\theta_{n,1}^{1/2}} \widetilde{\mathcal{W}}_y(s - \theta_{n,2}, \tau_n) = \mathcal{W}_x(s, \gamma(\tau_n)), \quad (18)$$

For fixed scale s_m , $\mathbf{w}_{x,s_m} = \mathcal{W}_x(s_m, \gamma(\tau)) \in \mathbb{C}^{N_\tau}$ is a zero mean random circular Gaussian vector with time independent variance (as a realization of the wavelet transform of a stationary process). Hence, the empirical variance is an unbiased estimator of the variance. We then obtain

$$\begin{aligned} \mathbb{E} \left\{ \frac{1}{N_\tau \|\psi\|_2^2} \|\mathbf{w}_{x,s_m}\|^2 \right\} &= \frac{1}{\|\psi\|_2^2} \int_0^\infty \mathcal{S}_X(\xi) q^{s_m} |\hat{\psi}(q^{s_m} \xi)|^2 d\xi \\ &\stackrel{\Delta}{=} \mathcal{S}_{X,\psi}(q^{-s_m} \omega_0), \end{aligned} \quad (19)$$

where ω_0 is the central frequency of $|\hat{\psi}|^2$. $\mathcal{S}_{X,\psi}$ is a band-pass filtered version of \mathcal{S}_X centered around frequency $\nu_m = q^{-s_m} \omega_0$. Besides, the bandwidth of the filter is proportional to the frequency ν_m . This motivates the introduction of the following estimator $\tilde{\mathcal{S}}_X$ of \mathcal{S}_X

$$\tilde{\mathcal{S}}_X(q^{-s_m} \omega_0) = \frac{1}{N_\tau \|\psi\|_2^2} \|\mathbf{w}_{x,s_m}\|^2. \quad (20)$$

Finally, the estimate \mathcal{S}_X is extended to all $\xi \in [0, F_s/2]$ by linear interpolation.

2) *Algorithm:* The estimation procedure is implemented in an iterative alternate optimization algorithm, whose pseudo-code is given as Algorithm 1. The initialization requires an initial guess for the power spectrum \mathcal{S}_X of X . We use for this the spectrum estimator (20) applied to the original observations Y .

After k iterations of the algorithm, estimates $\tilde{\Theta}_n^{(k)}$ and $\tilde{\mathcal{S}}_X^{(k)}$ for Θ_n and \mathcal{S}_X are available. Hence we can only evaluate the plug-in estimate $\tilde{\mathbf{C}}_0^{(k)}$ of \mathbf{C}_0 , obtained by replacing the power spectrum with its estimate in the covariance matrix (15). This yields an approximate expression $\mathcal{L}^{(k)}$ for the log-likelihood, which is used in place of \mathcal{L} in (16) for maximum likelihood estimation. The influence of such approximations on the performances of the algorithm are discussed in section III-C.

To assess the convergence of the algorithm, the relative update of the parameters is chosen as stopping criterion:

$$\frac{\left\| \tilde{\theta}_j^{(k)} - \tilde{\theta}_j^{(k-1)} \right\|_2^2}{\left\| \tilde{\theta}_j^{(k-1)} \right\|_2^2} < T, \text{ for } j = 1 \text{ and } 2, \quad (21)$$

where $0 < T < 1$ is a given threshold.

Finally, after convergence of the algorithm to the estimated value $\tilde{\Theta}^{(k)}$, $\log_q(\gamma')$ and a^2 are estimated for all time by cubic spline interpolation. Besides, γ is given by numerical integration assuming that $\gamma(0) = 0$.

Algorithm 1 Joint spectrum and deformations estimation

Initialization: Compute an estimate $\tilde{\mathcal{S}}_Y$ of the power spectrum of Y as an initial guess $\tilde{\mathcal{S}}_X^{(0)}$ for \mathcal{S}_X . Initialize the estimator of the squared amplitude modulation with $\tilde{\theta}_{n,1}^{(0)} = 1, \forall n$.

Compute the wavelet transform \mathbf{W}_y of y .

$k := 1$

while criterion (21) is false and $k \leq k_{max}$ **do**

- For each n , subsample \mathbf{w}_{y,τ_n} on scales \mathbf{s}_p , and estimate $\tilde{\theta}_{n,2}^{(k+1)}$ by maximizing the approximate log-likelihood $\mathcal{L}^{(k)}(\tilde{\theta}_{n,1}^{(k)}, \theta_{n,2})$ in (16).
- For each n , estimate $\tilde{\theta}_{n,1}^{(k+1)}$ by maximizing the approximate log-likelihood $\mathcal{L}^{(k)}(\theta_{n,1}, \tilde{\theta}_{n,1}^{(k+1)})$ with respect to $\theta_{n,1}$ in (16). Or, in absence of noise, directly apply equation (17) using the regularized covariance matrix given by (22).
- Construct the estimated wavelet transform \mathbf{W}_x of the underlying stationary signal by interpolation from \mathbf{W}_y and $\tilde{\theta}^{(k)}$ with equation (18). Estimate the corresponding power spectrum $\tilde{\mathcal{S}}_X^{(k+1)}$ with (20).
- $k := k + 1$

end while

- Compute \tilde{a} and $\tilde{\gamma}$ by interpolation from $\tilde{\Theta}^{(k)}$.
-

Remark 4: In order to control the variances of the estimators, and the computational cost, two different discretizations of the scale axis are used for $\tilde{\theta}_1$ or $\tilde{\theta}_2$. Indeed, the computation of the log-likelihood involves the evaluation of the inverse covariance matrix. In [20], a sufficient condition for invertibility was given in the presence of

noise. The major consequence induced by this condition is that when δ_s is close to zero (i.e. the sampling period of scales is small), the covariance matrix could not be numerically invertible. The scale discretization must then be sufficiently coarse to ensure good conditioning for the matrix. While this condition can be reasonably fulfilled to estimate $\theta_{n,2}$ without impairing the performances of the estimator, it cannot be applied to the estimation of $\theta_{n,1}$ because of the influence of M_s on its Cramér-Rao bound (see section III-C below). The choice we made is to maximize $\mathcal{L}(\Theta_n)$ for $\theta_{n,2}$ with \mathbf{w}_{y,τ_n} corresponding to a coarse sampling \mathbf{s}_p which is a subsampled version of the original vector \mathbf{s} , the scale sampling step and the size of \mathbf{s}_p being respectively $p\delta_s$ and $\lfloor M_s/p \rfloor$ for some $p \in \mathbb{N}^*$. While $\mathcal{L}(\Theta_n)$ is maximized for $\theta_{n,1}$ on the original fine sampling \mathbf{s} , a regularization of the covariance matrix has to be done to ensure its invertibility. The regularized matrix is constructed by replacing covariance matrix $\mathbf{C}_0(\theta_{n,2})$ given by (15) by its regularized version $\mathbf{C}_{0,r}(\theta_{n,2})$, given by

$$\mathbf{C}_{0,r}(\theta_{n,2}) = (1 - r)\mathbf{C}_0(\theta_{n,2}) + r\mathbf{I} , \quad (22)$$

for some regularization parameter $0 \leq r \leq 1$.

Remark 5: After convergence of the estimation algorithm, the estimated functions \tilde{a} and $\tilde{\gamma}$ allow constructing a “stationarized” signal

$$\tilde{x} = \mathcal{D}_{\tilde{\gamma}^{-1}} \mathcal{A}_{\tilde{a}^{-1}} \mathbf{y} .$$

\tilde{x} is an estimation of the original underlying stationary signal x . Furthermore, the Welch periodogram [23] may be computed from \tilde{x} to obtain an estimator of \mathcal{S}_X whose bias is not depending on frequency (unlike the estimator used within the iterative algorithm).

Remark 6: In order to accelerate the speed of the algorithm, the estimation can be done only on a subsampled time. The main effect of this choice on the algorithm concerns the final estimation of a and γ which is more sensitive to the interpolation operation.

In the following section, we analyze quantities that enable the evaluation of the expected performances of the estimators, and their influence of the algorithm. The reader who is not directly interested in the statistical background may skip these section and jump directly to the numerical results in part IV.

C. Performances of the estimators and the algorithm

(a) *Bias.* For $\theta_{n,1}$, the estimator is unbiased when the actual values of $\theta_{n,2}$ and \mathcal{S}_X are known. In our case, the bias $b_{n,1}^{(k)}(\theta_{n,1}) = \mathbb{E} \left\{ \tilde{\theta}_{n,1}^{(k)} \right\} - \theta_{n,1}$ is written as

$$b_{n,1}^{(k)}(\theta_{n,1}) = \frac{\theta_{n,1}}{M_s} \text{Trace} \left\{ \tilde{\mathbf{C}}_0^{(k)} \left(\tilde{\theta}_{n,2}^{(k)} \right)^{-1} \mathbf{C}_0(\theta_{n,2}) - \mathbf{I} \right\}. \quad (23)$$

As expected, the better the covariance matrix estimation, the lower the bias $\left| b_{n,1}^{(k)} \right|$.

For $\theta_{n,2}$, as we do not have a closed-form expression for the estimator we are not able to give an expression of the bias. Nevertheless, if we assume that the two other true variables are known, as a maximum likelihood estimator we make sure that $\tilde{\theta}_{n,2}$ is asymptotically unbiased (i.e. $\tilde{\theta}_{n,2} \rightarrow \theta_{n,2}$ when $M_s \rightarrow \infty$).

Regarding \mathcal{S}_X , equation (19) shows that the estimator yields a smoothed, thus biased version of the spectrum. Besides, proposition 1 below shows that the estimated spectrum converges to this biased version when the deformation parameters converge to their actual values.

Proposition 1: Let $\psi \in H^2(\mathbb{R})$ be an analytic wavelet such that $\hat{\psi}$ is bounded and $|\hat{\psi}(u)| = \mathcal{O}_{u \rightarrow \infty}(u^{-\eta})$ with $\eta > 2$. Let φ_1 and φ_2 be bounded functions defined on \mathbb{R}^+ by $\varphi_1(u) = u |\hat{\psi}(u)|^2$ and $\varphi_2(u) = u^2 |\hat{\psi}(u)|$. Assume \mathcal{S}_X is such that

$$J_X = \int_0^\infty \xi^{-1} \mathcal{S}_X(\xi) d\xi < \infty.$$

Let $\mathcal{S}_X^{(k)}$ denote the estimation of the spectrum after k iterations of the algorithm. Let $b_{\mathcal{S}_X}^{(k)}$ denote the bias defined for all $m \in \llbracket 1, M_s \rrbracket$ by

$$b_{\mathcal{S}_X}^{(k)}(m) = \mathbb{E} \left\{ \tilde{\mathcal{S}}_X^{(k)}(q^{-sm} \omega_0) \right\} - \mathcal{S}_{X,\psi}(q^{-sm} \omega_0).$$

Assume there exists a constant $c_{\theta_1} > 0$ such that $\theta_{n,1}^{(k)} > c_{\theta_1}, \forall n, k$. Then

$$\left\| b_{\mathcal{S}_X}^{(k)} \right\|_\infty \leq \frac{J_X}{\|\psi\|_2^2} \left(K'_1 \left\| \boldsymbol{\theta}_1 - \tilde{\boldsymbol{\theta}}_1^{(k)} \right\|_\infty + K'_2 \left\| \tilde{\boldsymbol{\theta}}_2^{(k)} - \boldsymbol{\theta}_2 \right\|_\infty \right), \quad (24)$$

where

$$K'_1 = \frac{\|\varphi_1\|_\infty}{c_{\theta_1}} < \infty,$$

$$K'_2 = \ln(q) (\|\varphi_1\|_\infty + 2\|\hat{\psi}'\|_\infty \|\varphi_2\|_\infty) < \infty.$$

The proof is given in appendix B.

Remark 7: If $\theta_1^{(k)} \rightarrow \theta_1$ and $\theta_2^{(k)} \rightarrow \theta_2$ as $k \rightarrow \infty$, we have $\mathbb{E} \left\{ \tilde{\mathcal{I}}_X^{(k)}(\nu_m) \right\} \xrightarrow[k]{} \mathcal{I}_{X,\psi}(\nu_m)$ which is the expected property.

Formula (24) enables the control of the bias of the spectrum at frequencies $\nu_m = q^{-s_m}\omega_0$ only. We can also notice the required property $J_X < \infty$ forces \mathcal{I}_X to vanish at zero frequency.

(b) *Variance.* The Cramér-Rao lower bound (CRLB) gives the minimum variance that can be attained by unbiased estimators. The Slepian-Bangs formula (see [24]) directly gives the following CRLB for component $\theta_{n,i}$

$$CRLB(\theta_{n,i}) = 2 \left(\text{Trace} \left\{ \left(\mathbf{C}(\Theta_n)^{-1} \frac{\partial \mathbf{C}(\Theta_n)}{\partial \theta_{n,i}} \right)^2 \right\} \right)^{-1}.$$

This bound gives information about the variance of the estimator at convergence of the algorithm, i.e. when both \mathcal{I}_X and the other parameters are well estimated.

Applying this formula to $\theta_{n,1}$ gives

$$\mathbb{E} \left\{ (\tilde{\theta}_{n,1} - \mathbb{E} \{ \tilde{\theta}_{n,1} \})^2 \right\} \geq CRLB(\theta_{n,1}) = \frac{2\theta_{n,1}^2}{M_s}.$$

This implies that the number of scales M_s of the wavelet transform must be large enough to yield an estimator with sufficiently small variance.

For $\theta_{n,2}$, no closed-form expression is available for the CRLB. Therefore, the evaluation of this bound and its comparison with the variance of the estimator $\tilde{\theta}_{n,2}$ can only be based on numerical results, see section IV.

(c) *Robustness to noise.* Assume now the observations are corrupted by a random Gaussian white noise W with variance σ_W^2 (supposed to be known). The model becomes

$$Y = \mathcal{A}_a \mathcal{D}_\gamma X + W. \quad (25)$$

The estimator $\tilde{\theta}_{n,1}$ is not robust to noise. Indeed, if the maximum likelihood estimator of model (7) in the presence of such white noise, a new term $b_{n,1|W}^{(k)}(\theta_{n,1})$ must be added to the bias expression (23), which becomes

$$b_{n,1|W}^{(k)}(\theta_{n,1}) = \frac{1}{M_s} \text{Trace} \left\{ \tilde{\mathbf{C}}_0^{(k)} \left(\tilde{\theta}_{n,2}^{(k)} \right)^{-1} \mathbf{C}_{wn} \right\},$$

where $(\mathbf{C}_{wn})_{ij} = \sigma_W^2 q^{(s_i+s_j)/2} \int_0^\infty \bar{\psi}(q^{s_i}\xi) \hat{\psi}(q^{s_j}\xi) d\xi$. In practice, this term can take large values, therefore noise has to be taken into account. To do so, the covariance matrix is now written as

$$\mathbf{C}(\Theta_n)_{ij} = q^{\frac{s_i+s_j}{2}} \int_0^\infty (\theta_{n,2} \mathcal{I}_X(q^{-\theta_{n,1}}\xi) + \sigma_W^2) \bar{\psi}(q^{s_i}\xi) \hat{\psi}(q^{s_j}\xi) d\xi, \quad (26)$$

and the likelihood is modified accordingly. Formula (17) is no longer true and no closed-form expression can be derived any more, the maximum likelihood estimate $\tilde{\theta}_{n,1}$ must be computed by a numerical scheme (here we use a simple gradient ascent).

The estimator $\tilde{\theta}_{n,2}$ is very robust to noise. Indeed, equation (26) shows that the only change in the covariance matrix formula is to replace the power spectrum \mathcal{S}_X by $\mathcal{S}_Z = \mathcal{S}_X + \frac{\sigma_W^2}{\theta_{n,2}}$. The additive constant term is not impairing the estimator as long as it is small in comparison with the maximum values of \mathcal{S}_X .

Moreover, the estimator $\tilde{\mathcal{S}}_X$ is modified because when computing $\frac{1}{\theta_{n,1}^{1/2}} \tilde{\mathcal{W}}_y(s - \theta_{n,2}, \tau_n)$ on scale s_m , we compute:

$$\mathbf{w}_{Z,s_m} = \mathbf{w}_{X,s_m} + \mathbf{w}_{w_*,s_m} ,$$

where $\mathbf{w}_{w_*,s_m} = \frac{1}{\theta_1^{1/2}} \tilde{\mathcal{W}}_w(s_m - \theta_2, \tau)$ is the wavelet transform of a white noise modulated in amplitude by a^{-1} . Thus a constant term $\tilde{\sigma}_W$ independent of frequency is added to the new spectrum estimator $\tilde{\mathcal{S}}_Z$, so that

$$\mathbb{E} \{ \tilde{\mathcal{S}}_Z \} = \mathcal{S}_{X,\psi} + \tilde{\sigma}_W^2 \quad \text{where} \quad \tilde{\sigma}_W^2 = \sigma_W^2 \frac{1}{N_\tau} \sum_{n=1}^{N_\tau} \frac{1}{\theta_{n,1}} .$$

D. Extension: estimation of other deformations

To describe other non-stationary behaviors of audio signals, other operators can be investigated. For example, combination of time warping and frequency modulation can be considered, as was done in [21], we shortly account for this case here for the sake of completeness. Let $\alpha \in C^2$ be a smooth function, and set

$$\mathcal{M}_\alpha : \quad \mathcal{M}_\alpha x(t) = e^{2i\pi\alpha(t)} x(t) , \quad (27)$$

The deformation model in [21] is of the form

$$Y = \mathcal{A}_a \mathcal{M}_\alpha \mathcal{D}_\gamma X . \quad (28)$$

To perform joint estimation of amplitude and frequency modulation and time warping for each time, a suitable time-scale-frequency transform \mathcal{V} is introduced, defined as $\mathcal{V}_X(s, \nu, \tau) = \langle X, \psi_{s\nu\tau} \rangle$, with $\psi_{s\nu\tau} = T_\tau M_\nu D_s \psi$. In that case, an approximation theorem similar to 1 can be obtained from which the corresponding log-likelihood can be written. At fixed time τ , the strategy of estimation is the same as before, but the parameter space is of higher dimension, and the extra parameter $\theta_3 = \alpha'(\tau)$ complicates the log-likelihood maximization. In particular, the choice of the discretization of the two scale

and frequency variables s and ν influences performances of the estimator, in particular the Cramér-Rao bound.

IV. NUMERICAL RESULTS

We now turn to numerical simulations and applications. A main ingredient is the choice of the wavelet transform. Here we shall always use the *sharp wavelet* ψ_{\sharp} defined in (2) and set the scale constant q to $q = 2$.

We will systematically compare our approach to simple estimators for amplitude modulation and time warping, commonly used in applications, defined below.

- Amplitude modulation: we use as baseline estimator of $a(\tau_n)^2$ the average energy $\tilde{\theta}_{n,1}^{(B)}$ defined as follows:

$$\tilde{\theta}_{n,1}^{(B)} = \frac{1}{M_s} \|\mathbf{w}_{y,\tau_n}\|^2.$$

This amounts to replace the estimated covariance matrix in (17) by the identity matrix. Notice that $\tilde{\theta}_{n,1}^{(B)}$ does not depend on the time warping estimator, and can be computed directly on the observation.

- Time warping: the baseline estimator $\tilde{\theta}_{n,2}^{(B)}$ is the scalogram scale center of mass defined as follows:

$$\tilde{\theta}_{n,2}^{(B)} = C_0 + \frac{1}{\|\mathbf{w}_{y,\tau_n}\|^2} \sum_{m=1}^{M_s} s[m] |\mathbf{w}_{y,\tau_n}[m]|^2.$$

C_0 is chosen such that $\tilde{\theta}_{n,2}^{(B)}$ is a zero mean vector.

Numerical evaluation is performed on both synthetic signals and deformations and real audio signals.

A. Synthetic signal

We first evaluate the performances of the algorithm on a synthetic signal. This allows us to compare variance and bias with their theoretical values.

The simulated signal has length $N_\tau = 2^{16}$ samples, sampled at $F_s = 8$ kHz (meaning the signal duration is $t_F = (N_\tau - 1)/F_s \approx 8.2$ s). The spectrum \mathcal{S}_X is written as $\mathcal{S}_X = S_1 + S_2$ where $S_l(\nu) = 1 + \cos\left(2\pi(\nu - \nu_0^{(l)})/\Delta_\nu^{(l)}\right)$ if $|\nu - \nu_0^{(l)}| < \Delta_\nu^{(l)}/2$ and vanishes elsewhere (for $l \in \{1, 2\}$). The amplitude modulation a is a sine wave $a(t) = a_0(1 + a_1 \cos(2\pi t/T_1))$, where a_0 is chosen such that $t_F^{-1} \int_0^{t_F} a^2(t) dt = 1$. The

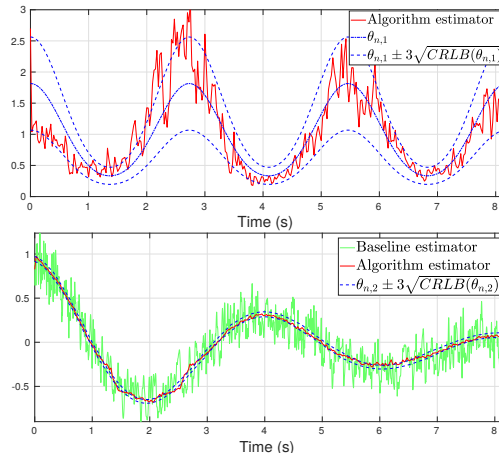


Fig. 1. Joint amplitude modulation/time warping estimation on a synthetic signal. Top: amplitude modulation estimation ($a_1 = 0.4$ and $T_1 = t_F/3$). Bottom: warping estimation ($T_2 = t_F/2$ and $T_3 = t_F/2$).

Estimation method	Amplitude modulation	Time warping
Baseline	2.01×10^{-1}	2.32×10^{-2}
Algorithm 1	7.01×10^{-2}	4.91×10^{-4}

TABLE I

MEAN SQUARE ERROR OF THE ESTIMATION METHODS FOR BOTH DEFORMATIONS

time warping function γ is such that $\log_q(\gamma'(t)) = \Gamma + \cos(2\pi t/T_2)e^{-t/T_3}$, where Γ is chosen such that $t_F^{-1} \int_0^{t_F} \gamma'(t) dt = 1$.

The estimation algorithm is implemented with $M_s = 106$ and $p = 7$. Results are shown in Fig. 1 and compared with baseline estimations. For the sake of visibility, the baseline estimator of the amplitude modulation (which is very oscillatory) is not displayed, but numerical assessments are provided in Table I, which gives MSEs for the different estimations. The proposed algorithm is clearly more precise than the baseline algorithm, furthermore its precision is well accounted for by the Cramér-Rao bound: in Fig. 1, the estimate is essentially contained within the 95 % confidence interval provided by the CRLB (assuming gaussianity and unbiasedness).

The left hand side of Fig. 2 displays the estimated spectrum given by the algorithm with formula (20). The agreement with the actual spectrum is very good, with a slight enlargement effect due to the filtering by $|\hat{\psi}|^2$. The right hand side of Fig. 2 gives

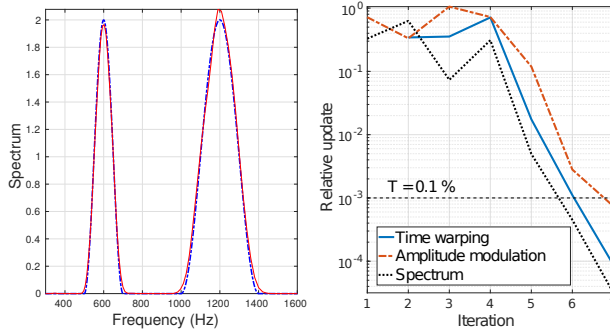


Fig. 2. Left: Spectrum estimation ($v_0^{(1)} = 600$ Hz, $\Delta_v^{(1)} = 200$ Hz, $v_0^{(2)} = 1.2$ kHz, $\Delta_v^{(2)} = 400$ Hz): actual (dash-dot blue line) and estimated (solid red line) spectra. Right: Stopping criterion evolution.

the evolution of the stopping criterion (21) with iterations. Numerical results show that time warping estimation converges faster than amplitude modulation estimation. Nevertheless, when fixing a stopping criterion to 0.1 % only 7 iterations are necessary for the algorithm to converge.

B. Application to Doppler estimation

After studying the influence of the various parameters, let us now turn to a real life audio example. The analyzed sound is a recording (from a fixed location) from a racing car, moving with constant speed. The car engine sound is then deformed by the Doppler effect, which results in time warping the sound emitted by the car. Besides, as the car is moving, the closer the car to the microphone, the larger the amplitude of the recorded sound. Thus, our model fits well this signal.

The wavelet transforms of the original signal and the two estimations of the underlying stationary signal are shown in Fig. 3. While the estimation of time warping only corrects the displacement of wavelet coefficients in the time-scale domain, the joint estimation of time warping and amplitude modulation also approximately corrects non-stationary variations of the amplitudes.

The physical relevance of the estimated time warping function can be verified. Indeed, denote by V the (constant) speed of the car and by c the sound velocity. Fixing the time origin to the time at which the car passes in front of the observer at distance d , the time

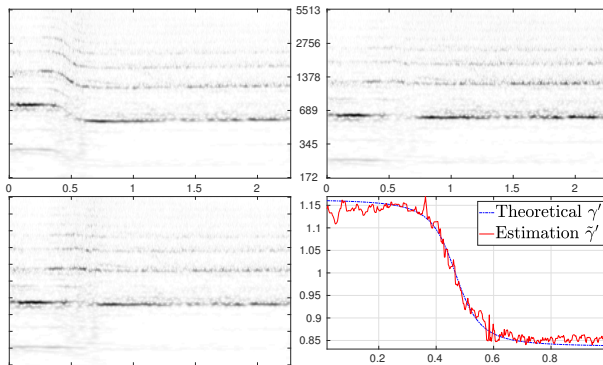


Fig. 3. Doppler estimation. Top left: Scalogram of the original signal. Top right: Scalogram of the unwarped and unmodulated signal. Bottom left: Scalogram of the unwarped signal. Bottom right: Estimated time warping compared with the theoretical value given in (29).

warping function due to Doppler effect can be shown to be

$$\gamma'(t) = \frac{c^2}{c^2 - V^2} \left(1 - \frac{V^2 t}{\sqrt{d^2(c^2 - V^2) + (cVt)^2}} \right). \quad (29)$$

We plot in Fig. 3 (bottom right) the estimation $\tilde{\gamma}'$ compared with its theoretical value where $d = 5$ m and $V = 54$ m/s. Clearly the estimate is close to the corresponding theoretical curve obtained with these data, which are therefore realistic values.

Nevertheless, a closer look at scalograms in Fig. 3 shows that the amplitude correction is not perfect, due to the presence of noise, and the fact that the model is still too simple: the amplitude modulation actually depends on frequency, which is not accounted for.

V. CONCLUSIONS

We have discussed in this paper extensions of methods and algorithms described earlier in [9], [22], [20], [21] for the joint estimation of deformation operator and power spectrum for deformed stationary signals, a problem already addressed in [19] with a different approach. The main improvements described in this paper concern the following two points

- 1) the extension of the algorithm to the joint estimation of deformations including amplitude modulation to the model and its estimation;
- 2) a statistical study of the estimators and of the performances of the algorithm.

The proposed approach was validated on numerical simulations and an application to Doppler estimation.

The results presented here show that the proposed extensions result in a significant improvement in terms of precision, and a better theoretical control. In particular, the continuous parameter estimation procedure avoids quantization effects that were present in [20] where the parameter space was discrete and the estimation based on exhaustive search. Regarding the approach of [19], its domain of validity seems to be limited to small scales (i.e. high frequency) signals, which is not the case here.

Contrary to [19] our approach is based on (approximate) maximum likelihood estimation in the Gaussian framework. Because of our choice to disregard time correlations, the estimates obtained here generally present spurious fluctuations, which can be smoothed out by appropriate filtering. A natural extension of our approach would be to introduce a smoothness prior that would avoid such filtering steps when necessary.

The code and datasets used to produce the numerical results of this paper are available at the web site

<https://cv.archives-ouvertes.fr/bruno-torresani>

APPENDIX A

PROOF OF THEOREM 1

To simplify notations, let \mathcal{B}_γ denote the operator $\mathcal{D}_\gamma / \sqrt{\gamma'}$. We split the approximation error as

$$\begin{aligned} \varepsilon(s, \tau) &= \langle \mathcal{A}_a \mathcal{D}_\gamma X, \psi_{s\tau} \rangle - \langle \widetilde{\mathcal{A}}_a^\tau \widetilde{\mathcal{D}}_\gamma^\tau X, \psi_{s\tau} \rangle \\ &= a(\tau) \left(\varepsilon^{(1)}(s, \tau) + \sqrt{\gamma'(\tau)} \varepsilon^{(2)}(s, \tau) + \varepsilon^{(3)}(s, \tau) \right), \end{aligned}$$

where

$$\begin{aligned} \varepsilon^{(1)}(s, \tau) &= \left\langle \left(\sqrt{\gamma'} - \sqrt{\gamma'(\tau)} \right) \mathcal{B}_\gamma X, \psi_{s\tau} \right\rangle \\ &= \left\langle X, \mathcal{B}_{\gamma^{-1}} \left(\sqrt{\gamma'} - \sqrt{\gamma'(\tau)} \right) \psi_{s\tau} \right\rangle, \\ \varepsilon^{(2)}(s, \tau) &= \left\langle \left(\mathcal{B}_\gamma - \widetilde{\mathcal{B}}_\gamma^\tau \right) X, \psi_{s\tau} \right\rangle \\ &= \left\langle X, \left(\mathcal{B}_{\gamma^{-1}} - \widetilde{\mathcal{B}}_\gamma^{\tau^{-1}} \right) \psi_{s\tau} \right\rangle, \end{aligned}$$

$$\begin{aligned}\varepsilon^{(3)}(s, \tau) &= \left\langle \left(\mathcal{A}_a \widetilde{\mathcal{A}}_a^\tau^{-1} - 1 \right) \mathcal{D}_\gamma X, \psi_{s\tau} \right\rangle \\ &= \left\langle X, \mathcal{D}_{\gamma^{-1}} \left(\mathcal{A}_a \widetilde{\mathcal{A}}_a^\tau^{-1} - 1 \right) \psi_{s\tau} \right\rangle.\end{aligned}$$

Furthermore, the triangle inequality gives:

$$\begin{aligned}\mathbb{E} \left\{ |\varepsilon(s, \tau)|^2 \right\} &\leq C_a^2 \left(\sqrt{\mathbb{E} \left\{ |\varepsilon^{(1)}(s, \tau)|^2 \right\}} + \sqrt{C_\gamma \mathbb{E} \left\{ |\varepsilon^{(2)}(s, \tau)|^2 \right\}} \right. \\ &\quad \left. + \sqrt{\mathbb{E} \left\{ |\varepsilon^{(3)}(s, \tau)|^2 \right\}} \right)^2.\end{aligned}\tag{30}$$

Let us now determine an upper bound for each error term. To this end, they are written as follows:

$$\mathbb{E} \left\{ |\varepsilon^{(k)}(s, \tau)|^2 \right\} = \int_0^\infty \mathcal{S}_X(\xi) \left| \hat{f}_{s\tau}^{(k)}(\xi) \right|^2 d\xi,$$

with $k \in \{1, 2, 3\}$.

Concerning the first error term, a Taylor expansion of $\sqrt{\gamma'}$ around τ gives

$$\begin{aligned}\left| \hat{f}_{s\tau}^{(1)}(\xi) \right| &= \left| \int_{\mathbb{R}} \left(\sqrt{\gamma'(t)} - \sqrt{\gamma'(\tau)} \right) \psi_{s\tau}(t) e^{-2i\pi\gamma(t)\xi} dt \right| \\ &\leq \int_{\mathbb{R}} \left\| \frac{\gamma''}{2\sqrt{\gamma'}} \right\|_\infty |t - \tau| |\psi_{s\tau}(t)| dt \leq q^{\frac{3s}{2}} \frac{\|\gamma''\|_\infty}{2\sqrt{c_\gamma}} I_\psi,\end{aligned}$$

where $I_\psi = \int_{\mathbb{R}} |t\psi(t)| dt$. Furthermore, the localization assumption on ψ allows us to write

$$I_\psi \leq 2 \int_0^\infty \frac{t}{1+t^\beta} dt \leq 2 \left(\int_0^1 t dt + \int_1^\infty \frac{1}{t^{\beta-1}} dt \right) = \frac{\beta}{\beta-2}.$$

Finally, we can control the first error term as follows:

$$\mathbb{E} \left\{ |\varepsilon^{(1)}(s, \tau)|^2 \right\} \leq \left(q^{3s/2} \frac{\|\gamma''\|_\infty}{\sqrt{c_\gamma}} \frac{\beta\sigma_X}{2(\beta-2)} \right)^2.$$

Concerning the second error term, we have

$$\begin{aligned}\left| \hat{f}_{s\tau}^{(2)}(\xi) \right| &= \left| \int_{\mathbb{R}} \left(e^{-2i\pi\gamma(t)\xi} - e^{-2i\pi(\gamma(\tau)+(t-\tau)\gamma'(\tau))\xi} \right) \psi_{s\tau}(t) dt \right| \\ &\leq \int_{\mathbb{R}} \left| 1 - e^{-2i\pi(\gamma(\tau)+(t-\tau)\gamma'(\tau)-\gamma(t))\xi} \right| |\psi_{s\tau}(t)| dt \\ &\leq \int_{\mathbb{R}} 2 \left| \sin \left(\frac{\pi}{2} \xi (t-\tau)^2 \gamma''(t_*) \right) \right| |\psi_{s\tau}(t)| dt,\end{aligned}$$

for some t_* between t and τ . Besides, we have $|\sin(u)| \leq |u|$ and $|\sin(u)| \leq 1$ so that:

$$\left| \hat{f}_{s\tau}^{(2)}(\xi) \right| \leq 2q^{s/2} \left(\int_J \frac{\pi}{2} \xi \|\gamma''\|_\infty q^{2s} t^2 |\psi(t)| dt + \int_{\mathbb{R} \setminus J} |\psi(t)| dt \right).$$

where $J = [-T, T]$. One can prove that the value of T minimizing the the right-hand side of the latter equation is $T = \left(\frac{\pi}{2}\xi\|\gamma''\|_\infty q^{2s}\right)^{-1/(\beta+2)}$. Therefore, we have:

$$\left|\hat{f}_{s\tau}^{(2)}(\xi)\right| \leq q^{\frac{5\beta-2}{2(\beta+2)}} \frac{4(\beta+2)}{3(\beta-1)} \left(\frac{\pi}{2}\xi\|\gamma''\|_\infty\right)^{\frac{\beta+2}{\beta-1}}.$$

Finally, we can control the second error term as follows:

$$\mathbb{E} \left\{ \left| \varepsilon^{(2)}(s, \tau) \right|^2 \right\} \leq \left(q^{\frac{5\beta-2}{2(\beta+2)}} \frac{4(\beta+2)}{3(\beta-1)} \left(\frac{\pi}{2}\|\gamma''\|_\infty\right)^{\frac{\beta+2}{\beta-1}} I_X^{(\rho)} \right)^2.$$

Concerning the third error term, we have

$$\begin{aligned} \left|\hat{f}_{s\tau}^{(3)}(\xi)\right| &= \left| \int_{\mathbb{R}} \sqrt{\gamma'(t)} \left(\frac{a(t)}{a(\tau)} - 1 \right) \psi_{s\tau}(t) e^{-2i\pi\gamma(t)\xi} dt \right| \\ &\leq C_\gamma^{\frac{1}{2}} \int_{\mathbb{R}} \frac{\|a'\|_\infty}{c_a} |t - \tau| |\psi_{s\tau}(t)| dt = q^{\frac{3s}{2}} C_\gamma^{\frac{1}{2}} \frac{\|a'\|_\infty}{c_a} I_\psi, \end{aligned}$$

Finally, we can control the third error term as follows:

$$\mathbb{E} \left\{ \left| \varepsilon^{(3)}(s, \tau) \right|^2 \right\} \leq \left(q^{3s/2} \frac{\sqrt{C_\gamma}}{c_a} \frac{\beta\sigma_X}{\beta-2} \|a'\|_\infty \right)^2.$$

To conclude the proof, the three errors terms in equation (30) are replaced by their upper bounds to obtain the approximation error given in the theorem. \blacksquare

APPENDIX B

PROOF OF PROPOSITION 1

Let $\tilde{\mathbf{w}}_{x,s_m}^{(k)} \in \mathbb{C}^{N_\tau}$ denote the estimation of \mathbf{w}_{x,s_m} after k iterations of the algorithm. Considering equation (18), we have $\tilde{\mathbf{w}}_{x,s_m}^{(k)} = \frac{1}{\sqrt{\tilde{\theta}_1^{(k)}}} \tilde{\mathcal{W}}_y \left(s_m - \tilde{\theta}_2^{(k)}, \boldsymbol{\tau} \right)$, thus

$$\mathbb{E} \left\{ \left| \tilde{\mathcal{S}}_X^{(k)}(q^{-s_m} \omega_0) \right|^2 \right\} = \frac{1}{N_\tau \|\boldsymbol{\psi}\|_2^2} \mathbb{E} \left\{ \left\| \tilde{\mathbf{w}}_{x,s_m}^{(k)} \right\|^2 \right\}.$$

To simplify notations, let us introduce some variables. We define $s_m^{(\xi)} = s_m + \log_q(\xi)$ and $h(x) = \varphi_1(q^x) = q^x |\hat{\psi}(q^x)|^2$ for $x \in \mathbb{R}$.

By means of the covariance expression given in equation (12) we can write

$$\begin{aligned} \mathbb{E} \left\{ \left\| \tilde{\mathbf{w}}_{x,s_m}^{(k)} \right\|^2 \right\} &= \sum_{n=1}^{N_\tau} \frac{1}{\tilde{\theta}_{n,1}^{(k)}} \mathbb{E} \left\{ \overline{\tilde{\mathcal{W}}_y \left(s_m - \tilde{\theta}_{n,2}^{(k)}, \tau_n \right)} \tilde{\mathcal{W}}_y \left(s_m - \tilde{\theta}_{n,2}^{(k)}, \tau_n \right) \right\} \\ &= \sum_{n=1}^{N_\tau} \frac{\theta_{n,1}}{\tilde{\theta}_{n,1}^{(k)}} \int_0^\infty \frac{\mathcal{S}_X(\xi)}{\xi} h \left(s_m^{(\xi)} + \theta_{n,2} - \tilde{\theta}_{n,2}^{(k)} \right) d\xi \end{aligned}$$

$$= \int_0^\infty \frac{\mathcal{S}_X(\xi)}{\xi} \left(\sum_{n=1}^{N_\tau} \frac{\theta_{n,1}}{\tilde{\theta}_{n,1}^{(k)}} h \left(s_m^{(\xi)} + \theta_{n,2} - \tilde{\theta}_{n,2}^{(k)} \right) \right) d\xi.$$

Let us now split the bias into two terms such that $b_{\mathcal{S}_X}^{(k)}(m) = g_1(m) + g_2(m)$, where g_1 and g_2 are defined as

$$g_1(m) = \frac{N_\tau^{-1}}{\|\psi\|_2^2} \int_0^\infty \frac{\mathcal{S}_X(\xi)}{\xi} \left(\sum_{n=1}^{N_\tau} \left(\frac{\theta_{n,1}}{\tilde{\theta}_{n,1}^{(k)}} - 1 \right) h \left(s_m^{(\xi)} + \theta_{n,2} - \tilde{\theta}_{n,2}^{(k)} \right) \right) d\xi,$$

$$g_2(m) = \frac{N_\tau^{-1}}{\|\psi\|_2^2} \int_0^\infty \frac{\mathcal{S}_X(\xi)}{\xi} \left(\sum_{n=1}^{N_\tau} \left(h \left(s_m^{(\xi)} + \theta_{n,2} - \tilde{\theta}_{n,2}^{(k)} \right) - h \left(s_m^{(\xi)} \right) \right) \right) d\xi.$$

Regarding the first term, we directly have

$$\begin{aligned} |g_1(m)| &\leq \frac{\|h\|_\infty}{\|\psi\|_2^2} \frac{1}{N_\tau} \sum_{n=1}^{N_\tau} \frac{|\theta_{n,1} - \tilde{\theta}_{n,1}^{(k)}|}{\tilde{\theta}_{n,1}^{(k)}} \int_0^\infty \frac{\mathcal{S}_X(\xi)}{\xi} d\xi \\ &\leq \frac{\|h\|_\infty}{\|\psi\|_2^2 c_{\theta_1}} \left\| \boldsymbol{\theta}_1 - \tilde{\boldsymbol{\theta}}_1^{(k)} \right\|_\infty J_X. \end{aligned}$$

Besides, we have $\|h\|_\infty = \|\varphi_1\|_\infty$ and the smoothness and decay assumptions on $\hat{\psi}$ allow us to write $\varphi_1(u) = \mathcal{O}_{u \rightarrow \infty}(u^{1-2\eta}) \xrightarrow{u \rightarrow \infty} 0$. Then φ_1 is bounded and $K'_1 < \infty$. This yields

$$|g_1(m)| \leq \frac{J_X K'_1}{\|\psi\|_2^2} \left\| \boldsymbol{\theta}_1 - \tilde{\boldsymbol{\theta}}_1^{(k)} \right\|_\infty.$$

Regarding the second term, a Taylor expansion of h around $s_m^{(\xi)}$ gives

$$\begin{aligned} |g_2(m)| &\leq \frac{N_\tau^{-1}}{\|\psi\|_2^2} \int_0^\infty \frac{\mathcal{S}_X(\xi)}{\xi} \sum_{n=1}^{N_\tau} \left| h \left(s_m^{(\xi)} + \theta_{n,2} - \tilde{\theta}_{n,2}^{(k)} \right) - h \left(s_m^{(\xi)} \right) \right| d\xi \\ &\leq \frac{\|h'\|_\infty}{\|\psi\|_2^2} \frac{1}{N_\tau} \sum_{n=1}^{N_\tau} \left| \theta_{n,2} - \tilde{\theta}_{n,2}^{(k)} \right| \int_0^\infty \frac{\mathcal{S}_X(\xi)}{\xi} d\xi \\ &\leq \frac{\|h'\|_\infty}{\|\psi\|_2^2} \left\| \boldsymbol{\theta}_2 - \tilde{\boldsymbol{\theta}}_2^{(k)} \right\|_\infty J_X. \end{aligned}$$

Furthermore, $\forall x \in \mathbb{R}$

$$\begin{aligned} |h'(x)| \ln(q)^{-1} &= |q^x \varphi_1'(q^x)| \leq q^x |\hat{\psi}(q^x)|^2 + 2q^{2x} |\hat{\psi}(q^x) \hat{\psi}'(q^x)| \\ &\leq \|\varphi_1\|_\infty + 2\|\hat{\psi}'\|_\infty \|\varphi_2\|_\infty = \ln(q)^{-1} K'_2. \end{aligned}$$

Besides, the decay assumption on $\hat{\psi}$ gives $|\varphi_2(u)| = \mathcal{O}_{u \rightarrow \infty}(u^{2-\eta}) \xrightarrow{u \rightarrow \infty} 0$ because $\eta > 2$.

Then φ_2 is bounded and $K'_2 < \infty$. This yields

$$|g_2(m)| \leq \frac{J_X K'_2}{\|\psi\|_2^2} \left\| \boldsymbol{\theta}_2 - \tilde{\boldsymbol{\theta}}_2^{(k)} \right\|_\infty.$$

The proof is concluded by summing up the upper bounds of $|g_1|$ and $|g_2|$ to obtain the upper bound of $b_{\mathcal{S}_X}^{(k)}$. Notice that this bound does not depends on m . \blacksquare

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