Discrete energy estimates for the abcd-systems

Cosmin Burtea, Clémentine Courtès

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1 Introduction

The Boussinesq regime deals with long wave, small amplitude gravity surface water waves: consider a layer of incompressible, irrotational, perfect fluid flowing through a canal with flat bottom represented by the plane:

$$\{(x,y,z) : z = -h\},$$

with $h > 0$. We suppose that perturbing the equilibrium state, the resulting free surface describes the graph of some function over the flat bottom. Let A be the maximum amplitude of the wave and l a typical wavelength. Furthermore, consider:

$$\epsilon = \frac{A}{h},\ \mu = \left(\frac{h}{l}\right)^2,\ S = \frac{\epsilon}{\mu}.$$

The following asymptotic regime

$$\epsilon \ll 1,\ \mu \ll 1 \text{ and } S \approx 1$$

is used to take into account small amplitude, long wavelength water waves. In order to study waves that fit into the Boussinesq regime, Bona, Chen and Saut, [9], derived the so-called *abcd*-system:

$$\begin{cases} (I - \mu b\Delta)\,\partial_t \eta + \operatorname{div} V + a\mu \operatorname{div} \Delta V + \epsilon \operatorname{div}(\eta V) = 0, \\ (I - \mu d\Delta)\,\partial_t V + \nabla \eta + c\mu \nabla \Delta \eta + \epsilon V \cdot \nabla V = 0, \end{cases} \tag{1.1}$$

with I the identity operator. In the system (1.1), $\eta = \eta(t,x) \in \mathbb{R}$ is an approximation of the deviation of the free surface of the water from the rest state while $V = V(t,x) \in \mathbb{R}^n$ is an approximation of the fluid velocity. The above family of systems is derived from the classical mathematical formulation of the water waves problem by a formal series expansion and by neglecting the second and higher order terms. In fact, one may regard the zeros on the right hand side of (1.1) as the second order terms (i.e. having one of ϵ^2, $\epsilon\mu$ or μ^2 as prefactor) neglected in establishing (1.1), from the modeling process. The parameters a, b, c, d are also restricted by the following relation:

$$a + b + c + d = \frac{1}{3}. \tag{1.2}$$

Nevertheless, if the surface tension is considered, the previous relation rewrites $a + b + c + d = \frac{1}{3} - \tau$ where $\tau \geq 0$ is the Bond number which characterizes the surface tension parameter, see [22] or [28]. This is why, in this paper, we consider a, b, c, and d which do not verify (1.2).

In [11], Bona, Colin and Lannes show that the systems (1.1) approximate the water waves problem proving that the error estimate between the solution of (1.1) and the water waves system at time t is of order $O\left(\epsilon^2 t\right)$, (see also [24]).

In [16], models taking into account more general topographies of the bottom are derived. One has to furthermore distinguish between two different regimes: small respectively strong topography variations. Time variating bottom-topographies are considered in [18]. For a systematic study of approximate models for the water waves problem along with their rigorous justification we refer the reader to the work of Lannes, [24]. We point out that the only values of n for which (1.1) is physically relevant are $n = 1, 2$.

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The *abcd* systems are well-posed locally in time in the following cases:

$$a \leq 0,\ c \leq 0,\ b \geq 0,\ d \geq 0 \tag{1.3}$$

$$\text{or } a = c \geq 0 \text{ and } b \geq 0,\ d \geq 0, \tag{1.4}$$

see for instance Bona, Chen and Saut [10], Anh [2], or Linares, Pilod and Saut [25]. Global existence is known to hold true in dimension 1 for the "classical" Boussinesq system:

$$a = b = c = 0,\ d > 0,$$

which was studied by Amick in [1] and Schonbek in [30] and for the so-called Bona-Smith systems:

$$b = d > 0,\ a \leq 0,\ c < 0,$$

assuming some smallness condition on the initial data, see [10] and the work of Bona and Smith, [13]. In other cases, the problem is still open.

Lower bounds on the time of existence of solutions of systems (1.1) in terms of the physical parameters are obtained in [29], [28], [26], [15] for initial data lying in Sobolev classes respectively in [14] for data manifesting nontrivial data at infinity.

Given the practical importance of systems (1.1) in concrete physical situations, works regarding their numerical aspects are abundant. One of the earliest papers devoted to numerical feature is the work of Peregrine, [27] who investigated undular bore propagation using the classical Boussinesq system, $a = b = c = 0$, $d = 1/3$.

In [7], Bona and Chen consider the boundary value problem for the BBM-BBM case (which corresponding to $a = c = 0$, $b = d = 1/6$). Using an integral reformulation of this problem, they implement a numerical algorithm which is fourth order accurate in both time and space. They employ it in order to track down generalized solitary waves and to study the collision between this numerical obtained generalized solitary waves.

In [12] and [6] Bona, Dougalis and Mitsotakis study the periodic value problem for the so-called KdV-KdV case (which corresponding to $b = d = 0$, $a = c = 1/6$) using an implicit Runge-Kutta for time discretization and a Galerkin method with periodic splines scheme for the space discretization. They track down generalized solitary wave solutions and they simulate the head-on collision of two such waves.

In [4], Antonopoulus, Dougalis and Mitsotakis study the periodic problem for a large class of values of the a, b, c, d parameters. They use Galerkin approximations combined with a Runge-Kutta method in order to provide a full time-space discretization of the systems and perform several interesting numerical experiments. In [5], the previous authors extend their results regarding the nonhomogeneous Dirichlet and reflection boundary conditions for the Bona-Smith family of systems corresponding to the case $a = 0$, $b = d > 0$, $c \leq 0$. They use their numerical algorithm to study soliton interactions. In [3], Antonopoulos and Dougalis provide error estimates using the semidiscrete and discrete Galerkin-finite element method for the boundary value problems for the classical Boussinesq system.

Regarding the two dimensional case, [23], Dougalis, Mitsotakis and Saut study the space semi-discretization using Galerkin methods. In [19], Chen, using a formal second-order semi-implicit Crank-Nicolson scheme along with spectral method studies the $2D$ case of the BBM-BBM system for 3D water waves over an uneven bottom.

A very recent result of Bona and Chen [8] provides numerical evidence of finite time blow-up for the BBM-BBM system. The blow-up phenomena seems to occur on head-on collision of some particular travelling waves solutions of the BBM system.

1.1 Statement of the main results

In this paper, attention is given to the numerical analysis for some of the *abcd* systems for the 1D case. We implement finite volume schemes, and we prove convergence and stability estimates, using a discrete variant of the "natural" energy functional associated to these systems. One practical advantage of our approach is that it allows us to treat explicitly the nonlinear terms.

Let us recall that the Cauchy problem associated with the one dimensional *abcd*-systems reads:

$$\begin{cases} \left(I - b\partial_{xx}^2\right)\partial_t \eta + \left(I + a\partial_{xx}^2\right)\partial_x u + \partial_x\left(\eta u\right) = 0, \\ \left(I - d\partial_{xx}^2\right)\partial_t u + \left(I + c\partial_{xx}^2\right)\partial_x \eta + \frac{1}{2}\partial_x u^2 = 0, \\ \eta_{|t=0} = \eta_0,\ u_{|t=0} = u_0. \end{cases} \tag{S_{abcd}}$$

We will treat the case where the parameters verify:

$$a \leq 0,\ c \leq 0,\ b \geq 0,\ d \geq 0, \tag{1.5}$$

excluding the five cases:

$$\begin{cases} a = b = 0,\ d > 0,\ c < 0, \\ a = b = c = d = 0^1, \\ a = d = 0,\ b > 0,\ c < 0, \\ a = b = d = 0,\ c < 0, \\ b = d = 0,\ c < 0,\ a < 0. \end{cases} \tag{1.6}$$

Given $s \in \mathbb{R}$ we will consider the following set of indices:

$$\begin{cases} s_{bc} = s + \operatorname{sgn}(b) - \operatorname{sgn}(c), \\ s_{ad} = s + \operatorname{sgn}(d) - \operatorname{sgn}(a), \end{cases} \tag{1.7}$$

where the sign function sgn is given by:

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if} \quad x > 0, \\ 0 & \text{if} \quad x = 0, \\ -1 & \text{if} \quad x < 0. \end{cases}$$

We will consider the set of indices defined by (1.7). The notation, $\mathcal{C}_T(H^{s_{bc}} \times H^{s_{ad}})$ stands for the space of continuous functions (η, u) on $[0, T]$ with values in the space $H^{s_{bc}}(\mathbb{R}) \times H^{s_{ad}}(\mathbb{R})$. Let us recall in the following lines, the existence result of regular solution that can be found for instance in [15] and [28]:

Theorem 1.1. *Consider $a, c \leq 0$ and $b, d \geq 0$ excluding the cases (1.6). Also, consider an integer s such that*

$$s > \frac{5}{2} - \operatorname{sgn}(b + d),$$

and s_{bc}, s_{ad} defined by (1.7). Let us consider $(\eta_0, u_0) \in H^{s_{bc}}(\mathbb{R}) \times H^{s_{ad}}(\mathbb{R})$. Then, there exists a positive time T and a unique solution

$$(\eta, u) \in \mathcal{C}_T(H^{s_{bc}} \times H^{s_{ad}})$$

of (S_{abcd}).

Remark 1.1. *Note that energy $\mathcal{E}_s$ is an energy for the system (S_{abcd}), see for exemple [15],*

$$\mathcal{E}_s(\eta, u) = ||\eta||^2_{H^s} + (b - c)||\partial_x \eta||^2_{H^s} + (-c)b||\partial_x^2 \eta||^2_{H^s} + ||u||^2_{H^s} + (d - a)||\partial_x u||^2_{H^s} + (-a)d||\partial_x^2 u||^2_{H^s}.$$

Consider Δt and Δx two positive real numbers. We endow $\ell^2(\mathbb{Z})$ with the following scalar product and norm

$$\langle v, w \rangle := \Delta x \sum_{j \in \mathbb{Z}} v_j w_j,\ \|v\|_{\ell^2_\Delta} := \langle v, v \rangle^{\frac{1}{2}}.$$

We sometimes denote this space $\ell^2_\Delta(\mathbb{Z})$.
For all $v = (v_j)_{j \in \mathbb{Z}} \in \ell^\infty(\mathbb{Z})$ we introduce the spatial shift operators:

$$(S_\pm v)_j := v_{j \pm 1},$$

If $v = (v_j)_{j \in \mathbb{Z}} \in \ell^\infty(\mathbb{Z})$ we denote by $D_+ v$, $D_- v$, Dv the discrete derivation operators:

$$\begin{cases} D_+ v = \frac{1}{\Delta x}(S_+ v - v), \\ D_- v = \frac{1}{\Delta x}(v - S_- v), \\ Dv = \frac{1}{2}(D_+ v + D_- v) = \frac{1}{2\Delta x}(S_+ v - S_- v). \end{cases}$$

[1] This case corresponds to a full hyperbolic system, then there exists a local in time solution. An example of discrete entropy for hyperbolic system are studied in [31].

Also, we consider the following discrete energy functional

$$\mathcal{E}(e,f) \overset{def.}{=} \|e\|^2_{\ell^2_\Delta} + (b-c)\|D_+e\|^2_{\ell^2_\Delta} + b(-c)\|D_+D_-e\|^2_{\ell^2_\Delta} + \|f\|^2_{\ell^2_\Delta} + (d-a)\|D_+f\|^2_{\ell^2_\Delta} + d(-a)\|D_+D_-f\|^2_{\ell^2_\Delta}. \tag{1.8}$$

We will now discuss the main results of the paper. From an initial datum $(\eta_0, u_0) \in H^{s_{bc}}(\mathbb{R}) \times H^{s_{ad}}(\mathbb{R})$ with s large enough, there exists a solution (η, u) of (S_{abcd}) which enables to define $\eta^n_\Delta, u^n_\Delta \in \ell^2_\Delta(\mathbb{Z})$ given by:

$$\begin{cases} \eta^n_{\Delta j} = \frac{1}{[\inf(t^{n+1},T)-t^n]\Delta x}\int_{t_n}^{\inf(t^{n+1},T)}\int_{x_j}^{x_{j+1}} \eta(s,y)\,dyds, \\ \\ u^n_{\Delta j} = \frac{1}{[\inf(t^{n+1},T)-t^n]\Delta x}\int_{t_n}^{\inf(t^{n+1},T)}\int_{x_j}^{x_{j+1}} u(s,y)\,dyds, \end{cases} \quad \text{for all } j \in \mathbb{Z}, \text{ and for } n \geq 1. \tag{1.9}$$

Moreover, we define

$$\begin{cases} \eta^0_{\Delta j} = \frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} \eta_0(y)\,dy, \\ \\ u^0_{\Delta j} = \frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} u_0(y)\,dy, \end{cases} \quad \text{for all } j \in \mathbb{Z}. \tag{1.10}$$

The results of the paper are gathered in two cases according to the values of the parameters b and d in (S_{abcd}).

The case when $b > 0$ and $d > 0$: The fact that $b, d > 0$ assures an ℓ^2-control on the discrete derivatives: this makes it possible to implement an energy method that mimics the one from the continuous case. Moreover, it allows us to close the estimates even without considering numerical viscosity.

We consider the following numerical scheme:

$$\begin{cases} \frac{1}{\Delta t}(I - bD_+D_-)(\eta^{n+1} - \eta^n) + (I + aD_+D_-)D\left((1-\theta)u^n + \theta u^{n+1}\right) + D(\eta^n u^n) = 0, \\ \\ \frac{1}{\Delta t}(I - dD_+D_-)(u^{n+1} - u^n) + (I + cD_+D_-)D\left((1-\theta)\eta^n + \theta\eta^{n+1}\right) + \frac{1}{2}D\left((u^n)^2\right) = 0, \end{cases} \tag{1.11}$$

for all $n \geq 0$ and $\theta = \frac{1}{2}$ or $\theta = 1$, with the discrete initial datum

$$\left(\eta^0, u^0\right) = \left(\eta^0_\Delta, u^0_\Delta\right). \tag{1.12}$$

The convergence error is defined as:

$$e^n = \eta^n - \eta^n_\Delta \quad \text{and} \quad f^n = u^n - u^n_\Delta. \tag{1.13}$$

Remark 1.2. *For $\theta = \frac{1}{2}$, we consider the Crank-Nicolson time discretization whereas for $\theta = 1$, we consider the implicit discretization.*

We are now in the position of stating our first result.

Theorem 1.2. *Let $a \leq 0$, $b > 0$, $c \leq 0$ and $d > 0$. Consider $s > 6$, $(\eta_0, u_0) \in H^{s_{bc}}(\mathbb{R}) \times H^{s_{ad}}(\mathbb{R})$ and $T > 0$ such that $(\eta, u) \in \mathcal{C}_T(H^{s_{bc}} \times H^{s_{ad}})$ is the solution on $[0,T]$ of the system (S_{abcd}). Let $N \in \mathbb{N}$, there exists a positive constant δ_0 depending on $\sup_{t\in[0,T]} \|(\eta(t), u(t))\|_{H^{s_{bc}} \times H^{s_{ad}}}$ such that if the number of time steps N and the space discretization step Δx are chosen such that*

$$\Delta t = T/N \leq \delta_0 \text{ and } \Delta x \leq \delta_0,$$

if we consider the numerical scheme (1.11) with $\theta = \frac{1}{2}$ or $\theta = 1$ along with the initial data (1.12) as well as the approximation $(\eta^n_\Delta, u^n_\Delta)_{n\in\overline{1,N}}$, defined by (1.9)-(1.10), where $\overline{1,N} = \{1, ...N\}$, then, the numerical scheme (1.11) is first order convergent in space and time (or second order in time for $\theta = \frac{1}{2}$) i.e. the convergence error defined in (1.13) satisfies:

$$\sup_{n\in\overline{0,N}} \mathcal{E}(e^n, f^n) \leq C_{abcd}\left\{\left|\theta - \frac{1}{2}\right|(\Delta t)^2 + (\Delta t)^4 + (\Delta x)^4\right\}, \tag{1.14}$$

where C_{abcd} depends on the parameters a,b,c,d, on $\sup_{t\in[0,T]} \|(\eta(t), u(t))\|_{H^{s_{bc}} \times H^{s_{ad}}}$ and on T.

Remark 1.3. *Note that no Courant-Friedrichs-Lewy-type condition (CFL-type condition hereafter) is needed. This is due to the regularity property of both terms* $(I - bD_+D_-)^{-1}$ *and* $(I - dD_+D_-)^{-1}$.

These two terms are also regularizing enough for the center scheme on the hyperbolic part to be stable.

Remark 1.4. *If we consider the Cauchy problem with small parameter* ϵ*:*

$$\begin{cases} \left(I - b\epsilon\partial_{xx}^2\right)\partial_t\eta + \left(I + a\epsilon\partial_{xx}^2\right)\partial_x u + \epsilon\partial_x(\eta u) = 0, \\ \left(I - d\epsilon\partial_{xx}^2\right)\partial_t u + \left(I + c\epsilon\partial_{xx}^2\right)\partial_x\eta + \frac{\epsilon}{2}\partial_x u^2 = 0, \\ \eta_{|t=0} = \eta_0,\ u_{|t=0} = u_0. \end{cases}$$

The previous scheme is transformed into

$$\begin{cases} \frac{1}{\Delta t}\left(I - b\epsilon D_+D_-\right)\left(\eta^{n+1} - \eta^n\right) + \left(I + a\epsilon D_+D_-\right)D\left((1-\theta)u^n + \theta u^{n+1}\right) + \epsilon D\left(\eta^n u^n\right) = 0, \\ \\ \frac{1}{\Delta t}\left(I - d\epsilon D_+D_-\right)\left(u^{n+1} - u^n\right) + \left(I + c\epsilon D_+D_-\right)D\left((1-\theta)\eta^n + \theta\eta^{n+1}\right) + \frac{\epsilon}{2}D\left(\left(u^n\right)^2\right) = 0, \end{cases}$$

for all $n \geq 0$ *and* $\theta = \frac{1}{2}$ *or* $\theta = 1$*. The conditions on time step and space discretization in Theorem 1.2 become then* $\frac{\Delta t}{\epsilon} \leq \delta_0$ *and* $\frac{\Delta x}{\epsilon} \leq \delta_0$ *and the order of convergence writes*

$$\sup_{n\in\overline{0,N}} \mathcal{E}\left(e^n, f^n\right) \leq C_{abcd}\left\{\left|\theta - \frac{1}{2}\right|\left(\frac{\Delta t}{\epsilon}\right)^2 + \left(\frac{\Delta t}{\epsilon}\right)^4 + \left(\frac{\Delta x}{\epsilon}\right)^4\right\},$$

which imposes the strong condition $\Delta x, \Delta t \leq \epsilon$ *and prevents the parameter* ϵ *to be too small.*

To avoid this strong restriction, we need to design a scheme with a uniformly rate of convergence with respect to ϵ *and in particular, we need to design an* asymptotic preserving *schemes (for which the stability requirement on* Δt *and* Δx *is independent of* ϵ*). In such a scheme, both limits* $\epsilon \to 0$ *and* $\Delta x, \Delta t \to 0$ *may be commuted without loss of generality for the accuracy of the scheme. Moreover, in such a scheme, the limit scheme when* $\epsilon \to 0$ *is consistent with the limit continuous system when* $\epsilon \to 0$*. In the case studied in this paper, the limit system when* $\epsilon \to 0$ *corresponds to the acoustic wave equation. The asymptotic scheme has to be consistent when* $\epsilon \to 0$ *with this acoustic equation.*
However, designing an asymptotic preserving scheme is not the aim of this paper and we will focus only on System (S_{abcd}) *with* $\epsilon = 1$

The case when $bd = 0$ We consider the following numerical scheme:

$$\begin{cases} \frac{1}{\Delta t}\left(I - bD_+D_-\right)\left(\eta^{n+1} - \eta^n\right) + \left(I + aD_+D_-\right)D\left(u^{n+1}\right) + D\left(\eta^n u^n\right) \\ = \frac{1}{2}\left(1 - \operatorname{sgn}(b)\right)\tau_1\Delta x D_+D_-\left(\eta^n\right), \\ \\ \frac{1}{\Delta t}\left(I - dD_+D_-\right)\left(u^{n+1} - u^n\right) + \left(I + cD_+D_-\right)D\left(\eta^{n+1}\right) + \frac{1}{2}D\left(\left(u^n\right)^2\right) \\ = \frac{1}{2}\left(1 - \operatorname{sgn}(d)\right)\tau_2\Delta x D_+D_-\left(u^n\right), \end{cases} \tag{1.15}$$

for all $n \geq 0$, with

$$\left(\eta^0, u^0\right) = \left(\eta_\Delta^0, u_\Delta^0\right) \tag{1.16}$$

Remark 1.5. *For the case* $bd = 0$*, we consider only the implicite time discretization.*

The convergence error is defined by (1.13). We are now in the position of stating our second main result.

Theorem 1.3. *Let* $a \leq 0$*,* $b \geq 0$*,* $c \leq 0$ *and* $d \geq 0$ *with* $bd = 0$*, excluding the cases* (1.6)*. Consider* $s > 8$*,* $(\eta_0, u_0) \in H^{s_{bc}}(\mathbb{R}) \times H^{s_{ad}}(\mathbb{R})$ *and* $T > 0$ *such that* $(\eta, u) \in \mathcal{C}_T(H^{s_{bc}} \times H^{s_{ad}})$ *is the solution on* $[0, T]$ *of the system* (S_{abcd})*.*

Choose $\alpha > 0, \tau_1 > 0$ *and* $\tau_2 > 0$ *such that:*

$$\|u\|_{L_t^\infty L_x^\infty} + \alpha \leq \tau_1 \text{ and } \|u\|_{L_t^\infty L_x^\infty} + \alpha \leq \tau_2.$$

There exists $\delta_0 > 0$ (depending on $\alpha, \tau_1,\ \tau_2$ and on $\sup_{t\in[0,T]} \|(\eta(t),u(t))\|_{H^{s_{bc}}\times H^{s_{ad}}}$) such that if the number of time steps $N \in \mathbb{N}$ and the space discretization step are chosen in order to verify

$$\Delta t = T/N \leq \delta_0,\ \Delta x \leq \delta_0,$$

and

$$\max\{(1-\mathrm{sgn}(b))\,\tau_1, (1-\mathrm{sgn}(d))\,\tau_2\}\Delta t \leq \Delta x, \tag{1.17}$$

if we consider the numerical scheme (1.15) *along with the initial data* (1.16) *with the numerical viscosities τ_1 and τ_2 as well as the approximation $(\eta^n_\Delta, u^n_\Delta)_{n\in\overline{1,N}}$ defined by* (1.9)*, then, the numerical scheme* (1.15) *is first order convergent i.e. the convergence error defined in* (1.13) *satisfies:*

$$\sup_{n\in\overline{0,N}} \mathcal{E}\left(e^n, f^n\right) \leq C_{abcd}\left(\Delta x\right)^2,$$

where C_{abcd} depends on the parameters a,b,c,d, on $\sup_{t\in[0,T]} \|(\eta(t),u(t))\|_{H^{s_{bc}}\times H^{s_{ad}}}$ and on T.

Remark 1.6. *In this case, one of the two equations of* (1.15) *does not contain the operator $(I - bD_+D_-)^{-1}$ or $(I - dD_+D_-)^{-1}$. A discrete diffusion together with a hyperbolic CFL-type condition are thus needed to stabilize the hyperbolic part of the equations. We then choose a center scheme combined with a Rusanov diffusion and the CFL-type condition corresponds to Relation* (1.17)*.*

Remark 1.7. *We could choose the viscosities $\tau_i = \tau_i^n$ depending on n such that*

$$||u^n_\Delta||_{\ell^\infty} + \alpha \leq \tau_i^n \quad with \quad \tau_i^n - \alpha > \|u\|_{L^\infty_t L^\infty_x}.$$

Remark 1.8. *As mentioned in Remark 1.4, Scheme* (1.15) *is not suitable if we consider System (S_{abcd}) with the small parameter ϵ. Indeed, the obtained convergence order is*

$$\sup_{n\in\overline{0,N}} \mathcal{E}\left(e^n, f^n\right) \leq C_{abcd}\left(\frac{\Delta x}{\epsilon}\right)^2,$$

provided $\frac{\Delta x}{\epsilon} \leq \delta_0$ and $\frac{\Delta t}{\epsilon} \leq \delta_0$. Once again, we have to use an asymptotic scheme to recover a rate of convergence which is uniform in ϵ.

Remark 1.9. *In the first case, $b > 0$ and $d > 0$, the discrete energy $\mathcal{E}$ corresponds to a H^2-discrete norm on both u and η, if a and c are non zero. In the case where $b = 0$ or $d = 0$, it corresponds to a H^1-discrete norm and possibly even a L^2-discrete norm according to the value of parameters a and c.*

Our results owes much to the technique developed by Courtès, Lagoutière and Rousset in [21]. Let us give some more details. For the Korteweg-de-Vries equation

$$\partial_t v + v\partial_x v + \partial^3_{xxx} v = 0$$

they employ the following θ-scheme, with $\theta \in [0,1]$

$$\frac{1}{\Delta t}\left(v^{n+1} - v^n\right) + D_+D_+D_-((1-\theta)v^n + \theta v^{n+1}) + D\left(\frac{(v^n)^2}{2}\right) = \frac{\tau\Delta x}{2} D_+D_-v^n.$$

With the aim to study the order of convergence, they consider the finite volume discrete operators:

$$(v_\Delta)^n_j = \frac{1}{[\inf(t^{n+1},T) - t^n]\Delta x}\int_{t^n}^{\inf(t^{n+1},T)}\int_{x_j}^{x_{j+1}} v\,(s,y)\,dsdy$$

and the convergence error

$$e^n = v^n - v^n_\Delta,$$

which obeys the following equation:

$$e^{n+1} + \theta \Delta t D_+ D_+ D_- (e^{n+1}) = \mathscr{B} e^n,$$

with

$$\mathscr{B} e^n = e^n - (1-\theta)\,\Delta t D_+ D_+ D_- (e^n) - \Delta t D\left(e^n v_\Delta^n\right) - \Delta t D\left(\frac{(e^n)^2}{2}\right) + \frac{\tau \Delta x \Delta t}{2} D_+ D_- e^n - \Delta t \epsilon^n,$$

where ϵ^n is the consistency error. In order to establish ℓ^2_Δ-estimates, they show that under the CFL condition

$$\begin{cases} \tau > \|v_\Delta^n\|_{\ell^\infty}, \\ \left[\tau + \frac{1}{2}\right] \Delta t < \Delta x, \\ (1-2\theta)\,\Delta t < \frac{(\Delta x)^3}{4}, \end{cases}$$

the following holds true:

$$\left\| e^{n+1} + \theta \Delta t D_+ D_+ D_- (e^{n+1}) \right\|^2_{\ell^2_\Delta} \leq \Delta t C \left\| \epsilon^n \right\|^2_{\ell^2_\Delta} + (1 + C\Delta t) \left\| e^n + \theta \Delta t D_+ D_+ D_- (e^n) \right\|^2_{\ell^2_\Delta}, \tag{1.18}$$

provided that $\|e^n\|_{\ell^\infty}$ is sufficiently small. Thus, they are able to use the discrete Gronwall lemma and loop their estimates. The proof of (1.18) is rather technical and tricky. Essentially, using some clever identities when computing $\|\mathscr{B} e^n\|^2_{\ell^2_\Delta}$ they get several negative terms wich, under the above CFL and smallness conditions, are used to balance the "bad" terms.

In Section 2.2.1 we study a discrete operator which appears in hyperbolic systems and we provide a bound on its ℓ^2_Δ norm, the proof being largely inspired from [21]. As a consequence, we establish a higher order estimate which proves crucial in the analysis of some of the *abcd* systems. Among the systems in view, we distinguish three situations:

- when $bd > 0$, establishing energy estimates for the convergence error can be done by imitating the approach from the continuous case. The structure of the equations provides enough control such that we do not need to impose numerical viscosity or CFL-type conditions.
- when at least one of the weakly dispersive operators does not appear, we have to work only with estimations established in Section 2.2.1 (see for instance the case $a < 0$,$b = c = d = 0$) either
- combine the techniques of Section 2.2.1 with "continuous-type" estimates like those established for the case when $bd > 0$. (see for instance the case $a = b = c = 0$, $d > 0$)

In order to validate the theoretical results regarding the order of convergence and reproduce the numerical experiences we compare the numerical solutions with the exact travelling wave solutions established by Chen in [17] and [7].

Finally, we use our results in order to study exact traveling waves interactions. We perform two such numerical experiments. Recently, in [8] Bona and Chen pointed out that finite time blow-up seems to occur at the head-on collision of the two exact solutions:

$$\begin{cases} \eta_\pm(t,x) = \frac{15}{2}\,\mathrm{sech}^2\left(\frac{3}{\sqrt{10}}\left(x - x_0 \mp \frac{5}{2}t\right)\right) - \frac{45}{4}\,\mathrm{sech}^4\left(\frac{3}{\sqrt{10}}\left(x - x_0 \mp \frac{5}{2}t\right)\right), \\ \\ u_\pm(t,x) = \pm\frac{15}{2}\,\mathrm{sech}^2\left(\frac{3}{\sqrt{10}}(x - x_0 \mp \frac{5}{2}t)\right). \end{cases}$$

In order to validate our codes, we repeated their experiment and roughly obtained similar results. We performed a second experiment by changing parameters a, b, c, d and do not observe any blow up.

The plan of the rest of this chapter is the following. In the next section we give a list of the main notations and identities that we use all along in this manuscript. Section 2.1 is devoted to the proof of Theorem 1.2. The proof of Theorem 1.3 is more involved as we cannot treat in a uniformly manner all the cases when $bd = 0$. First, we establish in Section 2.2.1 some estimates for discrete operators appearing in hyperbolic systems. The rest of Section 2.2 is dedicated to the proof corresponding to the different values of the *abcd* parameters appearing in Theorem 1.3. Finally, in Sections 3 and 4, we present our numerical simulations.

1.2 Notations

In the following we present the main notations that will be used throught the rest of the paper. Let us fix $\Delta x > 0$. For all $v = (v_j)_{j\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z})$ we introduce the spatial shift operators:

$$(S_\pm v)_j := v_{j\pm 1},$$

For $v = (v_j)_{j\in\mathbb{Z}}$, $w = (w_j)_{j\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ we define the product operator:

$$vw = (v_j w_j)_{j\in\mathbb{Z}} \in \ell^1(\mathbb{Z}).$$

Also, we denote by

$$v^2 = vv.$$

We list below some basic formulas, whose proofs can be found in [21].
The following identities describe the derivation law of a product, for $v, w \in \ell^2(\mathbb{Z})$

$$D(vw) = vDw + \frac{1}{2}(S_+ wD_+ v + S_- wD_- v), \tag{1.19}$$

$$D(vw) = S_+ vDw + DvS_- w, \tag{1.20}$$

$$D_+(vw) = S_+ vD_+ w + wD_+ v. \tag{1.21}$$

We observe that we dispose of the following basic integration by parts rules, for $w, v \in \ell^2_\Delta(\mathbb{Z})$:

$$\langle D_+ v, w\rangle = -\langle v, D_- w\rangle, \tag{1.22}$$

$$\langle Dv, w\rangle = -\langle v, Dw\rangle. \tag{1.23}$$

In particular, we see that, for $v \in \ell^2_\Delta(\mathbb{Z})$:

$$\langle Dv, v\rangle = 0,$$

and

$$\langle v, D_+ v\rangle = -\frac{\Delta x}{2}||D_+ v||^2_{\ell^2_\Delta}.$$

More elaborate integration by parts identities are the following, for $w, v \in \ell^2_\Delta(\mathbb{Z})$

$$\langle v, D(vw)\rangle = \frac{1}{2}\langle D_+ w, vS_+ v\rangle \tag{1.24}$$

respectively

$$\langle D_+ D_- v, D(vw)\rangle = -\frac{1}{\Delta x^2}\langle D_+ w, vS_+ v\rangle + \frac{1}{\Delta x^2}\langle Dw, S_- vS_+ v\rangle. \tag{1.25}$$

When $v = w \in \ell^2_\Delta(\mathbb{Z})$ in the previous equation, some additional simplifications occur and one has

$$\left\langle D_+ D_- v, D\left(v^2\right)\right\rangle = \frac{1}{3}\left\langle D_+ v, (D_+ v)^2\right\rangle - \frac{4}{3}\left\langle Dv, (Dv)^2\right\rangle. \tag{1.26}$$

Also, it holds true that, for $v \in \ell^2_\Delta(\mathbb{Z})$

$$\left\langle v, D\left(v^2\right)\right\rangle = -\frac{\Delta x^2}{6}\left\langle D_+ v, (D_+ v)^2\right\rangle, \tag{1.27}$$

$$||D(v^2)||^2_{\ell^2_\Delta} = \left\langle (Dv)^2, \left(\frac{S_+ v + S_- v}{2}\right)^2\right\rangle \tag{1.28}$$

and

$$||D_+ D_- v||^2_{\ell^2_\Delta} = \frac{4}{\Delta x^2}||D_+ v||^2_{\ell^2_\Delta} - \frac{4}{\Delta x^2}||Dv||^2_{\ell^2_\Delta}. \tag{1.29}$$

Finally, for $v, w \in \ell^2_\Delta(\mathbb{Z})$, we recall Lemma 5 of [21]

$$||D(vw)||^2_{\ell^2_\Delta} \leq \left\{||w||^2_{\ell^\infty} + \Delta t||D_+w||^2_{\ell^\infty}\right\}||Dv||^2_{\ell^2_\Delta} + \left\{\frac{1}{\Delta t}||w||^2_{\ell^\infty} + \frac{3}{4}||D_+w||^2_{\ell^\infty}\right\}||v||^2_{\ell^2_\Delta} \tag{1.30}$$

and Lemma 8 of [21]

$$\left\langle D(vw), D(v^2)\right\rangle \leq 2||v||_{\ell^\infty}||w||_{\ell^\infty}||Dv||^2_{\ell^2_\Delta} - \frac{8\Delta x^2}{3}\left\langle Dw, (Dv)^3\right\rangle - \frac{2}{3}\left\langle DDw, v^3\right\rangle. \tag{1.31}$$

In the following we will frequently use the notations:

$$\begin{cases} e^n = \eta^n - \eta^n_\Delta, & E^\pm(n) = e^{n+1} \pm e^n, \\ f^n = u^n - u^n_\Delta, & F^\pm(n) = f^{n+1} \pm f^n. \end{cases} \tag{1.32}$$

We end this section with the following discrete version of Gronwall's lemma, a proof of which can be found in [20]:

Lemma 1.1. *Let $v = (v^n)_{n\in\mathbb{N}}$, $a = (a^n)_{n\in\mathbb{N}}$, $b = (b^n)_{j\in\mathbb{N}}$ be sequences of real numbers with $b^n \geq 0$ for all $n \in \mathbb{N}$ which satisfies*

$$v^n \leq a^n + \sum_{j=0}^{n-1} b^j v^j,$$

for all $n \in \mathbb{N}$. Then, for any $n \in \mathbb{N}$, we have:

$$v^n \leq \max_{k\in\overline{1,n}} a^k \prod_{j=0}^{n-1}\left(1 + b^j\right).$$

2 The proof of the main results

2.1 The proof of Theorem 1.2

Let us recall the energy functional:

$$\begin{aligned}\mathcal{E}(e,f) \overset{def.}{=} &\|e\|^2_{\ell^2_\Delta} + (b-c)\|D_+e\|^2_{\ell^2_\Delta} + b(-c)\|D_+D_-e\|^2_{\ell^2_\Delta} \\ &+ \|f\|^2_{\ell^2_\Delta} + (d-a)\|D_+f\|^2_{\ell^2_\Delta} + d(-a)\|D_+D_-f\|^2_{\ell^2_\Delta}.\end{aligned} \tag{2.1}$$

For all $n \geq 0$, we will consider $(\epsilon^n_1, \epsilon^n_2) \in \left(\ell^2_\Delta(\mathbb{Z})\right)^2$ the consistency error defined as:

$$\begin{cases} \frac{1}{\Delta t}(I - bD_+D_-)(\eta^{n+1}_\Delta - \eta^n_\Delta) + \frac{1}{2}(I + aD_+D_-)D\left(u^n_\Delta + u^{n+1}_\Delta\right) + D\left(\eta^n_\Delta u^n_\Delta\right) = \epsilon^n_1, \\ \frac{1}{\Delta t}(I - dD_+D_-)(u^{n+1}_\Delta - u^n_\Delta) + \frac{1}{2}(I + cD_+D_-)D\left(\eta^n_\Delta + \eta^{n+1}_\Delta\right) + \frac{1}{2}D\left(\left(u^n_\Delta\right)^2\right) = \epsilon^n_2. \end{cases} \tag{2.2}$$

Proposition 2.1. *For all $n \in \overline{0, N-1}$, there exist two constants C_1 and C_2 depending on a, b, c, d, on the ℓ^∞-norm of $(\eta^n_\Delta, u^n_\Delta)$ and $(D(\eta^n_\Delta), D(u^n_\Delta))$ and proportional to $\max\{||e^n||_{\ell^\infty}, ||f^n||_{\ell^\infty}\}$, such that,*

$$\mathcal{E}\left(e^{n+1}, f^{n+1}\right) \leq 2\Delta t\|\epsilon^n\|^2_{\ell^2} + (1 + \Delta tC_1)\mathcal{E}(e^n, f^n) + \Delta tC_2\mathcal{E}\left(e^{n+1}, f^{n+1}\right), \tag{2.3}$$

with (e^n, f^n) the two convergence errors defined by (1.32) and $\|\epsilon^n\|_{\ell^2_\Delta} = \max\left\{\|\epsilon^n_1\|_{\ell^2_\Delta}, \|\epsilon^n_2\|_{\ell^2_\Delta}\right\}$, defined by (2.2).

Proof of Proposition 2.1. As we announced in the introduction, we will establish energy estimates imitating the approach from the continuous case.

For the Crank-Nicolson case ($\theta = \frac{1}{2}$). Using the notations introduced in (1.13) and (1.32) we see that the equations governing the convergence error (e^n, f^n) are the following:

$$\begin{cases} (I - bD_+D_-)(E^-(n)) + \frac{\Delta t}{2}(I + aD_+D_-)D(F^+(n)) + \Delta t D(e^n u_\Delta^n) \\ \qquad\qquad + \Delta t D(\eta_\Delta^n f^n) + \Delta t D(e^n f^n) = -\Delta t \epsilon_1^n, \\ (I - dD_+D_-)(F^-(n)) + \frac{\Delta t}{2}(I + cD_+D_-)D(E^+(n)) + \Delta t D(f^n u_\Delta^n) \\ \qquad\qquad + \frac{\Delta t}{2} D\left((f^n)^2\right) = -\Delta t \epsilon_2^n. \end{cases} \tag{2.4}$$

Let $n \in \overline{0, N-1}$ and observe that by multiplying the first equation of (2.4) by $(I + cD_+D_-)E^+(n)$, the second by $(I + aD_+D_-)F^+(n)$ and adding up the results, we find that

$$\begin{aligned} &\left\langle (I - bD_+D_-)(E^-(n)), (I + cD_+D_-)E^+(n) \right\rangle + \frac{\Delta t}{2}\left\langle (I + aD_+D_-)D(F^+(n)), (I + cD_+D_-)E^+(n) \right\rangle \\ &+ \left\langle (I - dD_+D_-)(F^-(n)), (I + aD_+D_-)F^+(n) \right\rangle + \frac{\Delta t}{2}\left\langle (I + cD_+D_-)D(E^+(n)), (I + aD_+D_-)F^+(n) \right\rangle \\ &= -\Delta t \left\langle \epsilon_1^n, (I + cD_+D_-)E^+(n) \right\rangle - \Delta t \left\langle \epsilon_2^n, (I + aD_+D_-)F^+(n) \right\rangle \\ &- \Delta t \left\langle D(e^n u_\Delta^n) + D(\eta_\Delta^n f^n), (I + cD_+D_-)E^+(n) \right\rangle \\ &- \Delta t \left\langle D(e^n f^n), (I + cD_+D_-)E^+(n) \right\rangle \\ &- \Delta t \left\langle D(f^n u_\Delta^n), (I + aD_+D_-)F^+(n) \right\rangle \\ &- \frac{\Delta t}{2}\left\langle D\left((f^n)^2\right), (I + aD_+D_-)F^+(n) \right\rangle \overset{not.}{=} \sum_{i=1}^{5} T_i. \end{aligned} \tag{2.5}$$

We begin by treating the left hand side of (2.5). Notice that

$$\left\langle \mathcal{L}\left(E^-(n)\right), \mathcal{L}\left(E^+(n)\right) \right\rangle = ||\mathcal{L}e^{n+1}||_{\ell^2_\Delta}^2 - ||\mathcal{L}e^n||_{\ell^2_\Delta}^2.$$

with $\mathcal{L}$ any linear operator. With this in mind together with Relations (1.22) and (1.23), it gives,

$$\begin{aligned} &\left\langle (I - bD_+D_-)(E^-(n)), (I + cD_+D_-)E^+(n) \right\rangle + \frac{\Delta t}{2}\left\langle (I + aD_+D_-)D(F^+(n)), (I + cD_+D_-)E^+(n) \right\rangle \\ &+ \left\langle (I - dD_+D_-)(F^-(n)), (I + aD_+D_-)F^+(n) \right\rangle + \frac{\Delta t}{2}\left\langle (I + cD_+D_-)D(E^+(n)), (I + aD_+D_-)F^+(n) \right\rangle \\ &= \mathcal{E}(e^{n+1}, f^{n+1}) - \mathcal{E}(e^n, f^n). \end{aligned} \tag{2.6}$$

• Let us now focus on T_1. We recall that

$$\|\epsilon^n\|_{\ell^2_\Delta} = \max\left\{ \|\epsilon_1^n\|_{\ell^2_\Delta}, \|\epsilon_2^n\|_{\ell^2_\Delta} \right\},$$

and we write that, thanks to Cauchy-Schwarz inequality (we recall that $c \leq 0$ and $a \leq 0$)

$$\begin{aligned} -\Delta t \left\langle \epsilon_1^n, (I + cD_+D_-)E^+(n) \right\rangle - \Delta t \left\langle \epsilon_2^n, (I + aD_+D_-)F^+(n) \right\rangle \leq &\Delta t \|\epsilon_1^n\|_{\ell^2_\Delta} \left(\left\|E^+(n)\right\|_{\ell^2_\Delta} - c \left\|D_+D_-E^+(n)\right\|_{\ell^2_\Delta} \right) \\ &+ \Delta t \|\epsilon_2^n\|_{\ell^2_\Delta} \left(\left\|F^+(n)\right\|_{\ell^2_\Delta} - a \left\|D_+D_-F^+(n)\right\|_{\ell^2_\Delta} \right). \end{aligned}$$

By applied Young inequality, we recover the ℓ^2_Δ-norm of e^n and f^n.

$$\begin{aligned} &- \Delta t \left\langle \epsilon_1^n, (I + cD_+D_-)E^+(n) \right\rangle - \Delta t \left\langle \epsilon_2^n, (I + aD_+D_-)F^+(n) \right\rangle \\ &\leq 2\Delta t \|\epsilon\|_{\ell^2_\Delta}^2 + \Delta t \left(\|e^n\|_{\ell^2_\Delta}^2 + \|f^n\|_{\ell^2_\Delta}^2 + c^2 ||D_+D_-e^n||_{\ell^2_\Delta}^2 + a^2 ||D_+D_-f^n||_{\ell^2_\Delta}^2 \right) \\ &+ \Delta t \left(\left\|e^{n+1}\right\|_{\ell^2_\Delta}^2 + \left\|f^{n+1}\right\|_{\ell^2_\Delta}^2 + c^2 ||D_+D_-e^{n+1}||_{\ell^2_\Delta}^2 + a^2 ||D_+D_-f^{n+1}||_{\ell^2_\Delta}^2 \right) \\ &\leq 2\Delta t \|\epsilon^n\|_{\ell^2_\Delta}^2 + \Delta t \max\left\{ 1, \frac{-c}{b}, \frac{-a}{d} \right\} \mathcal{E}(e^n, f^n) + \Delta t \max\left\{ 1, \frac{-c}{b}, \frac{-a}{d} \right\} \mathcal{E}\left(e^{n+1}, f^{n+1}\right). \end{aligned} \tag{2.7}$$

• Let us treat T_2. Using (1.20), we first write

$$\begin{aligned}\Delta t \left\| D\left(e^n u_\Delta^n\right)\right\|_{\ell^2_\Delta} &= \Delta t \left\| S_- e^n D u_\Delta^n + S_+ u_\Delta^n D e^n\right\|_{\ell^2_\Delta} \\ &\leq \Delta t \left\| D u_\Delta^n\right\|_{\ell^\infty} \left\| S_- e^n\right\|_{\ell^2_\Delta} + \Delta t \left\| S_+ u_\Delta^n\right\|_{\ell^\infty} \left\| D e^n\right\|_{\ell^2_\Delta} \\ &\leq \Delta t \max\left\{1, \frac{1}{\sqrt{b-c}}\right\}\left(\left\| D u_\Delta^n\right\|_{\ell^\infty} \left\| e^n\right\|_{\ell^2_\Delta} + \left\| u_\Delta^n\right\|_{\ell^\infty} \sqrt{b-c}\left\| D e^n\right\|_{\ell^2_\Delta}\right) \\ &\leq \Delta t \max\left\{\left\| u_\Delta^n\right\|_{\ell^\infty}, \left\| D u_\Delta^n\right\|_{\ell^\infty}\right\} \max\left\{1, \frac{1}{\sqrt{b-c}}\right\}\left(\left\| e^n\right\|_{\ell^2_\Delta} + \sqrt{b-c}\left\| D e^n\right\|_{\ell^2_\Delta}\right).\end{aligned}$$

Proceeding in a similar fashion with the other terms we arrive, thanks to the Cauchy-Schwarz inequality, at (we recall that $c \leq 0$)

$$\begin{aligned}&-\Delta t \left\langle D\left(e^n u_\Delta^n\right) + D\left(\eta_\Delta^n f^n\right), \left(I + cD_+D_-\right) E^+(n)\right\rangle \\ &\qquad \leq \Delta t \left(\left\| D\left(e^n u_\Delta^n\right)\right\|_{\ell^2_\Delta} + \left\| D\left(\eta_\Delta^n f^n\right)\right\|_{\ell^2_\Delta}\right)\left(\left\| E^+(n)\right\|_{\ell^2_\Delta} - c\left\| D_+D_-E^+(n)\right\|_{\ell^2_\Delta}\right).\end{aligned}$$

By Definition (1.32) of $E^+(n)$, one has

$$\begin{aligned}\left\| E^+(n)\right\|_{\ell^2_\Delta} - c\left\| D_+D_-E^+(n)\right\|_{\ell^2_\Delta} &\leq ||e^{n+1}||_{\ell^2_\Delta} + ||e^n||_{\ell^2_\Delta} - c||D_+D_-(e^{n+1})||_{\ell^2_\Delta} - c||D_+D_-(e^n)||_{\ell^2_\Delta} \\ &\leq \max\left\{1, \sqrt{-c/b}\right\} 2\left[\sqrt{\mathcal{E}(e^{n+1}, f^{n+1})} + \sqrt{\mathcal{E}(e^n, f^n)}\right].\end{aligned}$$

These together give

$$-\Delta t \left\langle D\left(e^n u_\Delta^n\right) + D\left(\eta_\Delta^n f^n\right), \left(I + cD_+D_-\right) E^+(n)\right\rangle \leq \Delta t C_{1,1}\mathcal{E}\left(e^n, f^n\right) + \Delta t C_{2,1}\mathcal{E}\left(e^{n+1}, f^{n+1}\right) \tag{2.8}$$

where $C_{1,1}$ and $C_{2,1}$ can be written, for example, as a numerical constants multiplied with:

$$\max\left\{1, \sqrt{-\frac{c}{b}}\right\}\left[\max\left\{\left\| u_\Delta^n\right\|_{\ell^\infty}, \left\| D u_\Delta^n\right\|_{\ell^\infty}\right\} \max\left\{1, \frac{1}{\sqrt{b-c}}\right\} + \max\left\{\left\| \eta_\Delta^n\right\|_{\ell^\infty}, \left\| D \eta_\Delta^n\right\|_{\ell^\infty}\right\} \max\left\{1, \frac{1}{\sqrt{d-a}}\right\}\right]. \tag{2.9}$$

• We treat T_4 in same spirit as above in order to obtain that

$$-\Delta t \left\langle D\left(f^n u_\Delta^n\right), \left(I + aD_+D_-\right) F^+(n)\right\rangle \leq \Delta t C_{1,2}\mathcal{E}\left(e^n, f^n\right) + \Delta t C_{2,2}\mathcal{E}\left(e^{n+1}, f^{n+1}\right), \tag{2.10}$$

where $C_{1,2}$ and $C_{2,2}$ are multiples of :

$$\max\left\{\left\| u_\Delta^n\right\|_{\ell^\infty}, \left\| D u_\Delta^n\right\|_{\ell^\infty}\right\} \max\left\{1, 1/\sqrt{d-a}\right\} \max\left\{1, \sqrt{-a/d}\right\}. \tag{2.11}$$

• In order to treat T_3, we first observe that

$$\begin{aligned}\left\| D\left(e^n f^n\right)\right\|_{\ell^2_\Delta} &= \left\| S_- e^n D f^n + S_+ f^n D e^n\right\|_{\ell^2_\Delta} \leq \left\| e^n\right\|_{\ell^\infty} \left\| D f^n\right\|_{\ell^2_\Delta} + \left\| f^n\right\|_{\ell^\infty} \left\| D e^n\right\|_{\ell^2_\Delta} \\ &\leq \max\left\{1/\sqrt{b-c}, 1/\sqrt{d-a}\right\}\left(||e^n||_{\ell^\infty}\sqrt{d-a}\left\| D f^n\right\|_{\ell^2_\Delta} + ||f^n||_{\ell^\infty}\sqrt{b-c}\left\| D e^n\right\|_{\ell^2_\Delta}\right).\end{aligned}$$

Thus, we obtain

$$-\Delta t \left\langle D\left(e^n f^n\right), \left(I + cD_+D_-\right) E^+(n)\right\rangle \leq \Delta t C_{1,3}\mathcal{E}\left(e^n, f^n\right) + \Delta t C_{2,3}\mathcal{E}\left(e^{n+1}, f^{n+1}\right), \tag{2.12}$$

where $C_{1,3}$ respectively $C_{2,3}$ are proportional with

$$\max\left\{1/\sqrt{b-c}, 1/\sqrt{d-a}\right\} \max\left\{1, \sqrt{-c/b}\right\} \max\left\{||e^n||_{\ell^\infty}, ||f^n||_{\ell^\infty}\right\}. \tag{2.13}$$

• The same holds for T_5, namely, we get that

$$-\frac{\Delta t}{2}\left\langle D\left(\left(f^n\right)^2\right), \left(I + aD_+D_-\right) F^+(n)\right\rangle \leq \Delta t C_{1,4}\mathcal{E}\left(e^n, f^n\right) + \Delta t C_{2,4}\mathcal{E}\left(e^{n+1}, f^{n+1}\right), \tag{2.14}$$

where $C_{1,4}$ respectively $C_{2,4}$ are proportional with

$$\frac{1}{\sqrt{d-a}}\max\left\{1,\sqrt{-a/d}\right\}||f^n||_{\ell^\infty}. \tag{2.15}$$

Gathering the informations from (2.5), (2.6), (2.7), (2.8), (2.10), (2.12) and (2.14) we obtain the existence of two constants C_1 and C_2 that depend on a,b,c,d, and the ℓ^∞-norm of $(\eta_\Delta^n,u_\Delta^n)$ and $(D\eta_\Delta^n,Du_\Delta^n)$ (dependence which we can track using relations (2.9), (2.11), (2.13) respectively (2.15)) and that are proportional to $\max\{||e^n||_{\ell^\infty},||f^n||_{\ell^\infty}\}$ such that

$$\mathcal{E}\left(e^{n+1},f^{n+1}\right)-\mathcal{E}\left(e^n,f^n\right)\leq 2\Delta t\left\|\epsilon^n\right\|_{\ell^2_\Delta}^2+\Delta tC_1\mathcal{E}\left(e^n,f^n\right)+\Delta tC_2\mathcal{E}\left(e^{n+1},f^{n+1}\right).$$

For the implicit case ($\theta=1$). The equations governing the convergence error (e^n,f^n) are the following

$$\begin{cases}(I-bD_+D_-)e^{n+1}+\Delta t(I+aD_+D_-)Df^{n+1}=(I-bD_+D_-)e^n-\Delta tD(e^nu_\Delta^n)-\Delta tD(\eta_\Delta^nf^n)\\ \qquad\qquad -\Delta tD(e^nf^n)-\Delta t\epsilon_1^n,\\ (I-dD_+D_-)f^{n+1}+\Delta t(I+cD_+D_-)De^{n+1}=(I-dD_+D_-)f^n-\Delta tD(f^nu_\Delta^n)-\dfrac{\Delta t}{2}D((f^n)^2)-\Delta t\epsilon_2^n.\end{cases} \tag{2.16}$$

For that case, we fix the discret energy

$$\begin{aligned}\mathcal{E}(e,f)&=(-c)d||(I-bD_+D_-)e||_{\ell^2_\Delta}^2+(-a)b||(I-dD_+D_-)f||_{\ell^2_\Delta}^2\\&=(-c)d||e||_{\ell^2_\Delta}^2+2b(-c)d||D_+e||_{\ell^2_\Delta}^2+b^2(-c)d||D_+D_-e||_{\ell^2_\Delta}^2\\&\quad+(-a)b||f||_{\ell^2_\Delta}^2+2(-a)bd||D_+f||_{\ell^2_\Delta}^2+(-a)bd^2||D_+D_-f||_{\ell^2_\Delta}^2.\end{aligned} \tag{2.17}$$

This energy is of course equivalent to the one from (2.1).

Let us multiply the first equation of (2.16) by $\sqrt{-cd}$ and the second one of (2.16) by $\sqrt{-ab}$. The sum of ℓ^2_Δ-norm gives in that case

$$\begin{aligned}&-cd||(I-bD_+D_-)e^{n+1}+\Delta t(I+aD_+D_-)Df^{n+1}||_{\ell^2_\Delta}^2-ab||(I-dD_+D_-)f^{n+1}+\Delta t(I+cD_+D_-)De^{n+1}||_{\ell^2_\Delta}^2\\=&-cd||(I-bD_+D_-)e^n-\Delta tD(e^nu_\Delta^n)-\Delta tD(\eta_\Delta^nf^n)-\Delta tD(e^nf^n)-\Delta t\epsilon_1^n||_{\ell^2_\Delta}^2\\&-ab||(I-dD_+D_-)f^n-\Delta tD(f^nu_\Delta^n)-\frac{\Delta t}{2}D((f^n)^2)-\Delta t\epsilon_2^n||_{\ell^2_\Delta}^2.\end{aligned} \tag{2.18}$$

The left hand side of the previous equality gives

$$\begin{aligned}&-cd||(I-bD_+D_-)e^{n+1}+\Delta t(I+aD_+D_-)Df^{n+1}||_{\ell^2_\Delta}^2-ab||(I-dD_+D_-)f^{n+1}+\Delta t(I+cD_+D_-)De^{n+1}||_{\ell^2_\Delta}^2\\=&-cd||(I-bD_+D_-)e^{n+1}||_{\ell^2_\Delta}^2-2cd\Delta t\left\langle(I-bD_+D_-)e^{n+1},(I+aD_+D_-)Df^{n+1}\right\rangle-cd\Delta t^2||(I+aD_+D_-)Df^{n+1}||_{\ell^2_\Delta}^2\\&-ab||(I-dD_+D_-)f^{n+1}||_{\ell^2_\Delta}^2-2ab\Delta t\left\langle(I-dD_+D_-)f^{n+1},(I+cD_+D_-)De^{n+1}\right\rangle-ab\Delta t^2||(I+cD_+D_-)De^{n+1}||_{\ell^2_\Delta}^2.\end{aligned}$$

For both cross products, it gives

$$\begin{aligned}&-2cd\Delta t\left\langle(I-bD_+D_-)e^{n+1},(I+aD_+D_-)Df^{n+1}\right\rangle-2ab\Delta t\left\langle(I-dD_+D_-)f^{n+1},(I+cD_+D_-)De^{n+1}\right\rangle\\&=2(-c)d\Delta t\left\langle e^{n+1},Df^{n+1}\right\rangle+2(-a)(-c)d\Delta t\left\langle De^{n+1},D_+D_-f^{n+1}\right\rangle+2b(-c)d\Delta t\left\langle De^{n+1},D_+D_-f^{n+1}\right\rangle\\&+2(-a)b(-c)d\Delta t\left\langle D_+D_-e^{n+1},D_+D_-Df^{n+1}\right\rangle\\&+2(-a)b\Delta t\left\langle f^{n+1},De^{n+1}\right\rangle+2(-a)b(-c)\Delta t\left\langle Df^{n+1},D_+D_-e^{n+1}\right\rangle+2(-a)bd\Delta t\left\langle Df^{n+1},D_+D_-e^{n+1}\right\rangle\\&+2(-a)b(-c)d\Delta t\left\langle D_+D_-f^{n+1},D_+D_-De^{n+1}\right\rangle.\end{aligned}$$

Thanks to integration by parts, Young's inequality together with Cauchy-Schwarz inequality, the previous equality simplifies into

$$\begin{aligned}&-2cd\Delta t\left\langle(I-bD_+D_-)e^{n+1},(I+aD_+D_-)Df^{n+1}\right\rangle-2ab\Delta t\left\langle(I-dD_+D_-)f^{n+1},(I+cD_+D_-)De^{n+1}\right\rangle\\&\geq-(-c)d\Delta t||e^{n+1}||_{\ell^2_\Delta}^2-(-a)b\Delta t||f^{n+1}||_{\ell^2_\Delta}^2\\&-\left[(-a)(-c)d+b(-c)d+(-a)b\right]\Delta t||De^{n+1}||_{\ell^2_\Delta}^2-\left[(-c)d+(-a)b(-c)+(-a)bd\right]\Delta t||Df^{n+1}||_{\ell^2_\Delta}^2\\&-\left[(-a)b(-c)+(-a)bd\right]\Delta t||D_+D_-e^{n+1}||_{\ell^2_\Delta}^2-\left[(-a)(-c)d+b(-c)d\right]\Delta t||D_+D_-f^{n+1}||_{\ell^2_\Delta}^2.\end{aligned}$$

The left hand side of (2.18) becomes

$$
\begin{aligned}
&-cd||(I-bD_+D_-)e^{n+1}+\Delta t(I+aD_+D_-)Df^{n+1}||^2_{\ell^2_\Delta}-ab||(I-dD_+D_-)f^{n+1}+\Delta t(I+cD_+D_-)De^{n+1}||^2_{\ell^2_\Delta}\\
&\geq(-c)d||(I-bD_+D_-)e^{n+1}||^2_{\ell^2_\Delta}+(-a)b||(I-dD_+D_-)f^{n+1}||^2_{\ell^2_\Delta}-(-c)d\Delta t||e^{n+1}||^2_{\ell^2_\Delta}-(-a)b\Delta t||f^{n+1}||^2_{\ell^2_\Delta}\\
&-\left[(-a)(-c)d+b(-c)d+(-a)b\right]\Delta t||De^{n+1}||^2_{\ell^2_\Delta}-\left[(-c)d+(-a)b(-c)+(-a)bd\right]\Delta t||Df^{n+1}||^2_{\ell^2_\Delta}\\
&-\left[(-a)b(-c)+(-a)bd\right]\Delta t||D_+D_-e^{n+1}||^2_{\ell^2_\Delta}-\left[(-a)(-c)d+b(-c)d\right]\Delta t||D_+D_-f^{n+1}||^2_{\ell^2_\Delta}\\
&+(-c)d\Delta t^2||(I+aD_+D_-)Df^{n+1}||^2_{\ell^2_\Delta}+(-a)b\Delta t^2||(I+cD_+D_-)De^{n+1}||^2_{\ell^2_\Delta}.
\end{aligned}
$$

Due to the definition of the energy (2.17), one has

$$
\begin{aligned}
&-cd||(I-bD_+D_-)e^{n+1}+\Delta t(I+aD_+D_-)Df^{n+1}||^2_{\ell^2_\Delta}-ab||(I-dD_+D_-)f^{n+1}+\Delta t(I+cD_+D_-)De^{n+1}||^2_{\ell^2_\Delta}\\
&\geq\mathcal{E}(e^{n+1},f^{n+1})-C_1\Delta t\mathcal{E}(e^{n+1},f^{n+1}),
\end{aligned}
\tag{2.19}
$$

with

$$
C_1=\max\left\{1,\frac{-a}{2b}+\frac{1}{2}+\frac{-a}{2(-c)d},\frac{-c}{2(-a)b}+\frac{-c}{2d}+\frac{1}{2},\frac{-a}{bd}+\frac{-a}{b(-c)},\frac{-c}{bd}+\frac{-c}{(-a)d}\right\}.
$$

Let us now focus on the right hand side of (2.18). The triangular inequality together with Young's inequality give the existence of a constant C_0 independent of Δt, such that

$$
\begin{aligned}
&-cd||(I-bD_+D_-)e^n-\Delta tD(e^nu^n_\Delta)-\Delta tD(\eta^n_\Delta f^n)-\Delta tD(e^nf^n)-\Delta t\epsilon^n_1||^2_{\ell^2_\Delta}\\
&-ab||(I-dD_+D_-)f^n-\Delta tD(f^nu^n_\Delta)-\frac{\Delta t}{2}D((f^n)^2)-\Delta t\epsilon^n_2||^2_{\ell^2_\Delta}\\
&\leq(-c)d||(I-bD_+D_-)e^n||^2_{\ell^2_\Delta}(1+C_0\Delta t)+(-c)dC_0(\Delta t+\Delta t^2)||D(e^nu^n_\Delta)||^2_{\ell^2_\Delta}+(-c)dC_0(\Delta t+\Delta t^2)||D(\eta^n_\Delta f^n)||^2_{\ell^2_\Delta}\\
&+(-c)dC_0(\Delta t+\Delta t^2)||D(e^nf^n)||^2_{\ell^2_\Delta}+(-c)dC_0(\Delta t+\Delta t^2)||\epsilon^n_1||^2_{\ell^2_\Delta}\\
&+(-a)b||(I-dD_+D_-)f^n||^2_{\ell^2_\Delta}(1+C_0\Delta t)+(-a)b\,C_0(\Delta t+\Delta t^2)||D(f^nu^n_\Delta)||^2_{\ell^2_\Delta}+(-a)b\,C_0(\Delta t+\Delta t^2)||D((f^n)^2)||^2_{\ell^2_\Delta}\\
&+(-a)b\,C_0(\Delta t+\Delta t^2)||\epsilon^n_2||^2_{\ell^2_\Delta}.
\end{aligned}
$$

Since

$$
||D(e^nf^n)||^2_{\ell^2_\Delta}\leq 2||e^n||^2_{\ell^\infty}||Df^n||^2_{\ell^2_\Delta}+2||f^n||^2_{\ell^\infty}||De^n||^2_{\ell^2_\Delta},
$$

it holds

$$
||D(e^nf^n)||^2_{\ell^2_\Delta}\leq\max\left\{\frac{1}{(-a)bd},\frac{1}{b(-c)d}\right\}\max\left\{||e^n||^2_{\ell^\infty},||f^n||^2_{\ell^\infty}\right\}\mathcal{E}(e^n,f^n).
\tag{2.20}
$$

The same holds for other terms to obtain

$$
||D(e^nu^n_\Delta)||^2_{\ell^2_\Delta}\leq\max\left\{\frac{2}{(-c)d},\frac{1}{b(-c)d}\right\}\max\left\{||u^n_\Delta||^2_{\ell^\infty},||Du^n_\Delta||^2_{\ell^\infty}\right\}\mathcal{E}(e^n,f^n),
\tag{2.21}
$$

and

$$
||D(\eta^n_\Delta f^n)||^2_{\ell^2_\Delta}\leq\max\left\{\frac{2}{(-a)b},\frac{1}{(-a)bd}\right\}\max\left\{||\eta^n_\Delta||^2_{\ell^\infty},||D\eta^n_\Delta||^2_{\ell^\infty}\right\}\mathcal{E}(e^n,f^n),
\tag{2.22}
$$

and

$$
||D(u^n_\Delta f^n)||^2_{\ell^2_\Delta}\leq\max\left\{\frac{2}{(-a)b},\frac{1}{(-a)bd}\right\}\max\left\{||u^n_\Delta||^2_{\ell^\infty},||Du^n_\Delta||^2_{\ell^\infty}\right\}\mathcal{E}(e^n,f^n),
\tag{2.23}
$$

and finally

$$
||D((f^n)^2)||^2_{\ell^2_\Delta}\leq||f^n||^2_{\ell^\infty}\frac{2}{(-a)bd}\mathcal{E}(e^n,f^n).
\tag{2.24}
$$

Thus, there exists a constant C_2 (which can be tracked by (2.20)-(2.24)), such that

$$\begin{aligned}
&- cd||(I - bD_+D_-)e^n - \Delta tD(e^n u^n_\Delta) - \Delta tD(\eta^n_\Delta f^n) - \Delta tD(e^n f^n) - \Delta t\epsilon^n_1||^2_{\ell^2_\Delta} \\
&- ab||(I - dD_+D_-)f^n - \Delta tD(f^n u^n_\Delta) - \frac{\Delta t}{2}D((f^n)^2) - \Delta t\epsilon^n_2||^2_{\ell^2_\Delta} \\
&\leq \mathcal{E}(e^n, f^n)(1 + C_0\Delta t) + C_2(\Delta t + \Delta t^2)\mathcal{E}(e^n, f^n) + (-c)dC_0(\Delta t + \Delta t^2)||\epsilon^n_1||^2_{\ell^2_\Delta} + (-a)b\, C_0(\Delta t + \Delta t^2)||\epsilon^n_2||^2_{\ell^2_\Delta}.
\end{aligned} \tag{2.25}$$

Gathering the informations from (2.19) and (2.25), there exists constants C_3 and C_4 such that

$$(1 - C_1\Delta t)\mathcal{E}(e^{n+1}, f^{n+1}) \leq (1 + C_3\Delta t)\mathcal{E}(e^n, f^n) + C_4\Delta t||\epsilon^n||^2_{\ell^2_\Delta},$$

with $||\epsilon^n||^2_{\ell^2_\Delta} = \max\left\{||\epsilon^n_1||^2_{\ell^2_\Delta}, ||\epsilon^n_2||^2_{\ell^2_\Delta}\right\}$. Proposition 2.1 is a straighforward consequence.

□

Proof of Theorem 1.2. Let us arbitrary fix $n \in \overline{0, N-1}$. Suppose the strong induction hypothesis

$$\left\|e^k\right\|_{\ell^\infty} \leq 1 \text{ and } \left\|f^k\right\|_{\ell^\infty} \leq 1, \tag{2.26}$$

for all $k \in \overline{0, n}$.
This is obviously true for $n = 0$, since $e^0_j = f^0_j$, for all $j \in \mathbb{Z}$. Let us prove that $||e^{n+1}||_{\ell^\infty} \leq 1$ and $||f^{n+1}||_{\ell^\infty} \leq 1$. Inequality (2.3) is thus available for any $k \in \overline{0, n}$ and constants C_1 and C_2 may be upper bounded by C_3 and C_4 independent of $\left\|e^k\right\|_{\ell^\infty}$ and $\left\|f^k\right\|_{\ell^\infty}$. One has, for all $k \in \overline{0, n}$

$$\mathcal{E}\left(e^{k+1}, f^{k+1}\right) \leq \frac{2\Delta t}{1 - \Delta tC_4}\left\|\epsilon^k\right\|^2_{\ell^2_\Delta} + \left(1 + \frac{\Delta t\,(C_3 + C_4)}{1 - \Delta tC_4}\right)\mathcal{E}\left(e^k, f^k\right).$$

Namely, it becomes

$$\mathcal{E}\left(e^{k+1}, f^{k+1}\right) - \mathcal{E}\left(e^k, f^k\right) \leq \frac{2\Delta t}{1 - \Delta tC_4}\left\|\epsilon^k\right\|^2_{\ell^2_\Delta} + \frac{\Delta t\,(C_3 + C_4)}{1 - \Delta tC_4}\mathcal{E}\left(e^k, f^k\right).$$

Thus, taking the sum of all these inequalities, and noticing that $e^0 = f^0 = 0$, we end up with

$$\mathcal{E}\left(e^{n+1}, f^{n+1}\right) \leq \frac{2\,(n+1)\,\Delta t}{1 - \Delta tC_4}\max_{k\in\overline{0,n}}\left\|\epsilon^k\right\|^2_{\ell^2_\Delta} + \frac{\Delta t\,(C_3 + C_4)}{1 - \Delta tC_4}\sum_{k=1}^{n}\mathcal{E}\left(e^k, f^k\right).$$

Applying the discrete Grönwall lemma 1.1 and using the fact that the consistency error is order one accurate in time and and space, see Appendix A, we get

$$\begin{aligned}
\mathcal{E}\left(e^{n+1}, f^{n+1}\right) &\leq \frac{2\,(n+1)\,\Delta t}{1 - \Delta tC_4}\exp\left(\frac{(n+1)\,\Delta t\,(C_3 + C_4)}{1 - \Delta tC_4}\right)\max_{k\in\overline{0,n}}\left\|\epsilon^k\right\|^2_{\ell^2_\Delta} \\
&\leq \frac{TC\,(\eta_0, u_0)}{1 - \Delta tC_4}\exp\left(\frac{T\,(C_3 + C_4)}{1 - \Delta tC_4}\right)\left\{(\Delta x)^4 + \left|\theta - \frac{1}{2}\right|(\Delta t)^2 + (\Delta t)^4\right\},
\end{aligned}$$

where $C\,(\eta_0, u_0)$ is some constant depending on the initial data (η_0, u_0). Thus, for $\Delta x, \Delta t$ small enough and using the inequality:

$$\left\|e^{n+1}\right\|_{\ell^\infty} \leq C\left\|e^{n+1}\right\|^{\frac{1}{2}}_{\ell^2_\Delta}\left\|D_+e^{n+1}\right\|^{\frac{1}{2}}_{\ell^2_\Delta},$$

with C a constant, we get that for sufficient small Δx and Δt :

$$||e^{n+1}||_{\ell^\infty} \leq 1 \quad \text{and} \quad ||f^{n+1}||_{\ell^\infty} \leq 1.$$

We can assure that the inductive hypothesis (2.26) holds for all $k \in \overline{0, n+1}$.

Obviously, this allows to close the estimates and provide the desired bound. This concludes the proof of Theorem 1.2. □

2.2 The proof of Theorem 1.3

As announced in the introduction, in this section we aim at providing a proof for our second main result. As opposed to the previous result, the proof of Theorem 1.3 is rather sensitive to the different values of the *abcd* parameters. We recall that we will treat the case where the parameters verify:

$$a \leq 0,\ c \leq 0,\ b \geq 0,\ d \geq 0,\ \text{and } bd = 0,$$

excluding the five cases:

$$\begin{cases} a = b = 0,\ d > 0,\ c < 0, \\ a = b = c = d = 0, \\ a = d = 0,\ b > 0,\ c < 0, \\ a = b = d = 0,\ c < 0, \\ b = d = 0,\ c < 0,\ a < 0. \end{cases}$$

For all $n \geq 0$, we will consider $(\epsilon_1^n, \epsilon_2^n) \in \left(\ell^2_\Delta(\mathbb{Z})\right)^2$ the consistency error defined as:

$$\begin{cases} \frac{1}{\Delta t}\left(I - bD_+D_-\right)\left(\eta_\Delta^{n+1} - \eta_\Delta^n\right) + \left(I + aD_+D_-\right) D\left(u_\Delta^{n+1}\right) + D\left(\eta_\Delta^n u_\Delta^n\right) \\ = \epsilon_1^n + \frac{1}{2}\left(1 - \operatorname{sgn}(b)\right)\tau_1 \Delta x D_+D_-\left(\eta_\Delta^n\right), \\ \\ \frac{1}{\Delta t}\left(I - dD_+D_-\right)\left(u_\Delta^{n+1} - u_\Delta^n\right) + \left(I + cD_+D_-\right) D\left(\eta_\Delta^{n+1}\right) + \frac{1}{2} D\left(\left(u_\Delta^n\right)^2\right) \\ = \epsilon_2^n + \frac{1}{2}\left(1 - \operatorname{sgn}(d)\right)\tau_2 \Delta x D_+D_-\left(u_\Delta^n\right). \end{cases} \tag{2.27}$$

In this section, we only detail the derivation of the energy inequality for $\mathcal{E}$, the equivalent of (2.3). This inequality is summarized as follows.

Proposition 2.2. *Assume* $||e^n||_{\ell^\infty}, ||f^n||_{\ell^\infty}, |\operatorname{sgn}(a)|||D_+(f)^n||_{\ell^\infty} \leq \Delta x^{\frac{1}{2}-\gamma}$, *with* $\gamma \in (0, \frac{1}{2})$, *for all* $n \in \overline{0, N}$. *Then the following energy estimate holds true, for* $n \in \overline{0, N-1}$

$$\begin{aligned} &\left(1 - \max\left\{\operatorname{sgn}(b), \operatorname{sgn}(d)\right\} C\Delta t\right) \mathcal{E}\left(e^{n+1}, f^{n+1}\right) \leq \left(1 + C\Delta t\right) \mathcal{E}\left(e^n, f^n\right) \\ &+ \left(\Delta t + \max\left\{|\operatorname{sgn}(c)|, 1 - \operatorname{sgn}(b)\right\} \Delta t^2\right) C||\epsilon_1^n||^2_{\ell^2_\Delta} \\ &+ \left(\Delta t + \max\left\{|\operatorname{sgn}(a)|, 1 - \operatorname{sgn}(d)\right\} \Delta t^2\right) C||\epsilon_2^n||^2_{\ell^2_\Delta} \\ &+ \left(\Delta t + \Delta t^2\right) \max\left\{|\operatorname{sgn}(cd)|, |\operatorname{sgn}(c)|\left(1 - \max\left\{\operatorname{sgn}(b), \operatorname{sgn}(d)\right\}\right)\right\} C||D_+\left(\epsilon_1\right)^n||^2_{\ell^2_\Delta} \\ &+ \left(\Delta t + \Delta t^2\right) \max\left\{|\operatorname{sgn}(ab)|, |\operatorname{sgn}(a)|\left(1 - \max\left\{\operatorname{sgn}(b), \operatorname{sgn}(d)\right\}\right)\right\} C||D_+\left(\epsilon_2\right)^n||^2_{\ell^2_\Delta}, \end{aligned} \tag{2.28}$$

with C *a positive constant depending on* $||\eta_\Delta^n||_{\ell^\infty}, ||D(\eta_\Delta)^n||_{\ell^\infty}, ||u_\Delta^n||_{\ell^\infty}$, $||D(u_\Delta)^n||_{\ell^\infty}$ *and* $||D_+D(u_\Delta)^n||_{\ell^\infty}$.

In order to close the estimates and ensure the convergence proof (as the one made in Subsection 2.1, for the case $b > 0$ and $d > 0$) we perform as usual an induction hypothesis on the smallness of $||e^n||_{\ell^\infty}, ||f^n||_{\ell^\infty}, ||D_+(f)^n||_{\ell^\infty}$ according the cases. It is sufficient to assume by induction

$$||e^n||_{\ell^\infty}, ||f^n||_{\ell^\infty}, |\operatorname{sgn}(a)|||D_+(f)^n||_{\ell^\infty} \leq \Delta x^{\frac{1}{2}-\gamma}, \quad \text{with } \gamma \in (0, \frac{1}{2}). \tag{2.29}$$

Hypothesis (2.29) is sufficient to assure the hypothesis of Proposition 2.2. The energy estimate (2.28) is thus satisfied and the convergence rate (Theorem 1.3) is a consequence of the discrete strong Grönwall inequality, Lemma 1.1 and the study of the consistency error (2.27) detailed in Appendix A (all the previous guidelines are detailed in Subsection 2.1 for the case $b > 0$ and $d > 0$).

First, we establish a technical result that interfers in a crucial manner in establishing the *a priori* estimates.

2.2.1 Burgers-type estimates

Let us state the first result of this section:

Proposition 2.3. *Let $u \in \ell^\infty(\mathbb{Z})$ such that $(D_+(u)_j)_{j\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z})$ and $\lambda \geq 0$. Fix $\alpha > 0$ and τ such that*

$$\|u\|_{\ell^\infty} + \alpha < \tau.$$

Then, there exists a sufficiently small positive number δ_0 such that the following holds true. Consider two positive reals $\Delta t, \Delta x$ such that

$$\frac{\tau\Delta t}{\Delta x} \leq 1, \quad \Delta x \leq \delta_0$$

and $a \in \ell^2(\mathbb{Z})$, such that

$$\lambda \|a\|_{\ell^\infty} \leq \Delta x^{\frac{1}{2}-\gamma}, \text{ with } \gamma \in (0, \frac{1}{2}).$$

Then, there exists a positive constant C depending on the ℓ^∞-norm of u and $D_+(u)$ such that

$$\left\| a - \Delta t D\left(\left(u + \lambda\frac{a}{2}\right)a\right) + \frac{\tau}{2}\Delta x\Delta t D_+D_-a \right\|_{\ell^2_\Delta} \leq (1 + C\Delta t)\|a\|_{\ell^2_\Delta}.$$

Proof. We define

$$\mathcal{B}a = a - \Delta t D\left(a\left(u + \lambda\frac{a}{2}\right)\right) + \frac{\tau}{2}\Delta x\Delta t D_+D_-a.$$

We compute the ℓ^2_Δ-norm of $\mathcal{B}a$:

$$\begin{aligned}
||\mathcal{B}a||^2_{\ell^2_\Delta} - ||a||^2_{\ell^2_\Delta} - \Delta t^2||D(au)||^2_{\ell^2_\Delta} - \Delta t^2\lambda^2||D\left(\frac{a^2}{2}\right)||^2_{\ell^2_\Delta} - \frac{\tau^2}{4}\Delta t^2\Delta x^2||D_+D_-a||^2_{\ell^2_\Delta} \\
= -2\Delta t\left\langle a, D\left(au + \lambda\frac{a^2}{2}\right)\right\rangle + \tau\Delta x\Delta t\langle a, D_+D_-a\rangle \\
+2\lambda\Delta t^2\left\langle D(au), D\left(\frac{a^2}{2}\right)\right\rangle - \tau\Delta x\Delta t^2\langle D(au), D_+D_-a\rangle \\
-\tau\lambda\Delta x\Delta t^2\left\langle D\left(\frac{a^2}{2}\right), D_+D_-a\right\rangle \overset{not.}{=} \sum_{i=1}^{3} R_i.
\end{aligned} \tag{2.30}$$

• For R_1, Relations (1.24), (1.27) and (1.22) give

$$\begin{aligned}
-2\Delta t\left\langle a, D\left(au + \lambda\frac{a^2}{2}\right)\right\rangle + \tau\Delta x\Delta t\langle a, D_+D_-a\rangle &= -\Delta t\langle D_+u, aS_+a\rangle + \frac{\Delta x^2\Delta t}{6}\lambda\left\langle D_+a, (D_+a)^2\right\rangle - \tau\Delta t\Delta x||D_+a||^2_{\ell^2_\Delta} \\
&\leq \Delta t||D_+u||_{\ell^\infty}||a||^2_{\ell^2_\Delta} + \frac{\Delta x^2\Delta t}{6}\lambda\left\langle D_+a, (D_+a)^2\right\rangle - \tau\Delta t\Delta x||D_+a||^2_{\ell^2_\Delta}.
\end{aligned}$$

• For the R_2-term, one has, thanks to Relations (1.31) and (1.25),

$$\begin{aligned}
&2\lambda\Delta t^2\left\langle D(au), D\left(\frac{a^2}{2}\right)\right\rangle - \tau\Delta x\Delta t^2\langle D(au), D_+D_-a\rangle \\
&\leq 2\lambda\Delta t^2||a||_{\ell^\infty}||u||_{\ell^\infty}||Da||^2_{\ell^2_\Delta} - \frac{8\Delta t^2\Delta x^2}{3}\lambda\left\langle Du, (Da)^3\right\rangle - \frac{2\Delta t^2}{3}\lambda\langle DDu, a^3\rangle \\
&+ \frac{\tau\Delta t^2}{\Delta x}||D_+u||_{\ell^\infty}||a||^2_{\ell^2_\Delta} + \frac{\tau\Delta t^2}{\Delta x}||Du||_{\ell^\infty}||a||^2_{\ell^2_\Delta} \\
&\leq 2\lambda\Delta t^2||a||_{\ell^\infty}||u||_{\ell^\infty}||Da||^2_{\ell^2_\Delta} + \frac{8\Delta x^2\Delta t^2}{3}\lambda||Du||_{\ell^\infty}||Da||_{\ell^\infty}||Da||^2_{\ell^2_\Delta} + \frac{2\Delta t^2}{3\Delta x}\lambda||a||^2_{\ell^2_\Delta}||Du||_{\ell^\infty}||a||_{\ell^\infty} \\
&+ \frac{\tau\Delta t^2}{\Delta x}||D_+u||_{\ell^\infty}||a||^2_{\ell^2_\Delta} + \frac{\tau\Delta t^2}{\Delta x}||Du||_{\ell^\infty}||a||^2_{\ell^2_\Delta}.
\end{aligned}$$

• Eventually, for R_3, one has, thanks to (1.26)

$$-\tau\lambda\Delta x\Delta t^2\left\langle D\left(\frac{a^2}{2}\right), D_+D_-a\right\rangle = -\lambda\frac{\Delta t^2\Delta x\tau}{6}\left\langle D_+a, (D_+a)^2\right\rangle + \frac{2\Delta t^2\Delta x\tau}{3}\lambda\left\langle Da, (Da)^2\right\rangle.$$

For the left hand side of the ℓ^2_Δ-norm of $\mathcal{B}a$, Equation (2.30), we know, thanks to (1.30)

$$\Delta t^2 ||D\left(au\right)||^2_{\ell^2_\Delta} \leq \Delta t^2 \left\{||u||^2_{\ell^\infty} + \Delta t||D_+u||^2_{\ell^\infty}\right\} ||Da||^2_{\ell^2_\Delta} + \Delta t \left\{||u||^2_{\ell^\infty} + \frac{3\Delta t}{4}||D_+u||^2_{\ell^\infty}\right\} ||a||^2_{\ell^2_\Delta}.$$

Thanks to (1.29), one has

$$\frac{\tau^2}{4}\Delta t^2\Delta x^2||D_+D_-a||^2_{\ell^2_\Delta} = \tau^2\Delta t^2||D_+a||^2_{\ell^2_\Delta} - \tau^2\Delta t^2||Da||^2_{\ell^2_\Delta},$$

and, thanks to (1.28),

$$\Delta t^2\lambda^2||D\left(\frac{a^2}{2}\right)||^2_{\ell^2_\Delta} \leq \Delta t^2\lambda^2||Da||^2_{\ell^2_\Delta}||a||^2_{\ell^\infty}.$$

Finally, we gather all these results

$$\begin{aligned}
||\mathcal{B}a||^2_{\ell^2_\Delta} \leq ||a||^2_{\ell^2_\Delta} \Big\{ & 1 + \Delta t||D_+u||_{\ell^\infty} + \Delta t||u||^2_{\ell^\infty} + \frac{3\Delta t^2}{4}||D_+u||^2_{\ell^\infty} + \frac{2\Delta t^2}{3\Delta x}\lambda||Du||_{\ell^\infty}||a||_{\ell^\infty} \\
& + \frac{\tau\Delta t^2}{\Delta x}||D_+u||_{\ell^\infty} + \frac{\tau\Delta t^2}{\Delta x}||Du||_{\ell^\infty}\Big\} + \frac{\Delta x^2\Delta t}{6}\lambda\left\langle D_+a, \left(D_+a\right)^2\right\rangle - \tau\Delta t\Delta x||D_+a||^2_{\ell^2_\Delta} \\
& - \frac{\Delta t^2\Delta x\tau}{6}\lambda\left\langle D_+a, \left(D_+a\right)^2\right\rangle + \tau^2\Delta t^2||D_+a||^2_{\ell^2_\Delta} + \Delta t^2||Da||^2_{\ell^2_\Delta}\left\{||u||^2_{\ell^\infty} + \Delta t||D_+u||^2_{\ell^\infty}\right. \\
& \left. + 2\lambda||a||_{\ell^\infty}||u||_{\ell^\infty} + \frac{8\Delta x^2}{3}\lambda||Du||_{\ell^\infty}||Da||_{\ell^\infty} + \lambda^2||a||^2_{\ell^\infty} + \frac{2\tau\Delta x}{3}\lambda||Da||_{\ell^\infty} - \tau^2\right\}.
\end{aligned}$$

Thus

$$\begin{aligned}
||\mathcal{B}a||^2_{\ell^2_\Delta} \leq ||a||^2_{\ell^2_\Delta} \Big\{ & 1 + \Delta t||D_+u||_{\ell^\infty} + \Delta t||u||^2_{\ell^\infty} + \frac{3\Delta t^2}{4}||D_+u||^2_{\ell^\infty} + \frac{2\Delta t^2}{3\Delta x}\lambda||Du||_{\ell^\infty}||a||_{\ell^\infty} \\
& + \frac{\tau\Delta t^2}{\Delta x}||D_+u||_{\ell^\infty} + \frac{\tau\Delta t^2}{\Delta x}||Du||_{\ell^\infty}\Big\} + \left\langle \frac{\Delta x^2\Delta t}{6}\lambda D_+a - \tau\Delta t\Delta x\mathbf{1} - \frac{\Delta t^2\Delta x\tau}{6}\lambda D_+a + \tau^2\Delta t^2\mathbf{1}, \left(D_+a\right)^2\right\rangle \\
& + \Delta t^2||Da||^2_{\ell^2_\Delta}\Big\{||u||^2_{\ell^\infty} + \Delta t||D_+u||^2_{\ell^\infty} + 2\lambda||a||_{\ell^\infty}||u||_{\ell^\infty} + \frac{8\Delta x}{3}\lambda||Du||_{\ell^\infty}||a||_{\ell^\infty} + \lambda^2||a||^2_{\ell^\infty} \\
& + \frac{2\tau}{3}\lambda||a||_{\ell^\infty} - \tau^2\Big\},
\end{aligned}$$

where

$$\mathbf{1} = \left(...1, 1, 1...\right).$$

However,

$$\frac{\Delta x^2\Delta t}{6}\lambda D_+a - \tau\Delta t\Delta x\mathbf{1} - \frac{\Delta t^2\Delta x\tau}{6}\lambda D_+a + \tau^2\Delta t^2\mathbf{1} = \left(\frac{\Delta x}{6}\lambda D_+a - \tau\mathbf{1}\right)\left(\Delta x - \tau\Delta t\right)\Delta t.$$

By hypothesis, $\tau\Delta t \leq \Delta x$ and $||u||_{\ell^\infty} + \alpha < \tau$, thus for δ_0 such that

$$\delta_0 \leq \left(3||u||_{\ell^\infty}\right)^{\frac{2}{1-2\gamma}}, \tag{2.31}$$

one has

$$\frac{\Delta x}{6}\lambda D_+a \leq \frac{\lambda||a||_{\ell^\infty}}{3} \leq \frac{\Delta x^{\frac{1}{2}-\gamma}}{3} \leq \frac{\delta_0^{\frac{1}{2}-\gamma}}{3} \leq ||u||_{\ell^\infty} < \tau.$$

This implies

$$\left\langle \frac{\Delta x^2\Delta t}{6}\lambda D_+a - \tau\Delta t\Delta x\mathbf{1} - \frac{\Delta t^2\Delta x\tau}{6}\lambda D_+a + \tau^2\Delta t^2\mathbf{1}, \left(D_+a\right)^2\right\rangle \leq 0.$$

In the same way, for Δt and Δx small enough and $||a||_{\ell^\infty}$ small enough, for δ_0 satisfying

$$||D_+(u)||^2_{\ell^\infty}\frac{\delta_0}{\tau} + 2\delta_0^{\frac{1}{2}-\gamma}||u||_{\ell^\infty} + \frac{8}{3}||D(u)||_{\ell^\infty}\delta_0^{\frac{3}{2}-\gamma} + \delta_0^{1-2\gamma} + \frac{2\tau}{3}\delta_0^{\frac{1}{2}-\gamma} \leq \alpha^2, \tag{2.32}$$

the condition $||u||_{\ell^\infty} + \alpha \leq \tau$ implies

$$||u||_{\ell^\infty}^2 + \Delta t||D_+u||_{\ell^\infty}^2 + 2\lambda||a||_{\ell^\infty}||u||_{\ell^\infty} + \frac{8\Delta x}{3}\lambda||Du||_{\ell^\infty}||a||_{\ell^\infty} + \lambda^2||a||_{\ell^\infty}^2 + \frac{2\tau}{3}\lambda||a||_{\ell^\infty} - \tau^2 \leq 0.$$

Proposition 2.3 results from the fact that there exists C_1 such that $\sqrt{1+C\Delta t} \leq 1 + C_1\Delta t$ with

$$C = ||D_+u||_{\ell^\infty} + ||u||_{\ell^\infty}^2 + \frac{3\Delta t}{4}||D_+u||_{\ell^\infty}^2 + \frac{2\Delta t}{3\Delta x}\lambda||Du||_{\ell^\infty}||a||_{\ell^\infty} + \frac{\tau\Delta t}{\Delta x}||D_+u||_{\ell^\infty} + \frac{\tau\Delta t}{\Delta x}||Du||_{\ell^\infty}.$$

The upper bound δ_0 must be chosen such that Conditions (2.31) and (2.32) be satisfied. □

The next result is an immediate consequence of the preceding one.

Proposition 2.4. *Consider $u \in \ell^\infty(\mathbb{Z})$ such that $(D_+(u)_j)_{j\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z})$ and $\lambda \geq 0$. Fix τ such that*

$$\|u\|_{\ell^\infty} < \tau.$$

Then, there exist two sufficiently small positive numbers δ_0, δ_1 such that the following holds true. Consider two positive reals $\Delta t, \Delta x$ such that

$$\frac{\tau\Delta t}{\Delta x} \leq 1, \quad \Delta x \leq \delta_0$$

and $a \in \ell^2(\mathbb{Z})$, $b \in \ell^\infty(\mathbb{Z})$ such that

$$\lambda \|a\|_{\ell^\infty} \leq \Delta x^{\frac{1}{2}-\gamma}, \text{ with } \gamma \in (0, \frac{1}{2}), \quad \|b\|_{\ell^\infty} \leq \delta_1 \text{ and } \|D_+b\|_{\ell^\infty} \leq 1.$$

Then, there exists a positive constant C depending on the ℓ^∞-norm of u and $D_+(u)$ such that:

$$\left\|a - \Delta t D\left(a\left(u+b+\lambda\frac{a}{2}\right)\right) + \frac{\tau}{2}\Delta x\Delta t D_+D_-a\right\|_{\ell^2_\Delta} \leq (1+C\Delta t)\|a\|_{\ell^2_\Delta}. \tag{2.33}$$

Proof. Let us consider $\tau > \|u\|_{\ell^\infty}$. Let us suppose that δ_1 is chosen small enough such that

$$\|u+b\|_{\ell^\infty} \leq \delta_1 + \|u\|_{\ell^\infty} < \tau.$$

Then, taking a smaller Δt and Δx if neccesary, we may apply Proposition 2.3 with $u+b$ instead of u in order to establish the estimate (2.33). □

Proposition 2.5. *Consider $u \in \ell^\infty(\mathbb{Z})$ such that $(D_+(u)_j)_{j\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z})$ and $\lambda \geq 0$. Fix τ such that*

$$\|u\|_{\ell^\infty} < \tau.$$

Then, there exist two sufficiently small positive numbers δ_0, δ_1 such that the following holds true. Consider two positive reals $\Delta t, \Delta x$ such that

$$\frac{\tau\Delta t}{\Delta x} \leq 1, \quad \Delta x \leq \delta_0$$

and $a \in \ell^2(\mathbb{Z})$, $b \in \ell^\infty(\mathbb{Z})$ such that

$$\lambda\|a\|_{\ell^\infty}, \lambda\|D_+a\|_{\ell^\infty} \leq \Delta x^{\frac{1}{2}-\gamma}, \text{ with } \gamma \in (0, \frac{1}{2}), \tag{2.34}$$

$$\|b\|_{\ell^\infty} < \delta_1 \text{ and } \|D_+b\|_{\ell^\infty}, \|D_+D_-(b)\|_{\ell^\infty}\|D_+D_-(u)\|_{\ell^\infty} \leq 1. \tag{2.35}$$

Then, there exist positive constants C_1,C_2 depending on the ℓ^∞-norm of u, $D_+(u)$ and $DD_+(u)$ such that:

$$\left\|D_+a - \Delta t D_+D\left(a\left(u+b+\lambda\frac{a}{2}\right)\right) + \frac{\tau}{2}\Delta x\Delta t D_+D_+D_-a\right\|_{\ell^2_\Delta} \leq C_1\Delta t\|a\|_{\ell^2_\Delta} + (1+C_2\Delta t)\|D_+a\|_{\ell^2_\Delta}.$$

Proof. Let us observe that

$$\begin{aligned}
&D_+a-\Delta tD_+D\left(a\left(u+b+\lambda\frac{a}{2}\right)\right)+\frac{\tau}{2}\Delta x\Delta tD_+D_+D_-\left(a\right)\\
&=D_+a-\Delta tD\left(D_+\left(a\right)\left(u+b+\lambda\frac{a}{2}\right)\right)-\Delta tD\left(S_+\left(a\right)D_+\left(u+b+\lambda\frac{a}{2}\right)\right)+\frac{\tau}{2}\Delta x\Delta tD_+D_-\left(D_+\left(a\right)\right)\\
&=D_+a-\Delta tD\left(D_+\left(a\right)\left(u+b+\frac{\lambda}{2}S_+\left(a\right)+\lambda\frac{a}{2}\right)\right)+\frac{\tau}{2}\Delta x\Delta tD_+D_-\left(D_+\left(a\right)\right)-\Delta tD\left(S_+\left(a\right)D_+\left(u\right)\right)\\
&\qquad\qquad-\Delta tD\left(S_+\left(a\right)D_+\left(b\right)\right).
\end{aligned}$$

Owing to the hypothesis (2.34)-(2.35) and Proposition 2.4 with $b+\frac{\lambda}{2}S_+a+\lambda\frac{a}{2}$ instead of b and 0 instead of λ, we may choose δ_0 small enough which ensures the existence of a positive constant C_2 such that:

$$\left\|D_+a-\Delta tD\left(D_+\left(a\right)\left(u+b+\frac{\lambda}{2}S_+\left(a\right)+\frac{\lambda}{2}a\right)\right)+\frac{\tau}{2}\Delta x\Delta tD_+D_-\left(D_+\left(a\right)\right)\right\|_{\ell^2_\Delta}\leq\left(1+C_2\Delta t\right)\left\|D_+\left(a\right)\right\|_{\ell^2_\Delta}. \tag{2.36}$$

Moreover, using the derivation formula (1.20), it transpires that

$$\begin{aligned}
&\left\|-\Delta tD\left(S_+\left(a\right)D_+\left(u\right)\right)-\Delta tD\left(S_+\left(a\right)D_+\left(b\right)\right)\right\|_{\ell^2_\Delta}\\
&\leq\Delta t\left(\left\|DD_+\left(u\right)\right\|_{\ell^\infty}+\left\|DD_+\left(b\right)\right\|_{\ell^\infty}\right)\left\|a\right\|_{\ell^2_\Delta}+\Delta t\left(\left\|D_+\left(u\right)\right\|_{\ell^\infty}+\left\|D_+\left(b\right)\right\|_{\ell^\infty}\right)\left\|D_+\left(a\right)\right\|_{\ell^2_\Delta}.
\end{aligned} \tag{2.37}$$

The conclusion follows from estimates (2.36) and (2.37). □

2.2.2 The case $b=d=0$

The case $a<0$, $b=c=d=0$ The convergence error satisfies:

$$\begin{cases}
e^{n+1}+\Delta tDf^{n+1}+a\Delta tD_+D_-Df^{n+1}\\
=e^n-\Delta tD\left(e^nu^n_\Delta\right)-\Delta tD\left(e^nf^n\right)-\Delta tD\left(\eta^n_\Delta f^n\right)+\frac{\tau_1}{2}\Delta t\Delta xD_+D_-e^n-\Delta t\epsilon^n_1,\\
\\
f^{n+1}+\Delta tDe^{n+1}=f^n-\Delta tD\left(f^n\left(u^n_\Delta+\frac{1}{2}f^n\right)\right)+\frac{\tau_2}{2}\Delta t\Delta xD_+D_-f^n-\Delta t\epsilon^n_2,
\end{cases} \tag{2.38}$$

with

$$\begin{cases}
\max\left\{\tau_1,\tau_2\right\}\Delta t<\Delta x,\\
\left\|u^n_\Delta\right\|_{\ell^\infty}<\min\left\{\tau_1^n\ ,\tau_2\right\},
\end{cases}\qquad n\in\overline{0,N}.$$

We consider the energy functional

$$\mathcal{E}\left(e,f\right)\overset{def.}{=}\left\|e\right\|^2_{\ell^2_\Delta}+\left\|f\right\|^2_{\ell^2_\Delta}+\left(-a\right)\left\|D_+f\right\|^2_{\ell^2_\Delta}.$$

Proof of Proposition 2.2 in the case $a<0$, $b=c=d=0$. In order to recover a H^1-type control for f^n (necessary for the control of $\mathcal{E}(e,f)$), let us apply $\sqrt{-a}D_+$ to the second equation of (2.38). We get that:

$$\begin{aligned}
\sqrt{-a}D_+f^{n+1}+\Delta t\sqrt{-a}D_+De^{n+1}&=\sqrt{-a}D_+f^n-\sqrt{-a}\Delta tDD_+\left(f^n\left(u^n_\Delta+\frac{1}{2}f^n\right)\right)\\
&\quad+\sqrt{-a}\frac{\tau_2}{2}\Delta t\Delta xD_+D_+D_-f^n-\sqrt{-a}\Delta tD_+\epsilon^n_2.
\end{aligned} \tag{2.39}$$

We consider now the three equations system comprised of system (2.38) with added equation (2.39). As previously, we square the equations, and add them together. Young's inequality enables us to obtain, with C_0 a constant

$$\begin{aligned}
&\left\|e^{n+1}+\Delta t\left(I+aD_+D_-\right)Df^{n+1}\right\|^2_{\ell^2_\Delta}+\left\|f^{n+1}+\Delta tDe^{n+1}\right\|^2_{\ell^2_\Delta}+\left\|\sqrt{-a}D_+f^{n+1}+\Delta t\sqrt{-a}D_+D(e^{n+1})\right\|^2_{\ell^2_\Delta}\\
&\leq\left(\Delta t+\Delta t^2\right)C_0\left\|\epsilon^n_1\right\|^2_{\ell^2_\Delta}+\left(\Delta t+\Delta t^2\right)C_0\left\|\epsilon^n_2\right\|^2_{\ell^2_\Delta}+\left(-a\right)\left(\Delta t+\left(\Delta t\right)^2\right)C_0\left\|D_+(\epsilon_2)^n\right\|^2_{\ell^2_\Delta}\\
&+\left(1+C_0\Delta t\right)\left\|e^n-\Delta tD\left(e^nu^n_\Delta\right)-\Delta tD\left(e^nf^n\right)+\frac{\tau_1^n}{2}\Delta t\Delta xD_+D_-e^n\right\|^2_{\ell^2_\Delta}+\left(\Delta t+\left(\Delta t\right)^2\right)C_0\left\|D\left(\eta^n_\Delta f^n\right)\right\|^2_{\ell^2_\Delta}
\end{aligned}$$

$$+ (1 + C_0\Delta t) \left\| f^n - \Delta t D\left(f^n\left(u_\Delta^n + \frac{1}{2}f^n\right)\right) + \frac{\tau_2^n}{2}\Delta t \Delta x D_+ D_- f^n \right\|_{\ell^2_\Delta}^2$$
$$+ (1 + C_0\Delta t)(-a) \left\| D_+ f^n - \Delta t D D_+\left(f^n\left(u_\Delta^n + \frac{1}{2}f^n\right)\right) + \frac{\tau_2^n}{2}\Delta t \Delta x D_+ D_+ D_- f^n \right\|_{\ell^2_\Delta}^2 . \quad (2.40)$$

Let us consider the right hand side of (2.40). Owing to Proposition (2.4) and Proposition (2.3) we have that, thanks hypotheses of Proposition 2.2 on $||e^n||_{\ell^\infty}, ||f^n||_{\ell^\infty}, ||D_+ f^n||_{\ell^\infty} \leq \Delta x^{\frac{1}{2}-\gamma}$

$$\left\| e^n - \Delta t D\left(e^n u_\Delta^n\right) - \Delta t D\left(e^n f^n\right) + \frac{\tau_1^n}{2}\Delta t \Delta x D_+ D_- e^n \right\|_{\ell^2_\Delta}^2 \leq (1 + C_1\Delta t) \left\| e^n \right\|_{\ell^2_\Delta}^2 , \quad (2.41)$$

respectively that

$$\left\| f^n - \Delta t D\left(f^n\left(u_\Delta^n + \frac{1}{2}f^n\right)\right) + \frac{\tau_2^n}{2}\Delta t \Delta x D_+ D_- f^n \right\|_{\ell^2_\Delta}^2 \leq (1 + C_2\Delta t) \left\| f^n \right\|_{\ell^2_\Delta}^2 . \quad (2.42)$$

Proposition 2.5 ensures the existence of constants C_3, C_4 such that:

$$(-a) \left\| D_+ f^n - \Delta t D_+ D\left(f^n\left(u_\Delta^n + \frac{f^n}{2}\right)\right) + \frac{\tau_2^n}{2}\Delta t \Delta x D_+ D_+ D_- f^n \right\|_{\ell^2_\Delta}^2$$
$$\leq (1 + C_3\Delta t)(-a) \left\| D_+ f^n \right\|_{\ell^2_\Delta}^2 + C_4(-a)\Delta t \left\| f^n \right\|_{\ell^2_\Delta}^2 . \quad (2.43)$$

Finally, we have that, thanks Relation (1.20)

$$\begin{aligned} \left\| D\left(\eta_\Delta^n f^n\right) \right\|_{\ell^2_\Delta}^2 &\leq 2 \left\| D\eta_\Delta^n \right\|_{\ell^\infty}^2 \left\| f^n \right\|_{\ell^2_\Delta}^2 + 2 \left\| \eta_\Delta^n \right\|_{\ell^\infty}^2 \left\| D f^n \right\|_{\ell^2_\Delta}^2 \\ &\leq 2 \max\left\{ \left\| D\eta_\Delta^n \right\|_{\ell^\infty}^2 , \frac{\left\| \eta_\Delta^n \right\|_{\ell^\infty}^2}{-a} \right\} \mathcal{E}\left(e^n, f^n\right) . \end{aligned} \quad (2.44)$$

Next, we compute the left hand side of (2.40). First of all, we observe that

$$\left\| \sqrt{-a} D_+ f^{n+1} + \Delta t \sqrt{-a} D_+ D e^{n+1} \right\|_{\ell^2_\Delta}^2$$
$$= -a \left\| D_+ f^{n+1} \right\|_{\ell^2_\Delta}^2 - a\left(\Delta t\right)^2 \left\| D_+ D e^{n+1} \right\|_{\ell^2_\Delta}^2 + 2a\Delta t \left\langle D_+ D_- f^{n+1}, D e^{n+1} \right\rangle .$$

Moreover, one has

$$\left\| e^{n+1} + \Delta t \left(I + a D_+ D_-\right) D f^{n+1} \right\|_{\ell^2_\Delta}^2 + \left\| f^{n+1} + \Delta t D e^{n+1} \right\|_{\ell^2_\Delta}^2 = ||e^{n+1}||_{\ell^2_\Delta}^2$$
$$+ 2a\Delta t \left\langle e^{n+1}, D_+ D_- D f^{n+1} \right\rangle + \Delta t^2 ||(I + a D_+ D_-) D f^{n+1}||_{\ell^2_\Delta}^2 + ||f^{n+1}||_{\ell^2_\Delta}^2 + \Delta t^2 ||D e^{n+1}||_{\ell^2_\Delta}^2 .$$

Eventually, we get, for the left hand side,

$$\begin{aligned} &\left\| e^{n+1} + \Delta t \left(I + a D_+ D_-\right) D f^{n+1} \right\|_{\ell^2_\Delta}^2 + \left\| f^{n+1} + \Delta t D e^{n+1} \right\|_{\ell^2_\Delta}^2 + \left\| \sqrt{-a} D_+ f^{n+1} + \Delta t \sqrt{-a} D_+ D e^{n+1} \right\|_{\ell^2_\Delta}^2 \\ &= \mathcal{E}\left(e^{n+1}, f^{n+1}\right) + \Delta t^2 \left\| \left(I + a D_+ D_-\right) D f^{n+1} \right\|_{\ell^2_\Delta}^2 + \Delta t^2 \left\| D e^{n+1} \right\|_{\ell^2_\Delta}^2 - a\Delta t^2 \left\| D_+ D e^{n+1} \right\|_{\ell^2_\Delta}^2 \\ &\geq \mathcal{E}\left(e^{n+1}, f^{n+1}\right) . \end{aligned} \quad (2.45)$$

Gathering relations (2.40), (2.41), (2.42), (2.43), (2.44) and (2.45) yields

$$\mathcal{E}\left(e^{n+1}, f^{n+1}\right) \leq \left(\Delta t + \left(\Delta t\right)^2\right) C_0 \left(\left\| \epsilon_1^n \right\|_{\ell^2_\Delta}^2 + \left\| \epsilon_2^n \right\|_{\ell^2_\Delta}^2 + (-a) \left\| D_+ \epsilon_2^n \right\|_{\ell^2_\Delta}^2 \right) + (1 + C_1\Delta t) \mathcal{E}\left(e^n, f^n\right) .$$

□

2.2.3 The case $b = 0, d > 0$

The case $a = b = c = 0, d > 0$. In this case, without any difficulties, we are able to prove a more general result. Indeed, we will show that the following general θ-scheme:

$$\begin{cases} \frac{1}{\Delta t}(\eta^{n+1} - \eta^n) + D\left((1-\theta)u^n + \theta u^{n+1}\right) + D\left(\eta^n u^n\right) = \frac{\tau_1 \Delta x}{2} D_+ D_- \left(\eta^n\right), \\ \\ \frac{1}{\Delta t}\left(I - dD_+D_-\right)\left(u^{n+1} - u^n\right) + D\left((1-\theta)\eta^n + \theta\eta^{n+1}\right) + \frac{1}{2}D\left(\left(u^n\right)^2\right) = 0. \end{cases} \tag{2.46}$$

is adapted for studying the classical Boussinesq system, with $\theta \in [0,1]$. The convergence error verifies:

$$\begin{cases} E^-(n) + \Delta t D\left((1-\theta) f^n + \theta f^{n+1}\right) + \Delta t D\left(e^n u_\Delta^n\right) + \Delta t D\left(\eta_\Delta^n f^n\right) \\ \qquad\qquad + \Delta t D\left(e^n f^n\right) = \frac{\tau_1}{2}\Delta t \Delta x D_+ D_-\left(e^n\right) - \Delta t \epsilon_1^n, \\ \\ \left(I - dD_+D_-\right)F^-(n) + \Delta t D\left((1-\theta)e^n + \theta e^{n+1}\right) + \Delta t D\left(f^n u_\Delta^n\right) \\ \qquad\qquad + \frac{\Delta t}{2} D\left(\left(f^n\right)^2\right) = -\Delta t \epsilon_2^n. \end{cases} \tag{2.47}$$

Recall that in this case:

$$\mathcal{E}(e,f) = \|e\|_{\ell^2_\Delta}^2 + \|f\|_{\ell^2_\Delta}^2 + d\left\|D_+ f\right\|_{\ell^2_\Delta}^2. \tag{2.48}$$

Proof of Proposition 2.2 in the case $a = b = c = 0, d > 0$. Multiply the second equation of (2.47) with $F^+(n)$ and proceeding as we did in Section 2.1 (Identity (2.14) with $a = 0$), we obtain that there exist two constants $C_{1,1}$ and $C_{1,2}$ depending on d, $\|u_\Delta^n\|_{\ell^\infty}$, $\|Du_\Delta^n\|_{\ell^\infty}$ and θ, such that:

$$\begin{aligned} &\left\|f^{n+1}\right\|_{\ell^2_\Delta}^2 + d\left\|D_+ f^{n+1}\right\|_{\ell^2_\Delta}^2 - \left\|f^n\right\|_{\ell^2_\Delta}^2 - d\left\|D_+ f^n\right\|_{\ell^2_\Delta}^2 \\ &= -\left\langle F^+(n), \Delta t D\left((1-\theta)e^n + \theta e^{n+1}\right) + \Delta t D\left(f^n u_\Delta^n\right) + \frac{\Delta t}{2} D\left(\left(f^n\right)^2\right) - \Delta t \epsilon_2^n \right\rangle \\ &\leq \Delta t \left\|\epsilon_2^n\right\|_{\ell^2_\Delta}^2 + \Delta t C_{1,1}\mathcal{E}\left(e^n, f^n\right) + \Delta t C_{1,2}\mathcal{E}\left(e^{n+1}, f^{n+1}\right) - \Delta t(1-\theta)\left\langle De^n, F^+(n)\right\rangle - \theta\Delta t\left\langle De^{n+1}, F^+(n)\right\rangle. \end{aligned} \tag{2.49}$$

Notice that Relation (1.22) combined with the Cauchy-Schwarz inequality, the Young's inequality and the upper bound $||D(.)||_{\ell^2_\Delta} \leq ||D_+(.)||_{\ell^2_\Delta}$ simplifies the previous last term

$$\begin{aligned} \left\langle De^n, F^+(n)\right\rangle &\leq ||e^n||_{\ell^2_\Delta}^2 + \frac{1}{2}||Df^n||_{\ell^2_\Delta}^2 + \frac{1}{2}||Df^{n+1}||_{\ell^2_\Delta}^2 \\ &\leq \max\left\{1 + \frac{1}{2d}\right\}\mathcal{E}\left(e^n, f^n\right) + \frac{1}{2d}\mathcal{E}\left(e^{n+1}, f^{n+1}\right). \end{aligned}$$

A similar inequality holds true for $\left\langle De^{n+1}, F^+(n)\right\rangle$.
Next we rewrite the first equation of (2.47) as

$$e^{n+1} = e^n - \Delta t D\left(e^n\left(u_\Delta^n + f^n\right)\right) + \frac{\tau_1}{2}\Delta t \Delta x D_+ D_-\left(e^n\right) - \Delta t D\left((1-\theta)f^n + \theta f^{n+1}\right) - \Delta t D\left(\eta_\Delta^n f^n\right) - \Delta t \epsilon_1^n.$$

Thus, using the Proposition (2.4) and Relation (1.20) we get that

$$\begin{aligned} \left\|e^{n+1}\right\|_{\ell^2_\Delta} &\leq \left\|e^n - \Delta t D\left(e^n\left(u_\Delta^n + f^n\right)\right) + \frac{\tau_1}{2}\Delta t \Delta x D_+ D_-\left(e^n\right)\right\|_{\ell^2_\Delta} \\ &\quad + \Delta t\left\|D\left((1-\theta)f^n + \theta f^{n+1}\right)\right\|_{\ell^2_\Delta} + \Delta t\left\|D\left(\eta_\Delta^n f^n\right)\right\|_{\ell^2_\Delta} + \Delta t\left\|\epsilon_1^n\right\|_{\ell^2_\Delta} \\ &\leq \Delta t\left\|\epsilon_1^n\right\|_{\ell^2_\Delta} + (1 + C\Delta t)\left\|e^n\right\|_{\ell^2_\Delta} + \Delta t\left((1-\theta)\left\|D\left(f^n\right)\right\|_{\ell^2_\Delta} + \theta\left\|D\left(f^{n+1}\right)\right\|_{\ell^2_\Delta}\right) \\ &\quad + \Delta t \max\left\{\left\|\eta_\Delta^n\right\|_{\ell^\infty}, \left\|D\left(\eta_\Delta^n\right)\right\|_{\ell^\infty}\right\}\left(\left\|f^n\right\|_{\ell^2_\Delta} + \left\|D\left(f^n\right)\right\|_{\ell^2_\Delta}\right) \\ &\leq \Delta t\left\|\epsilon_1^n\right\|_{\ell^2_\Delta} + (1 + C\Delta t)\left\|e^n\right\|_{\ell^2_\Delta} + \Delta t C_{1,2}\mathcal{E}^{\frac{1}{2}}\left(e^n, f^n\right) + C_{2,2}\Delta t\mathcal{E}^{\frac{1}{2}}\left(e^{n+1}, f^{n+1}\right), \end{aligned} \tag{2.50}$$

with $C_{1,2}$ and $C_{2,2}$ depending on $||\eta_\Delta^n||_{\ell^\infty}$ and $||D\eta_\Delta^n||_{\ell^\infty}$. From (2.50) we deduce that

$$\begin{aligned}\left\|e^{n+1}\right\|_{\ell^2_\Delta}^2 \leq & \left(\Delta t + (\Delta t)^2\right) C_0 \left\|\epsilon_1^n\right\|_{\ell^2_\Delta}^2 + (1 + C_{1,3}\Delta t) \left\|e^n\right\|_{\ell^2_\Delta}^2 \\ & + \left(\Delta t + (\Delta t)^2\right) C_{1,3}\mathcal{E}\left(e^n, f^n\right) + \left(\Delta t + (\Delta t)^2\right) C_{2,3}\mathcal{E}\left(e^{n+1}, f^{n+1}\right). \end{aligned} \tag{2.51}$$

Adding up the estimates yields

$$(1 - C_1\Delta t)\,\mathcal{E}\left(e^{n+1}, f^{n+1}\right) \leq \left(\Delta t + (\Delta t)^2\right) C_0 \left\|\epsilon_1^n\right\|_{\ell^2_\Delta}^2 + \Delta t \left\|\epsilon_2^n\right\|_{\ell^2_\Delta}^2 + (1 + C_2\Delta t)\,\mathcal{E}\left(e^n, f^n\right).$$

□

The case $a < 0, b = 0, c = 0, d > 0$ In this case, the convergence error satisfies:

$$\begin{cases} e^{n+1} + \Delta t D f^{n+1} + a\Delta t D_+ D_- D f^{n+1} \\ = e^n - \Delta t D\left(e^n u_\Delta^n\right) - \Delta t D\left(e^n f^n\right) - \Delta t D\left(\eta_\Delta^n f^n\right) + \frac{\tau_1}{2}\Delta t \Delta x D_+ D_- e^n - \Delta t \epsilon_1^n, \\ \\ \left(I - dD_+D_-\right) f^{n+1} + \Delta t D e^{n+1} = \left(I - dD_+D_-\right) f^n - \Delta t D\left(f^n\left(u_\Delta^n + \frac{1}{2}f^n\right)\right) - \Delta t \epsilon_2^n. \end{cases}$$

In this section, we will work in the following with the energy functional:

$$\mathcal{E}\left(e, f\right) = d\left\|e\right\|_{\ell^2_\Delta}^2 + (-a)\left\|\left(I - dD_+D_-\right) f\right\|_{\ell^2_\Delta}^2,$$

which is better adapted to the system corresponding to the particular values of the parameters in view here. Of course, $\mathcal{E}$ is equivalent to the energy from (2.1).

Proof of Proposition 2.2 in the case $a < 0, b = c = 0, d > 0$. By summing up the square of the ℓ^2_Δ-norm of the first equation by d, the square of the ℓ^2_Δ norm of the second one by $(-a)$, we get that, thanks to Relations (1.22) and (1.23),

$$\begin{aligned} & \mathcal{E}\left(e^{n+1}, f^{n+1}\right) + 2(d+a)\Delta t\left\langle Df^{n+1}, e^{n+1}\right\rangle + d\Delta t^2 ||Df^{n+1}||_{\ell^2_\Delta}^2 + 2\Delta t^2(-a)d||D_+Df^{n+1}||_{\ell^2_\Delta}^2 + (-a)\Delta t^2||De^{n+1}||_{\ell^2_\Delta}^2 \\ & + \Delta t^2 a^2 d||D_+D_-Df^{n+1}||_{\ell^2_\Delta}^2 \leq (\Delta t + \Delta t^2)C_0 d\left\|\epsilon_1^n\right\|_{\ell^2_\Delta}^2 + (\Delta t + \Delta t^2)C_0\,(-a)\left\|\epsilon_2^n\right\|_{\ell^2_\Delta}^2 \\ & + (1 + \Delta t C_0)\, d\left\|e^n - \Delta t D\left(e^n\left(u_\Delta^n + f^n\right)\right) + \frac{\tau_1}{2}\Delta t\Delta x D_+D_-e^n\right\|_{\ell^2_\Delta}^2 \\ & + \left(\Delta t + \Delta t^2\right) C_0 d\,||D\left(\eta_\Delta^n f^n\right)||_{\ell^2_\Delta}^2 \\ & + (1 + \Delta t C_0)\,(-a)\left\|\left(I - dD_+D_-\right) f^n\right\|_{\ell^2_\Delta}^2 \\ & + \left(\Delta t + \Delta t^2\right) C_0\,(-a)\left\|D\left(f^n\left(u_\Delta^n + \frac{1}{2}f^n\right)\right)\right\|_{\ell^2_\Delta}^2. \end{aligned}$$

We notice that

$$\begin{aligned} 2\Delta t(d+a)\left\langle f^{n+1}, De^{n+1}\right\rangle & \geq -\Delta t(d-a)||Df^{n+1}||_{\ell^2_\Delta}^2 - \Delta t(d-a)||e^{n+1}||_{\ell^2_\Delta}^2 \\ & \geq -\Delta t C_1 \mathcal{E}\left(e^{n+1}, f^{n+1}\right). \end{aligned}$$

Using Proposition 2.4 we get that

$$(1 + \Delta t C_0)\, d\left\|e^n - \Delta t D\left(e^n\left(u_\Delta^n + f^n\right)\right) + \frac{\tau_1}{2}\Delta t\Delta x D_+D_-e^n\right\|_{\ell^2_\Delta}^2 \leq (1 + C_2\Delta t)\, d||e^n||_{\ell^2_\Delta}^2.$$

Also, we have, due to (1.20)

$$||D\left(\eta_\Delta^n f^n\right)||_{\ell^2_\Delta}^2 \leq ||D\left(\eta_\Delta^n\right)||_{\ell^\infty}^2 ||f^n||_{\ell^2_\Delta}^2 + ||\eta_\Delta^n||_{\ell^\infty}^2 ||D\left(f^n\right)||_{\ell^2_\Delta}^2 \leq C_3\mathcal{E}\left(e^n, f^n\right).$$

Proceeding as above, we get that:

$$\left\| D\left(f^n\left(u_\Delta^n + \frac{1}{2} f^n \right) \right) \right\|_{\ell^2_\Delta}^2 \leq \max\left\{ ||Du_\Delta^n||_{\ell^\infty}^2, ||u_\Delta^n||_{\ell^\infty}^2, ||Df^n||_{\ell^\infty}^2, ||f^n||_{\ell^\infty}^2 \right\} C_4 \mathcal{E}\left(e^n, f^n\right).$$

Adding up the above estimate gives us:

$$\begin{aligned}\left(1 - C_1\Delta t\right)\mathcal{E}\left(e^{n+1}, f^{n+1}\right) \leq \left(\Delta t + (\Delta t)^2\right) C_0 d \left\|\epsilon_1^n\right\|_{\ell^2_\Delta}^2 + \left(\Delta t + (\Delta t)^2\right) C_0 \left(-a\right) \left\|\epsilon_2^n\right\|_{\ell^2_\Delta}^2 \\ + \left(1 + C_5\Delta t\right)\mathcal{E}\left(e^n, f^n\right).\end{aligned}$$

□

The case $a < 0, b = 0, c < 0, d > 0$ In this case, the convergence error satisfies:

$$\begin{cases} e^{n+1} + \Delta t D f^{n+1} + a\Delta t D_+ D_- D f^{n+1} \\ = e^n - \Delta t D\left(e^n u_\Delta^n\right) - \Delta t D\left(e^n f^n\right) - \Delta t D\left(\eta_\Delta^n f^n\right) + \frac{\tau_1}{2}\Delta t \Delta x D_+ D_- e^n - \Delta t \epsilon_1^n, \\ \\ \left(I - dD_+D_-\right) f^{n+1} + \Delta t D e^{n+1} + c\Delta t D_+ D_- D e^{n+1} = \left(I - dD_+D_-\right) f^n - \Delta t D\left(f^n\left(u_\Delta^n + \frac{1}{2} f^n\right)\right) - \Delta t \epsilon_2^n. \end{cases} \tag{2.52}$$

Again, in order to close the estimates and prove the convergence of the scheme, we will be using the following energy functional:

$$\mathcal{E}\left(e, f\right) \overset{def.}{=} \left(-a\right)\left\|e\right\|_{\ell^2_\Delta}^2 + \left(-cd\right)\left\|D_+ e\right\|_{\ell^2_\Delta}^2 + \left(-a\right)\left\|\left(I - dD_+D_-\right) f\right\|_{\ell^2_\Delta}^2,$$

which is equivalent to that one from (2.1).

Proof of Proposition 2.2 in the case $a < 0, b = 0, c < 0, d > 0$. In order to derive a H^1 control on e^{n+1}, let us apply $\sqrt{-cd}D_+$ in the first equation, to obtain

$$\begin{aligned}&\sqrt{-cd}D_+ e^{n+1} + \Delta t\sqrt{-cd}D_+ D f^{n+1} + \sqrt{-cd}a\Delta t D_+ D_+ D_- D f^{n+1} = \sqrt{-cd}D_+ e^n - \Delta t\sqrt{-cd}D_+ D\left(e^n u_\Delta^n\right) \\ &- \Delta t\sqrt{-cd}D_+ D\left(e^n f^n\right) + \frac{\tau_1}{2}\sqrt{-cd}\Delta t \Delta x D_+ D_+ D_- e^n - \sqrt{-cd}\Delta t D_+ D\left(\eta_\Delta^n f^n\right) - \sqrt{-cd}\Delta t D_+ \epsilon_1^n.\end{aligned} \tag{2.53}$$

We focus now on the system composed of both equations of (2.52) with Equation (2.53) in addition. We will multiply the first and second equations of (2.52) by $(-a)$ thereafter. As before, we square the three equalities to compute the ℓ^2_Δ-norm. Let us observe that

$$\begin{aligned}I_1 &= \left(-a\right)\left\|e^{n+1} + \Delta t D f^{n+1} + a\Delta t D_+ D_- D f^{n+1}\right\|_{\ell^2_\Delta}^2 \\ &= \left(-a\right)\left\|e^{n+1}\right\|_{\ell^2_\Delta}^2 + \left(-a\right)\left(\Delta t\right)^2\left\|D f^{n+1}\right\|_{\ell^2_\Delta}^2 - a^3\left(\Delta t\right)^2\left\|D_+ D_- D f^{n+1}\right\|_{\ell^2_\Delta}^2 \\ &+ 2a^2(\Delta t)^2\left\|D_+ D f^{n+1}\right\|_{\ell^2_\Delta}^2 + 2(-a)\Delta t\left\langle e^{n+1}, D f^{n+1}\right\rangle - 2a^2\Delta t\left\langle e^{n+1}, D_+ D_- D f^{n+1}\right\rangle,\end{aligned} \tag{2.54}$$

along with

$$\begin{aligned}I_2 &= \left(-cd\right)\left\|D_+ e^{n+1} + \Delta t D_+ D f^{n+1} + a\Delta t D_+ D_+ D_- D f^{n+1}\right\|_{\ell^2_\Delta}^2 \\ &= \left(-cd\right)\left\|D_+ e^{n+1}\right\|_{\ell^2_\Delta}^2 + \left(-cd\right)\left(\Delta t\right)^2\left\|D_+ D f^{n+1}\right\|_{\ell^2_\Delta}^2 + \left(-cd\right)a^2\left(\Delta t\right)^2\left\|D_+ D_+ D_- D f^{n+1}\right\|_{\ell^2_\Delta} \\ &+ 2acd(\Delta t)^2\left\|D_+ D_- D f^{n+1}\right\|_{\ell^2_\Delta}^2 - 2cd\Delta t\left\langle D_+ e^{n+1}, D_+ D f^{n+1}\right\rangle \\ &- 2acd\Delta t\left\langle D_+ e^{n+1}, D_+ D_+ D_- D f^{n+1}\right\rangle,\end{aligned} \tag{2.55}$$

and

$$\begin{aligned}I_3 &= \left(-a\right)\left\|\left(I - dD_+D_-\right) f^{n+1} + \Delta t D e^{n+1} + c\Delta t D_+ D_- D e^{n+1}\right\|_{\ell^2_\Delta}^2 \\ &= \left(-a\right)\left\|\left(I - dD_+D_-\right) f^{n+1}\right\|_{\ell^2_\Delta}^2 + \left(-a\right)c^2\left(\Delta t\right)^2\left\|D_+ D_- D e^{n+1}\right\|_{\ell^2_\Delta}^2 + \left(-a\right)\left(\Delta t\right)^2\left\|D e^{n+1}\right\|_{\ell^2_\Delta}^2\end{aligned}$$

$$+ 2ac\left(\Delta t\right)^2 \left\|D_+De^{n+1}\right\|_{\ell^2_\Delta} + 2(-a)\Delta t \left\langle \left(I - dD_+D_-\right) f^{n+1}, De^{n+1}\right\rangle$$
$$- 2ac\Delta t \left\langle f^{n+1}, D_+D_-De^{n+1}\right\rangle + 2acd\Delta t \left\langle D_+D_-f^{n+1}, D_+D_-De^{n+1}\right\rangle. \tag{2.56}$$

Observe that the last term from (2.55) cancels with the last term of (2.56), according to Relations (1.22) and (1.23). The same is true for $2\Delta t \left\langle e^{n+1}, Df^{n+1}\right\rangle$ in (2.54) and $2\Delta t \left\langle f^{n+1}, De^{n+1}\right\rangle$ in (2.56). Therefore, by integrating by part $2a\Delta t \left\langle e^{n+1}, D_+D_-Df^{n+1}\right\rangle$ in (2.54) and $-2ac\Delta t \left\langle f^{n+1}, D_+D_-De^{n+1}\right\rangle$ in (2.56), it yieds

$$\begin{aligned} I_1 + I_2 + I_3 \geq \mathcal{E}\left(e^{n+1}, f^{n+1}\right) &+ 2a^2\Delta t \left\langle De^{n+1}, D_+D_-f^{n+1}\right\rangle \\ &- 2cd\Delta t \left\langle D_+e^{n+1}, D_+Df^{n+1}\right\rangle + 2da\Delta t \left\langle D_+D_-f^{n+1}, De^{n+1}\right\rangle \\ &- 2ac\Delta t \left\langle D_+D_-f^{n+1}, De^{n+1}\right\rangle. \end{aligned}$$

Young's inequality enables to lower bound the previous inequality:

$$\begin{aligned} I_1 + I_2 + I_3 &\geq \mathcal{E}\left(e^{n+1}, f^{n+1}\right) - (-a^2 - cd + d(-a) + ac)\Delta t \left\{||D_+e^{n+1}||^2_{\ell^2_\Delta} + ||D_+D_-f^{n+1}||^2_{\ell^2_\Delta}\right\} \\ &\geq (1 - C_1\Delta t)\,\mathcal{E}\left(e^{n+1}, f^{n+1}\right). \end{aligned}$$

Let us now focus on the right hand side of the squared equations. Using Cauchy-Schwarz inequality and Proposition 2.4, we get that

$$\begin{aligned} I_1 &= (-a)\left\|e^n - \Delta tD\left(e^nu^n_\Delta\right) - \Delta tD\left(e^nf^n\right) + \frac{\tau_1}{2}\Delta t\Delta xD_+D_-e^n - \Delta tD(\eta^n_\Delta f^n) - \Delta t\epsilon^n_1\right\|^2_{\ell^2_\Delta} \\ &\leq (-a)\left\|e^n - \Delta tD\left(e^nu^n_\Delta\right) - \Delta tD\left(e^nf^n\right) + \frac{\tau_1}{2}\Delta t\Delta xD_+D_-e^n\right\|^2_{\ell^2_\Delta}(1 + C_0\Delta t) \\ &+ (-a)(\Delta t + \Delta t^2)C_0\left\|D\left(\eta^n_\Delta f^n\right)\right\|^2_{\ell^2_\Delta} + (-a)(\Delta t + \Delta t^2)C_0\left\|\epsilon^n_1\right\|^2_{\ell^2_\Delta} \\ &\leq (1 + C\Delta t)(-a)\left\|e^n\right\|^2_{\ell^2} + (\Delta t + \Delta t^2)(-a)C_0\left\|D\left(\eta^n_\Delta f^n\right)\right\|^2_{\ell^2_\Delta} + (\Delta t + \Delta t^2)(-a)C_0\left\|\epsilon^n_1\right\|^2_{\ell^2_\Delta}. \end{aligned} \tag{2.57}$$

Using Proposition 2.5 and Young's inequality, we get that

$$I_2 \leq -cd\left\|D_+e^n - \Delta tD_+D\left(e^n(u^n_\Delta + f^n)\right) + \frac{\tau_1}{2}\Delta t\Delta xD_+D_+D_-e^n - \Delta tD_+D(\eta^n_\Delta f^n) - \Delta tD_+\left(\epsilon^n_1\right)\right\|^2_{\ell^2_\Delta} \tag{2.58}$$

$$\begin{aligned} &\leq -cd(1 + C_0\Delta t)\left\|D_+e^n - \Delta tD_+D\left(e^nu^n_\Delta\right) - \Delta tD_+D\left(e^nf^n\right) + \frac{\tau_1}{2}\Delta t\Delta xD_+D_+D_-e^n\right\|^2_{\ell^2_\Delta} \\ &- cd(\Delta t + \Delta t^2)C_0\left\|D_+D\left(\eta^n_\Delta f^n\right)\right\|^2_{\ell^2_\Delta} - cd(\Delta t + \Delta t^2)C_0\left\|D_+\epsilon^n_1\right\|^2_{\ell^2_\Delta} \\ &\leq -cd(\Delta t + \Delta t^2)C_0\left\|D_+\epsilon^n_1\right\|^2_{\ell^2_\Delta} - cd\left(\Delta tC_1\left\|e^n\right\|^2_{\ell^2_\Delta} + (1 + C_2\Delta t)\left\|D_+e^n\right\|^2_{\ell^2_\Delta}\right) \\ &- cd(\Delta t + \Delta t^2)\max\left\{||\eta^n_\Delta||^2_{\ell^\infty}, ||D_+\eta^n_\Delta||^2_{\ell^\infty}, ||D_+D\eta^n_\Delta||^2_{\ell^\infty}\right\}\left(||f^n||^2_{\ell^2_\Delta} + ||D_+f^n||^2_{\ell^2_\Delta} + ||D_+Df^n||^2_{\ell^2_\Delta}\right) \\ &\leq -cd(\Delta t + \Delta t^2)C_0\left\|D_+\epsilon^n_1\right\|^2_{\ell^2_\Delta} + \Delta tC_3\mathcal{E}\left(e^n, f^n\right) + (-cd)||D_+e^n||^2_{\ell^2_\Delta}. \end{aligned} \tag{2.59}$$

Using once again the Young's inequality, we get that

$$\begin{aligned} I_3 &= -a\left\|\left(I - dD_+D_-\right)f^n - \Delta tD\left(f^n\left(u^n_\Delta + \frac{f^n}{2}\right)\right) - \Delta t\epsilon^n_2\right\|^2_{\ell^2_\Delta} \\ &\leq -a(1 + C_0\Delta t)\left\|\left(I - dD_+D_-\right)f^n\right\|^2_{\ell^2_\Delta} - a(\Delta t + \Delta t^2)C_0\left\|D\left(f^n\left(u^n_\Delta + \frac{1}{2}f^n\right)\right)\right\|^2_{\ell^2_\Delta} - a(\Delta t + \Delta t^2)C_0\left\|\epsilon^n_2\right\|^2_{\ell^2_\Delta} \\ &\leq -a(\Delta t + \Delta t^2)C_0\left\|\epsilon^n_2\right\|^2_{\ell^2_\Delta} + C\Delta t\max\left\{1, ||u^n_\Delta||_{\ell^\infty}, ||D_+u^n_\Delta||_{\ell^\infty}\right\}\mathcal{E}\left(e^n, f^n\right) + (-a)||(I - dD_+D_-)f^n||^2_{\ell^2_\Delta}. \end{aligned} \tag{2.60}$$

From (2.54) - (2.60) we get that

$$\mathcal{E}\left(e^{n+1}, f^{n+1}\right)(1 - C_1\Delta t) \leq \left(\Delta t + \Delta t^2\right)\left((-a)\left\|\epsilon^n_1\right\|^2_{\ell^2_\Delta} + (-a)\left\|\epsilon^n_2\right\|^2_{\ell^2_\Delta} + (-c)d\left\|D_+\epsilon^n_1\right\|^2_{\ell^2_\Delta}\right) + (1 + \Delta tC_2)\,\mathcal{E}\left(e^n, f^n\right).$$

□

2.2.4 The case $d = 0, b > 0$

The case $a = c = d = 0, b > 0$. In this case, as for the classical Boussinesq system (the case $a = b = c = 0, d > 0$, page 21) we derive estimates for a more general scheme: the following θ-scheme where the advection term is discretized according to a convex combination of n and $n+1$

$$\begin{cases} \frac{1}{\Delta t}\left(I - bD_+D_-\right)\left(\eta^{n+1} - \eta^n\right) + D\left((1-\theta)u^n + \theta u^{n+1}\right) + D\left(\eta^n u^n\right) = 0, \\ \\ \frac{1}{\Delta t}(u^{n+1} - u^n) + D\left((1-\theta)\eta^n + \theta\eta^{n+1}\right) + \frac{1}{2}D\left(\left(u^n\right)^2\right) = \frac{\tau_2 \Delta x}{2} D_+D_-\left(\eta^n\right). \end{cases} \tag{2.61}$$

The convergence error verifies:

$$\begin{cases} \left(I - bD_+D_-\right)E^-(n) + \Delta t D\left((1-\theta)f^n + \theta f^{n+1}\right) + \Delta t D\left(e^n u^n_\Delta\right) + \Delta t D\left(\eta^n_\Delta f^n\right) + \Delta t D\left(e^n f^n\right) = -\Delta t \epsilon_1^n, \\ \\ F^-(n) + \Delta t D\left((1-\theta)e^n + \theta e^{n+1}\right) + \Delta t D\left(f^n u^n_\Delta\right) + \frac{\Delta t}{2} D\left(\left(f^n\right)^2\right) = \frac{\tau_2}{2}\Delta t \Delta x D_+D_-\left(e^n\right) - \Delta t \epsilon_2^n. \end{cases} \tag{2.62}$$

We work with

$$\mathcal{E}(e,f) = \|e\|^2_{\ell^2_\Delta} + b\left\|D_+(e)\right\|^2_{\ell^2_\Delta} + \|f\|^2_{\ell^2_\Delta}. \tag{2.63}$$

Proof of Proposition 2.2 in the case $a = c = d = 0, b > 0$. We multiply the first equation by $E^+(n)$ in order to get that

$$\begin{aligned} &\left(\left\|e^{n+1}\right\|^2_{\ell^2} + b\left\|D_+e^{n+1}\right\|^2_{\ell^2}\right) - \left(\|e^n\|^2_{\ell^2} + b\|D_+e^n\|^2_{\ell^2}\right) \\ &= -\Delta t\left\langle D\left((1-\theta)f^n + \theta f^{n+1}\right) + D\left(e^n u^n_\Delta\right) + D\left(\eta^n_\Delta f^n\right) + D\left(e^n f^n\right) + \epsilon_1^n, E^+(n)\right\rangle. \end{aligned}$$

Integration by parts (1.22) gives

$$\begin{aligned} &\left(\left\|e^{n+1}\right\|^2_{\ell^2} + b\left\|D_+e^{n+1}\right\|^2_{\ell^2}\right) - \left(\|e^n\|^2_{\ell^2} + b\|D_+e^n\|^2_{\ell^2}\right) \\ &= -\Delta t\left\langle \epsilon_1^n, E^+(n)\right\rangle + \Delta t\left\langle (1-\theta)f^n + \theta f^{n+1} + e^n u^n_\Delta + \eta^n_\Delta f^n + e^n f^n, DE^+(n)\right\rangle \\ &\leq \Delta t\|\epsilon_1^n\|^2_{\ell^2_\Delta} + \Delta t C_1 \mathcal{E}\left(e^n, f^n\right) + \Delta t C_2 \mathcal{E}\left(e^{n+1}, f^{n+1}\right), \end{aligned} \tag{2.64}$$

with C_1 and C_2 two constants proportional to

$$1 + \frac{1}{b}\max\left\{||u^n_\Delta||_{\ell^\infty}, ||\eta^n_\Delta||_{\ell^\infty}\right\}.$$

Using the second equation of (2.62), the triangle inequality along with Proposition 2.3 we get that:

$$\begin{aligned} \left\|f^{n+1}\right\|_{\ell^2} &\leq \left\|f^n - \Delta t D\left(f^n u^n_\Delta\right) - \frac{\Delta t}{2} D\left(\left(f^n\right)^2\right) + \frac{\tau_2}{2}\Delta t \Delta x D_+D_-\left(f^n\right)\right\|_{\ell^2} \\ &\quad + \Delta t\|\epsilon_2^n\|_{\ell^2_\Delta} + \Delta t(1-\theta)\left\|D\left(e^n\right)\right\|_{\ell^2_\Delta} + \Delta t\theta\left\|D\left(e^{n+1}\right)\right\|_{\ell^2_\Delta} \\ &\leq \Delta t\|\epsilon_2^n\|_{\ell^2_\Delta} + (1 + C\Delta t)\|f^n\|_{\ell^2_\Delta} + \Delta t(1-\theta)\left\|D\left(e^n\right)\right\|_{\ell^2_\Delta} + \Delta t\theta\left\|D\left(e^{n+1}\right)\right\|_{\ell^2_\Delta}. \end{aligned} \tag{2.65}$$

Thus, by adding up estimate (2.64) with the square of the estimate (2.65) we get that

$$(1 - C_3\Delta t)\,\mathcal{E}\left(e^{n+1}, f^{n+1}\right) \leq \Delta t\|\epsilon_1^n\|^2_{\ell^2_\Delta} + (\Delta t + \Delta t^2)C_0\|\epsilon_2^n\|^2_{\ell^2_\Delta} + (1 + C_4\Delta t)\,\mathcal{E}\left(e^n, f^n\right).$$

□

The case $a < 0, b > 0, c = d = 0$. The convergence error satisfies:

$$\begin{cases} \left(I - bD_+D_-\right)e^{n+1} + \Delta t D f^{n+1} + a\Delta t D_+D_-Df^{n+1} = \left(I - bD_+D_-\right)e^n \\ -\Delta t D\left(e^n u^n_\Delta\right) - \Delta t D\left(e^n f^n\right) - \Delta t D\left(\eta^n_\Delta f^n\right) - \Delta t \epsilon_1^n, \\ \\ f^{n+1} + \Delta t D e^{n+1} = f^n - \Delta t D\left(f^n\left(u^n_\Delta + \frac{1}{2}f^n\right)\right) + \frac{\tau_2}{2}\Delta t\Delta x D_+D_-\left(f^n\right) - \Delta t\epsilon_2^n. \end{cases} \tag{2.66}$$

Consider the energy functional we will work with will be

$$\mathcal{E}(e,f) = \|e\|^2_{\ell^2_\Delta} + b\|D_+e\|^2_{\ell^2_\Delta} + \|f\|^2_{\ell^2_\Delta} + (-a)\|D_+f\|^2_{\ell^2_\Delta}.$$

Proof of Proposition 2.2 in the case $a<0, b>0, c=d=0$. Let us multiply the first equation of (2.66) with $2e^{n+1}$ to obtain:

$$\begin{aligned}
&2\left\|e^{n+1}\right\|_{\ell^2_\Delta}^2+2b\left\|D_+e^{n+1}\right\|_{\ell^2_\Delta}^2+2\Delta t\left\langle Df^{n+1},e^{n+1}\right\rangle+2a\Delta t\left\langle D_+D_-Df^{n+1},e^{n+1}\right\rangle\\
&=2\left\langle\left(I-bD_+D_-\right)e^n,e^{n+1}\right\rangle-2\Delta t\left\langle D\left(e^nu^n_\Delta\right)+D\left(e^nf^n\right)+D\left(\eta^n_\Delta f^n\right)+\epsilon_1^n,e^{n+1}\right\rangle\\
&\leq\left\|e^{n+1}\right\|_{\ell^2_\Delta}^2+b\left\|D_+e^{n+1}\right\|_{\ell^2_\Delta}^2+\left\|e^n\right\|_{\ell^2_\Delta}^2+b\left\|D_+e^n\right\|_{\ell^2_\Delta}^2\\
&+2\Delta t||D(e^nu^n_\Delta)+D(e^nf^n)+D(\eta^n_\Delta f^n)||_{\ell^2_\Delta}||e^{n+1}||_{\ell^2_\Delta}+\Delta t||\epsilon_1^n||^2_{\ell^2_\Delta}+\Delta t||e^{n+1}||^2_{\ell^2_\Delta}. \qquad (2.67)
\end{aligned}$$

The last inequality is due to Cauchy-Schwarz and Young's inequalities. Notice that

$$||D(e^nu^n_\Delta)+D(e^nf^n)+D(\eta^n_\Delta f^n)||_{\ell^2_\Delta}\leq C_1\mathcal{E}^{1/2}(e^n,f^n),$$

with C_1 proportional to

$$\max\left\{||u^n_\Delta||_{\ell^\infty},||Du^n_\Delta||_{\ell^\infty},||\eta^n_\Delta||_{\ell^\infty},||D\eta^n_\Delta||_{\ell^\infty}\right\}\max\left\{1,\frac{1}{b},\frac{1}{-a}\right\}.$$

Thus, Equation (2.67) becomes

$$\begin{aligned}
&\left\|e^{n+1}\right\|_{\ell^2_\Delta}^2+b\left\|D_+e^{n+1}\right\|_{\ell^2_\Delta}^2+2\Delta t\left\langle Df^{n+1},e^{n+1}\right\rangle+2a\Delta t\left\langle D_+D_-Df^{n+1},e^{n+1}\right\rangle\\
&\leq\left\|e^n\right\|_{\ell^2_\Delta}^2+b\left\|D_+e^n\right\|_{\ell^2_\Delta}^2+C_1\Delta t\mathcal{E}^{\frac{1}{2}}\left(e^n,f^n\right)\left\|e^{n+1}\right\|_{\ell^2_\Delta}+\Delta t\left\|\epsilon_1^n\right\|_{\ell^2_\Delta}^2+\Delta t\left\|e^{n+1}\right\|_{\ell^2_\Delta}^2. \qquad (2.68)
\end{aligned}$$

Next, taking the square of the ℓ^2_Δ-norm of the second equation of (2.66) and using Proposition 2.4 we get that

$$\begin{aligned}
&\left\|f^{n+1}\right\|_{\ell^2_\Delta}^2+2\Delta t\left\langle f^{n+1},De^{n+1}\right\rangle+(\Delta t)^2\left\|De^{n+1}\right\|_{\ell^2_\Delta}^2\\
&\leq\left(\Delta t+(\Delta t)^2\right)C_0\left\|\epsilon_2^n\right\|_{\ell^2_\Delta}^2+(1+C_0\Delta t)\left\|f^n-\Delta tD\left(f^n\left(u^n_\Delta+\frac{1}{2}f^n\right)\right)+\frac{\tau_2}{2}\Delta t\Delta xD_+D_-\left(f^n\right)\right\|_{\ell^2_\Delta}^2\\
&\leq\left(\Delta t+(\Delta t)^2\right)C_0\left\|\epsilon_2^n\right\|_{\ell^2_\Delta}^2+(1+C_2\Delta t)\left\|f^n\right\|_{\ell^2_\Delta}^2. \qquad (2.69)
\end{aligned}$$

When we will sum up Equations (2.68) and (2.69), both terms

$$2\Delta t\left\langle Df^{n+1},e^{n+1}\right\rangle\quad\text{and}\quad2\Delta t\left\langle f^{n+1},De^{n+1}\right\rangle$$

will cancel each other, thanks to Relation (1.22). We have to cancel $2a\Delta t\left\langle D_+D_-Df^{n+1},e^{n+1}\right\rangle$ in (2.68) too. This is the aim of the following computation.
Applying $\sqrt{-a}D_+$ into the second equation of (2.66) and taking the square of the ℓ^2_Δ-norm of the resulting equation yields (with Proposition 2.5)

$$\begin{aligned}
&(-a)\left\|D_+f^{n+1}\right\|_{\ell^2_\Delta}^2+2(-a)\Delta t\left\langle D_+f^{n+1},D_+De^{n+1}\right\rangle+(-a)(\Delta t)^2\left\|D_+De^{n+1}\right\|_{\ell^2_\Delta}^2\\
&\leq\left(\Delta t+(\Delta t)^2\right)C_0(-a)\left\|D_+\epsilon_2^n\right\|_{\ell^2_\Delta}^2\\
&+(1+C_0\Delta t)(-a)\left\|D_+f^n-\Delta tD_+D\left(f^n\left(u^n_\Delta+\frac{1}{2}f^n\right)\right)+\frac{\tau_2}{2}\Delta t\Delta xD_+D_+D_-\left(f^n\right)\right\|_{\ell^2_\Delta}^2\\
&\leq\left(\Delta t+(\Delta t)^2\right)C_0(-a)\left\|D_+\epsilon_2^n\right\|_{\ell^2_\Delta}^2+(1+C_3\Delta t)(-a)\left\|D_+f^n\right\|_{\ell^2_\Delta}^2+\Delta tC_4(-a)\left\|f^n\right\|_{\ell^2_\Delta}^2. \qquad (2.70)
\end{aligned}$$

Adding up the estimates (2.68), (2.69) and (2.70) leads to

$$\begin{aligned}
(1-C_5\Delta t)\mathcal{E}\left(e^{n+1},f^{n+1}\right)\leq\Delta t\left\|\epsilon_1^n\right\|_{\ell^2_\Delta}^2+\left(\Delta t+(\Delta t)^2\right)C_0\left(\left\|\epsilon_2^n\right\|_{\ell^2_\Delta}^2+(-a)\left\|D_+\epsilon_2^n\right\|_{\ell^2_\Delta}^2\right)\\
+(1+C_6\Delta t)\mathcal{E}\left(e^n,f^n\right).
\end{aligned}$$

□

The case $a<0, b>0, c<0, d=0$. Finally, in this case, the convergence error satisfies:

$$\begin{cases} (I-bD_+D_-)\,e^{n+1}+\Delta tDf^{n+1}+a\Delta tD_+D_-Df^{n+1}=(I-bD_+D_-)\,e^n \\ -\Delta tD\left(e^nu_\Delta^n\right)-\Delta tD\left(e^nf^n\right)-\Delta tD\left(\eta_\Delta^nf^n\right)-\Delta t\epsilon_1^n, \\ \\ f^{n+1}+\Delta tDe^{n+1}+c\Delta tD_+D_-De^{n+1}=f^n-\Delta tD\left(f^n\left(u_\Delta^n+\frac{1}{2}f^n\right)\right)+\frac{\tau_2}{2}\Delta t\Delta xD_+D_-\left(f^n\right)-\Delta t\epsilon_2^n. \end{cases} \tag{2.71}$$

In this case, we will use the energy functional:

$$\mathcal{E}\left(e,f\right)=(-c)\left\|(I-bD_+D_-)\,e\right\|_{\ell^2_\Delta}^2+\left\|f\right\|_{\ell^2_\Delta}^2+(-a)\,b\left\|D_+f\right\|_{\ell^2_\Delta}^2.$$

Proof of Proposition 2.2 in the case $a<0, b>0, c<0, d=0$. Taking the square of the first equations of (2.71) and multiplying the result with $(-c)$ yields

$$\begin{aligned} J_1=&(-c)\left\|(I-bD_+D_-)\,e^{n+1}\right\|_{\ell^2_\Delta}^2+(-c)\left(\Delta t\right)^2\left\|Df^{n+1}\right\|_{\ell^2_\Delta}^2+(-c)\,a^2\left(\Delta t\right)^2\left\|D_+D_-Df^{n+1}\right\|_{\ell^2_\Delta}^2 \\ &+2ac\left(\Delta t\right)^2\left\|D_+Df^{n+1}\right\|_{\ell^2_\Delta}^2-2c\Delta t\left\langle(I-bD_+D_-)\,e^{n+1},Df^{n+1}\right\rangle \\ &-2ac\Delta t\left\langle e^{n+1},D_+D_-Df^{n+1}\right\rangle+2acb\Delta t\left\langle D_+D_-e^{n+1},D_+D_-Df^{n+1}\right\rangle. \end{aligned} \tag{2.72}$$

Taking the square of the ℓ^2_Δ-norm of the second equation of (2.71), we get that

$$\begin{aligned} J_2=&\left\|f^{n+1}\right\|_{\ell^2_\Delta}^2+\left(\Delta t\right)^2\left\|De^{n+1}\right\|_{\ell^2_\Delta}^2+\left(c\Delta t\right)^2\left\|D_+D_-De^{n+1}\right\|_{\ell^2_\Delta}^2-2c\left(\Delta t\right)^2\left\|D_+De^{n+1}\right\|_{\ell^2_\Delta}^2 \\ &+2\Delta t\left\langle f^{n+1},De^{n+1}\right\rangle+2c\Delta t\left\langle f^{n+1},D_+D_-De^{n+1}\right\rangle. \end{aligned} \tag{2.73}$$

Let us observe that the terms

$$2\Delta t\left\langle f^{n+1},De^{n+1}\right\rangle \quad\text{and}\quad 2c\Delta t\left\langle f^{n+1},D_+D_-De^{n+1}\right\rangle, \quad \text{in (2.73)}$$

and

$$-2ac\Delta t\left\langle e^{n+1},D_+D_-Df^{n+1}\right\rangle \quad\text{and}\quad -2c\Delta t\left\langle(I-bD_+D_-)e^{n+1},Df^{n+1}\right\rangle, \quad \text{in (2.72)}$$

can be lower controlled by the energy $-\Delta tC\mathcal{E}(e^{n+1},f^{n+1})$ (by using integration by parts (1.22) and Cauchy-Schwarz inequality) such that they do not raise any issues. In order to get rid of the term

$$2acb\Delta t\left\langle D_+D_-e^{n+1},D_+D_-Df^{n+1}\right\rangle \quad \text{in (2.72)} \tag{2.74}$$

we will apply $\sqrt{(-a)\,b}D_+$ into the second equation of (2.71) and consider the square of the ℓ^2_Δ-norm of the result. First, let us compute:

$$\begin{aligned} J_3=&(-a)\,b\left\|D_+f^{n+1}+\Delta tD_+De^{n+1}+c\Delta tD_+D_+D_-De^{n+1}\right\|_{\ell^2_\Delta}^2 \\ =&(-a)\,b\left\|D_+f^{n+1}\right\|_{\ell^2_\Delta}^2+(-a)\,b\left(\Delta t\right)^2\left\|D_+De^{n+1}\right\|_{\ell^2_\Delta}^2+(-a)\,b\left(c\Delta t\right)^2\left\|D_+D_+D_-De^{n+1}\right\|_{\ell^2_\Delta}^2 \\ &+2\left(-a\right)b\Delta t\left\langle D_+f^{n+1},D_+De^{n+1}\right\rangle+2(-a)bc\Delta t\left\langle D_+f^{n+1},D_+D_+D_-De^{n+1}\right\rangle \\ &+2abc\left(\Delta t\right)^2\left\|D_+D_-De^{n+1}\right\|_{\ell^2_\Delta}^2. \end{aligned} \tag{2.75}$$

Of course the problematic term from (2.74) will cancel with

$$-2abc\Delta t\left\langle D_+f^{n+1},D_+D_+D_-De^{n+1}\right\rangle$$

appearing in (2.75). The additional term $2\left(-a\right)b\Delta t\left\langle D_+f^{n+1},D_+D(e)^{n+1}\right\rangle$ in (2.75) can be once again controlled by the energy $-\Delta tC\mathcal{E}(e^{n+1},f^{n+1})$ (by using integrations by parts (1.22) and Cauchy-Schwarz inequality).
Let us now interest to the right hand side. For the first equation of (2.71), we obtain by Young inequality

$$J_1\leq(-c)\left(\Delta t+\Delta t^2\right)C_0\left\|\epsilon_1^n\right\|_{\ell^2_\Delta}^2+\left(1+C_0\Delta t\right)(-c)\left\|(I-bD_+D_-)\,e^n\right\|_{\ell^2_\Delta}^2+C_1\Delta t\mathcal{E}\left(e^n,f^n\right). \tag{2.76}$$

In the previous inequality, we have upper bounded $\Delta t||-D(e^n u_\Delta^n)-D(e^n f^n)-D(\eta_\Delta^n f^n)||_{\ell^2_\Delta}^2$ by

$$\max\left\{||u_\Delta^n||_{\ell^\infty},||Du_\Delta^n||_{\ell^\infty},||\eta_\Delta^n||_{\ell^\infty},||D\eta_\Delta^n||_{\ell^\infty},\right\}\max\left\{\frac{1}{(-a)b},\frac{1}{2b(-c)}\right\}\mathcal{E}(e^n,f^n).$$

The second equation of (2.71) gives

$$\begin{aligned}J_2&\leq\left(\Delta t+\Delta t^2\right)C_0\left\|\epsilon_2^n\right\|_{\ell^2_\Delta}^2+(1+C_0\Delta t)\left\|f^n-\Delta tD\left(f^n\left(u_\Delta^n+\frac{1}{2}f^n\right)\right)+\frac{\tau_2^n}{2}\Delta t\Delta xD_+D_-\left(f^n\right)\right\|_{\ell^2_\Delta}^2\\&\leq\left(\Delta t+\Delta t^2\right)C_0\left\|\epsilon_2^n\right\|_{\ell^2_\Delta}^2+(1+C_1\Delta t)\left\|f^n\right\|_{\ell^2_\Delta}^2.\end{aligned}\tag{2.77}$$

Moreover, using Proposition (2.5) we get

$$J_3\leq(-a)\,b\left(\Delta t+(\Delta t)^2\right)C_0\left\|D_+\epsilon_2^n\right\|_{\ell^2_\Delta}^2+C_2\Delta t(-a)b\left\|f^n\right\|_{\ell^2_\Delta}^2+(1+C_3\Delta t)\,(-a)\,b\left\|D_+f^n\right\|_{\ell^2_\Delta}^2\tag{2.78}$$

Puting togheter Estimates (2.72), (2.73) and (2.75) with (2.76), (2.77) and (2.78) we get that

$$\begin{aligned}(1-C_4\Delta t)\,\mathcal{E}\left(e^{n+1},f^{n+1}\right)\leq&\left(\Delta t+\Delta t^2\right)C_0\,(-c)\left\|\epsilon_1^n\right\|_{\ell^2_\Delta}^2+\left(\Delta t+(\Delta t)^2\right)C_0\left(\left\|\epsilon_2^n\right\|_{\ell^2_\Delta}^2+(-ab)\left\|D_+\epsilon_2^n\right\|_{\ell^2_\Delta}^2\right)\\&+(1+C_5\Delta t)\,\mathcal{E}\left(e^n,f^n\right).\end{aligned}$$

□

3 Experimental results

To illustrate our numerical results, we compare in this paragraph the exact solution and the numerical one, for several values of a, b, c and d. The simulation is done by fixing $[0,L]$ the space domain with $L=40$. Moreover, we use periodic boundary conditions. Those conditions are not absorbing boundary conditions, which would mimic perfectly the behavior on $\mathbb{Z}$, but we fix the final time T small enough and we take the initial conditions localized enough in order to minimize boundary effects.

The comparisons between the exact solution and the numerical one (Figures 2-7) are done with the space cell size $\Delta x=\frac{1}{2^6}$ and a time step $\Delta t=0.001$. The convergence results (Tables 1-5) are computed with a number of cells J satisfying $J\in\{640,1280,2560,5120,10240\}$. In these tables, we choose the final time $T=2$ and the time step is $\Delta t^n=\frac{\Delta x}{||u_\Delta^n||_{\ell^\infty}}$ to verify the CFL-type condition. In the case where $bd=0$, we chose the Rusanov scheme for the hyperbolic part and the Rusanov coefficients τ_i^n verify $\tau_1^n\geq||u_\Delta^n||_{\ell^\infty}$ and $\tau_2^n\geq||u_\Delta^n||_{\ell^\infty}$, when they are needed. In the case where $bd\neq0$, the choice $\theta=1$ combined with the relation $\Delta t^n=\frac{\Delta x}{||u_\Delta^n||_{\ell^\infty}}$ impose that the expected experimental order is 1.

Linear case

First of all, we test the linear case

$$\begin{cases}\left(I-b\partial_{xx}^2\right)\partial_t\eta+\left(I+a\partial_{xx}^2\right)\partial_xu=0,\\\left(I-d\partial_{xx}^2\right)\partial_tu+\left(I+c\partial_{xx}^2\right)\partial_x\eta=0,\\\eta_{|t=0}=\eta_0,\ u_{|t=0}=u_0.\end{cases}\tag{L_{abcd}}$$

For that case, the discrete energy is conserved when b and d are non zero and when $\theta=\frac{1}{2}$ (*ie* for the Crank-Nicolson scheme), and indeed, we illustrate this conservation in Figure 1. We choose $a=-\frac{7}{30}$, $b=\frac{7}{15}$, $c=-\frac{2}{5}$ and $d=\frac{1}{2}$ and initialize the scheme with

$$\begin{cases}\eta_0(x)=\dfrac{3}{8}\mathrm{sech}^2\left(\dfrac{1}{2}\sqrt{\dfrac{5}{7}}\left(x-\dfrac{L}{2}\right)\right),\\u_0(x)=\dfrac{1}{2\sqrt{2}}\mathrm{sech}^2\left(\dfrac{1}{2}\sqrt{\dfrac{5}{7}}\left(x-\dfrac{L}{2}\right)\right).\end{cases}$$

For $\Delta x = \frac{1}{2^5}$, $\Delta t = 0.001$ and nor τ_1^n neither τ_2^n we obtain the conservation of the discret energy, more or less 10^{-11}, as illustrated in Figure 1.

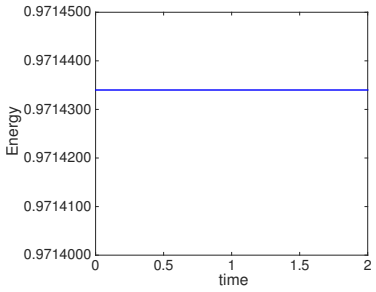

Figure 1 – Energy conservation in the linear case

Case $b > 0$, $d > 0$

First we consider the BBM-BBM system ($a = 0$, $b = \frac{1}{6}$, $c = 0$, $d = \frac{1}{6}$) for which two different exact solutions are known. The first of them is obtained in [7] and corresponds to the exact traveling-wave solution of [17] :

$$(A)\begin{cases} \eta(t,x) = \dfrac{15}{4}\left\{-2+\cosh\left(3\sqrt{\dfrac{2}{5}}\left(x-\dfrac{L}{2}-\dfrac{5}{2}t\right)\right)\right\}\operatorname{sech}^4\left(\dfrac{3}{\sqrt{10}}\left(x-\dfrac{L}{2}-\dfrac{5}{2}t\right)\right), \\ u(t,x) = \dfrac{15}{2}\operatorname{sech}^2\left(\dfrac{3}{\sqrt{10}}\left(x-\dfrac{L}{2}-\dfrac{5}{2}t\right)\right). \end{cases}$$

The results are represented in Figure 2.

Next, we consider the exact solution obtained in [17], whose expression is

$$(B)\begin{cases} \eta(t,x) = -1, \\ u(t,x) = C_s\left(1-\dfrac{\rho}{6}\right) + \dfrac{C_s\rho}{2}\operatorname{sech}^2\left(\dfrac{\sqrt{\rho}}{2}\left(x-\dfrac{L}{2}-C_st\right)\right). \end{cases}$$

We choose $C_s = 2$ and $\rho = 1.1$ in order to construct the numerical solution. The corresponding result can be found in Figure 3.

In order to illustrate our theoretical rate of convergence, we perform the previous examples with increasingly smaller space meshes. The experimental rates are gathered in Table 1.

Δx	Case (A)		Case (B)	
	energy error	exp. rate	energy error	exp. rate
$6.25000.10^{-2}$	$4.48993.10^{0}$		$8.51815.10^{-2}$	
$3.12500.10^{-2}$	$2.05132.10^{0}$	1.13270	$4.14750.10^{-2}$	1.03830
$1.56250.10^{-2}$	$9.80969.10^{-1}$	1.06450	$2.04332.10^{-2}$	1.02133
$7.81250.10^{-3}$	$4.79738.10^{-1}$	1.03181	$1.01409.10^{-2}$	1.01073
$3.90625.10^{-3}$	$2.37234.10^{-1}$	1.01580	$5.05189.10^{-3}$	1.00529

Table 1 – Experimental rates of convergence for $b, d \neq 0$

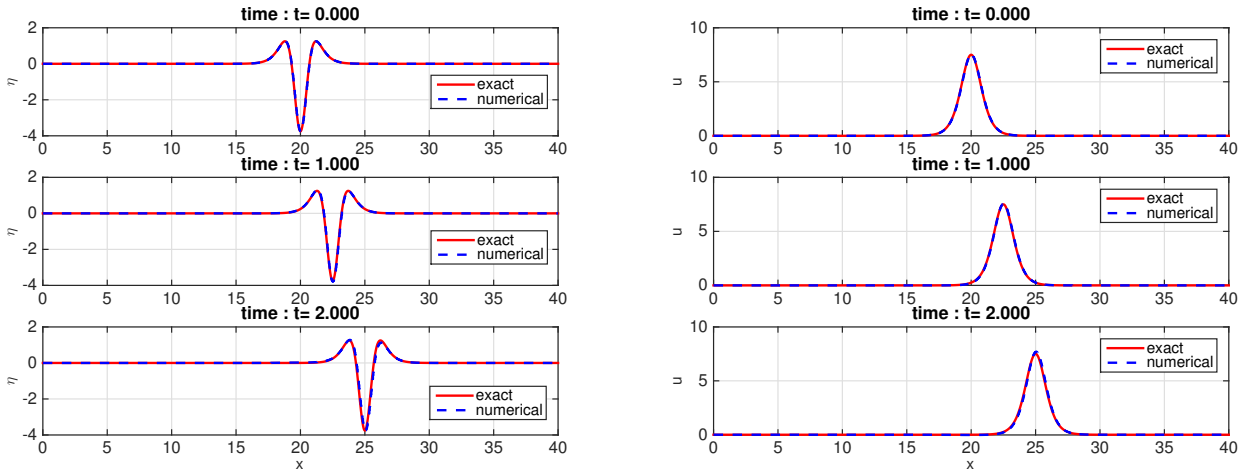

Figure 2 – Case (A) where $a = 0$, $b = \frac{1}{6}$, $c = 0$, $d = \frac{1}{6}$: results for η (left) and u (right)

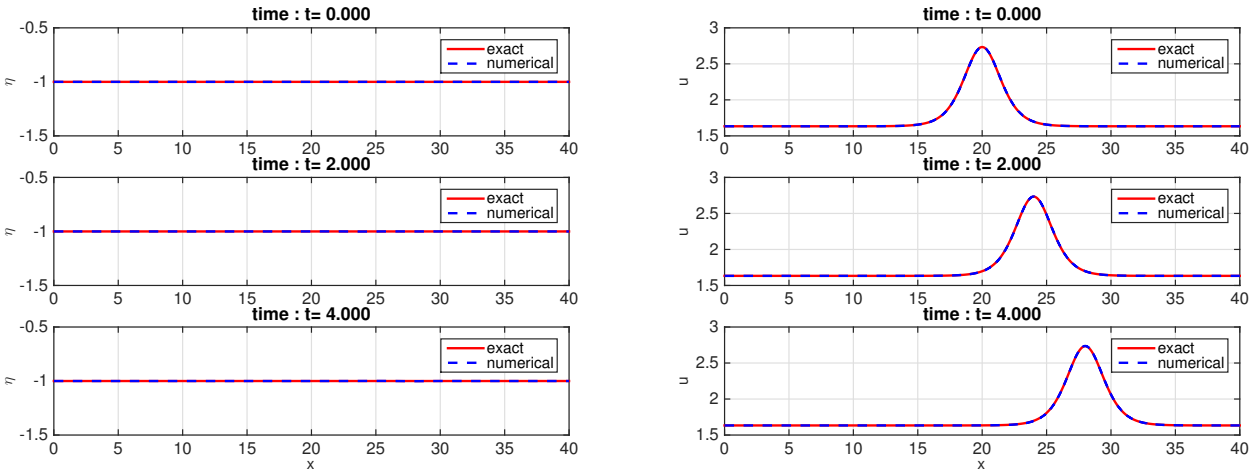

Figure 3 – Case (B) where $a = 0$, $b = \frac{1}{6}$, $c = 0$, $d = \frac{1}{6}$: results for η (left) and u (right)

Another example regards the case $a = -\frac{7}{30}$, $b = \frac{7}{15}$, $c = -\frac{2}{5}$ and $d = \frac{1}{2}$ which is discussed in [17]. The exact solution is

$$(C)\begin{cases} \eta(t,x) = \dfrac{3}{8}\mathrm{sech}^2\left(\dfrac{1}{2}\sqrt{\dfrac{5}{7}}\left(x - \dfrac{L}{2} - 5\dfrac{\sqrt{2}}{6}t\right)\right), \\ u(t,x) = \dfrac{1}{2\sqrt{2}}\mathrm{sech}^2\left(\dfrac{1}{2}\sqrt{\dfrac{5}{7}}\left(x - \dfrac{L}{2} - 5\dfrac{\sqrt{2}}{6}t\right)\right). \end{cases}$$

The results are gathered in Figure 4 and Table 2.

The last example of this paragraph is $a = 0$, $b = \frac{1}{3}$, $c = -\frac{1}{3}$ and $d = \frac{1}{3}$, studied in [17]. The exact solution writes

$$(D)\begin{cases} \eta(t,x) = -1, \\ u(t,x) = (1 - \dfrac{\rho}{3})C_s + C_s\rho\mathrm{sech}^2\left(\dfrac{\sqrt{\rho}}{2}\left(x - \dfrac{L}{2} - C_s t\right)\right), \end{cases}$$

with $C_s = 3$ and $\rho = 2$ to obtain Figure 5 and Table 3.

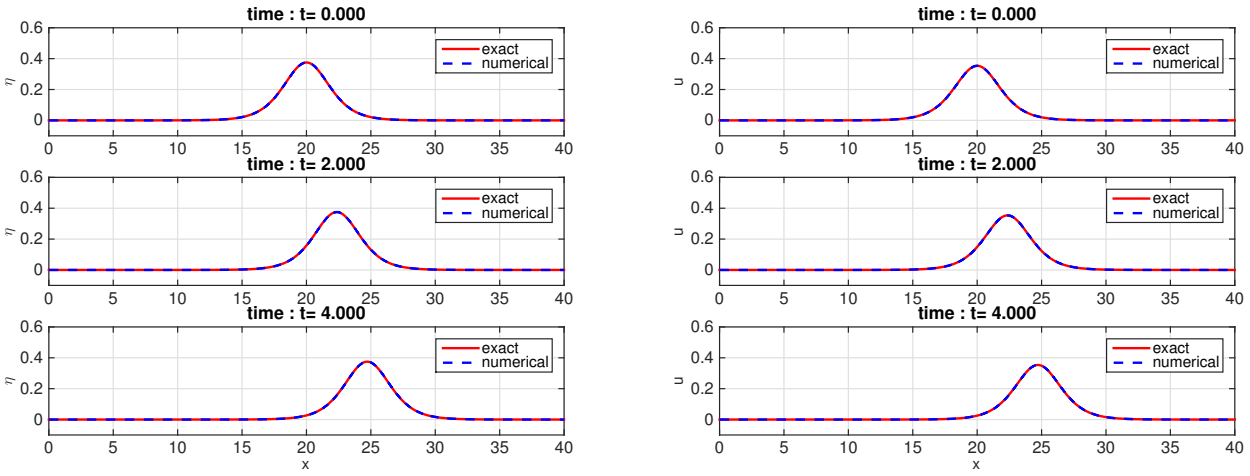

Figure 4 – Case (C) where $a = -\frac{7}{30}$, $b = \frac{7}{15}$, $c = -\frac{2}{5}$, $d = \frac{1}{2}$: results for η (left) and u (right)

Δx	Case (C)	
	energy error	exp. rate
$6.25000.10^{-2}$	$2.27860.10^{-2}$	
$3.12500.10^{-2}$	$1.126019.10^{-2}$	1.01692
$1.56250.10^{-2}$	$5.612993.10^{-3}$	1.00439
$7.81250.10^{-3}$	$2.847910.10^{-3}$	0.97887

Table 2 – Experimental rates of convergence for $a, b, c, d \neq 0$

Remark 3.1. *In this four examples, there is nor viscosity τ_1^n neither τ_2^n. The terms b and d enable to control and stabilize the scheme, even for the nonlinear Rusanov term, as shown in Figures 2-5.*
With the previous choices of Δt^n and Δx, we recover the first order convergence, but notice that the schemes would be second order accurate in space and first order in time. When $\Delta t = \Delta x^2$, the second order is well found as illustrated in Table 4.

Case $bd = 0$

We decide to cancel b by testing two examples. First, we take $a = b = c = 0$, $d = \frac{1}{6}$. The exact solution is detailed in [17] :

$$(E) \begin{cases} \eta(t,x) = -1, \\ u(t,x) = \left(1 - \dfrac{\rho}{6}\right) C_s + \dfrac{C_s \rho}{2} \operatorname{sech}^2 \left(\dfrac{\sqrt{\rho}}{2} \left(x - \dfrac{L}{2} - C_s t \right) \right). \end{cases}$$

For Figure 6, we have fixed $C_s = 1$ and $\rho = 2$.

In the second example, we choose the case $a = -\frac{1}{6}$, $b = c = 0$ and $d = \frac{1}{2}$, for which we know the exact solution developed in [17]

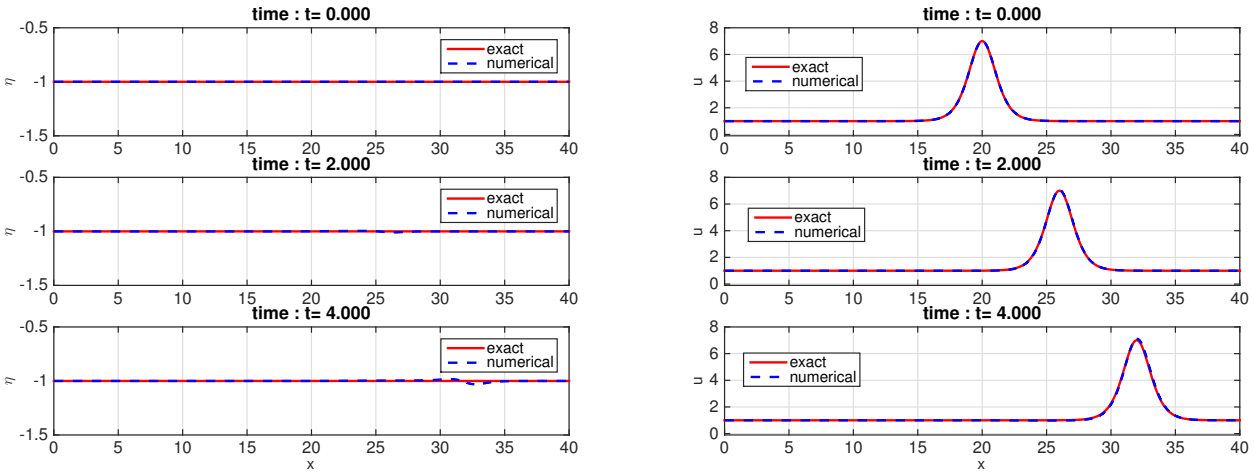

Figure 5 – Case (D) where $a = 0$, $b = \frac{1}{3}$, $c = -\frac{1}{3}$, $d = \frac{1}{3}$: results for η (left) and u (right)

Δx	Case (D)	
	energy error	exp. rate
$6.25000.10^{-2}$	$6.39353.10^{-1}$	
$3.12500.10^{-2}$	$3.09159.10^{-1}$	1.04826
$1.56250.10^{-2}$	$1.52031.10^{-1}$	1.02399
$7.81250.10^{-3}$	$7.53884.10^{-2}$	1.01195
$3.90625.10^{-3}$	$3.75388.10^{-2}$	1.00596

Table 3 – Experimental rates of convergence for $b, c, d \neq 0$

$$(F)\begin{cases} \eta(t,x) = -\frac{7}{4}\text{sech}^2\left(\frac{\sqrt{7}}{2}\left(x - \frac{L}{2} + \frac{1}{\sqrt{15}}t\right)\right), \\ u(t,x) = -\frac{7}{2}\sqrt{\frac{3}{5}}\text{sech}^2\left(\frac{\sqrt{7}}{2}\left(x - \frac{L}{2} + \frac{1}{\sqrt{15}}t\right)\right). \end{cases}$$

We obtain Figure 7.

The experimental rates of convergence in the two cases $b = 0$ are gathered in Table 5

In all the previous experimental results, the first order convergence is recovered, which matches to the theoretical results.

4 Traveling waves collision

Recently, in [8], the authors simulate the collision of two traveling waves moving in opposite directions in $[-L, L]$ in the BBM-BBM case ($a = 0$, $b = \frac{1}{6}$, $c = 0$, $d = \frac{1}{6}$). Motivated, by their results, we simulated the same phenomena but for different values of the *abcd* parameters.

First, we used our numerical results and performed the same experiment described in [8]. The initial condition

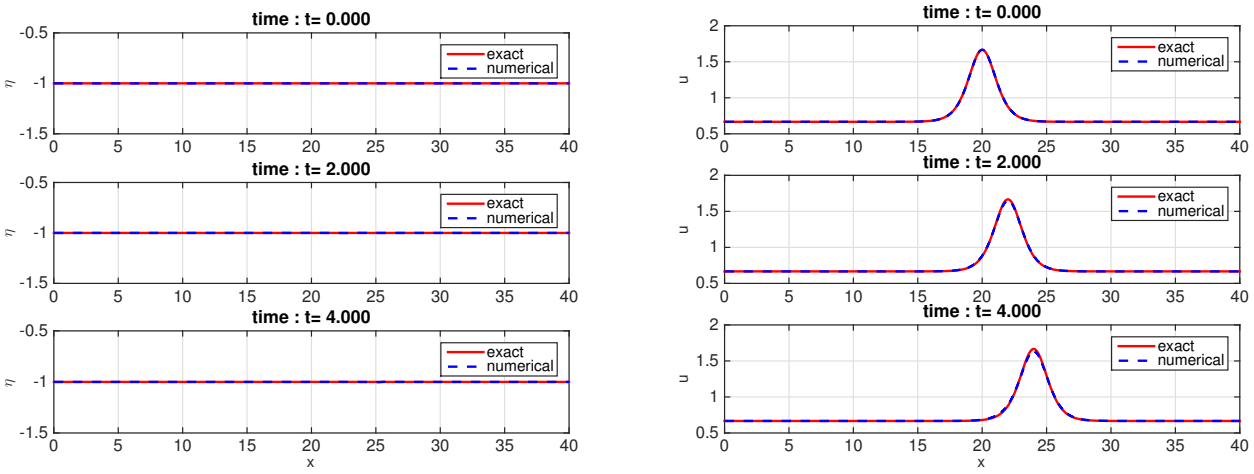

Figure 6 – Case (E) where $a = b = c = 0$, $d = \frac{1}{6}$: results for η (left) and u (right)

is fixed to

$$\begin{cases} \eta(t,x) = \eta_+(t,x) + \eta_-(t,x), \\ u(t,x) = u_+(t,x) + u_-(t,x), \end{cases}$$

with

$$\begin{cases} \eta_\pm(t,x) = \dfrac{15}{2}\mathrm{sech}^2\left(\dfrac{3}{\sqrt{10}}\left(x - x_\pm \pm \dfrac{5}{2}t\right)\right) - \dfrac{45}{4}\mathrm{sech}^4\left(\dfrac{3}{\sqrt{10}}\left(x - x_\pm \pm \dfrac{5}{2}t\right)\right), \\ u_\pm(t,x) = \mp\dfrac{15}{2}\mathrm{sech}^2\left(\dfrac{3}{\sqrt{10}}\left(x - x_\pm \pm \dfrac{5}{2}t\right)\right), \end{cases}$$

where $x_\pm = \pm\frac{L}{2}$.
The space domain is fixed at $[-14, 14]$ and initially, the traveling-waves are centered in $x_+ = 7$ and $x_- = -7$. We choose the same space size and time step as in [8], namely $\Delta x = 0.02$ and $\Delta t = 0.0001$. The simulation suggest that blow-up occurs while the explosion time appears to be around $t = 4.5$ as shown in Figure 8, result which is very close to the one obtained in [8].

We believe that it is an interesting feature that performing the same kind of experiments for different values of the parameters suggests different qualitative behaviour. Indeed, for $(a = -\frac{7}{30},\ b = \frac{7}{15},\ c = -\frac{2}{5},\ d = \frac{1}{2})$, Figure 9 suggests that there is no blow-up resulting from the collision of two traveling waves. The experiment is performed

Δx	Case (A)		Case (C)	
	energy error	exp. rate	energy error	exp. rate
$2.5000.10^{-1}$	$3.29137.10^{\ 0}$		$2.52768.10^{-2}$	
$1.2500.10^{-1}$	$7.75742.10^{-1}$	2.08504	$6.08893.10^{-3}$	2.05355
$6.2500.10^{-2}$	$1.90828.10^{-1}$	2.02330	$1.50692.10^{-3}$	2.01459
$3.1250.10^{-2}$	$4.75112.10^{-2}$	2.00594	$3.81640.10^{-4}$	1.98131

Table 4 – Experimental rates of convergence with $\Delta t = \Delta x^2$

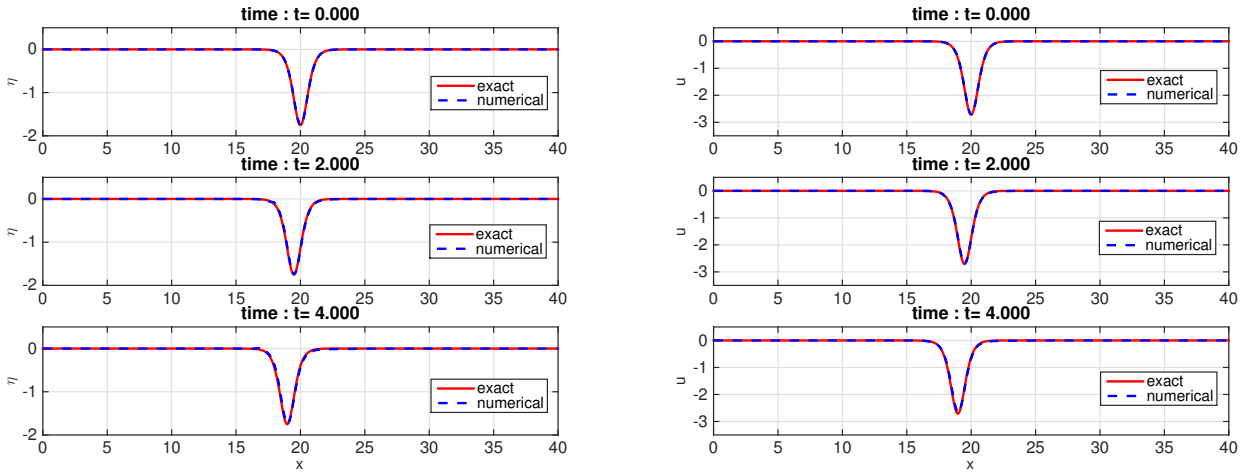

Figure 7 – Case (F) where $a = -\frac{1}{6}$, $b = c = 0$, $d = \frac{1}{2}$: results for η (left) and u (right)

Δx	Case (E)		Case (F)	
	energy error	exp. rate	energy error	exp. rate
$6.25000.10^{-2}$	$5.94214.10^{-2}$		$3.62176.10^{-1}$	
$3.12500.10^{-2}$	$3.01052.10^{-2}$	0.98097	$1.92823.10^{-1}$	0.93164
$1.56250.10^{-2}$	$1.51573.10^{-2}$	0.99000	$1.00366.10^{-1}$	0.94780
$7.81250.10^{-3}$	$7.60581.10^{-3}$	0.99483	$5.16267.10^{-2}$	0.95784
$3.90625.10^{-3}$	$3.80985.10^{-3}$	0.99737	$2.63453.10^{-2}$	0.96715

Table 5 – Experimental rates of convergence when $b = 0$ and $d > 0$

for the same value of Δt and Δx as in the previous case. This time, the initial value is

$$\begin{cases} \eta(t,x) = \eta_+(t,x) + \eta_-(t,x), \\ u(t,x) = u_+(t,x) + u_-(t,x), \end{cases}$$

with

$$\begin{cases} \eta_\pm(t,x) = \dfrac{3}{8}\mathrm{sech}^2\left(\dfrac{1}{2}\sqrt{\dfrac{5}{7}}\left(x - x_\pm \mp \dfrac{5\sqrt{2}}{6}t\right)\right), \\ u_\pm(t,x) = \pm\dfrac{1}{2\sqrt{2}}\mathrm{sech}^2\left(\dfrac{1}{2}\sqrt{\dfrac{5}{7}}\left(x - x_\pm \mp \dfrac{5\sqrt{2}}{6}t\right)\right), \end{cases}$$

where $x_\pm = \pm\frac{L}{2}$.

For $(a = 0, b = \frac{3}{5}, c = 0, d = 0)$, Figure 10 suggests that the blow up is due to the derivative of u. The experiment is performed with $\Delta t = 0.001$, $\Delta x = 0.01$ and the initial value is

$$\begin{cases} \eta(t,x) = \eta_+(t,x) + \eta_-(t,x), \\ u(t,x) = u_+(t,x) + u_-(t,x), \end{cases}$$

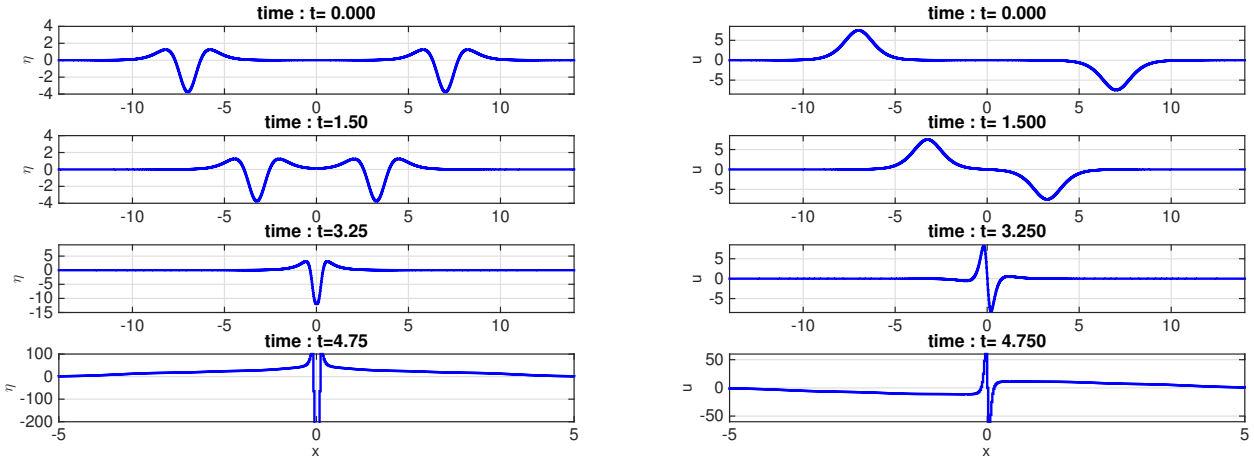

Figure 8 – Explosion for ($a = 0$, $b = \frac{1}{6}$, $c = 0$, $d = \frac{1}{6}$) : results for η (left) and u (right)

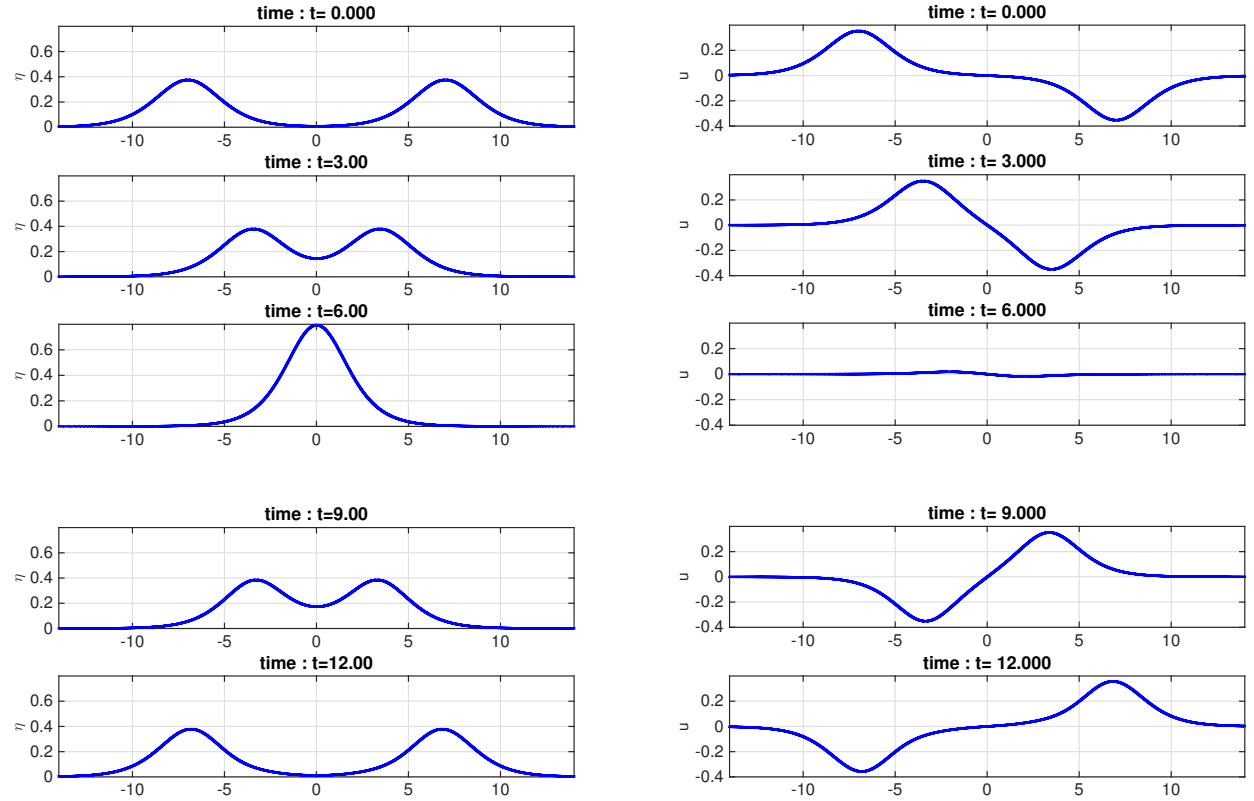

Figure 9 – No collision for ($a = -\frac{7}{30}$, $b = \frac{7}{15}$, $c = -\frac{2}{5}$, $d = \frac{1}{6}$) : results for η (left) and u (right)

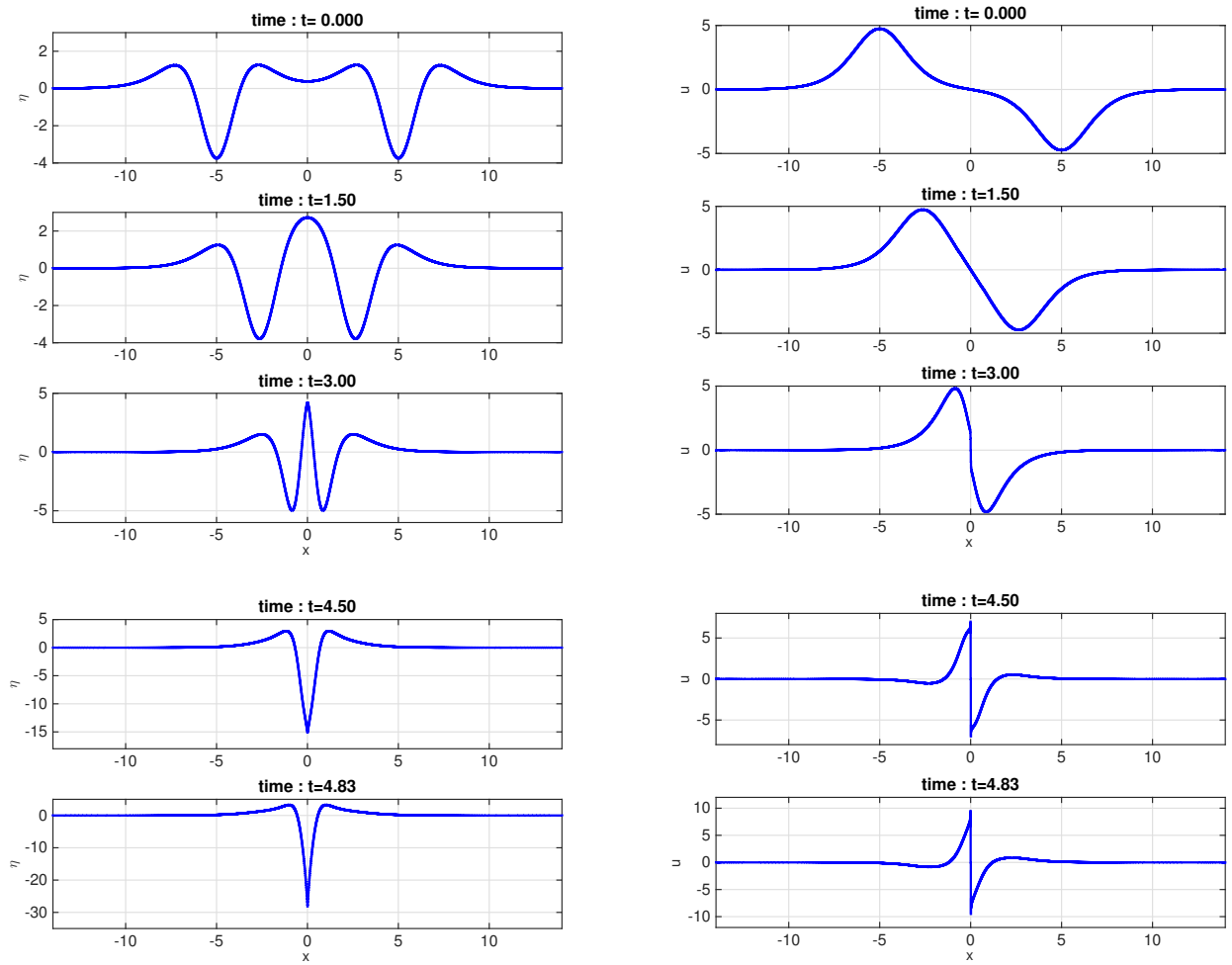

Figure 10 – Explosion for the derivative of u for ($a = 0$, $b = \frac{3}{5}$, $c = 0$, $d = 0$) : results for η (left) and u (right)

with

$$\begin{cases} \eta_{\pm}(t,x) = \dfrac{15}{2}\mathrm{sech}^2\left(\dfrac{1}{2}\left(x - x_{\pm} \mp \dfrac{\sqrt{10}}{2}t\right)\right) - \dfrac{45}{4}\mathrm{sech}^4\left(\dfrac{1}{2}\left(x - x_{\pm} \mp \dfrac{\sqrt{10}}{2}t\right)\right), \\ u_{\pm}(t,x) = \pm\dfrac{3\sqrt{10}}{2}\mathrm{sech}^2\left(\dfrac{1}{2}\left(x - x_{\pm} \mp \dfrac{\sqrt{10}}{2}t\right)\right), \end{cases}$$

where $x_{\pm} = \pm 5$.

A Appendix : Consistency error

1.1 Consistency error ϵ_1^n

By definition of the consistency error, on has

$$\epsilon_1^n = (I - bD_+D_-)\left(\frac{\eta_\Delta^{n+1} - \eta_\Delta^n}{\Delta t}\right) + (I + aD_+D_-)\,D\left(\theta u_\Delta^{n+1} + (1-\theta)u_\Delta^n\right) + D\left(\eta_\Delta^n u_\Delta^n\right) - \frac{\tau_1}{2}\Delta x D_+D_-\eta_\Delta^n.$$

We define

$$\begin{cases} E_{\text{time}} = (I - bD_+D_-)\left(\dfrac{\eta_\Delta^{n+1} - \eta_\Delta^n}{\Delta t}\right), \\ E_{\text{linear}} = (I + aD_+D_-)\, D\left(\theta u_\Delta^{n+1} + (1-\theta)u_\Delta^n\right), \\ E_{\text{non linear}} = D\left(\eta_\Delta^n u_\Delta^n\right), \\ E_{\text{viscosity}} = \dfrac{\tau_1}{2}\Delta x D_+D_-\eta_\Delta^n. \end{cases}$$

We define $Q = [x_j, x_{j+1}] \times [t^n, t^{n+1}]$. We will only develop the non linear part and enonce the results for the other parts. By Taylor expansions and Cauchy-Schwarz inequality, one has

$$E_{\text{time}} \lesssim \frac{1}{\Delta t \Delta x}\int_{x_j}^{x_{j+1}}\int_{t^n}^{t^{n+1}} \partial_t\eta(s,y) - b\partial_x^2\partial_t\eta(s,y)dsdy + \sqrt{\frac{\Delta t}{\Delta x}}||\partial_t^2\eta||_{L^2(Q)} + b\Delta x\sqrt{\frac{\Delta x}{\Delta t}}||\partial_x^4\partial_t\eta||_{L^2(Q)} + b\sqrt{\frac{\Delta t}{\Delta x}}||\partial_x^2\partial_t^2\eta||_{L^2(Q)}.$$

In the same way, we develop the E_{linear}-term.

$$\begin{aligned} E_{\text{linear}} \lesssim{}& \frac{1}{\Delta t\Delta x}\int_{x_j}^{x_{j+1}}\int_{t^n}^{t^{n+1}} \partial_x u(s,y) + a\partial_x^3 u(s,y)dsdy + \theta\sqrt{\frac{\Delta t}{\Delta x}}||\partial_x\partial_t u||_{L^2(Q)} + \Delta x\sqrt{\frac{\Delta x}{\Delta t}}||\partial_x^3 u||_{L^2(Q)} \\ &+ |a|\Delta x\sqrt{\frac{\Delta x}{\Delta t}}||\partial_x^5 u||_{L^2(Q)} + |a|\theta\sqrt{\frac{\Delta t}{\Delta x}}||\partial_t\partial_x^3 u||_{L^2(Q)} + |a|\Delta x\sqrt{\frac{\Delta x}{\Delta t}}||\partial_x^5 u||_{L^2(Q)}. \end{aligned}$$

We will develop the non linear part. We denote K the function on $[0,1]$

$$K(\nu) = \int_{x_j}^{x_{j+1}}\int_{t^n}^{t^{n+1}}\int_{x_j}^{x_{j+1}}\int_{t^n}^{t^{n+1}} \eta(s, y+\nu\Delta x)u(t, x+\nu\Delta x)dxdydsdt.$$

Thus,

$$\begin{aligned} K'(\nu) &= \int_{x_j}^{x_{j+1}}\int_{t^n}^{t^{n+1}}\int_{x_j}^{x_{j+1}}\int_{t^n}^{t^{n+1}} \Delta x\partial_x\eta(s,y+\nu\Delta x)u(t,x+\nu\Delta x) + \Delta x\eta(s,y+\nu\Delta x)\partial_x u(t,x+\nu\Delta x)dxdydsdt \\ &= \int_{x_j}^{x_{j+1}}\int_{t^n}^{t^{n+1}}\int_{x_j}^{x_{j+1}}\int_{t^n}^{t^{n+1}} \Delta x\partial_x\eta(s,y+\nu\Delta x)\left[u(s,x+\nu\Delta x) + \int_s^t \partial_t u(\tau, x+\nu\Delta x)d\tau\right] \\ &+ \Delta x\partial_x u(t,x+\nu\Delta x)\left[\eta(t,y+\nu\Delta x) + \int_t^s \partial_t\eta(\tau,y+\nu\Delta x)d\tau\right]dxdydsdt \\ &= \int_{x_j}^{x_{j+1}}\int_{t^n}^{t^{n+1}}\int_{x_j}^{x_{j+1}}\int_{t^n}^{t^{n+1}} \Delta x\partial_x\eta(s,y+\nu\Delta x)\left[u(s,y+\nu\Delta x) - (y-x)\partial_x u(s,x+\nu\Delta x)\right. \\ &\qquad\qquad \left. - \int_x^y \partial_x^2 u(s,z+\nu\Delta x)(y-z)dz + \int_s^t \partial_t u(\tau,x+\nu\Delta x)d\tau\right] \\ &+ \Delta x\partial_x u(t,x+\nu\Delta x)\left[\eta(t,x+\nu\Delta x) - (x-y)\partial_x\eta(t,y+\nu\Delta x) - \int_y^x \partial_x^2\eta(t,z+\nu\Delta x)(x-z)dz\right. \\ &\qquad\qquad \left. + \int_t^s \partial_t\eta(\tau,y+\nu\Delta x)d\tau\right]dxdydsdt \\ &= \Delta x^2\Delta t\int_{x_j}^{x_{j+1}}\int_{t^n}^{t^{n+1}} \partial_x(\eta u)(s,y+\nu\Delta x)dsdy \\ &- \Delta x\Delta t\int_{x_j}^{x_{j+1}}\int_{t^n}^{t^{n+1}}\int_{x_j}^{x_{j+1}}\int_x^y \partial_x\eta(s,y+\nu\Delta x)\partial_x^2 u(s,z+\nu\Delta x)(y-z)dzdxdyds \\ &- \Delta x\Delta t\int_{x_j}^{x_{j+1}}\int_{t^n}^{t^{n+1}}\int_{x_j}^{x_{j+1}}\int_y^x \partial_x u(t,x+\nu\Delta x)\partial_x^2\eta(t,z+\nu\Delta x)(x-z)dzdxdyds \\ &+ \Delta x\int_{x_j}^{x_{j+1}}\int_{t^n}^{t^{n+1}}\int_{x_j}^{x_{j+1}}\int_{t^n}^{t^{n+1}}\int_s^t \partial_x\eta(s,y+\nu\Delta x)\partial_t u(\tau,x+\nu\Delta x)d\tau dxdydsdt \end{aligned}$$

$$+\Delta x\int_{x_j}^{x_{j+1}}\int_{t^n}^{t^{n+1}}\int_{x_j}^{x_{j+1}}\int_{t^n}^{t^{n+1}}\int_t^s \partial_x u(t,x+\nu\Delta x)\partial_t\eta(\tau,y+\nu\Delta x)d\tau dxdydsdt$$

Moreover,

$$K''(\nu)=\Delta x^2\Delta t\int_{x_j}^{x_{j+1}}\int_{t^n}^{t^{n+1}} 2\Delta x\left(\partial_x u\partial_x\eta\right)(s,y+\nu\Delta x)+\Delta x\left(u\partial_x^2\eta\right)(s,y+\nu\Delta x)+\Delta x\left(\eta\partial_x^2 u\right)(s,y+\nu\Delta x)dsdy$$
$$+\Delta x\Delta t\int_{x_j}^{x_{j+1}}\int_{t^n}^{t^{n+1}}\int_{x_j}^{x_{j+1}}\int_y^x \Delta x\partial_x^2\eta(s,y+\nu\Delta x)\partial_x u(s,z+\nu\Delta x)+\Delta x\partial_x\eta(s,y+\nu\Delta x)\partial_x^2 u(s,z+\nu\Delta x)dzdxdyds$$
$$+\Delta x\Delta t\int_{x_j}^{x_{j+1}}\int_{t^n}^{t^{n+1}}\int_{x_j}^{x_{j+1}}\int_x^y \Delta x\partial_x^2 u(t,s+\nu\Delta x)\partial_x\eta(t,z+\nu\Delta x)+\Delta x\partial_x u(t,x+\nu\Delta x)\partial_x^2\eta(t,z+\nu\Delta x)dzdxdyds$$
$$+\Delta x\int_{x_j}^{x_{j+1}}\int_{t^n}^{t^{n+1}}\int_{x_j}^{x_{j+1}}\int_{t^n}^{t^{n+1}}\int_s^t \Delta x\partial_x^2\eta(s,y+\nu\Delta x)\partial_t u(\tau,x+\nu\Delta x)+\Delta x\partial_x\eta(s,y+\nu\Delta x)\partial_{tx}u(\tau,x+\nu\Delta x)d\tau dxdydsdt$$
$$+\Delta x\int_{x_j}^{x_{j+1}}\int_{t^n}^{t^{n+1}}\int_{x_j}^{x_{j+1}}\int_{t^n}^{t^{n+1}}\int_t^s \Delta x\partial_x^2 u(t,x+\nu\Delta x)\partial_t\eta(\tau,y+\nu\Delta x)+\Delta x\partial_x u(t,x+\nu\Delta x)\partial_{tx}\eta(\tau,y+\nu\Delta x)d\tau dxdydsdt.$$

We have the same type of equality for K'''. Applying once again the Cauchy-Schwarz inequality gives

$$\begin{aligned}|K'''(\nu)|\lesssim{}&\Delta x^4\Delta t||\partial_x^2 u||_{L^2(Q)}||\partial_x\eta||_{L^2(Q)}+\Delta x^4\Delta t||\partial_x u||_{L^2(Q)}||\partial_x^2\eta||_{L^2(Q)}+\Delta x^4\Delta t||u||_{L^2(Q)}||\partial_x^3\eta||_{L^2(Q)}\\&+\Delta x^4\Delta t||\partial_x^3 u||_{L^2(Q)}||\eta||_{L^2(Q)}+\Delta x^5\Delta t||\partial_x u||_{L^2(Q)}||\partial_x^3\eta||_{L^2(Q)}+\Delta x^5\Delta t||\partial_x^2 u||_{L^2(Q)}||\partial_x^2\eta||_{L^2(Q)}\\&+\Delta x^5\Delta t||\partial_x^3 u||_{L^2(Q)}||\partial_x\eta||_{L^2(Q)}+\Delta x^4\Delta t^2||\partial_{xt}u||_{L^2(Q)}||\partial_x^2\eta||_{L^2(Q)}+\Delta x^4\Delta t^2||\partial_t u||_{L^2(Q)}||\partial_x^3\eta||_{L^2(Q)}\\&+\Delta x^4\Delta t^2||\partial_t\partial_x^2 u||_{L^2(Q)}||\partial_x\eta||_{L^2(Q)}+\Delta x^4\Delta t^2||\partial_x^3 u||_{L^2(Q)}||\partial_t\eta||_{L^2(Q)}+\Delta x^4\Delta t^2||\partial_x^2 u||_{L^2(Q)}||\partial_{xt}\eta||_{L^2(Q)}\\&+\Delta x^4\Delta t^2||\partial_x u||_{L^2(Q)}||\partial_t\partial_x^2\eta||_{L^2(Q)}\end{aligned}$$

Thus, the $E_{\text{non linear}}$-term rewrites

$$E_{\text{non linear}}\lesssim\frac{1}{2\Delta x^3\Delta t^2}\left(K(1)-K(-1)\right)=\frac{1}{2\Delta x^3\Delta t^2}\left(2K'(0)+\int_0^1 K'''(w)\frac{(1-w)^2}{2}dw+\int_0^1 K'''(-w)\frac{(-1-w)^2}{2}dw\right)$$
$$\begin{aligned}&\leq\frac{1}{\Delta x\Delta t}\int_{x_j}^{x_{j+1}}\int_{t^n}^{t^{n+1}}\partial_x(\eta u)(s,y)dsdy+\Delta x\sqrt{\frac{\Delta x}{\Delta t}}||\partial_x\eta||_{L^\infty(Q)}||\partial_x^2 u||_{L^2(Q)}+\Delta x\sqrt{\frac{\Delta x}{\Delta t}}||\partial_x^2\eta||_{L^2(Q)}||\partial_x u||_{L^\infty(Q)}\\&+\sqrt{\frac{\Delta t}{\Delta x}}||\partial_x\eta||_{L^\infty(Q)}||\partial_t u||_{L^2(Q)}+\sqrt{\frac{\Delta t}{\Delta x}}||\partial_t\eta||_{L^2(Q)}||\partial_x u||_{L^\infty(Q)}+\Delta x\sqrt{\frac{\Delta x}{\Delta t}}||\partial_x^2 u||_{L^2(Q)}||\partial_x\eta||_{L^\infty(Q)}\\&+\Delta x\sqrt{\frac{\Delta x}{\Delta t}}||\partial_x u||_{L^\infty(Q)}||\partial_x^2\eta||_{L^2(Q)}+\Delta x\sqrt{\frac{\Delta x}{\Delta t}}||u||_{L^\infty(Q)}||\partial_x^3\eta||_{L^2(Q)}+\Delta x\sqrt{\frac{\Delta x}{\Delta t}}||\partial_x^3 u||_{L^2(Q)}||\eta||_{L^\infty(Q)}\\&+\Delta x^2\sqrt{\frac{\Delta x}{\Delta t}}||\partial_x u||_{L^\infty(Q)}||\partial_x^3\eta||_{L^2(Q)}+\Delta x^2\sqrt{\frac{\Delta x}{\Delta t}}||\partial_x^2 u||_{L^\infty(Q)}||\partial_x^2\eta||_{L^2(Q)}+\Delta x^2\sqrt{\frac{\Delta x}{\Delta t}}||\partial_x^3 u||_{L^2(Q)}||\partial_x\eta||_{L^\infty(Q)}\\&+\Delta x\sqrt{\Delta x\Delta t}||\partial_{xt}u||_{L^2(Q)}||\partial_x^2\eta||_{L^\infty(Q)}+\Delta x\sqrt{\Delta x\Delta t}||\partial_t u||_{L^2(Q)}||\partial_x^3\eta||_{L^\infty(Q)}+\Delta x\sqrt{\Delta x\Delta t}||\partial_t\partial_x^2 u||_{L^2(Q)}||\partial_x\eta||_{L^\infty(Q)}\\&+\Delta x\sqrt{\Delta x\Delta t}||\partial_x^3 u||_{L^\infty(Q)}||\partial_t\eta||_{L^2(Q)}+\Delta x\sqrt{\Delta x\Delta t}||\partial_x^2 u||_{L^\infty(Q)}||\partial_{xt}\eta||_{L^2(Q)}+\Delta x\sqrt{\Delta x\Delta t}||\partial_x u||_{L^\infty(Q)}||\partial_t\partial_x^2\eta||_{L^2(Q)}.\end{aligned}$$

Finally, one has

$$E_{\text{viscosity}}\lesssim\frac{\tau_1\sqrt{\Delta x}}{2\sqrt{\Delta t}}||\partial_x^2\eta||_{L^2(Q)}.$$

Then, when we sum up all the previous results, we obtain

$$\epsilon_1^n=\frac{1}{\Delta t\Delta x}\int_{x_j}^{x_{j+1}}\int_{t^n}^{t^{n+1}}\partial_t\eta(s,y)-b\partial_x^2\partial_t\eta(s,y)+\partial_x u(s,y)+a\partial_x^3 u(s,y)+\partial_x(\eta u)(s,y)dsdy$$
$$+\sqrt{\frac{\Delta t}{\Delta x}}||\partial_t^2\eta||_{L^2(Q)}+b\Delta x\sqrt{\frac{\Delta x}{\Delta t}}||\partial_x^4\partial_t\eta||_{L^2(Q)}+b\sqrt{\frac{\Delta t}{\Delta x}}||\partial_x^2\partial_t^2\eta||_{L^2(Q)}+\theta\sqrt{\frac{\Delta t}{\Delta x}}||\partial_x\partial_t u||_{L^2(Q)}+\Delta x\sqrt{\frac{\Delta x}{\Delta t}}||\partial_x^3 u||_{L^2(Q)}$$

$$
\begin{aligned}
&+|a|\Delta x\sqrt{\frac{\Delta x}{\Delta t}}||\partial_x^5 u||_{L^2(Q)}+|a|\theta\sqrt{\frac{\Delta t}{\Delta x}}||\partial_t\partial_x^3 u||_{L^2(Q)}+|a|\Delta x\sqrt{\frac{\Delta x}{\Delta t}}||\partial_x^5 u||_{L^2(Q)}\\
&+\Delta x\sqrt{\frac{\Delta x}{\Delta t}}||\partial_x\eta||_{L^\infty(Q)}||\partial_x^2 u||_{L^2(Q)}+\Delta x\sqrt{\frac{\Delta x}{\Delta t}}||\partial_x^2\eta||_{L^2(Q)}||\partial_x u||_{L^\infty(Q)}\\
&+\sqrt{\frac{\Delta t}{\Delta x}}||\partial_x\eta||_{L^\infty(Q)}||\partial_t u||_{L^2(Q)}+\sqrt{\frac{\Delta t}{\Delta x}}||\partial_t\eta||_{L^2(Q)}||\partial_x u||_{L^\infty(Q)}\\
&+\Delta x\sqrt{\frac{\Delta x}{\Delta t}}||\partial_x^2 u||_{L^2(Q)}||\partial_x\eta||_{L^\infty(Q)}+\Delta x\sqrt{\frac{\Delta x}{\Delta t}}||\partial_x u||_{L^\infty(Q)}||\partial_x^2\eta||_{L^2(Q)}\\
&+\Delta x\sqrt{\frac{\Delta x}{\Delta t}}||u||_{L^\infty(Q)}||\partial_x^3\eta||_{L^2(Q)}+\Delta x\sqrt{\frac{\Delta x}{\Delta t}}||\partial_x^3 u||_{L^2(Q)}||\eta||_{L^\infty(Q)}+\Delta x^2\sqrt{\frac{\Delta x}{\Delta t}}||\partial_x u||_{L^\infty(Q)}||\partial_x^3\eta||_{L^2(Q)}\\
&+\Delta x^2\sqrt{\frac{\Delta x}{\Delta t}}||\partial_x^2 u||_{L^\infty(Q)}||\partial_x^2\eta||_{L^2(Q)}+\Delta x^2\sqrt{\frac{\Delta x}{\Delta t}}||\partial_x^3 u||_{L^2(Q)}||\partial_x\eta||_{L^\infty(Q)}+\Delta x\sqrt{\Delta x\Delta t}||\partial_{xt}u||_{L^2(Q)}||\partial_x^2\eta||_{L^\infty(Q)}\\
&+\Delta x\sqrt{\Delta x\Delta t}||\partial_t u||_{L^2(Q)}||\partial_x^3\eta||_{L^\infty(Q)}+\Delta x\sqrt{\Delta x\Delta t}||\partial_t\partial_x^2 u||_{L^2(Q)}||\partial_x\eta||_{L^\infty(Q)}+\Delta x\sqrt{\Delta x\Delta t}||\partial_x^3 u||_{L^\infty(Q)}||\partial_t\eta||_{L^2(Q)}\\
&+\Delta x\sqrt{\Delta x\Delta t}||\partial_x^2 u||_{L^\infty(Q)}||\partial_{xt}\eta||_{L^2(Q)}+\Delta x\sqrt{\Delta x\Delta t}||\partial_x u||_{L^\infty(Q)}||\partial_t\partial_x^2\eta||_{L^2(Q)}+\frac{\tau_1\sqrt{\Delta x}}{2\sqrt{\Delta t}}||\partial_x^2\eta||_{L^2(Q)}.
\end{aligned}
$$

We recognize the initial equation on the first line. We recall the relation

$$
\sum_{j\in\mathbb{Z}}\Delta x||f||^2_{L^2(Q)}=\Delta x\int_{t^n}^{t^{n+1}}\int_{\mathbb{R}}|f(s,y)|^2dyds\leq\Delta x\Delta t\sup_{t\in[0,T]}||f(t,.)||^2_{L^2(\mathbb{R})}=\Delta x\Delta t||f(t,.)||^2_{L^\infty_t L^2_x}. \tag{A.1}
$$

Eventually, when we compute the ℓ^2_Δ-norm, we obtain

$$
\begin{aligned}
&||\epsilon_1^n||^2_{\ell^2_\Delta}=\Delta t^2||\partial_t^2\eta||^2_{L^\infty_t L^2_x}+b^2\Delta x^4||\partial_x^4\partial_t\eta||^2_{L^\infty_t L^2_x}+b^2\Delta t^2||\partial_x^2\partial_t^2\eta||^2_{L^\infty_t L^2_x}+\theta^2\Delta t^2||\partial_x\partial_t u||^2_{L^\infty_t L^2_x}+\Delta x^4||\partial_x^3 u||^2_{L^\infty_t L^2_x}\\
&+|a|^2\Delta x^4||\partial_x^5 u||^2_{L^\infty_t L^2_x}+|a|^2\theta^2\Delta t^2||\partial_t\partial_x^3 u||^2_{L^\infty_t L^2_x}+|a|^2\Delta x^4||\partial_x^5 u||^2_{L^\infty_t L^2_x}+\Delta x^4||\partial_x\eta||_{L^\infty_t L^\infty_x}||\partial_x^2 u||_{L^\infty_t L^2_x}\\
&+\Delta x^4||\partial_x^2\eta||_{L^\infty_t L^2_x}||\partial_x u||_{L^\infty_t L^\infty_x}+\Delta t^2||\partial_x\eta||_{L^\infty_t L^\infty_x}||\partial_t u||_{L^\infty_t L^2_x}+\Delta t^2||\partial_x u||_{L^\infty_t L^\infty_x}||\partial_t\eta||_{L^\infty_t L^2_x}+\Delta x^4||\partial_x^2 u||_{L^\infty_t L^2_x}||\partial_x\eta||_{L^\infty_t L^\infty_x}\\
&+\Delta x^4||\partial_x u||_{L^\infty_t L^\infty_x}||\partial_x^2\eta||_{L^\infty_t L^2_x}+\Delta x^4||u||_{L^\infty_t L^\infty_x}||\partial_x^3\eta||_{L^\infty_t L^2_x}+\Delta x^4||\partial_x^3 u||_{L^\infty_t L^2_x}||\eta||_{L^\infty_t L^\infty_x}\\
&+\Delta x^6||\partial_x u||_{L^\infty_t L^\infty_x}||\partial_x^3\eta||_{L^\infty_t L^2_x}+\Delta x^6||\partial_x^2 u||_{L^\infty_t L^\infty_x}||\partial_x^2\eta||_{L^\infty_t L^2_x}+\Delta x^6||\partial_x^3 u||_{L^\infty_t L^2_x}||\partial_x\eta||_{L^\infty_t L^\infty_x}\\
&+\Delta x^4\Delta t^2||\partial_{xt}u||_{L^\infty_t L^2_x}||\partial_x^2\eta||_{L^\infty_t L^\infty_x}+\Delta x^4\Delta t^2||\partial_t u||_{L^\infty_t L^2_x}||\partial_x^3\eta||_{L^\infty_t L^\infty_x}+\Delta x^4\Delta t^2||\partial_t\partial_x^2 u||_{L^\infty_t L^2_x}||\partial_x\eta||_{L^\infty_t L^\infty_x}\\
&+\Delta x^4\Delta t^2||\partial_x^3 u||_{L^\infty_t L^\infty_x}||\partial_t\eta||_{L^\infty_t L^2_x}+\Delta x^4\Delta t^2||\partial_x^2 u||_{L^\infty_t L^\infty_x}||\partial_{xt}\eta||_{L^\infty_t L^2_x}\\
&+\Delta x^4\Delta t^2||\partial_x u||_{L^\infty_t L^\infty_x}||\partial_t\partial_x^2\eta||_{L^\infty_t L^2_x}+(\tau_1)^2\Delta x^2||\partial_x^2\eta||^2_{L^\infty_t L^2_x}.
\end{aligned}
$$

Thus, one has

$$
||\epsilon_1^n||_{\ell^2_\Delta}\leq\begin{cases}C(\Delta t+\Delta x), & \text{if } \tau_1\neq 0 \text{ } ie \text{ in the case } bd=0,\\ C(\Delta t+\Delta x^2), & \text{if } \tau_1=0 \text{ } ie \text{ in the case } bd\neq 0.\end{cases}
$$

with C a constant depending of u, η and their derivatives.

Remark A.1. *One can prove that, for $\theta=\frac{1}{2}$,*

$$
||\epsilon_1^n||_{\ell^2_\Delta}\leq\begin{cases}C(\Delta t^2+\Delta x), & \textit{if } \tau_1\neq 0 \textit{ ie in the case } bd=0,\\ C(\Delta t^2+\Delta x^2), & \textit{if } \tau_1=0 \textit{ ie in the case } bd\neq 0.\end{cases}
$$

In some cases, $D_+\epsilon_1^n$ is needed. To obtain an upper bound, we perform the same computations and find

$$
\begin{aligned}
&||D_+\epsilon_1^n||^2_{\ell^2_\Delta}\lesssim\Delta x^2||\partial_x^2\partial_t\eta||^2_{L^\infty_t L^2_x}+\Delta t^2||\partial_x\partial_t^2\eta||^2_{L^\infty_t L^2_x}+b^2\Delta x^4||\partial_x^4\partial_t\eta||^2_{L^\infty_t L^2_x}+b^2\Delta t^2||\partial_x^3\partial_t^2\eta||^2_{L^\infty_t L^2_x}\\
&+\Delta x^2||\partial_x^3 u||^2_{L^\infty_t L^2_x}+|a|^2\Delta x^2||\partial_x^5 u||^2_{L^\infty_t L^2_x}+\theta^2\Delta t^2||\partial_x^2\partial_t u||^2_{L^\infty_t L^2_x}+|a|^2\theta^2\Delta t^2||\partial_t\partial_x^4 u||^2_{L^\infty_t L^2_x}+\Delta x^4||\partial_x^4 u||^2_{L^\infty_t L^2_x}\\
&+|a|^2\Delta x^4||\partial_x^6 u||^2_{L^\infty_t L^2_x}+\Delta x^2||\partial_x^2\eta||^2_{L^\infty_t L^2_x}||\partial_x u||_{L^\infty(Q)}+\Delta x^2||\partial_x\eta||^2_{L^\infty_t L^\infty(Q)}||\partial_x^2 u||^2_{L^\infty_t L^2_x}
\end{aligned}
$$

$$
\begin{aligned}
&+\Delta t^2||\partial_x^2\eta||^2_{L_t^\infty L_x^\infty}||\partial_t u||^x_{L_t^\infty L_x^2}+\Delta t^2||\partial_x\eta||^2_{L_t^\infty L_x^\infty}||\partial_{tx}u||^2_{L_t^\infty L_x^2}+\Delta t^2|\partial_x^2 u||^2_{L_t^\infty L_x^\infty}||\partial_t\eta||^2_{L_t^\infty L_x^2}\\
&+\Delta t^2||\partial_x u||^2_{L_t^\infty L_x^\infty}||\partial_{tx}\eta||^2_{L_t^\infty L_x^2}+\Delta x^2||\partial_x^2 u||^2_{L_t^\infty L_x^2}||\partial_x\eta||^2_{L_t^\infty L_x^\infty}\\
&+\Delta x^2||\partial_x u||^2_{L_t^\infty L_x^\infty}||\partial_x^2\eta||^2_{L_t^\infty L_x^2}+\Delta x^2||u||^2_{L_t^\infty L_x^\infty}||\partial_x^3\eta||^2_{L_t^\infty L_x^2}\\
&+\Delta x^2||\eta||^2_{L_t^\infty L_x^\infty}||\partial_x^3 u||^2_{L_t^\infty L_x^2}+\Delta x^4||\partial_x^3\eta||^2_{L_t^\infty L_x^2}||\partial_x u||^2_{L_t^\infty L_x^\infty}+\Delta x^4||\partial_x^2 u||^2_{L_t^\infty L_x^2}||\partial_x^2\eta||^2_{L_t^\infty L_x^\infty}\\
&+\Delta x^4||\partial_x\eta||^2_{L_t^\infty L_x^\infty}||\partial_x^3 u||^2_{L_t^\infty L_x^2}+\Delta t^2\Delta x^2||\partial_x^3\eta||^2_{L_t^\infty L_x^\infty}||\partial_t u||^2_{L_t^\infty L_x^2}+\Delta t^2\Delta x^2||\partial_x^2\eta||^2_{L_t^\infty L_x^\infty}||\partial_t\partial_x u||^2_{L_t^\infty L_x^2}\\
&+\Delta t^2\Delta x^2||\partial_x\eta||^2_{L_t^\infty L_x^\infty}||\partial_t\partial_x^2 u||^2_{L_t^\infty L_x^2}+\Delta t^2\Delta x^2||\partial_t\eta||^2_{L_t^\infty L_x^2}||\partial_x^3 u||^2_{L_t^\infty L_x^\infty}+\Delta t^2\Delta x^2||\partial_t\partial_x\eta||^2_{L_t^\infty L_x^2}||\partial_x^2 u||^2_{L_t^\infty L_x^\infty}\\
&+\Delta t^2\Delta x^2||\partial_x u||^2_{L_t^\infty L_x^\infty}||\partial_t\partial_x^2\eta||^2_{L_t^\infty L_x^2}+\left(\tau_1\right)^2\Delta x^2||\partial_x^3\eta||^2_{L_t^\infty L_x^2}.
\end{aligned}
$$

1.2 Consistency error ϵ_2^n

By definition of the consistency error, on has

$$
\epsilon_2^n=\left(I-dD_+D_-\right)\left(\frac{u_\Delta^{n+1}-u_\Delta^n}{\Delta t}\right)+\left(I+cD_+D_-\right)D\left(\theta\eta_\Delta^{n+1}+(1-\theta)\eta_\Delta^n\right)+D\left(\frac{\left(u_\Delta^n\right)^2}{2}\right)-\frac{\tau_2}{2}\Delta xD_+D_-u_\Delta^n.
$$

We adapt the previous computations with $(d,c,\eta_\Delta^n,u_\Delta^n)$ instead of $(b,a,u_\Delta^n,\eta_\Delta^n)$. The only difference is concerning the non linear term

$$
K_2(\nu)=\frac{1}{2}\int_{x_j}^{x_{j+1}}\int_{t^n}^{t^{n+1}}\int_{x_j}^{x_{j+1}}\int_{t^n}^{t^{n+1}}u(s,y+\nu\Delta x)u(t,x+\nu\Delta x)dxdydsdt.
$$

So one has

$$
\begin{aligned}
E_{\text{non linear}}\lesssim&\frac{1}{\Delta x\Delta t}\int_{x_j}^{x_{j+1}}\int_{t^n}^{t^{n+1}}\partial_x(\frac{u^2}{2})(s,y)dsdy+\Delta x\sqrt{\frac{\Delta x}{\Delta t}}||\partial_x u||_{L^\infty(Q)}||\partial_x^2 u||_{L^2(Q)}+\sqrt{\frac{\Delta t}{\Delta x}}||\partial_x u||_{L^\infty(Q)}||\partial_t u||_{L^2(Q)}\\
&+\Delta x\sqrt{\frac{\Delta x}{\Delta t}}||\partial_x^2 u||_{L^2(Q)}||\partial_x u||_{L^\infty(Q)}+\Delta x\sqrt{\frac{\Delta x}{\Delta t}}||u||_{L^\infty(Q)}||\partial_x^3 u||_{L^2(Q)}+\Delta x^2\sqrt{\frac{\Delta x}{\Delta t}}||\partial_x u||_{L^\infty(Q)}||\partial_x^3 u||_{L^2(Q)}\\
&+\Delta x^2\sqrt{\frac{\Delta x}{\Delta t}}||\partial_x^2 u||_{L^\infty(Q)}||\partial_x^2 u||_{L^2(Q)}+\Delta x\sqrt{\Delta x\Delta t}||\partial_{xt}u||_{L^2(Q)}||\partial_x^2 u||_{L^\infty(Q)}\\
&+\Delta x\sqrt{\Delta x\Delta t}||\partial_t u||_{L^2(Q)}||\partial_x^3 u||_{L^\infty(Q)}+\Delta x\sqrt{\Delta x\Delta t}||\partial_t\partial_x^2 u||_{L^2(Q)}||\partial_x u||_{L^\infty(Q)}.
\end{aligned}
$$

The consistency error verifies

$$
\begin{aligned}
\epsilon_2^n\lesssim&\frac{1}{\Delta t\Delta x}\int_{x_j}^{x_{j+1}}\int_{t^n}^{t^{n+1}}\partial_t u(s,y)-d\partial_x^2\partial_t u(s,y)+\partial_x\eta(s,y)+c\partial_x^3\eta(s,y)+\partial_x\frac{u^2}{2}(s,y)dsdy\\
&+\sqrt{\frac{\Delta t}{\Delta x}}||\partial_t^2 u||_{L^2(Q)}+d\Delta x\sqrt{\frac{\Delta x}{\Delta t}}||\partial_x^4\partial_t u||_{L^2(Q)}+d\sqrt{\frac{\Delta t}{\Delta x}}||\partial_x^2\partial_t^2 u||_{L^2(Q)}+\theta\sqrt{\frac{\Delta t}{\Delta x}}||\partial_x\partial_t\eta||_{L^2(Q)}+\Delta x\sqrt{\frac{\Delta x}{\Delta t}}||\partial_x^3\eta||_{L^2(Q)}\\
&+|c|\Delta x\sqrt{\frac{\Delta x}{\Delta t}}||\partial_x^5\eta||_{L^2(Q)}+|c|\theta\sqrt{\frac{\Delta t}{\Delta x}}||\partial_t\partial_x^3\eta||_{L^2(Q)}+|c|\Delta x\sqrt{\frac{\Delta x}{\Delta t}}||\partial_x^5\eta||_{L^2(Q)}+\Delta x\sqrt{\frac{\Delta x}{\Delta t}}||\partial_x^2 u||_{L^2(Q)}||\partial_x u||_{L^\infty(Q)}\\
&+\sqrt{\frac{\Delta t}{\Delta x}}||\partial_x u||_{L^\infty(Q)}||\partial_t u||_{L^2(Q)}+\Delta x\sqrt{\frac{\Delta x}{\Delta t}}||\partial_x^2 u||_{L^2(Q)}||\partial_x u||_{L^\infty(Q)}+\Delta x\sqrt{\frac{\Delta x}{\Delta t}}||u||_{L^\infty(Q)}||\partial_x^3 u||_{L^2(Q)}\\
&+\Delta x^2\sqrt{\frac{\Delta x}{\Delta t}}||\partial_x u||_{L^\infty(Q)}||\partial_x^3 u||_{L^2(Q)}+\Delta x^2\sqrt{\frac{\Delta x}{\Delta t}}||\partial_x^2 u||_{L^\infty(Q)}||\partial_x^2 u||_{L^2(Q)}+\Delta x\sqrt{\Delta x\Delta t}||\partial_{xt}u||_{L^2(Q)}||\partial_x^2 u||_{L^\infty(Q)}\\
&+\Delta x\sqrt{\Delta x\Delta t}||\partial_t u||_{L^2(Q)}||\partial_x^3 u||_{L^\infty(Q)}+\Delta x\sqrt{\Delta x\Delta t}||\partial_t\partial_x^2 u||_{L^2(Q)}||\partial_x u||_{L^\infty(Q)}+\frac{\tau_2\sqrt{\Delta x}}{2\sqrt{\Delta t}}||\partial_x^2 u||_{L^2(Q)}.
\end{aligned}
$$

Finally, one has, thanks to the relation (A.1)

$$
||\epsilon_2^n||^2_{\ell^2_\Delta}\lesssim\Delta t^2||\partial_t^2 u||^2_{L_t^\infty L_x^2}+d^2\Delta x^4||\partial_x^4\partial_t u||^2_{L_t^\infty L_x^2}+d^2\Delta t^2||\partial_x^2\partial_t^2 u||^2_{L_t^\infty L_x^2}+\theta^2\Delta t^2||\partial_x\partial_t\eta||^2_{L_t^\infty L_x^2}+\Delta x^4||\partial_x^3\eta||^2_{L_t^\infty L_x^2}
$$

$$+|c|^2\Delta x^4||\partial_x^5\eta||^2_{L_t^\infty L_x^2}+|c|^2\theta^2\Delta t^2||\partial_t\partial_x^3\eta||^2_{L_t^\infty L_x^2}+|c|^2\Delta x^4||\partial_x^5\eta||^2_{L_t^\infty L_x^2}+\Delta x^4||\partial_x^2 u||^2_{L_t^\infty L_x^2}||\partial_x u||^2_{L_t^\infty L_x^\infty}$$
$$+\Delta t^2||\partial_x u||^2_{L_t^\infty L_x^\infty}||\partial_t u||^2_{L_t^\infty L_x^2}+\Delta x^4||\partial_x^2 u||_{L_t^\infty L_x^2}||\partial_x u||_{L_t^\infty L_x^\infty}+\Delta x^4||u||_{L_t^\infty L_x^\infty}||\partial_x^3 u||_{L_t^\infty L_x^2}$$
$$+\Delta x^6||\partial_x u||_{L_t^\infty L_x^\infty}||\partial_x^3 u||_{L_t^\infty L_x^2}+\Delta x^6||\partial_x^2 u||_{L_t^\infty L_x^\infty}||\partial_x^2 u||_{L_t^\infty L_x^2}+\Delta x^4\Delta t^2||\partial_{xt} u||_{L_t^\infty L_x^2}||\partial_x^2 u||_{L_t^\infty L_x^\infty}$$
$$+\Delta x^4\Delta t^2||\partial_t u||_{L_t^\infty L_x^2}||\partial_x^3 u||_{L_t^\infty L_x^\infty}+\Delta x^4\Delta t^2||\partial_t\partial_x^2 u||_{L_t^\infty L_x^2}||\partial_x u||_{L_t^\infty L_x^\infty}+(\tau_2)^2\Delta x^2||\partial_x^2 u||^2_{L_t^\infty L_x^2}.$$

As for the ϵ_1^n case, there exists a constant C depending on u, η and their derivatives such that

$$||\epsilon_2^n||_{\ell^2_\Delta}\leq\begin{cases}C(\Delta t+\Delta x), & \text{if } \tau_2\neq 0 \text{ } ie \text{ in the case } bd=0,\\ C(\Delta t+\Delta x^2), & \text{if } \tau_2=0 \text{ } ie \text{ in the case } bd\neq 0.\end{cases}$$

Remark A.2. *As for the ϵ_1^n case, one can prove that, for $\theta=\frac{1}{2}$,*

$$||\epsilon_2^n||_{\ell^2_\Delta}\leq\begin{cases}C(\Delta t^2+\Delta x), & \textit{if } \tau_2\neq 0 \textit{ ie in the case } bd=0,\\ C(\Delta t^2+\Delta x^2), & \textit{if } \tau_2=0 \textit{ ie in the case } bd\neq 0.\end{cases}$$

For $D_+\epsilon_2^n$, the results are similar to those for $D_+\epsilon_1^n$.

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