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# A theory of functions of several variables applied to square matrices

Laurent Veysseire

In this paper, we give one possible definition for functions of several variables applied to endomorphisms of finite dimensional  $\mathbb{C}$ -vector spaces. This definition is consistent with the usual notion of a function of a square matrix. The fact that with this definition,  $f^{\otimes}(A, B)$  is not a square matrix of size  $n$  anymore when  $A$  and  $B$  are square matrices of size  $n$  (well, except in the trivial case where  $n = 1$ ) can seem weird and unsatisfying, but this objects naturally appear when one differentiates a smooth function of a matrix.

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## 1 definitions and notations

As for defining functions in one variable applied to a single matrix (as done in [1]), we start with the simple case of polynomials.

**Definition 1** *Let  $k \in \mathbb{N}^*$ , and let  $M_1, \dots, M_k$  be endomorphisms of the finite dimensional  $\mathbb{C}$ -vector spaces  $E_1, \dots, E_k$  (their dimensions  $n_1, \dots, n_k$  may be different). Let*

$$P(x_1, \dots, x_k) = \sum_{\alpha \in F} c_{\alpha} x_1^{\alpha_1} \dots x_k^{\alpha_k}$$

*be a polynomial of  $k$  variables (here  $F$  is a finite subset of  $\mathbb{N}^k$ ,  $c_{\alpha} \in \mathbb{C}$  and the  $\alpha_l$ 's are the coordinates of  $\alpha$ ). We set:*

$$P^{\otimes}(M_1, \dots, M_k) := \sum_{\alpha \in F} c_{\alpha} M_1^{\alpha_1} \otimes \dots \otimes M_k^{\alpha_k} \in \bigotimes_{l=1}^k E_l \otimes E_l^*.$$

**Remark 2** In this definition, the tensor products are implicitly taken over  $\mathbb{C}$ . We can use a similar definition if  $E_1, \dots, E_k$  are  $\mathbb{R}$ -vector spaces and if  $P$  has real coefficients, but with tensor products over  $\mathbb{R}$ . In the cases when some of the  $E_l$ 's are  $\mathbb{R}$ -vector spaces and the other ones are  $\mathbb{C}$ -vector spaces, or if all the  $E_l$ 's are  $\mathbb{R}$ -vector spaces and  $P$  has complex coefficients, we can complexify the real vector spaces by replacing the concerned  $E_l$ 's by  $E'_l = \mathbb{C} \otimes_{\mathbb{R}} E_l$  and  $M_l$  by its natural  $\mathbb{C}$ -linear extension  $M'_l$  to  $E'_l$ , and use the classical version of definition 1.

**Remark 3** We use the notation  $P^{\otimes}(M_1, \dots, M_k)$  and not  $P(M_1, \dots, M_k)$ , because doing so would be confusing, and because the  $\otimes$  sign reminds you it is a tensor of higher order. For example, we have  $((x, y) \mapsto x + y)^{\otimes}(A, B) = A \otimes I + I \otimes B \neq A + B$  in general (even in simple cases where  $A = B$  or  $B = 0$ ).

**Lemma 4** Let  $k \in \mathbb{N}^*$ , and let  $M_1, \dots, M_k$  be endomorphisms of the finite dimensional  $\mathbb{C}$ -vector spaces  $E_1, \dots, E_k$ . For  $1 \leq l \leq k$ , let  $A_{(l)}$  be an invertible  $\mathbb{C}$ -linear application from  $E_l$  to another  $\mathbb{C}$ -vector space  $F_l$  of same dimension than  $E_l$ . We set  $M'_l = A_{(l)} M_l A_{(l)}^{-1} \in L(F_l, F_l)$ . Let  $P$  be a polynomial of  $k$  variables. Then we have:

$$P^{\otimes}(M'_1, \dots, M'_k)^{i_1}_{j_1} \dots^{i_k}_{j_k} = A_{(1)}^{i_1}_{i'_1} A_{(1)}^{-1 j'_1}_{j_1} \dots A_{(k)}^{i_k}_{i'_k} A_{(k)}^{-1 j'_k}_{j_k} P^{\otimes}(M_1, \dots, M_k)^{i'_1}_{j'_1} \dots^{i'_k}_{j'_k}.$$

**Proof:** In the simple case where  $P$  is a monomial, it follows from the fact that  $(AMA^{-1})^n = AM^n A^{-1}$ . The more general case where  $P$  is any polynomial easily follows by linearity.  $\square$

Any complex square matrix can be put in a Jordan form by conjugation by an invertible matrix. Let us see what one gets when all the matrices  $M_1, \dots, M_k$  are Jordan matrices.

**Example 5** Let  $M_1, \dots, M_k$  be Jordan matrices, i.e, for  $1 \leq l \leq k$ , the matrix  $M_l$  has the following form:

$$M_l = \begin{pmatrix} J_{\lambda_{l1}, r_{l1}} & 0 & \dots & 0 \\ 0 & J_{\lambda_{l2}, r_{l2}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & J_{\lambda_{lb_l}, r_{lb_l}} \end{pmatrix},$$

where  $b_l$  is the number of jordan blocks in  $M_l$ ,  $J_{\lambda,r}$  is the  $r \times r$  square matrix

$$J_{\lambda,r} = \begin{pmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \lambda & 1 \\ 0 & \dots & \dots & 0 & \lambda \end{pmatrix},$$

the  $\lambda_{lm}$  are the eigenvalues of  $M_l$  and the  $r_{lm}$  are the sizes of the Jordan blocks of  $M_l$ . Let  $P$  be a polynomial on  $k$  variables.

Let us choose the values of the indexes  $i_1, \dots, i_k$  and  $j_1, \dots, j_k$ . Then for each  $1 \leq l \leq k$ , there exist  $1 \leq c_l, d_l \leq b_l$  such that  $\sum_{m=1}^{c_l-1} r_{lm} < i_l \leq \sum_{m=1}^{c_l} r_{lm}$  and  $\sum_{m=1}^{d_l-1} r_{lm} < j_l \leq \sum_{m=1}^{d_l} r_{lm}$ . Then the coefficient  $P^{\otimes}(M_1, \dots, M_k)^{i_1 \dots i_k}_{j_1 \dots j_k}$  is given by the following formulas:

- If for some  $1 \leq l \leq k$ , one has  $i_l > j_l$  or  $c_l \neq d_l$ , then the coefficient  $P^{\otimes}(M_1, \dots, M_k)^{i_1 \dots i_k}_{j_1 \dots j_k}$  is 0.
- If for all  $1 \leq l \leq k$ , one has  $i_l \leq j_l$  and  $c_l = d_l$ , then the coefficient  $P^{\otimes}(M_1, \dots, M_k)^{i_1 \dots i_k}_{j_1 \dots j_k}$  is  $\left[ \prod_{l=1}^k \frac{1}{(j_l - i_l)!} \partial_l^{j_l - i_l} \right] P(\lambda_{1c_1}, \dots, \lambda_{kc_k})$ .

**Proof:** Like in the proof of Lemma 4, we start with the simple case where  $P$  is a monomial. In this case,  $P(x_1, \dots, x_k) = x_1^{\alpha_1} \dots x_k^{\alpha_k}$  and  $P^{\otimes}(M_1, \dots, M_k)^{i_1 \dots i_k}_{j_1 \dots j_k} = \prod_{l=1}^k M_l^{\alpha_l i_l}_{j_l}$ .

One easily gets the expression given in Example 5 for the coefficients of  $P^{\otimes}(M_1, \dots, M_k)$  by using the special form of the coefficients of powers of Jordan matrices, and the fact that  $\left[ \prod_{l=1}^k \frac{\partial}{\partial x_l} \right]^{\alpha_l} \left( \prod_{l=1}^k f_l(x_l) \right) = \prod_{l=1}^k f_l^{(\alpha_l)}(x_l)$ .

The case of a more general polynomial  $P$  trivially follows by linearity.  $\square$

**Proposition 6** Let  $k \in \mathbb{N}^*$ , and let  $M_1, \dots, M_k$  be square matrices with complex coefficients. Let  $P_1, \dots, P_k$  be their minimal polynomials.

Then the set of polynomials  $P$  of  $k$  variables such that

$$P^{\otimes}(M_1, \dots, M_k) = 0$$

is the ideal generated by the polynomials  $P_l(x_l)$ , for  $1 \leq k \leq l$ .

**Proof:** There exist  $M'_1, \dots, M'_k$  Jordan matrices and  $A_1, \dots, A_k$  invertibles such that  $M'_l = A_l M_l A_l^{-1}$ . According to Lemma 4, we have

$$P^{\otimes}(M_1, \dots, M_k) = 0 \Leftrightarrow P^{\otimes}(M'_1, \dots, M'_k) = 0.$$

For  $1 \leq l \leq k$ , we set  $\lambda_{lm}$  the eigenvalues of  $M_l$ , and  $r_{lm}$  their multiplicity as root of the minimal polynomial  $P_l$  of  $M_l$  (here the index  $m$  varies from 1 to the number  $e_l$  of distinct eigenvalues of  $M_l$ ). Then according to example 5,  $P^{\otimes}(M'_1, \dots, M'_k) = 0$  is equivalent to: for all  $(m_l)_{1 \leq l \leq k}, (j_l)_{1 \leq l \leq k}$  satisfying  $1 \leq m_l \leq e_l$  and  $0 \leq j_l < r_{lm_l}$ , one has

$$\left[ \prod_{l=1}^k \partial_l^{j_l} \right] P(\lambda_{1m_1}, \dots, \lambda_{km_k}) = 0.$$

Assume that there exists  $l$  such that  $P(x_1, \dots, x_k) = P_l(x_l)Q(x_1, \dots, x_k)$ , where  $Q$  is a polynomial. Then for any  $1 \leq m \leq e_l$  and any  $0 \leq j < m_{lm}$ ,  $x_l - \lambda_{lm}$  divides  $\partial_l^j P(x_1, \dots, x_k)$ , so it also divides all the derivatives of  $\partial_l^j P(x_1, \dots, x_k)$  with respect to all variables but  $x_l$ , thus they cancel when  $x_l = \lambda_{lm}$ . Thus  $P(M_1, \dots, M_k) = 0$ . By linearity, for any polynomial  $P$  in the ideal generated by  $P_1(x_1), \dots, P_k(x_k)$ , we have  $P(M_1, \dots, M_k) = 0$ .

Conversely, let  $P$  be a polynomial such that  $P(M_1, \dots, M_k) = 0$ . There exists two polynomials  $Q$  and  $R$  such that  $P = Q + R$ ,  $Q$  belongs to the ideal generated by  $P_1(x_1), \dots, P_k(x_k)$  and the degree of  $R$  with respect to the variable  $x_l$  is lower than  $\deg(P_l)$  (in the sense that any monomial  $x_1^{\alpha_1} \dots x_k^{\alpha_k}$  in  $R$  satisfies  $\alpha_l < \deg(P_l)$ ). This fact can be shown by performing Euclidean division by the Gröbner basis  $P_1(x_1), \dots, P_k(x_k)$  (this is a Gröbner basis for any monomial order) in the space of polynomials on  $k$  variables. For any  $1 \leq l \leq k$ , any  $1 \leq m \leq e_l$  and any  $0 \leq j < r_{lm}$ , there exists a polynomial of one variable  $P_{lmj}$  of degree less than  $\deg(P_l)$  such that for any  $1 \leq m' \leq e_l$  and any  $0 \leq j' < r_{lm'}$ , one has  $P_{lmj}^{(j')}(\lambda_{lm'}) = \mathbb{1}_{m=m', j=j'}$  (this follows from the classical theory of Lagrange–Sylvester interpolation polynomials). For any  $m_1, \dots, m_k$  and  $j_1, \dots, j_k$  such that  $1 \leq m_l \leq e_l$  and  $0 \leq j_l < r_{lm_l}$ , we set  $P_{m_1 j_1 \dots m_k j_k}(x_1, \dots, x_k) = \prod_{l=1}^k P_{lm_l j_l}(x_l)$ .

Let us consider the linear map  $\nu$  which associates to any polynomial  $S(x_1, \dots, x_k)$  of  $k$  variables such that its degree with respect to  $x_l$  is less than  $\deg(P_l)$ , the following  $\prod_{l=1}^k \deg(P_l)$  values: for any  $m_1, \dots, m_k$  and  $j_1, \dots, j_k$  satisfying  $1 \leq m_l \leq e_l$  and  $0 \leq j_l < r_{lm_l}$ , we set

$$\nu_{m_1 j_1 \dots m_k j_k}(S) := \left[ \prod_{l=1}^k \partial_l^{j_l} \right] S(\lambda_{1m_1}, \dots, \lambda_{km_k}).$$

Then we have  $\nu_{m_1 j_1 \dots m_k j_k}(P_{m'_1 j'_1 \dots m'_k j'_k}) = 1$  if  $m_l = m'_l$  and  $j_l = j'_l$  for every  $1 \leq l \leq k$  and  $\nu_{m_1 j_1 \dots m_k j_k}(P_{m'_1 j'_1 \dots m'_k j'_k}) = 0$  otherwise, so the linear map  $\nu$  is surjective. So because of the equality of the (finite) dimensions of its

domain and its image,  $\nu$  is also injective. So since  $\nu(R) = 0$ , we have  $R = 0$  and thus  $P = Q$  belongs to the ideal generated by the  $P_l(x_l)$ .  $\square$

So one does see from Proposition 6 that the dependancy in  $P$  of  $P(M_1, \dots, M_k)$  only relies on the values of  $P$  and of some of its derivatives at the eigenvalues of the matrices  $M_l$ ,  $1 \leq l \leq k$ . This justifies the following definition for more general functions than polynomials.

**Definition 7** Let  $k \in \mathbb{N}^*$ , and let  $M_1, \dots, M_k$  be endomorphisms of the finite dimensional  $\mathbb{C}$ -vector spaces  $E_1, \dots, E_k$ . For  $1 \leq l \leq k$ , let  $e_l$  be the number of different eigenvalues of  $M_l$ ,  $\lambda_{lm}$  be the eigenvalues of  $M_l$ , for  $1 \leq m \leq e_l$ , and  $r_{lm}$  be the multiplicity of the root  $\lambda_{lm}$  in the minimal polynomial of  $M_l$ . Let  $f$  be a function of  $k$  complex variables. Assume that for any  $k$ -uple of integers  $(m_1, \dots, m_k)$  such that  $1 \leq m_l \leq e_l$  for  $1 \leq l \leq k$ , the function  $f(x_1, \dots, x_k)$  is holomorphic with respect to the variables  $x_l$  with  $l$  satisfying  $r_{lm_l} \geq 2$  near  $(\lambda_{1m_1}, \dots, \lambda_{km_k})$ . That is, if we denote by  $l_1, \dots, l_p$  the indexes such that  $r_{l_q} \geq 2$  for  $1, \leq q \leq p$ , the function

$$g : \begin{cases} \mathbb{C}^p & \mapsto \mathbb{C} \\ (y_1, \dots, y_p) & \rightarrow f\left((\lambda_{1m_1}, \dots, \lambda_{km_k}) + \sum_{q=1}^p y_q v_{l_q}\right) \end{cases}$$

admits a continuous  $\mathbb{C}$ -linear differential on a neighborhood of 0, where  $v_l$  is the  $l$ -th vector of the canonical basis of  $\mathbb{C}^k$ . Then we set:

$$f^\otimes(M_1, \dots, M_k) := P^\otimes(M_1, \dots, M_k),$$

with  $P$  any complex polynomial of  $k$  variables such that for all  $(m_1, \dots, m_k)$  and  $(j_1, \dots, j_k)$  such that  $1 \leq m_l \leq e_l$  and  $0 \leq j_l < r_{lm_l}$  for  $1 \leq l \leq k$ , we have:

$$\left[ \prod_{l=1}^k \partial_l^{j_l} \right] f(\lambda_{1m_1}, \dots, \lambda_{km_k}) = \left[ \prod_{l=1}^k \partial_l^{j_l} \right] P(\lambda_{1m_1}, \dots, \lambda_{km_k}).$$

**Remark 8** The function  $f$  does not really need to be defined on the whole  $\mathbb{C}^k$ , but only on the points whose coordinates are the eigenvalues of the  $M_l$ 's and on some neighborhoods of those points intersected with some affine subspaces.

**Remark 9** If one wants to generalize this definition in a real framework, as said in Remark 2, one can not just assume  $f$  is a real function of real variables, because real matrices may have non-real eigenvalues.

The right thing to do is to chose  $f$  such that  $f(\bar{z}_1, \dots, \bar{z}_k) = \overline{f(z_1, \dots, z_k)}$  to be sure there exists a polynomial  $P$  with real coefficients satisfying all the equalities required.

**Remark 10** *The polynomial  $P$  can be chosen to be*

$$\sum \left( \left[ \prod_{l=1}^k \partial_l^{j_l} \right] f(\lambda_{1m_1}, \dots, \lambda_{km_k}) \right) P_{m_1 j_1 \dots m_k j_k}(x_1, \dots, x_k),$$

*with the same notation as in the proof of Proposition 6. In this case, we say  $P$  is the Lagrange–Sylvester interpolation polynomial of  $f$  associated to the product of multisets  $\prod_{l=1}^k \text{Root}(P_l)$ , where  $\text{Root}(P_l)$  is the multiset of the roots of  $P_l$ , counted with their multiplicity.*

## 2 Some rules for computations

In this section, we show that  $f^\otimes(M_1, \dots, M_k)$  behaves like functions of several variables do, and we show that some contractions of  $f^\otimes(M_1, \dots, M_k)$  can have a simplified expression. Finally, we show the main result of this paper, which is the expression of the derivative of  $f^\otimes(M_1, \dots, M_k)$  with respect to the matrices  $M_l$ .

The tensor  $f^\otimes(M_1, \dots, M_k)$  applied to eigenvectors of  $M_1, \dots, M_k$  has a simplified expression. This allows to give an alternative expression for  $f^\otimes(M_1, \dots, M_k)$  when  $M_1, \dots, M_k$  are all diagonalizable.

**Proposition 11** *Let  $k \in \mathbb{N}^*$ , and let  $M_1, \dots, M_k$  be endomorphisms of the finite dimensional  $\mathbb{C}$ -vector spaces  $E_1, \dots, E_k$ . For  $1 \leq l \leq k$ , let  $u_l \in E_l$  be an eigenvector of  $M_l$  for the eigenvalue  $\lambda_l$ . then we have*

$$f^\otimes(M_1, \dots, M_k)^{i_1 j_1 \dots i_k j_k}(u_1)^{j_1} \dots (u_k)^{j_k} = f(\lambda_1, \dots, \lambda_k)(u_1)^{i_1} \dots (u_k)^{i_k}.$$

**Proof:** In the simple case where  $f$  is a monomial, we have  $f(x_1, \dots, x_k) = x_1^{\alpha_1} \dots x_k^{\alpha_k}$ , with  $\alpha_l \in \mathbb{N}$  for  $1 \leq l \leq k$ . We get

$$\begin{aligned} f^\otimes(M_1, \dots, M_k)^{i_1 j_1 \dots i_k j_k}(u_1)^{j_1} \dots (u_k)^{j_k} &= (M_1^{\alpha_1})^{i_1 j_1} \dots (M_k^{\alpha_k})^{i_k j_k}(u_1)^{j_1} \dots (u_k)^{j_k} \\ &= \lambda_1^{\alpha_1}(u_1)^{i_1} \dots \lambda_k^{\alpha_k}(u_k)^{i_k} \\ &= f(\lambda_1, \dots, \lambda_k)(u_1)^{i_1} \dots (u_k)^{i_k}. \end{aligned}$$

By linearity, this extends to the case where  $f$  is a polynomial. For the more general case, we have  $f^\otimes(M_1, \dots, M_k) = P^\otimes(M_1, \dots, M_k)$  where  $P$  is a polynomial of  $k$  variables satisfying the conditions described in Definition 7. In particular,  $P$  has the same values  $f$  has on the eigenvalues of  $M_1, \dots, M_k$ , so the desired equality holds for a general function  $f$ .  $\square$

**Remark 12** In the case where  $M_1, \dots, M_k$  are all diagonalizable, for all  $1 \leq l \leq k$  we can take a basis of  $E_l$  made of eigenvectors of  $M_l$ . Let us denote by  $u_{lm}$ , for  $1 \leq m \leq \dim(E_l) = d_l$  the vectors of this basis, by  $\lambda_{lm}$  the corresponding eigenvalue and by  $u_{lm}^* \in E_l^*$  the corresponding vector of the dual basis.

Then the family of tensors  $\bigotimes_{l=1}^k u_{lm_l} \otimes u_{l m_l}^*$ , for  $1 \leq m_l \leq d_l$  and  $1 \leq n_l \leq d_l$ , is a basis of  $\bigotimes_{l=1}^k E_l \otimes E_l^*$ . According to Proposition 11, the decomposition of the tensor  $f^\otimes(M_1, \dots, M_k)$  in this basis can only be

$$f^\otimes(M_1, \dots, M_k) = \sum_{\forall 1 \leq l \leq k, 1 \leq m_l \leq d_l} f(\lambda_{1m_1}, \dots, \lambda_{km_k}) \bigotimes_{l=1}^k u_{lm_l} \otimes u_{l m_l}^*.$$

One of the simplest properties of  $f^\otimes(M_1, \dots, M_k)$  is its linearity with respect to  $f$ .

**Proposition 13** Let  $k \in \mathbb{N}^*$ , and let  $M_1, \dots, M_k$  be endomorphisms of the finite dimensional  $\mathbb{C}$ -vector spaces  $E_1, \dots, E_k$ . Let  $\lambda$  and  $\mu$  be two complex numbers, and  $f$  and  $g$  be two functions from  $\mathbb{C}^k$  to  $\mathbb{C}$ , regular enough such that  $f^\otimes(M_1, \dots, M_k)$  and  $g^\otimes(M_1, \dots, M_k)$  can be defined. Then the function  $\lambda f + \mu g$  has the same regularity, and we have:

$$(\lambda f + \mu g)^\otimes(M_1, \dots, M_k) = \lambda f^\otimes(M_1, \dots, M_k) + \mu g^\otimes(M_1, \dots, M_k).$$

**Proof:** If  $P$  and  $Q$  are interpolation polynomials of  $f$  and  $g$  as required in Definition 7 to compute  $f^\otimes(M_1, \dots, M_k)$  and  $g^\otimes(M_1, \dots, M_k)$ , then  $\lambda P + \mu Q$  is an interpolation polynomial of  $\lambda f + \mu g$  because partial derivatives are linear.

So we have

$$\begin{aligned} (\lambda f + \mu g)^\otimes(M_1, \dots, M_k) &= (\lambda P + \mu Q)^\otimes(M_1, \dots, M_k) \\ &= \lambda P^\otimes(M_1, \dots, M_k) + \mu Q^\otimes(M_1, \dots, M_k) \\ &= \lambda f^\otimes(M_1, \dots, M_k) + \mu g^\otimes(M_1, \dots, M_k), \end{aligned}$$

where the second equality trivially follows from Definition 1.  $\square$

Another trivial property is the nice behaviour of  $f^\otimes(M_1, \dots, M_k)$  with respect to transpositions.

**Proposition 14** Let  $k \in \mathbb{N}^*$ , and let  $M_1, \dots, M_k$  be endomorphisms of the finite dimensional  $\mathbb{C}$ -vector spaces  $E_1, \dots, E_k$ . Let  $f$  be a function from



$\mathbb{C}^k$  to  $\mathbb{C}$  such that  $f^\otimes(M_1, \dots, M_k)$  is well defined. Let  $1 \leq l \leq k$ . The transposition of  $M_l$  is the endomorphism of  $E_l^*$  defined by

$$\langle M_l^T(\zeta), v \rangle = \langle \zeta, M_l(v) \rangle,$$

for any  $\zeta$  in  $E_l^*$  and  $v \in E_l$ . More explicitly, we can write  $(M_l^T)_{j_i}^{i_i} = (M_l)_{j_i}^{i_i}$ .

Then we have:

$$f^\otimes(M_1, \dots, M_{l-1}, M_l^T, M_{l+1}, \dots, M_k)_{j_1 \dots j_{l-1} j_{l+1} \dots j_k}^{i_1 \dots i_{l-1} i_{l+1} \dots i_k} = f^\otimes(M_1, \dots, M_k)_{j_1 \dots j_k}^{i_1 \dots i_k}.$$

**Remark 15** In particular, if the matrix of  $M_l$  is symmetric in some basis  $\mathcal{B}$  of  $E_l$ , then the coefficients of  $f^\otimes(M_1, \dots, M_k)$  in a product basis where the one chosen for  $E_l$  is  $\mathcal{B}$  are invariant by swapping the corresponding indexes.

**Proof:** The matrices  $M_l$  and  $M_l^T$  have the same eigenvalues and the same minimal polynomial, thus if  $P$  is a suitable interpolation polynomial so that  $f^\otimes(M_1, \dots, M_k) = P^\otimes(M_1, \dots, M_k)$ , then we have  $f^\otimes(M_1, \dots, M_{l-1}, M_l^T, M_{l+1}, \dots, M_k) = P^\otimes(M_1, \dots, M_{l-1}, M_l^T, M_{l+1}, \dots, M_k)$ . Proposition 14 for polynomials trivially follows from the fact that  $(M^T)^a = (M^a)^T$ .  $\square$

The tensor  $f^\otimes(M_1, \dots, M_k)$  can also be seen as an endomorphism on the space  $E_1 \otimes E_2 \otimes \dots \otimes E_k$ . This allows to take products of such tensors, or to apply functions of several variables on them.

**Theorem 16** Let  $M_1, \dots, M_k$  be endomorphisms of finite dimensional  $\mathbb{C}$ -vector spaces. Let  $f_1$  and  $f_2$  be two functions from  $\mathbb{C}^k$  to  $\mathbb{C}$ . As in Definition 7, we set  $e_l$  the number of different eigenvalues of  $M_l$ , for  $1 \leq l \leq k$ ,  $\lambda_{lm}$  the eigenvalues of  $M_l$  and  $r_{lm}$  their multiplicity as roots of the minimal polynomial of  $M_l$ , for  $1 \leq m \leq e_l$ . Assume that  $f_1$  and  $f_2$  are holomorphic with respect to all the variables  $x_l$  such that  $r_{lm_l} \geq 2$  near  $(\lambda_{1m_1}, \dots, \lambda_{km_k})$ . Then we can define  $\bar{M}_1 = f_1^\otimes(M_1, \dots, M_k)$  and  $\bar{M}_2 = f_2^\otimes(M_1, \dots, M_k)$  and see them as endomorphisms of  $E_1 \otimes \dots \otimes E_k$ . Then we have

$$\bar{M}_1 \bar{M}_2 = g^\otimes(M_1, \dots, M_k),$$

with  $g(x_1, \dots, x_k) = f_1(x_1, \dots, x_k) f_2(x_1, \dots, x_k)$ .

**Remark 17** In particular,  $\bar{M}_1$  and  $\bar{M}_2$  commute, since one does get the same function  $g$  by swapping  $f_1$  and  $f_2$ .

**Proof:** We first notice that the function  $g$  is regular enough so that  $g^{\otimes}(M_1, \dots, M_k)$  is well defined, so the statement of Proposition 16 has a sense.

Let  $P_1$  and  $P_2$  be suitable interpolation polynomials of  $f_1$  and  $f_2$ , in the sense that all the partial derivatives at  $(\lambda_{1m_1}, \dots, \lambda_{km_k})$  such that one derives less than  $r_{lm_l}$  times with respect to  $x_l$  are equal for  $f_i$  and  $P_i$ .

Then according to Definition 7, one has  $f_i^{\otimes}(M_1, \dots, M_k) = P_i^{\otimes}(M_1, \dots, M_k)$ . The polynomials  $P_1$  and  $P_2$  can be written as

$$P_i = \sum_{\alpha \in F_i} a_{i\alpha} x_1^{\alpha_1} \dots x_k^{\alpha_k},$$

with  $F_i$  a finite subset of  $\mathbb{N}^k$  and  $a_{i\alpha}$  the complex coefficients of the polynomial  $P_i$ . Then we have:

$$\begin{aligned} (\bar{M}_1 \bar{M}_2)^{i_1 \dots i_k}_{j_1 \dots j_k} &= (\bar{M}_1)^{i_1}_{m_1} \dots (\bar{M}_2)^{i_k}_{m_k} (\bar{M}_2)^{m_1}_{j_1} \dots (\bar{M}_2)^{m_k}_{j_k} \\ &= \sum_{\alpha \in F_1} a_{1\alpha} (M_1^{\alpha_1})^{i_1}_{m_1} \dots (M_k^{\alpha_k})^{i_k}_{m_k} \sum_{\beta \in F_2} a_{2\beta} (M_1^{\beta_1})^{m_1}_{j_1} \dots (M_k^{\beta_k})^{m_k}_{j_k} \\ &= \sum_{\alpha \in F_1, \beta \in F_2} a_{1\alpha} a_{2\beta} (M_1^{\alpha_1 + \beta_1})^{i_1}_{j_1} \dots (M_k^{\alpha_k + \beta_k})^{i_k}_{j_k} \\ &= (P_1 P_2)^{\otimes}(M_1, \dots, M_k)^{i_1 \dots i_k}_{j_1 \dots j_k}. \end{aligned}$$

Finally, we have

$$\begin{aligned} \partial_1^{a_1}, \dots, \partial_k^{a_k} (P_1 P_2)(x_1, \dots, x_k) &= \\ \sum_{\substack{0 \leq b_1 \leq a_1 \\ \dots \\ 0 \leq b_k \leq a_k}} \binom{a_1}{b_1} \dots \binom{a_k}{b_k} \partial_1^{b_1} \dots \partial_k^{b_k} P_1(x_1, \dots, x_k) \partial_1^{a_1 - b_1} \dots \partial_k^{a_k - b_k} P_2(x_1, \dots, x_k). \end{aligned}$$

the same formula holds when we replace  $P_1$  with  $f_1$ ,  $P_2$  with  $f_2$ ,  $P_1 P_2$  with  $g$ , and  $x_l$  with  $\lambda_{lm_l}$ , for all  $k$ -tuple  $(m_1, \dots, m_k)$  such that  $1 \leq m_l \leq e_l$ , provided  $0 \leq a_l < r_{lm_l}$  for all  $1 \leq l \leq k$ . Thus we have

$$\partial_1^{a_1} \dots \partial_k^{a_k} g(\lambda_{1m_1}, \dots, \lambda_{km_k}) = \partial_1^{a_1} \dots \partial_k^{a_k} (P_1 P_2)(\lambda_{1m_1}, \dots, \lambda_{km_k}),$$

for all  $k$ -tuple  $(a_1, \dots, a_k)$  such that  $0 \leq a_l < r_{lm_l}$ . So we get

$$\bar{M}_1 \bar{M}_2 = (P_1 P_2)^{\otimes}(M_1, \dots, M_k) = g^{\otimes}(M_1, \dots, M_k),$$

as stated.  $\square$

**Theorem 18** Let  $r \in \mathbb{N}^*$ , and  $k_q \in \mathbb{N}^*$  for  $1 \leq q \leq r$ . Let  $E_{ql}$  be a family of finite dimensional  $\mathbb{C}$ -vector spaces, where  $1 \leq l \leq r$  and  $1 \leq l \leq k_q$ , and let  $M_{ql}$  be an endomorphism of  $E_{ql}$ .

We set  $e_{ql}$  the number of different eigenvalues of  $M_{ql}$ ,  $\lambda_{qlm}$  these eigenvalues, where  $1 \leq m \leq e_{ql}$ , and  $r_{qlm}$  the multiplicity of the root  $\lambda_{qlm}$  in the minimal polynomial of  $M_{ql}$ . For  $1 \leq q \leq r$ , let  $f_q$  be a function from  $\mathbb{C}^{k_q}$  to  $\mathbb{C}$ , such that  $f_q$  is holomorphic with respect to all the variables  $x_l$  such that  $r_{qlm_l} \geq 2$  near  $(\lambda_{q1m_1}, \dots, \lambda_{qlm_l}, \dots, \lambda_{qk_qm_{k_q}})$ , for every  $k_q$ -tuple  $(m_1, \dots, m_{k_q})$  with  $1 \leq m_l \leq e_{ql}$ .

Let  $\bar{e}_q$  be the number of different values that  $f_q(\lambda_{q1m_1}, \dots, \lambda_{qk_qm_{k_q}})$  takes, and  $\mu_{qp}$  be these values for  $1 \leq p \leq \bar{e}_q$ .

We set  $\bar{r}_{qp} := \max\{1 + \sum_{l=1}^{k_q} (r_{qlm_l} - 1), 1 \leq m_l \leq e_{ql} \mid f_q(\lambda_{q1m_1}, \dots, \lambda_{qk_qm_{k_q}}) = \mu_{qp}\}$ .

We set  $\bar{M}_q = f_q^{\otimes}(M_{q1}, \dots, M_{qk_q})$ , seen as an endomorphism of  $\bar{E}_q := E_{q1} \otimes \dots \otimes E_{qk_q}$ .

Let  $g$  be a function from  $\mathbb{C}^r$  to  $\mathbb{C}$ , such that for every  $r$ -tuple  $(p_1, \dots, p_r)$ ,  $g$  is holomorphic with respect to all the variables  $x_q$  such that  $\bar{r}_{qp_q} \geq 2$ , near the point  $(\mu_{1p_1}, \dots, \mu_{rp_r})$ .

Then for all  $1 \leq q \leq k$ , and all  $1 \leq p \leq \bar{e}_q$ , the  $\mu_{qp}$ 's are the eigenvalues of  $\bar{M}_q$ , and the multiplicity of the root  $\mu_{qp}$  in the minimal polynomial of  $\bar{M}_q$  is at most  $\bar{r}_{qp}$ .

So  $g^{\otimes}(\bar{M}_1, \dots, \bar{M}_r)$  is well defined and furthermore, we have:

$$g^{\otimes}(\bar{M}_1, \dots, \bar{M}_r) = h(M_{11}, \dots, M_{rk_r}),$$

where  $h$  is the function of  $\sum_{q=1}^r k_q$  variables defined by

$$h(x_{11}, \dots, x_{rk_r}) := g(f_1(x_{11}, \dots, x_{1k_1}), \dots, f_r(x_{r1}, \dots, x_{rk_r})).$$

**Proof:** Let us prove that the  $\mu_{qp}$ 's are the eigenvalues of  $\bar{M}_q$  and have multiplicity at most  $\bar{r}_{qp}$  in the minimal polynomial of  $\bar{M}_q$ .

Let  $1 \leq q \leq r$ . Let  $P(x_1, \dots, x_{k_q})$  be a suitable interpolation polynomial of  $f_q$ , such that  $\bar{M}_q = P^{\otimes}(M_{q1}, \dots, M_{qk_q})$ . For any  $m_1, \dots, m_{k_q}$ , if  $v_1, \dots, v_{k_q}$  are eigenvectors of  $M_{q1}, \dots, M_{qk_q}$  for the eigenvalues  $\lambda_{q1m_1}, \dots, \lambda_{qk_qm_{k_q}}$ , then  $v_1 \otimes \dots \otimes v_{k_q}$  is an eigenvector of  $\bar{M}_q$  for the eigenvalue  $P(\lambda_{q1m_1}, \dots, \lambda_{qk_qm_{k_q}}) = f_q(\lambda_{q1m_1}, \dots, \lambda_{qk_qm_{k_q}})$ , so the  $\mu_{qp}$ 's are eigenvalues of  $\bar{M}_q$ . Let  $Q(x) = \prod_{p=1}^{\bar{e}_q} (x - \mu_{qp})^{\bar{r}_{qp}}$ . It remains to check that  $Q(\bar{M}_q) = 0$ .

We set  $R(x_1, \dots, x_{k_q}) = Q(P(x_1, \dots, x_{k_q}))$ . Then it easily follows from Theorem 16 that  $Q(\bar{M}_q) = R^{\otimes}(M_{q1}, \dots, M_{qk_q})$ . For  $1 \leq l \leq k_q$ , let  $1 \leq$

$m_l \leq e_{ql}$ , and  $0 \leq j_l < r_{qlm_l}$ . One has, according to the Faa di Bruno formula,

$$\left[ \prod_{l=1}^{k_q} \partial_l^{j_l} \right] R(\lambda_{q1m_1}, \dots, \lambda_{qk_qm_{k_q}}) = \sum_{j=0}^{\sum_{l=1}^{k_q} j_l} Q^{(j)}(P(\lambda_{q1m_1}, \dots, \lambda_{qk_qm_{k_q}})) A_j,$$

where the term  $A_j$  can be written as

$$A_j = \sum_{\substack{n \leq j \\ (i_1, \dots, i_n) \in (\mathbb{N}^{k_q})^n \\ 0 \prec i_1 \prec i_2 \prec \dots \prec i_n \\ (a_1, \dots, a_n) \in \mathbb{N}^{*n} \\ a_1 + \dots + a_n = j \\ a_1 i_1 + \dots + a_n i_n = (j_1, \dots, j_{k_q})}} \left( \frac{\prod_{l=1}^{k_q} j_l!}{\prod_{k=1}^n a_k! \prod_{l=1}^{k_q} (i_{kl}!)^{a_k}} \right) \prod_{k=1}^n \left( \left[ \prod_{l=1}^{k_q} \partial_l^{i_{kl}} \right] P(\lambda_{q1m_1}, \dots, \lambda_{qk_qm_{k_q}}) \right)^{a_k},$$

where  $\prec$  can be any total order on  $\mathbb{N}^{k_q}$  such that 0 is its minimal element (one can take the lexicographical order, for example).

But actually the value of  $A_j$  does not matter, since  $Q^{(j)}(P(\lambda_{q1m_1}, \dots, \lambda_{qk_qm_{k_q}})) = 0$ , because  $P(\lambda_{q1m_1}, \dots, \lambda_{qk_qm_{k_q}})$  is equal to some  $\mu_{qp}$  such that  $\bar{r}_{qp} > j$ .

Thus, we have  $\left[ \prod_{l=1}^{k_q} \partial_l^{j_l} \right] R(\lambda_{q1m_1}, \dots, \lambda_{qk_qm_{k_q}}) = 0$  and then by Proposition 6, one has  $R^{\otimes}(M_{q1}, \dots, M_{qk_q})$ , and hence  $Q(\bar{M}_q) = 0$  as stated.

Now let  $P$  be a suitable interpolation polynomial of  $g$ , i.e. such that

$$\left[ \prod_{q=1}^r \partial_q^{j_q} \right] P(\mu_{1p_1}, \dots, \mu_{rp_r}) = \left[ \prod_{q=1}^r \partial_q^{j_q} \right] g(\mu_{1p_1}, \dots, \mu_{rp_r}),$$

for all  $p_1, \dots, p_r$  and  $j_1, \dots, j_r$  satisfying  $1 \leq p_q \leq \bar{e}_q$  and  $0 \leq j_q < \bar{r}_{qp_q}$ . Then we have  $P^{\otimes}(\bar{M}_1, \dots, \bar{M}_r) = g^{\otimes}(\bar{M}_1, \dots, \bar{M}_r)$ . It easily follows from Definition 1 and Theorem 16 that

$$P^{\otimes}(\bar{M}_1, \dots, \bar{M}_r) = H^{\otimes}(M_{11}, \dots, M_{rk_r}),$$

where  $H$  is the function of  $\sum_{q=1}^r k_q$  variables defined by

$$H(x_{11}, \dots, x_{rk_r}) = P(f_1(x_{11}, \dots, x_{1k_1}), \dots, f_r(x_{r1}, \dots, x_{rk_r})).$$

To complete the proof of the theorem, it remains to check that

$$\left[ \prod_{q=1}^r \prod_{l=1}^{k_q} \partial_{ql}^{j_{ql}} \right] H(\lambda_{11m_{11}}, \dots, \lambda_{rk_r m_{rk_r}}) = \left[ \prod_{q=1}^r \prod_{l=1}^{k_q} \partial_{ql}^{j_{ql}} \right] h(\lambda_{11m_{11}}, \dots, \lambda_{rk_r m_{rk_r}}),$$

for all  $1 \leq m_{ql} \leq e_{ql}$  and  $0 \leq j_{ql} < r_{qlm_{ql}}$ . This is easily done by using the Faa di Bruno formula and the fact that

$$\left[ \prod_{q=1}^r \partial_q^{j_q} \right] P(\mu_{1p_1}, \dots, \mu_{rp_r}) = \left[ \prod_{q=1}^r \partial_q^{j_q} \right] g(\mu_{1p_1}, \dots, \mu_{rp_r}),$$

for all  $1 \leq p_q \leq \bar{e}_q$  and  $0 \leq j_q < \bar{r}_{qp_q}$ .  $\square$

In some cases, contractions of the tensor  $f^\otimes(M_1, \dots, M_k)$  have a simplified expression.

**Theorem 19** *Let  $M_1, \dots, M_k$  be endomorphisms of finite dimensional  $\mathbb{C}$ -vector spaces. Let  $f$  be a function from  $\mathbb{C}^k$  to  $\mathbb{C}$ , such that  $f^\otimes(M_1, \dots, M_k)$  is well defined. Let  $1 \leq p \leq k$ . Then we have the following equality:*

$$f^\otimes(M_1, \dots, M_k)^{i_1}_{j_1} \dots^{i_{p-1}}_{j_{p-1}} i_p i_{p+1} j_{p+1} \dots^{i_k}_{j_k} = g^\otimes(M_1, \dots, M_{p-1}, M_{p+1}, \dots, M_k)^{i_1}_{j_1} \dots^{i_{p-1}}_{j_{p-1}} i_p i_{p+1} j_{p+1} \dots^{i_k}_{j_k},$$

where  $g$  is the function of  $k-1$  variables defined by

$$g(x_1, \dots, x_{p-1}, x_{p+1}, \dots, x_k) = \sum_{m=1}^{e_p} s_{pm} f(x_1, \dots, x_{p-1}, \lambda_{pm}, x_{p+1}, \dots, x_k),$$

where the  $\lambda_{pm}$ , for  $1 \leq m \leq e_p$  are all the different eigenvalues of  $M_p$  and  $s_{pm}$  is the multiplicity of the eigenvalue  $\lambda_{pm}$ , i.e. its multiplicity as root of the characteristic polynomial of  $M_p$ .

**Proof:** In the simple case where  $f(x_1, \dots, x_k)$  is a monomial, the equality of Theorem 19 easily follows from the classical fact that  $\text{Tr}(M^a) = \sum_{\lambda \text{ eigenvalue of } M} \lambda^a$  (where the same  $\lambda$  appears several times in the sum if it is a multiple eigenvalue of  $M$ ) for any square matrix  $M$ . By linearity, the equality of Theorem 19 extends to the case where  $f$  is a polynomial.

For a more general  $f$ , we set  $P$  a polynomial such that  $\left[ \prod_{l=1}^k \partial_l^{a_l} \right] (f - P)(\lambda_{1m_1}, \dots, \lambda_{km_k}) = 0$  for any sequences  $m_1, \dots, m_k$  and  $a_1, \dots, a_k$  such that  $1 \leq m_l \leq e_l$  and  $0 \leq a_l < r_{lm_l}$ , with  $e_l$  the number of different eigenvalues of  $M_l$ ,  $\lambda_{lm}$  these eigenvalues for  $1 \leq m \leq e_l$  and  $r_{lm}$  their multiplicities in the minimal polynomial of  $M_l$ . Then we have  $f^\otimes(M_1, \dots, M_k) = P^\otimes(M_1, \dots, M_k)$ . Thus we get

$$\begin{aligned} f^\otimes(M_1, \dots, M_k)^{i_1}_{j_1} \dots^{i_{p-1}}_{j_{p-1}} i_p i_{p+1} j_{p+1} \dots^{i_k}_{j_k} &= \\ P^\otimes(M_1, \dots, M_k)^{i_1}_{j_1} \dots^{i_{p-1}}_{j_{p-1}} i_p i_{p+1} j_{p+1} \dots^{i_k}_{j_k} &= \\ Q^\otimes(M_1, \dots, M_{p-1}, M_{p+1}, \dots, M_k)^{i_1}_{j_1} \dots^{i_{p-1}}_{j_{p-1}} i_p i_{p+1} j_{p+1} \dots^{i_k}_{j_k}, & \end{aligned}$$

with

$$Q(x_1, \dots, x_{p-1}, x_{p+1}, \dots, x_k) = \sum_{m=1}^{e_p} s_{pm} P(x_1, \dots, x_{p-1}, \lambda_{pm}, x_{p+1}, \dots, x_k).$$

To conclude the proof, it remains to check that

$$\left[ \prod_{\substack{1 \leq l \leq k \\ l \neq p}} \left( \frac{\partial}{\partial x_l} \right)^{a_l} (g - Q)(x_1, \dots, x_{p-1}, x_{p+1}, \dots, x_k) \right]_{(x_1, \dots, x_k) = (\lambda_{1m_1}, \dots, \lambda_{km_k})} = 0.$$

This quantity is just

$$\sum_{m=1}^{e_p} \left[ \prod_{\substack{1 \leq l \leq k \\ l \neq p}} \left( \frac{\partial}{\partial x_l} \right)^{a_l} (f - P)(\lambda_{1m_1}, \dots, \lambda_{p-1m_{p-1}}, \lambda_{pm}, \lambda_{p+1m_{p+1}}, \dots, \lambda_{km_k}), \right]$$

which is trivially 0.

So  $Q^{\otimes}(M_1, \dots, M_{p-1}, M_{p+1}, \dots, M_k) = g^{\otimes}(M_1, \dots, M_{p-1}, M_{p+1}, \dots, M_k)$  and the theorem is proved.  $\square$

**Theorem 20** *Let  $M_1, \dots, M_k$  be endomorphisms of finite dimensional  $\mathbb{C}$ -vector spaces  $E_1, \dots, E_k$ . Assume that for two indexes  $1 \leq p < q \leq k$ , we have  $E_p = E_q$  and  $M_p = M_q$ . Let  $f$  be a function from  $\mathbb{C}^k$  to  $\mathbb{C}$ , such that  $f^{\otimes}(M_1, \dots, M_k)$  is well defined. Then we have:*

$$\begin{aligned} f^{\otimes}(M_1, \dots, M_k)^{i_1 j_1 \dots i_{p-1} j_{p-1} \dots i_{p+1} j_{p+1} \dots i_{q-1} j_{q-1} \dots i_q j_q \dots i_k j_k} &= \\ f^{\otimes}(M_1, \dots, M_k)^{i_1 j_1 \dots i_{p-1} j_{p-1} \dots i_{p+1} j_{p+1} \dots i_{q-1} j_{q-1} \dots i_{q+1} j_{q+1} \dots i_k j_k} &= \\ g^{\otimes}(M_1, \dots, M_{q-1}, M_{q+1}, \dots, M_k)^{i_1 j_1 \dots i_{p-1} j_{p-1} \dots i_{p+1} j_{p+1} \dots i_{q-1} j_{q-1} \dots i_{q+1} j_{q+1} \dots i_k j_k}, \end{aligned}$$

where the function  $g$  of  $k - 1$  variables is defined by

$$g(x_1, \dots, x_{q-1}, x_{q+1}, \dots, x_k) = f(x_1, \dots, x_{q-1}, x_p, x_{q+1}, \dots, x_k).$$

**Proof:** We proceed as in the proof of Theorem 19. If  $f$  is a monomial, the result easily follows from the trivial fact that  $M^a M^b = M^b M^a = M^{a+b}$  for any square matrix  $M$ . So by linearity, it also holds for polynomials.

For a more general  $f$ , we set again  $P$  an interpolation polynomial such that  $f^{\otimes}(M_1, \dots, M_k) = P^{\otimes}(M_1, \dots, M_k)$ . And so we have:

$$\begin{aligned} f^{\otimes}(M_1, \dots, M_k)^{i_1 j_1 \dots i_{p-1} j_{p-1} \dots i_{p+1} j_{p+1} \dots i_{q-1} j_{q-1} \dots i_q j_q \dots i_k j_k} &= \\ P^{\otimes}(M_1, \dots, M_k)^{i_1 j_1 \dots i_{p-1} j_{p-1} \dots i_{p+1} j_{p+1} \dots i_{q-1} j_{q-1} \dots i_q j_q \dots i_k j_k} &= \\ P^{\otimes}(M_1, \dots, M_k)^{i_1 j_1 \dots i_{p-1} j_{p-1} \dots i_{p+1} j_{p+1} \dots i_{q-1} j_{q-1} \dots i_{q+1} j_{q+1} \dots i_k j_k} &= \\ f^{\otimes}(M_1, \dots, M_k)^{i_1 j_1 \dots i_{p-1} j_{p-1} \dots i_{p+1} j_{p+1} \dots i_{q-1} j_{q-1} \dots i_{q+1} j_{q+1} \dots i_k j_k} &= \\ Q^{\otimes}(M_1, \dots, M_{q-1}, M_{q+1}, \dots, M_k)^{i_1 j_1 \dots i_{p-1} j_{p-1} \dots i_{p+1} j_{p+1} \dots i_{q-1} j_{q-1} \dots i_{q+1} j_{q+1} \dots i_k j_k}, \end{aligned}$$

where the polynomial  $Q$  is defined by

$$Q(x_1, \dots, x_{q-1}, x_{q+1}, \dots, x_k) = P(x_1, \dots, x_{q-1}, x_p, x_{q+1}, \dots, x_k).$$

It remains to check that

$$\left[ \prod_{\substack{1 \leq l \leq k \\ l \neq q}} \left( \frac{\partial}{\partial x_l} \right)^{a_l} \right] (g - Q)(x_1, \dots, x_{q-1}, x_{q+1}, \dots, x_k) \Big|_{(x_1, \dots, x_k) = (\lambda_{1m_1}, \dots, \lambda_{km_k})} = 0,$$

with the same notation as above for the  $\lambda_{lm}$ 's and the same conditions for the  $m_l$ 's and  $a_l$ 's.

The left-hand side is equal to

$$\sum_{h=0}^{a_p} \binom{a_p}{h} \left[ \partial_p^h \partial_q^{a_p-h} \prod_{\substack{1 \leq l \leq k \\ l \neq p \\ l \neq q}} \partial_l^{a_l} \right] (f - P)(\lambda_{1m_1}, \dots, \lambda_{q-1m_{q-1}}, \lambda_{pm_p}, \lambda_{q+1m_{q+1}}, \dots, \lambda_{km_k}).$$

It is 0 because  $P$  is a nice interpolation polynomial of  $f$ .

So we get  $Q^\otimes(M_1, \dots, M_{q-1}, M_{q+1}, \dots, M_k) = g^\otimes(M_1, \dots, M_{q-1}, M_{q+1}, \dots, M_k)$  and the theorem is proved.  $\square$

In the case where  $E_p = E_q$  but  $M_p$  and  $M_q$  are different, there is no such simplified expression for the contraction, but one has a kind of commutation property if  $M_p$  and  $M_q$  commute.

**Proposition 21** *Let  $M_1, \dots, M_k$  be endomorphisms of finite dimensional  $\mathbb{C}$ -vector spaces  $E_1, \dots, E_k$ . Assume that for two indexes  $1 \leq p < q \leq k$ , we have  $E_p = E_q$  and the endomorphisms  $M_p$  and  $M_q$  commute. Let  $f$  be a function from  $\mathbb{C}^k$  to  $\mathbb{C}$ , such that  $f^\otimes(M_1, \dots, M_k)$  is well defined. Then we have:*

$$\begin{aligned} f^\otimes(M_1, \dots, M_k)^{i_1}_{j_1} \dots^{i_{p-1}}_{j_{p-1}} a^{i_{p+1}}_{j_{p+1}} \dots^{i_{q-1}}_{j_{q-1}} a^{i_{q+1}}_{j_{q+1}} \dots^{i_k}_{j_k} = \\ f^\otimes(M_1, \dots, M_k)^{i_1}_{j_1} \dots^{i_{p-1}}_{j_{p-1}} a^{i_{p+1}}_{j_{p+1}} \dots^{i_{q-1}}_{j_{q-1}} i^{i_{q+1}}_{j_{q+1}} \dots^{i_k}_{j_k}. \end{aligned}$$

**Proof:** One replaces  $f$  with a suitable interpolation polynomial  $P$  such that  $f^\otimes(M_1, \dots, M_k) = P^\otimes(M_1, \dots, M_k)$ . Then using linearity, we only have to check the property for monomials, which easily follows from the well known fact that if  $M_p$  and  $M_q$  commute, then  $M_p^{n_1}$  and  $M_q^{n_2}$  also commute (the commutation of two endomorphisms  $M$  and  $N$  can be written  $M^i_a N^j = M^j_a N^i$ ).  $\square$

**Remark 22** We have a generalization of Remark 17. If  $M_1, \dots, M_k$  and  $M'_1, \dots, M'_k$  are endomorphisms of  $E_1, \dots, E_k$  such that for all  $1 \leq l \leq k$ ,  $M_l$  and  $M'_l$  commute, and if  $f$  and  $g$  are two functions from  $\mathbb{C}^k$  to  $\mathbb{C}$  such that  $\bar{M} = f^\otimes(M_1, \dots, M_k)$  and  $\bar{M}' = g^\otimes(M'_1, \dots, M'_k)$  are well defined, then  $\bar{M}$  and  $\bar{M}'$  commute as endomorphisms of  $E_1 \otimes \dots \otimes E_k$ .

**Proof:** The tensors  $\bar{M}\bar{M}'$  and  $\bar{M}'\bar{M}$  can both be obtained by  $k$  contractions from the tensor  $h^\otimes(M_1, \dots, M_k, M'_1, \dots, M'_k)$ , where  $h(x_1, \dots, x_k, y_1, \dots, y_k) = f(x_1, \dots, x_k)g(y_1, \dots, y_k)$ . The fact you get the same result follows just from applying  $k$  times proposition 21.  $\square$

For a given holomorphic function  $f$ , the tensor  $f^\otimes(M_1, \dots, M_k)$  is an holomorphic function of  $M_1, \dots, M_k$ . The following theorem gives the expression of its derivatives.

**Theorem 23** Let  $k \in \mathbb{N}^*$ , and  $M_1, \dots, M_k$  be endomorphisms of the finite dimensional  $\mathbb{C}$ -vector spaces  $E_1, \dots, E_k$ . For any  $1 \leq l \leq k$ , we set  $e_l$  the number of different eigenvalues of  $M_l$ ,  $\lambda_{lm}$  these eigenvalues for  $1 \leq m \leq e_l$ ,  $r_{lm}$  their multiplicity as roots of the minimal polynomial of  $M_l$ . Let  $f$  be a function from  $\mathbb{C}^k$  to  $\mathbb{C}$ . Let  $1 \leq p \leq k$ . Assume that for all  $k$ -tuple  $(m_1, \dots, m_k)$ ,  $f(x_1, \dots, x_k)$  is holomorphic with respect to  $x_p$  and all the other variables  $x_l$  such that  $r_{lm_l} \geq 2$  near  $(\lambda_{1m_1}, \dots, \lambda_{km_k})$ .

Then for any endomorphism  $H$  of  $E_p$ , we have the following:

$$\lim_{\varepsilon \rightarrow 0} \frac{f^\otimes(M_1, \dots, M_{p-1}, M_{p+\varepsilon H}, M_{p+1}, \dots, M_k)^{i_1}_{j_1} \dots^{i_k}_{j_k} - f^\otimes(M_1, \dots, M_k)^{i_1}_{j_1} \dots^{i_k}_{j_k}}{\varepsilon} = g^\otimes(M_1, \dots, M_{p-1}, M_p, M_p, M_{p+1}, \dots, M_k)^{i_1}_{j_1} \dots^{i_{p-1}}_{j_{p-1}} \overset{i_p}{i} \overset{j}{j_p} \overset{i_{p+1}}{j_{p+1}} \dots^{i_k}_{j_k} H^i_j,$$

with  $g$  the function of  $k+1$  variables defined by

$$g(x_1, \dots, x_{p-1}, x_p, y, x_{p+1}, \dots, x_k) = \begin{cases} \frac{f(x_1, \dots, x_{p-1}, y, x_{p+1}, \dots, x_k) - f(x_1, \dots, x_k)}{y - x_p} & \text{if } y \neq x_p \\ \partial_p f(x_1, \dots, x_k) & \text{if } y = x_p. \end{cases}$$

**Remark 24** When  $y$  is close to  $x$ , we can write

$$g(x_1, \dots, x_{p-1}, x_p, y, x_{p+1}, \dots, x_k) = \int_0^1 \partial_p f(x_1, \dots, x_{p-1}, (1-t)x_p + ty, x_{p+1}, \dots, x_k) dt.$$

So we don't have regularity issues with  $g$  near the hyperplane  $x_p = y$ .

**Proof:** In the simple case of a monomial  $f(x_1, \dots, x_k) = \prod_{l=1}^k x_l^{\alpha_l}$ , it follows



from the fact that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \frac{((M_p + \varepsilon H)^{\alpha_p})^i_j - (M_p^{\alpha_p})^i_j}{\varepsilon} &= \sum_{h=0}^{\alpha_p-1} (M_p^h H M_p^{\alpha_p-1-h})^i_j \\ &= \left( (x, y) \rightarrow \sum_{h=0}^{\alpha_p-1} x^h y^{\alpha_p-1-h} \right)^{\otimes} (M_p, M_p)^i_a{}^b_j H^a_b \end{aligned}$$

and that we have

$$\sum_{h=0}^{\alpha_p-1} x^h y^{\alpha_p-1-h} = \begin{cases} \frac{y^{\alpha_p} - x^{\alpha_p}}{y-x} & \text{if } x \neq y \\ \frac{d}{dx} x^{\alpha_p} & \text{if } x = y. \end{cases}$$

By linearity, this result extends to polynomials.

For a more general  $f$ , for any  $1 \leq l \leq k$ , we set  $P_l$  the minimal polynomial of  $M_l$ , and we set  $P^{(\varepsilon)}$  the characteristic polynomial of  $M_p + \varepsilon H$ . Let  $Q_\varepsilon$  be the Lagrange–Sylvester interpolation polynomial of  $f$  associated to the product of multisets  $\prod_{l=1}^{p-1} \text{Root}(P_l) \times \text{Root}(P_p P^{(\varepsilon)}) \times \prod_{l=p+1}^k \text{Root}(P_l)$ . That is to say, if we set  $e_l$  the number of different roots of  $P_l$ ,  $\lambda_{lm}$  these roots for  $1 \leq m \leq e_l$  and  $r_{lm}$  their multiplicities, and if we set likewise  $e^{(\varepsilon)}$  the number of different roots of  $P_p P^{(\varepsilon)}$ ,  $\lambda_m^{(\varepsilon)}$  these roots for  $1 \leq m \leq e^{(\varepsilon)}$  and  $r_m^{(\varepsilon)}$  their multiplicities, then  $Q_\varepsilon(x_1, \dots, x_k)$  is the unique polynomial of  $k$  variables whose degree with respect to the variable  $x_l$  is at most  $\deg(P_l)$ , except for the variable  $x_p$  for which it is at most  $\deg(P_p) + \dim(E_p)$ , and wich satisfies the equalities

$$\left[ \prod_{l=1}^k \partial_l^{a_l} \right] (Q_\varepsilon - f)(\lambda_{1m_1}, \dots, \lambda_{p-1m_{p-1}}, \lambda_{m_p}^{(\varepsilon)}, \lambda_{p+1m_{p+1}}, \dots, \lambda_{km_k}) = 0$$

for all  $k$ -tuples  $(m_1, \dots, m_k)$  and  $(a_1, \dots, a_k)$  such that  $1 \leq m_l \leq e_l$  and  $0 \leq a_l < r_{lm_l}$  for  $l \neq p$  and  $1 \leq m_p \leq e^{(\varepsilon)}$  and  $0 \leq a_p < r_{m_p}^{(\varepsilon)}$ .

Because of the continuity of the Lagrange–Sylvester interpolation, we have  $Q_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} Q_0$ . Furthermore, we have

$$f^{\otimes}(M_1, \dots, M_{p-1}, M_p + \varepsilon H, M_{p+1}, \dots, M_k) = Q_\varepsilon^{\otimes}(M_1, \dots, M_{p-1}, M_p + \varepsilon H, M_{p+1}, \dots, M_k).$$

So we get

$$\begin{aligned} f^{\otimes}(M_1, \dots, M_{p-1}, M_p + \varepsilon H, M_{p+1}, \dots, M_k) - f^{\otimes}(M_1, \dots, M_k) &= \\ Q_\varepsilon^{\otimes}(M_1, \dots, M_{p-1}, M_p + \varepsilon H, M_{p+1}, \dots, M_k) - Q_0^{\otimes}(M_1, \dots, M_k) &= \\ = (Q_\varepsilon - Q_0)^{\otimes}(M_1, \dots, M_{p-1}, M_p + \varepsilon H, M_{p+1}, \dots, M_k) &+ \\ + Q_0^{\otimes}(M_1, \dots, M_{p-1}, M_p + \varepsilon H, M_{p+1}, \dots, M_k) - Q_0^{\otimes}(M_1, \dots, M_k). & \end{aligned}$$

For a fixed  $\varepsilon$ , the difference  $(Q_\varepsilon - Q_0)^\otimes(M_1, \dots, M_{p-1}, M_p + \varepsilon H, M_{p+1}, \dots, M_k)$  is a polynomial of  $\delta$  whose coefficients tend to 0 when  $\varepsilon$  tends to 0. The coefficient of this polynomial corresponding to  $\delta^0$  is  $(Q_\varepsilon - Q_0)^\otimes(M_1, \dots, M_k) = 0$ . Thus we have

$$(Q_\varepsilon - Q_0)^\otimes(M_1, \dots, M_{p-1}, M_p + \varepsilon H, M_{p+1}, \dots, M_k) = o(\varepsilon).$$

Hence, using the Theorem for the polynomial  $Q_0$ , we have

$$\lim_{\varepsilon \rightarrow 0} \frac{f^\otimes(M_1, \dots, M_{p-1}, M_p + \varepsilon H, M_{p+1}, \dots, M_k)^{i_1 j_1 \dots i_k j_k} - f^\otimes(M_1, \dots, M_k)^{i_1 j_1 \dots i_k j_k}}{\varepsilon} = R^\otimes(M_1, \dots, M_{p-1}, M_p, M_p, M_{p+1}, \dots, M_k)^{i_1 j_1 \dots i_{p-1} j_{p-1} i_p j_p \dots i_{p+1} j_{p+1} \dots i_k j_k} H^i_j,$$

with  $R$  the polynomial of  $k + 1$  variables given by:

$$R(x_1, \dots, x_{p-1}, x_p, y, x_{p+1}, \dots, x_k) = \begin{cases} \frac{Q_0(x_1, \dots, x_{p-1}, y, x_{p+1}, \dots, x_k) - Q_0(x_1, \dots, x_k)}{y - x_p} & \text{if } y \neq x_p \\ \partial_p Q_0(x_1, \dots, x_k) & \text{if } y = x_p. \end{cases}$$

Let  $(m_1, \dots, m_p, m'_p, m_{p+1}, \dots, m_k)$  and  $(a_1, \dots, a_p, a'_p, a_{p+1}, \dots, a_k)$  be two  $k + 1$ -tuples satisfying  $1 \leq m_l \leq e_l$ ,  $1 \leq m'_p \leq e_p$ ,  $0 \leq a_l < r_{lm_l}$  and  $0 \leq a'_p < r_{pm'_p}$ . We want to show that

$$\left[ \frac{\partial^{a'_p}}{\partial y} \prod_{l=1}^k \frac{\partial^{a_l}}{\partial x_l} \right] (g - R)(x_1, \dots, x_p, y, x_{p+1}, \dots, x_k) \Big|_{(x_1, \dots, x_p, y, x_{p+1}, \dots, x_k) = (\lambda_{1m_1}, \dots, \lambda_{pm_p}, \lambda_{pm'_p}, \lambda_{p+1m_{p+1}}, \dots, \lambda_{km_k})} = 0.$$

If  $m_p \neq m'_p$ , this quantity is

$$\sum_{h=0}^{a'_p} \binom{a'_p}{h} \frac{(-1)^{a'_p - h} (a_p + a'_p - h)! \left[ \frac{\partial^h}{\partial_p} \prod_{\substack{1 \leq l \leq k \\ l \neq p}} \partial_l^{a_l} \right] (f - Q_0)(\lambda_{1m_1}, \dots, \lambda_{p-1m_{p-1}}, \lambda_{pm'_p}, \lambda_{p+1m_{p+1}}, \dots, \lambda_{km_k})}{(\lambda_{pm'_p} - \lambda_{pm_p})^{a_p + a'_p - h + 1}} - \sum_{h=0}^{a_p} \binom{a_p}{h} \frac{(-1)^{a'_p} (a_p + a'_p - h)! \left[ \frac{\partial^h}{\partial_p} \prod_{\substack{1 \leq l \leq k \\ l \neq p}} \partial_l^{a_l} \right] (f - Q_0)(\lambda_{1m_1}, \dots, \lambda_{km_k})}{(\lambda_{pm'_p} - \lambda_{pm_p})^{a_p + a'_p - h + 1}}$$

which is 0 as wanted, since each term of both sums is 0. If  $m_p = m'_p$ , using the formula of Remark 24, the quantity we want to compute is

$$\left[ \partial_p^{a_p + a'_p + 1} \prod_{\substack{1 \leq l \leq k \\ l \neq p}} \partial_l^{a_l} \right] (f - Q_0)(\lambda_{1m_1}, \dots, \lambda_{km_k}) \int_0^1 (1-t)^{a'_p} t^{a_p} dt$$

which is 0 too because  $a_p + a'_p + 1 < 2r_{pm_p} \leq r_{pm_p} + s_{pm_p}$ , where  $s_{pm_p}$  is the multiplicity of  $\lambda_{pm_p}$  as root of the characteristic polynomial of  $M_p$ .

Thus  $(g - R)^\otimes(M_1, \dots, M_{p-1}, M_p, M_p, M_{p+1}, \dots, M_k) = 0$  and the theorem is proved.  $\square$

### 3 Some possible applications

In the particular case of symmetric square matrices with real coefficients, we know that such matrices have real eigenvalues and are diagonalizable in an orthonormal basis. So we can apply real-valued functions of real variables to them, without having the regularity concerns for the definitions. In this framework, we have the following result.

**Proposition 25** *We denote by  $\text{Sym}_n(\mathbb{R})$  the vector space of symmetric  $n \times n$  matrices with real coefficients, equipped with the Hilbert–Schmidt norm*

$$\|M\|_{HS} = \sqrt{\text{Tr}(M^T M)} = \sqrt{\text{Tr}(M^2)}.$$

*Let  $f : \mathbb{R} \mapsto \mathbb{R}$  be a function. Assume that  $f$  is  $k$ -Lipschitz for some  $k > 0$ .*

*Then the function*

$$F : \begin{cases} \text{Sym}_n(\mathbb{R}) & \mapsto \text{Sym}_n(\mathbb{R}) \\ M & \mapsto f(M) \end{cases}$$

*is also  $k$ -Lipschitz.*

**Proof:** We first prove that the result holds when  $f$  is a  $\mathcal{C}^1$  function. In that case, a modified version of Theorem 23 holds, so  $F$  is differentiable, and its derivative at  $M$  is given by

$$(\text{d}F(M).H)^{i_j} = \lim_{\varepsilon \rightarrow 0} \frac{F(M + \varepsilon H)^{i_j} - F(M)^{i_j}}{\varepsilon} = f_1(M, M)^{i_k l_j} H^k l_j,$$

where we have

$$f_1(x, y) = \begin{cases} \frac{f(y) - f(x)}{y - x} & \text{if } x \neq y \\ f'(x) & \text{if } x = y. \end{cases}$$

For a given  $M \in \text{Sym}_n(\mathbb{R})$ , we denote by  $\lambda_1, \lambda_2, \dots, \lambda_n$  its eigenvalues and  $u_1, \dots, u_n$  a set of orthonormal eigenvectors of  $M$ . Then, the matrices  $u_i \otimes u_i^*$  for  $1 \leq i \leq n$  and  $\frac{u_i \otimes u_j^* + u_j \otimes u_i^*}{\sqrt{2}}$  for  $1 \leq i < j \leq n$  form an orthonormal basis of  $\text{Sym}_n(\mathbb{R})$  and are eigenvectors of  $\text{d}F(M)$  with eigenvalues  $f_1(\lambda_i, \lambda_i)$  and  $f_1(\lambda_i, \lambda_j)$  respectively, according to Proposition 11. Using the fact that  $f$  is  $k$ -Lipschitz, we have  $\forall x, y \in \mathbb{R}, |f_1(x, y)| \leq k$ .

Thus  $\text{d}F(M)$  is a  $k$ -Lipschitz linear function of  $\text{Sym}_n(\mathbb{R})$ . This being true for every  $M \in \text{Sym}_n(\mathbb{R})$ , the function  $F$  itself is also  $k$ -Lipschitz.

If the function  $f$  is not  $\mathcal{C}^1$ , one can approximate  $f$  by a  $\mathcal{C}^1$  which is also  $k$ -Lipschitz. For example, we set

$$g(x) = \begin{cases} 0 & \text{if } |x| \geq 1 \\ \frac{1}{e^{1-x^2}} & \text{if } |x| < 1, \end{cases}$$

$$g_\varepsilon(x) = \frac{1}{\varepsilon} g\left(\frac{x}{\varepsilon}\right)$$

and

$$f_\varepsilon(x) = \int_{\mathbb{R}} f(x-y)g_\varepsilon(y)dy = \int_{\mathbb{R}} f(y)g_\varepsilon(x-y)dy.$$

Then we have

$$|f_\varepsilon(x) - f_\varepsilon(y)| = \left| \int_{\mathbb{R}} (f(x-z) - f(y-z))g_\varepsilon(z)dz \right| \leq k|x-y| \int_{\mathbb{R}} |g_\varepsilon(z)|dz = k|x-y|,$$

so  $f_\varepsilon$  is  $k$ -Lipschitz. And  $f_\varepsilon$  is differentiable, its derivative being

$$f'_\varepsilon(x) = \int_{\mathbb{R}} f(y)g'_\varepsilon(x-y)dy = \int_{\mathbb{R}} f(x-y)g'_\varepsilon(y)dy,$$

which is continuous so  $f_\varepsilon$  is  $\mathcal{C}^1$ .

Furthermore, we have

$$|f_\varepsilon(x) - f(x)| = \left| \int_{\mathbb{R}} (f(y) - f(x))g_\varepsilon(x-y)dy \right| \leq k \int_{\mathbb{R}} |y|g_\varepsilon(y)dy = k\varepsilon \int_{\mathbb{R}} |y|g(y)dy.$$

So for any  $M \in \text{Sym}_n(\mathbb{R})$ , we have

$$\begin{aligned} \|f_\varepsilon(M) - f(M)\|_{HS}^2 &= \|(f_\varepsilon - f)(M)\|_{HS}^2 = \text{Tr}((f_\varepsilon - f)^2(M)) = \sum_{i=1}^n (f_\varepsilon - f)^2(\lambda_i) \\ &\leq k^2 \varepsilon^2 n \left( \int_{\mathbb{R}} |x|g(x)dx \right)^2. \end{aligned}$$

So, for  $M_1$  and  $M_2$  in  $\text{Sym}_n(\mathbb{R})$ , we have

$$\begin{aligned} \|F(M_1) - F(M_2)\|_{HS} &\leq \|F(M_1) - f_\varepsilon(M_1)\|_{HS} + \|f_\varepsilon(M_1) - f_\varepsilon(M_2)\|_{HS} + \|f_\varepsilon(M_2) - F(M_2)\|_{HS} \\ &\leq k\varepsilon\sqrt{n} \int_{\mathbb{R}} |x|g(x)dx + k\|M_2 - M_1\|_{HS} + k\varepsilon\sqrt{n} \int_{\mathbb{R}} |x|g(x)dx. \end{aligned}$$

This inequality being true for every  $\varepsilon > 0$ , we finally get  $\|F(M_1) - F(M_2)\|_{HS} \leq k\|M_2 - M_1\|_{HS}$ , so  $F$  is  $k$ -Lipschitz.  $\square$

The main reason the author thinks it is a good idea to introduce  $f^\otimes(M_1, \dots, M_k)$  is that it allows to give a rather simple expression of derivatives of a function of a square matrix. If we iterate Theorem 23, we can get the following expression for the  $n$ -th derivative of  $f(M) = f^\otimes(M)$ .

**Proposition 26** *Let  $M$  and  $H$  be two endomorphisms of a finite-dimensional  $\mathbb{C}$ -vector space  $E$ . Let  $f : \mathbb{C} \mapsto \mathbb{C}$  be a function which is holomorphic in the neighborhood of each eigenvalue of  $M$ . Then the function  $F : U \subset \mathbb{C} \mapsto L(E, E)$  given by  $F(z) = f(M + zH)$  is holomorphic in a neighborhood of 0 and furthermore, we have:*

$$F^{(n)}(z)^i_j = n! f_n^\otimes \underbrace{(M + zH, M + zH, \dots, M + zH)}_{n+1 \text{ times}}^{i_{j_1} i_{j_2} \dots i_{j_{n-1}} i_{j_n}} H^{j_1}_{i_1} \dots H^{j_n}_{i_n},$$

where  $U$  is an open subset of  $\mathbb{C}$  which contains 0, and

$$f_n(x_0, \dots, x_n) = f[x_0, \dots, x_n]$$

is the (generalized) divided difference of the function  $f$  on the nodes  $x_0, \dots, x_n$ .

**Proof:** We proceed by induction on  $n$ . The case  $n = 0$  is trivial.

Assume that

$$F^{(n)}(z)^i_j = n! f_n^\otimes (M + zH, \dots, M + zH)^{i_{j_1} i_{j_2} \dots i_{j_{n-1}} i_{j_n}} H^{j_1}_{i_1} \dots H^{j_n}_{i_n}.$$

then if we want to differentiate this expression one more time with respect to  $z$ , we have to differentiate  $f_n^\otimes (M + zH, \dots, M + zH)$  with respect to each (matricial) variable. So using Theorem 23, we get

$$\begin{aligned} F^{(n+1)}(z)^i_j &= n! \sum_{k=0}^n f_{n+1}^\otimes (M + zH, \dots, M + zH)^{i_{j_1} i_{j_2} \dots i_k i_{j_{k+1}} \dots i_{j_n}} H^{j_1}_{i_1} \dots H^{j_n}_{i_n} \\ &= n! \sum_{k=0}^n f_{n+1}^\otimes (M + zH, \dots, M + zH)^{i_{j_1} i_{j_2} \dots i_n i_{j_{n+1}}} H^{j_1}_{i_1} \dots H^{j_{n+1}}_{i_{n+1}} \\ &= (n+1)! f_{n+1}^\otimes (M + zH, \dots, M + zH)^{i_{j_1} i_{j_2} \dots i_n i_{j_{n+1}}} H^{j_1}_{i_1} \dots H^{j_{n+1}}_{i_{n+1}}, \end{aligned}$$

where we just relabelled the indexes to derive line 2 from line 1. So the induction hypothesis is true at the rank  $n + 1$ .  $\square$

**Remark 27** *In the case where  $H$  and  $M$  commute, we can get a simpler expression. Indeed, we can write*

$$F^{(n)}(z)^i_j = n! g_n^\otimes (M + zH, \dots, M + zH, H, \dots, H)^{i_{j_1} i_{j_2} \dots i_{j_{n-1}} i_n i_{j_1} \dots i_n},$$

where  $g_n$  is the function of  $2n + 1$  variables defined by

$$g_n(x_0, \dots, x_n, y_1, \dots, y_n) = f[x_0, \dots, x_n]y_1 \dots y_n.$$

Since  $H$  and  $M$  commute,  $H$  and  $M + zH$  commute too. Using Proposition 21 sufficiently many times, we can get

$$F^{(n)}(z)^i_j = n!g_n^{\otimes}(M + zH, \dots, M + zH, H, \dots, H)^i_{j_1 j_2 \dots j_n} j_1^{i_1} j_2^{i_2} \dots j_n^{i_n}.$$

Using Theorem 20, one gets:

$$F^{(n)}(z)^i_j = n!\bar{g}_n^{\otimes}(M + zH, H)^i_k k_j$$

where  $\bar{g}_n(x, y) = g_n(x, \dots, x, y, \dots, y) = f[x, \dots, x]y^n = \frac{f^{(n)}(x)}{n!}y^n$ . So finally, we get:

$$F^{(n)}(z) = f^{(n)}(M + zH)H^n.$$

**Remark 28** If we differentiate  $\text{Tr}(f(M + zH))$ , one gets

$$F^{(n)}(z)^i_i = n!f_n^{\otimes}(M + zH, \dots, M + zH)^i_{j_1 j_2 \dots j_n} i_1^{j_1} i_2^{j_2} \dots i_n^{j_n} H^{j_1}_{i_1} \dots H^{j_n}_{i_n}.$$

Using Theorem 20, we get

$$F^{(n)}(z)^i_i = n!h_{n,n}^{\otimes}(M + zH, \dots, M + zH)^i_{j_1 j_2 \dots j_n} i_1^{j_1} i_2^{j_2} \dots i_n^{j_n} H^{j_1}_{i_1} \dots H^{j_n}_{i_n},$$

with  $h_{n,n}$  the function of  $n$  variables defined by

$$h_{n,n}(x_1, \dots, x_n) = f_n(x_n, x_1, \dots, x_n).$$

With  $n = 1$ , one gets the classical result that

$$\frac{d}{dz} \text{Tr}(f(M + zH)) = \text{Tr}(f'(M + zH)H).$$

If one differentiates  $n - 1$  extra times this formula, one gets

$$F^{(n)}(z)^i_i = (n-1)!f_{n-1}^{\otimes}(M + zH, \dots, M + zH)^i_{j_1 j_2 \dots j_n} i_1^{j_1} i_2^{j_2} \dots i_n^{j_n} H^{j_1}_{i_1} \dots H^{j_n}_{i_n},$$

which seems to be different from the other formula above. In fact we get the same thing because if we take  $h_{n,k}(x_1, \dots, x_n) = h_{n,n}(x_{k+1}, \dots, x_n, x_1, \dots, x_k)$ , we have

$$\begin{aligned} h_{n,n}^{\otimes}(M + zH, \dots, M + zH)^i_{j_1 j_2 \dots j_n} i_1^{j_1} i_2^{j_2} \dots i_n^{j_n} H^{j_1}_{i_1} \dots H^{j_n}_{i_n} = \\ h_{n,k}^{\otimes}(M + zH, \dots, M + zH)^i_{j_1 j_2 \dots j_n} i_1^{j_1} i_2^{j_2} \dots i_n^{j_n} H^{j_1}_{i_1} \dots H^{j_n}_{i_n} \end{aligned}$$

by relabelling the indexes, and the equality

$$f'_{n-1}(x_1, \dots, x_n) = \sum_{k=1}^n h_{n,k}(x_1, \dots, x_n)$$

holds.

**Remark 29** Despite the fact that the tensor  $f_n^\otimes(M, \dots, M)$  has a lot of symmetries (due to the fact that  $f_n$  is a symmetric function), the  $n$ -linear application  $F^{(n)}(0).(H_1, \dots, H_n)$  obtained by polarization is not in general given by  $n!$  times this tensor, but by a symmetrization of it.

$$\begin{aligned} F^{(n)}(0).(H_1, \dots, H_n)^i_j &= \left( \sum_{\sigma \in \mathfrak{S}_n} f_n^\otimes(M, \dots, M)^{i_{j_{\sigma(1)}} \dots i_{j_{\sigma(n)}}} \right) \\ &\quad (H_1)^{j_{i_1}} \dots (H_n)^{j_{i_n}} \\ &\neq n! f_n^\otimes(M, \dots, M)^{i_{j_1} \dots i_{j_n}} (H_1)^{j_{i_1}} \dots (H_n)^{j_{i_n}}, \end{aligned}$$

where  $\mathfrak{S}_n$  is the set of permutations of  $\{1, \dots, n\}$ .

For example, if  $f(z) = z^n$ , we have  $f_n = 1$  and then  $f_n^\otimes(M, \dots, M) = I^{\otimes n+1}$ , but we have

$$\sum_{\sigma \in \mathfrak{S}_n} H_{\sigma(1)} \dots H_{\sigma(n)} \neq n! H_1 \dots H_n.$$

If a function  $f : \mathbb{C} \mapsto \mathbb{C}$  is  $\mathbb{R}$ -differentiable but not holomorphic, then  $f(M)$  is well defined if  $M$  has no multiple eigenvalues. One can wonder if  $f(M)$  is  $\mathbb{R}$ -differentiable. The answer is yes, but to the author's knowledge, one does not have a nice expression of the derivative like we have in the holomorphic case. But we can use the fact that

$$f(M) = \sum_{\lambda \text{ eigenvalue of } M} f(\lambda) P_M(\lambda),$$

where  $P_M(\lambda)$  is the projection on the eigensubspace of  $M$  corresponding to the eigenvalue  $\lambda$ , parallelwise to all the other eigensubspaces of  $M$ .

We set  $M(t) = M + tH$ . Because of the continuity of eigenvalues, there exist continuous functions  $\lambda_k(t)$  for  $1 \leq k \leq \dim(E)$ , defined in a neighborhood of 0, such that for a given  $t$ , the  $\lambda_k(t)$  are the eigenvalues of  $M(t)$ . The projectors  $P_{M(t)}(\lambda_k(t))$  are also continuous. As we will see below, the  $\lambda_k(t)$  and  $P_{M(t)}(\lambda_k(t))$  are analytic. So if we set  $F(t) = f(M(t))$ , we have

$$F(t) = \sum_{k=1}^{\dim(E)} f(\lambda_k(t)) P_{M(t)}(\lambda_k(t))$$

Thus if  $f$  is  $n$  times  $\mathbb{R}$ -differentiable, we get

$$F^{(n)}(t) = \sum_{k=1}^{\dim(E)} \sum_{h=0}^n \binom{n}{h} \frac{d^h}{dt} f(\lambda_k(t)) \frac{d^{n-h}}{dt} P_{M(t)}(\lambda_k(t)).$$

The derivatives of  $f(\lambda_k(t))$  can be expressed with the derivatives of  $f$  and  $\lambda_k$  thanks to the Faa di Bruno formula.

Now we look at the behaviour of  $\lambda_k(t)$  and  $P_{M(t)}(\lambda_k(t))$  near 0. Since the eigenvalues of  $M$  are different, we can set  $2\delta > 0$  the minimum of the distance between two of them. So the balls  $\mathcal{B}(\lambda_k, \delta)$  do not overlap (with  $\lambda_k = \lambda_k(0)$ ). Because of the continuity of eigenvalues, there exists  $\varepsilon > 0$  such that for  $|t| \leq \varepsilon$ , we have  $\lambda_k(t) \in \mathcal{B}(\lambda_k, \delta)$  for all  $1 \leq k \leq \dim(E)$ . Let us denote by  $u_{(\delta)}^{(\lambda)}$  the characteristic function of  $\mathcal{B}(\lambda, \delta)$ , which is holomorphic everywhere except on the boundary of  $\mathcal{B}(\lambda, \varepsilon)$ . Then we have, for  $t < \varepsilon$ ,

$$\begin{aligned} \lambda_k(t) &= \text{Tr}(M(t)u_{(\delta)}^{(\lambda_k)}(M(t))) \\ P_{M(t)}(\lambda_k(t)) &= u_{(\delta)}^{(\lambda_k)}(M(t)). \end{aligned}$$

To get the derivatives of these two guys, one can use Proposition 26, but using the fact that the derivative of  $zu_{(\delta)}^{(\lambda_k)}(z)$  is  $u_{(\delta)}^{(\lambda_k)}(z)$  and Remark 28, we get  $\lambda_k^{(n)}(t) = \text{Tr}\left(\frac{d}{dt}^{n-1}(P_{M(t)}(\lambda_k(t)))H\right)$ , so we only have to look at the derivatives of  $P_{M(t)}(\lambda_k(t))$ . Using Proposition 26, we get:

$$\frac{d^n}{dt} (P_{M(t)}(\lambda_k(t)))^{i_j} = n! u_{(\delta)}^{(\lambda_k) \otimes n}(M(t), \dots, M(t))^{i_{j_1} i_{j_2} \dots i_{n-1} j_n i_n} H^{j_1 i_1} \dots H^{j_n i_n}.$$

If we compute that at  $t = 0$ , we can replace  $u_{(\delta)}^{(\lambda_k)}$  with  $u_{(\delta')}^{(\lambda_k)}$  with  $\delta' \leq \delta$ , since this two functions coincide on a neighborhood of the spectrum of  $M$ .

We have the following interesting fact: for all  $n$  the functions  $u_{(\delta)}^{(\lambda)}$  simply converge to a limit we will denote by  $u_n^{(\lambda)}$  when  $\delta$  tends to 0 (these functions are not defined on the whole space  $\mathbb{C}^{n+1}$ , but for any single point, there are only finitely many bad  $\delta$ 's, and furthermore, for any  $(x_0, \dots, x_n)$ , the function  $\delta \mapsto u_{(\delta)}^{(\lambda)}(x_0, \dots, x_n)$  is constant by parts).

The function  $u_n^{(\lambda)}$  is the symmetric function of  $n+1$  variables such that for all  $0 \leq m \leq n+1$  and  $z_0, \dots, z_{n-m} \neq \lambda$ , we have

$$u_n^{(\lambda)}(\underbrace{\lambda, \dots, \lambda}_{m \text{ times}}, z_0, \dots, z_{n-m}) = \begin{cases} 0 & \text{if } m = 0 \\ \frac{1}{(m-1)!} \frac{d^{m-1}}{dz} \prod_{h=0}^{n-m} \frac{1}{z-z_h} \Big|_{z=\lambda} & \text{else.} \\ = (-1)^{(m-1)} \sum_{k_0+\dots+k_{n-m}=m-1} \prod_{h=0}^{n-m} \frac{1}{(\lambda-z_h)^{1+k_h}} & \end{cases}$$



So we get

$$\frac{d}{dt}^n (P_{M(t)}(\lambda_k(t)))^i_j = n! u_n^{(\lambda_k(t)) \otimes} (M(t), \dots, M(t))^{i_{j_1} i_{j_2} \dots i_{j_{n-1}} i_{j_n}} H^{j_1}_{i_1} \dots H^{j_n}_{i_n}.$$

And finally, we have

$$\frac{d}{dt}^n (P_{M(t)}(\lambda_k(t))) = n! \sum_{1 \leq k_0, \dots, k_n \leq \dim(E)} u_n^{(\lambda_k(t))} (\lambda_{k_0}(t), \dots, \lambda_{k_n}(t)) P_{M(t)}(\lambda_{k_0}(t)) H P_{M(t)}(\lambda_{k_1}(t)) H \dots H P_{M(t)}(\lambda_{k_n}(t)) \\ \lambda_k^{(n)}(t) = (n-1)! \sum_{1 \leq k_0, \dots, k_{n-1} \leq \dim(E)} u_{n-1}^{(\lambda_k(t))} (\lambda_{k_0}(t), \dots, \lambda_{k_{n-1}}(t)) \text{Tr}(P_{M(t)}(\lambda_{k_0}(t)) H \dots H P_{M(t)}(\lambda_{k_{n-1}}(t)) H).$$

The following proposition shows how we can get  $\sum_{i_1 \neq i_2 \neq \dots \neq i_k} f(\lambda_{i_1}, \dots, \lambda_{i_k})$  from the tensor  $f^{\otimes}(M, \dots, M)$ , where the  $\lambda_i$ 's are the eigenvalues of  $M$ . This is a generalization of  $\text{Tr}(f(M)) = \sum_i f(\lambda_i)$ .

**Proposition 30** *Let  $M$  be an endomorphism of a  $\mathbb{C}$ -vector space  $E$  of finite dimension  $d$ . Let  $f$  be a function of  $k$  variables such that  $f^{\otimes}(M, \dots, M)$  is well defined. We set  $\lambda_1, \dots, \lambda_d$  the eigenvalues of  $M$  (appearing with their multiplicities). Then we have the following:*

$$\sum_{\substack{1 \leq n_1, \dots, n_k \leq d \\ \forall l \neq l', n_l \neq n_{l'}}} f(\lambda_{n_1}, \dots, \lambda_{n_k}) = k! f^{\otimes}(M, \dots, M)^{i_1 \dots i_k}_{j_1 \dots j_k} (\Pi_k^{\wedge})^{j_1 \dots j_k}_{i_1 \dots i_k},$$

where

$$(\Pi_k^{\wedge})^{j_1 \dots j_k}_{i_1 \dots i_k} = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \epsilon(\sigma) I^{j_1}_{i_{\sigma(1)}} I^{j_2}_{i_{\sigma(2)}} \dots I^{j_k}_{i_{\sigma(k)}},$$

with  $\epsilon(\sigma)$  the signature of the permutation  $\sigma$ . The tensor  $(\Pi_k^{\wedge})$  is the one corresponding to the canonical projection from  $E^{\otimes k}$  to the subspace  $E^{\wedge k}$  of antisymmetric tensors of order  $k$ .

**Proof:** Let  $\sigma \in \mathfrak{S}_k$ . We set  $\Omega(\sigma)$  the number of orbits of  $\sigma$ , and we denote by  $\omega_0, \dots, \omega_{\Omega(\sigma)}$  these orbits. For  $1 \leq l \leq k$ , we set  $1 \leq c(l) \leq \Omega(\sigma)$  the unique index such that  $l \in \omega_{c(l)}$ .

Then, using theorems 19 and 20, we have the following:

$$f^{\otimes}(M, \dots, M)^{i_1 \dots i_k}_{j_1 \dots j_k} I^{j_1}_{i_{\sigma(1)}} I^{j_2}_{i_{\sigma(2)}} \dots I^{j_k}_{i_{\sigma(k)}} = \sum_{1 \leq n_1, \dots, n_{\Omega(\sigma)} \leq d} f(\lambda_{n_{c(1)}}, \dots, \lambda_{n_{c(k)}}).$$

Now, for  $1 \leq n_1, \dots, n_k \leq d$ , we look at the number of times the term  $f(\lambda_{n_1}, \dots, \lambda_{n_k})$  appears when we use the formula above to compute

$$\sum_{\sigma \in \mathfrak{S}_k} \epsilon(\sigma) f^{\otimes}(M, \dots, M)^{i_1 \dots i_k}_{j_1 \dots j_k} I^{j_1}_{i_{\sigma(1)}} I^{j_2}_{i_{\sigma(2)}} \dots I^{j_k}_{i_{\sigma(k)}}.$$

This term appears once for each  $\sigma$  such that  $l \mapsto n_l$  is constant on all the orbits of  $\sigma$ , with the prefactor  $\epsilon(\sigma)$ . The set of such permutations is in fact a subgroup of  $\mathfrak{S}_k$ , more precisely the one of permutations which stabilize the equivalence classes of the relation  $l \sim l' \Leftrightarrow n_l = n_{l'}$ .

If there exist two different indexes  $p$  and  $q$  such that  $n_p$  and  $n_q$  are equal, then the above-mentioned subgroup contains the transposition  $(p, q)$  which has signature  $-1$ , and thus, since  $\epsilon$  is a group morphism, half of the elements of the subgroup has signature 1 and the other half has signature  $-1$ , so the sum of the signatures is 0.

If all the  $n_l$  are different, then the subgroup is just the identity, so our term only appears once.

Hence we have

$$\sum_{\sigma \in \mathfrak{S}_k} \epsilon(\sigma) f^{\otimes}(M, \dots, M)^{i_1 \dots i_k}_{j_1 \dots j_k} I^{j_1}_{i_{\sigma(1)}} I^{j_2}_{i_{\sigma(2)}} \dots I^{j_k}_{i_{\sigma(k)}} = \sum_{\substack{1 \leq n_1, \dots, n_k \leq d \\ \forall l \neq l', n_l \neq n_{l'}}} f(\lambda_{n_1}, \dots, \lambda_{n_k}),$$

as wanted.  $\square$

A classical example is given by taking  $k = d$ , and  $f(x_1, \dots, x_d) = x_1 \dots x_d$ , in which case we can get

$$\det(M) = \sum_{a_1 + 2a_2 + \dots + da_d = d} \frac{(-1)^{d - a_1 - a_2 - \dots - a_d}}{a_1! a_2! \dots a_d! 1^{a_1} 2^{a_2} \dots d^{a_d}} \text{Tr}(M)^{a_1} \text{Tr}(M^2)^{a_2} \dots \text{Tr}(M^d)^{a_d}.$$

The restriction to  $E^{\wedge k}$  of  $\Pi_k^{\wedge} f^{\otimes}(M, \dots, M)$ , where  $\Pi_k^{\wedge}$  and  $f^{\otimes}(M, \dots, M)$  are seen as endomorphisms of  $E^{\otimes k}$ , can also be itself interesting (and not only its trace). Let us denote it by  $f^{\wedge}(M, \dots, M)$ .

Indeed, since  $E^{\wedge k}$  has dimension  $\binom{d}{k}$ ,  $f^{\wedge}(M, \dots, M)$  lives in a space of dimension  $\binom{d}{k}^2$ , whereas  $f^{\otimes}(M, \dots, M)$  lives in a space of much greater dimension  $d^{2k}$ .

The eigenvalues of  $f^{\wedge}(M, \dots, M)$  are  $\frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} f(\lambda_{n_{\sigma(1)}}, \dots, \lambda_{n_{\sigma(k)}})$  for  $1 \leq n_1 < n_2 < \dots < n_k \leq d$  (the corresponding eigenvector is  $v_{n_1} \wedge \dots \wedge v_{n_k}$  if  $M$  is diagonalizable and if  $v_1, \dots, v_d$  is a basis of eigenvectors of  $M$ , such that  $v_n$  is an eigenvector of  $M$  for the eigenvalue  $\lambda_n$ ).

One could want to extend the definition of  $f^{\wedge}(M, \dots, M)$  to some cases in which  $f^{\otimes}(M, \dots, M)$  is not well defined. For example, for  $f(x, y) = \frac{1}{(x-y)^2}$ ,  $f^{\wedge}(M, M)$  should have a sense if  $M$  only has simple eigenvalues, whereas  $f^{\otimes}(M, M)$  has no sense, whatever  $M$  is. But the minimal conditions we should put on  $f$  and the  $M_i$ 's to extend the definition of  $f^{\wedge}(M_1, \dots, M_k)$  are not clear.

If the fact that  $f^\otimes(M_1, \dots, M_k)$  is a tensor and not a matrix is disturbing, there could be a way to define a kind of  $f(M_1, \dots, M_k)$  which would be a matrix. One way to get a matrix from the tensor  $f^\otimes(M_1, \dots, M_k)$  is to use contractions. Assume that one can write  $f(x_1, \dots, x_k) = \sum a_{n_1, \dots, n_k} x_1^{n_1} \dots x_k^{n_k}$ , where we have  $\sum |a_{n_1, \dots, n_k}| \rho_1^{n_1} \dots \rho_k^{n_k} < \infty$  for some  $\rho_1, \dots, \rho_k > 0$ . Assume that for every  $1 \leq l \leq k$ , we have  $\sup_{\lambda \text{ eigenvalue of } M_l} |\lambda| < \rho_l$ . Then the sum:

$$\sum a_{n_1, \dots, n_k} (M_1^{n_1} M_2^{n_2} \dots M_k^{n_k})^i_j$$

is convergent, and this matrix is exactly

$$f^\otimes(M_1, \dots, M_k)^i_{i_1 i_2 \dots i_{k-2} i_{k-1} j}.$$

It is what we expect  $f(M_1, \dots, M_k)$  to be when all the  $M_l$  commute. But if they do not commute, this may introduce a dissymmetry. It does still work well if  $f(x_1, \dots, x_k) = \sum_{l=1}^k f_l(x_l)$ .

But in the simple example  $f(x, y) = (x + y)^2$ , we have  $f^\otimes(A, B)^i_k{}^k_j = (A^2 + B^2 + 2AB)^i_j$  and not  $((A + B)^2)^i_j$  as one could want.

If  $f(x, y) = \frac{1}{x+y}$ , and provided none of the eigenvalues of  $A$  is the opposite of an eigenvalue of  $B$ , it gives you the unique matrix  $M$  such that  $AM + MB = I$ , and this matrix is not  $(A + B)^{-1}$  in general.

## References

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