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The Irredundance-related Ramsey Numbers

$$s(3, 8) = 21 \text{ and } w(3, 8) = 21$$

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The *irredundant Ramsey number* $s = s(m, n)$ is the smallest natural number s such that in any red-blue edge colouring (R, B) of the complete graph of order s , it holds that $IR(B) \geq m$ or $IR(R) \geq n$, where IR denotes the irredundance number of a graph. Furthermore, the *mixed irredundant Ramsey number* $t = t(m, n)$ [*irredundant-domination Ramsey number* $w = w(m, n)$, respectively] is the smallest natural number t [w , respectively] such that in any red-blue edge colouring (R, B) of the complete graph of order t [w , respectively], it holds that $IR(B) \geq m$ or $\alpha(R) \geq n$ [$IR(B) \geq m$ or $\Gamma(R) \geq n$, respectively], where α and Γ denote respectively the (vertex) independence number and the upper domination number of a graph. These three classes of Ramsey numbers have previously been studied in the literature. In this paper, we employ a computer search to determine complete sets of avoidance colourings for the Ramsey numbers $t(3, 7)$ and $t(3, 8)$. In the process the two previously unknown Ramsey numbers $s(3, 8) = 21$ and $w(3, 8) = 21$ are established.

Keywords: Irredundance, Ramsey number

1 Introduction

A subset \mathcal{I} of vertices of a graph G is *independent* if no two vertices in \mathcal{I} are adjacent in G . The cardinality of a largest independent set of G is denoted by $\alpha(G)$ and is called the *vertex independence number* of G .

A subset \mathcal{D} of the vertices of a graph G is *dominating* if every vertex v of G is an element of \mathcal{D} or is adjacent in G to an element of \mathcal{D} . A dominating set \mathcal{D} of G is *minimal* if no proper subset of \mathcal{D} is a dominating set in G . The cardinality of a largest minimal dominating set of G is denoted by $\Gamma(G)$ and is called the *upper domination number* of G .

The *closed neighbourhood* of a graph vertex v , denoted by $N[v]$, is the subset of all vertices of G that are adjacent to v in G , together with the vertex v itself. The *closed neighbourhood of a set of vertices* $\mathcal{S} = \{v_1, \dots, v_m\}$ of G , denoted by $N[\mathcal{S}]$, is simply $N[\mathcal{S}] = \cup_{i=1}^m N[v_i]$. The set of *private neighbours* of a vertex $v \in \mathcal{V}_G$ with respect to some subset $\mathcal{S} \subset \mathcal{V}_G$ is defined as $\text{PN}(v, \mathcal{S}) = N[v] \setminus N[\mathcal{S} \setminus \{v\}]$. A subset \mathcal{X} of the vertices of G is *irredundant* if every vertex $v \in \mathcal{X}$ has at least one private neighbour in G .

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with respect to \mathcal{X} (that is, if $\text{PN}(v, \mathcal{X}) \neq \emptyset$ for all $v \in \mathcal{X}$). Finally, the cardinality of a largest irredundant set of G is denoted by $IR(G)$ and is called the *irredundance number* of G .

It holds for any graph G that

$$\alpha(G) \leq \Gamma(G) \leq IR(G), \quad (1)$$

a result due to Cockayne et al. (1978).

A bi-colouring of the edges of a graph, using the colours red and blue, is called a *red-blue edge colouring* of the graph. If R and B are the subgraphs induced by respectively the red and the blue edges of such a colouring, the colouring is denoted by the *ordered pair* (R, B) , and R is referred to as the *red subgraph*, while B is called the *blue subgraph*.

The following six classes of Ramsey numbers have previously been defined in terms of the graph theoretic notions of independence, irredundance and domination.

The *independent Ramsey number* $r = r(m, n)$ is the smallest natural number r such that in any red-blue edge colouring (R, B) of the complete graph K_r of order r , it holds that $\alpha(B) \geq m$ or $\alpha(R) \geq n$ (or both). This definition is due to Ramsey (1930).

The *irredundant Ramsey number* $s = s(m, n)$ is the smallest natural number s such that in any red-blue edge colouring (R, B) of the complete graph K_s of order s , it holds that $IR(B) \geq m$ or $IR(R) \geq n$ (or both). This definition is due to Brewster et al. (1989).

The *mixed irredundant Ramsey number* $t = t(m, n)$ is the smallest natural number t such that in any red-blue edge colouring (R, B) of the complete graph K_t of order t , it holds that $IR(B) \geq m$ or $\alpha(R) \geq n$ (or both). This definition is due to Cockayne et al. (1990).

The *upper domination Ramsey number* $u = u(m, n)$ is the smallest natural number u such that in any red-blue edge colouring (R, B) of the complete graph K_u of order u , it holds that $\Gamma(B) \geq m$ or $\Gamma(R) \geq n$ (or both). This definition dates from the early 1990s and is due to Oellermann and Shreve.

The *mixed domination Ramsey number* $v = v(m, n)$ is the smallest natural number v such that in any red-blue edge colouring (R, B) of the complete graph K_v of order v , it holds that $\Gamma(B) \geq m$ or $\alpha(R) \geq n$ (or both). This definition also dates from the early 1990s and is due to Oellermann and Shreve.

The *irredundant-domination Ramsey number* $w = w(m, n)$ is the smallest natural number w such that in any red-blue edge colouring (R, B) of the complete graph K_w of order w , it holds that $IR(B) \geq m$ or $\Gamma(R) \geq n$ (or both). This definition is due to Burger and van Vuuren (2011).

The arguments of the independent, irredundant and upper domination Ramsey numbers may be interchanged without altering the values of these numbers, *i.e.* $r(m, n) = r(n, m)$, $s(m, n) = s(n, m)$ and $u(m, n) = u(n, m)$ for any integers $m, n \geq 2$. In the case of the mixed Ramsey numbers $t(m, n)$, $v(m, n)$ and $w(m, n)$, however, the arguments m and n do not necessarily commute, *i.e.* $t(m, n) \neq t(n, m)$, $v(m, n) \neq v(n, m)$ and $w(m, n) \neq w(n, m)$ in general. Furthermore, the Ramsey numbers are related to each other by the inequality chains

$$s(m, n) \leq w(m, n) \leq \left\{ \begin{array}{l} u(m, n) \\ t(m, n) \end{array} \right\} \leq v(m, n) \leq r(m, n), \quad (2)$$

which are easily shown to hold for all $m, n \geq 2$ in view of (1). It is unknown in what way (if any) the Ramsey numbers $u(m, n)$ and $t(m, n)$ are related.

Apart from the infinite classes of Ramsey numbers $x(2, n) = x(n, 2) = n$ for all $x \in \{r, s, t, u, v, w\}$ and all $n \geq 2$, only 45 members of the above-mentioned six classes of Ramsey numbers are known exactly. These numbers are shown in Table 1. Our aim in this paper is to determine the two Ramsey numbers $s(3, 8)$ and $w(3, 8)$. The paper is organised as follows. After reviewing a number of preliminary results in §2, we present a recursive enumeration algorithm in §3 for determining bounds on any of the six types of Ramsey numbers mentioned above. In §4 we then show how the algorithm may be applied to determine the values of the two Ramsey numbers $s(3, 8)$ and $w(3, 8)$. The numerical results returned by the algorithm are presented and analysed in §5, after which the paper closes with suggestions for further work in §6.

$s(m, n)$	$w(m, n)$	$u(m, n)$	$t(m, n)$	$v(m, n)$	$r(m, n)$
$s(3, 3) = 6^a$	$w(3, 3) = 6^b$	$u(3, 3) = 6^c$	$t(3, 3) = 6^d$	$v(3, 3) = 6^e$	$r(3, 3) = 6^f$
$s(3, 4) = 8^a$	$w(3, 4) = 8^b$	$u(3, 4) = 8^c$	$t(3, 4) = 9^d$	$v(3, 4) = 9^e$	$r(3, 4) = 9^f$
$s(3, 5) = 12^a$	$w(4, 3) = 8^b$	$u(3, 5) = 12^c$	$t(4, 3) = 8^d$	$v(4, 3) = 8^b$	$r(3, 5) = 14^f$
$s(3, 6) = 15^g$	$w(3, 5) = 12^b$	$u(3, 6) = 15^c$	$t(3, 5) = 12^d$	$v(3, 5) = 12^e$	$r(3, 6) = 18^i$
$s(3, 7) = 18^j$	$w(5, 3) = 12^b$	$u(4, 4) = 13^c$	$t(5, 4) = 13^d$	$v(5, 3) = 13^d$	$r(3, 7) = 23^l$
$s(4, 4) = 13^k$	$w(3, 6) = 15^b$		$t(3, 6) = 15^h$	$v(3, 6) = 15^e$	$r(3, 8) = 28^m$
	$w(6, 3) = 15^b$		$t(6, 3) = 15^b$	$v(6, 3) = 15^b$	$r(3, 9) = 36^n$
	$w(3, 7) = 18^o$		$t(3, 7) = 18^o$	$v(4, 4) = 14^b$	$r(4, 4) = 18^f$
	$w(4, 4) = 13^b$		$t(3, 8) = 22^o$		$r(4, 5) = 25^p$
			$t(4, 4) = 14^b$		

Tab. 1: Exactly known Ramsey numbers from the literature. Due to ^aBrewster et al. (1989), ^bBurger and van Vuuren (2011), ^cDzido and Zakrzewska (2006), ^dCockayne et al. (1990), ^eHenning and Oellermann (2002), ^fGreenwood and Gleason (1955), ^gBrewster et al. (1990), ^hGrobler (1992), ⁱGraver and Yackel (1968), ^jCockayne et al. (1991), ^kCockayne et al. (1992), ^lKalbfleisch (1966), ^mMcKay and Min (1992), ⁿGrinstead and Roberts (1982), ^oBurger et al. (2014), ^pMcKay and Radziszowski (1995).

2 Preliminary results

Let $x \in \{r, s, t, u, v, w\}$ be a Ramsey number. An $x(m, n, p)$ colouring is a red-blue edge colouring (R, B) of the complete graph K_p of order p which satisfies neither of the two sought-after chromatic conditions of the Ramsey number $x(m, n)$, hence showing that $x(m, n) > p$.

Burger and van Vuuren (2011) computed complete sets of $x(3, n, p)$ and $x(m, 3, p)$ colourings for all $x \in \{s, t, u, v, w\}$, all $m, n \in \{3, 4, 5, 6\}$ and all admissible values of p . In that paper, we also computed complete sets of $x(4, 4, p)$ colourings for all $x \in \{s, t, u, v, w\}$ and all admissible values of p . The red subgraphs of the complete sets of $t(3, 3, p)$ colourings of orders $p = 4$ and $p = 5$ are, for example, shown in Figure 1. The red subgraphs of the complete sets of $t(3, 4, p)$ colourings of orders 5, 6, 7 and 8 are similarly shown in Figures 2, 3, 4 and 5, respectively.

The following characterisation is due to Brewster et al. (1989).

Lemma 1 (Brewster et al. (1989)) *The blue subgraph of a red-blue edge colouring of a complete graph has an irredundant set of cardinality 3 if and only if there is a 3-cycle in the red subgraph or a 6-cycle $v_1v_2v_3v_4v_5v_6v_1$ in the red subgraph with the edges v_1v_4 , v_2v_5 and v_3v_6 in the blue subgraph.*

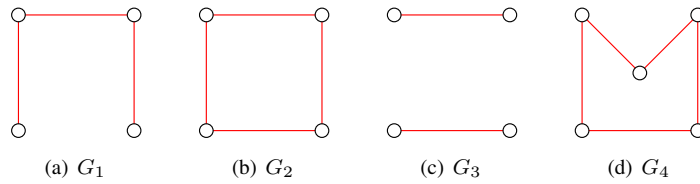


Fig. 1: (a)–(c) The red subgraphs of the complete set of $t(3, 3, 4)$ colourings, and (d) the red subgraph of the only $t(3, 3, 5)$ colouring.

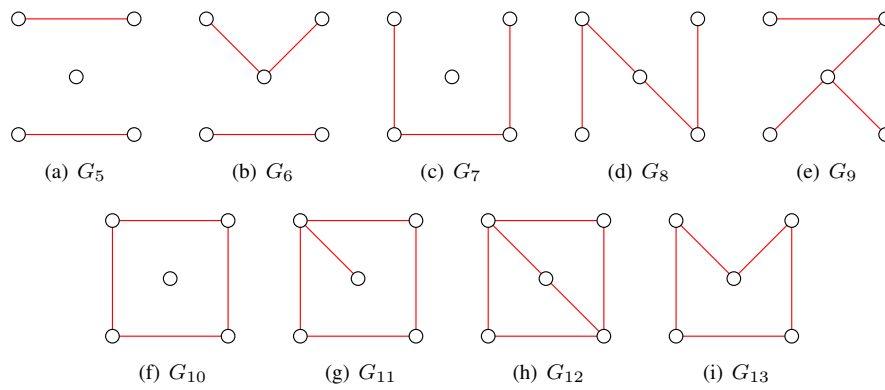


Fig. 2: The red subgraphs of the complete set of $t(3, 4, 5)$ colourings.

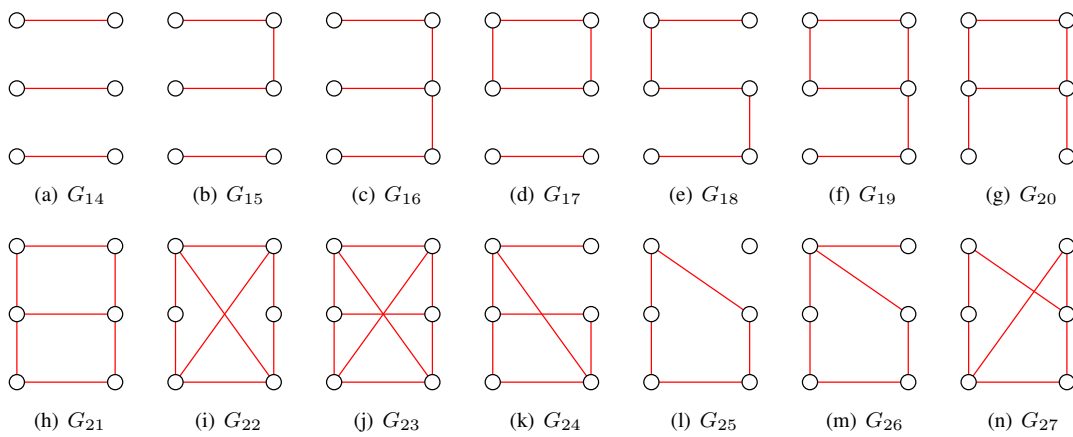


Fig. 3: The red subgraphs of the complete set of $t(3, 4, 6)$ colourings.

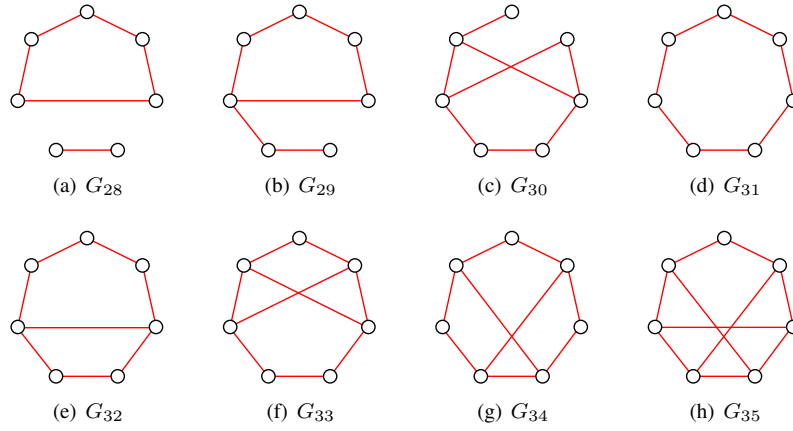


Fig. 4: The red subgraphs of the complete set of $t(3, 4, 7)$ colourings.

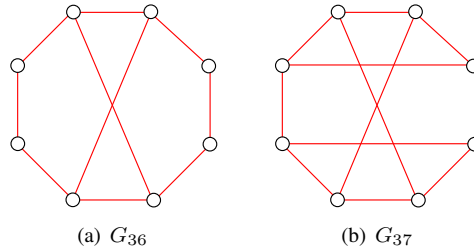


Fig. 5: The red subgraphs of the complete set of $t(3, 4, 8)$ colourings.

We call the second substructure in the above proposition a *red 6-cycle with blue diagonals*. The following result relates $x(m, n, p)$ colourings and $y(m, n, p)$ colourings for different types $x, y \in \{r, s, t, u, v, w\}$ of Ramsey numbers.

Lemma 2 (Burger and van Vuuren (2011)) *Let $m, n > 2$ be integers, and let $x(m, n)$ and $y(m, n)$ be two Ramsey numbers in (2) satisfying $x(m, n) \leq y(m, n)$. Then any $x(m, n, p)$ colouring is also a $y(m, n, p)$ colouring.*

The minimum degree of the subgraph R [B , resp.] in a $x(m, n, p)$ colouring (R, B) is referred to as the *minimum red* [*blue*, resp.] *degree* and is denoted by $\delta(R)$ [$\delta(B)$, resp.]. Similar definitions hold for the maximum degrees of R and B , denoted by $\Delta(R)$ and $\Delta(B)$, respectively.

Lemma 3 (Hattingh (1990)) *Let $m, n > 2$ be integers. Then, $\Delta(R) < s(m - 1, n)$ and $\Delta(B) < s(m, n - 1)$ in any $s(m, n, p)$ colouring (R, B) .*

The following corollary of the above result holds by Lemma 3.

Corollary 1 *Let $m, n \geq 2$ be integers and let $x \in \{r, s, t, u, v, w\}$ be a Ramsey number. Then, $\Delta(R) < x(m - 1, n)$ and $\Delta(B) < x(m, n - 1)$ in any $x(m, n, p)$ colouring (R, B) .*

Finally, the following useful result immediately follows from Corollary 1.

Corollary 2 *Let $m, n \geq 2$ be integers and let $x \in \{r, s, t, u, v, w\}$ be a Ramsey number. Then, $\delta(R) \geq p - x(m, n - 1)$ and $\delta(B) \geq p - x(m - 1, n)$ in any $x(m, n, p)$ colouring (R, B) .*

3 An algorithm for enumerating $x(m, n, p)$ colourings

We present a recursive algorithm for deciding whether or not a partial red-blue edge colouring of a complete graph of order p can be completed to form an $x(m, n, p)$ colouring, where $x \in \{r, s, t, u, v, w\}$ is a Ramsey number. A pseudo-code listing of the algorithm is presented as Algorithm 1.

Algorithm 1 Ramsey($\mathcal{R}, \mathcal{B}, \mathcal{U}$)

Input: A graph of order p (global variable) as well as a set \mathcal{R} of red edges, a set \mathcal{B} of blue edges and a set \mathcal{U} of uncoloured edges of a complete graph of order p (local variables).

Output: `True` if the colouring induced by the red edges in \mathcal{R} and the blue edges in \mathcal{B} can be completed to an $x(m, n, p)$ colouring, or `False` otherwise.

```

1: repeat
2:    $t \leftarrow |\mathcal{R}| + |\mathcal{B}|$ 
3:    $(\mathcal{R}, \mathcal{B}, \mathcal{U}, \text{OK}) \leftarrow \text{CheckUndecidedEdges}(\mathcal{R}, \mathcal{B}, \mathcal{U})$ 
4:   if OK = False then return(False)
5: until  $|\mathcal{R}| + |\mathcal{B}| = t$ 
6: if  $\mathcal{U} = \emptyset$  then return(True)
7:  $e \leftarrow$  any edge in  $\mathcal{U}$ 
8: if Ramsey( $\mathcal{R} \cup \{e\}, \mathcal{B}, \mathcal{U} \setminus \{e\}$ ) or Ramsey( $\mathcal{R}, \mathcal{B} \cup \{e\}, \mathcal{U} \setminus \{e\}$ ) then return(True)
9: else return(False)

```

The algorithm takes as global input parameter a natural number p and as local input variables a set \mathcal{R} of red edges, a set \mathcal{B} of blue edges and a set \mathcal{U} of uncoloured edges of a complete graph of order p and essentially follows a branch-and-bound approach to yield the boolean variable output `True` if the edges in \mathcal{U} can be coloured red or blue in some combination to form an $x(m, n, p)$ colouring, or `False` otherwise.

During the repeat-until loop spanning steps 1–5, an external function $\text{CheckUndecidedEdges}(\mathcal{R}, \mathcal{B}, \mathcal{U})$ is repeatedly called (step 3). This function examines each edge in \mathcal{U} in turn, determining whether perhaps it must necessarily be coloured either red or blue in order to avoid violating a so-called *bounding list* of properties of an $x(m, n, p)$ colouring. An example of such a list of properties in the context of the Ramsey number $r(3, 4)$ is the bounding list in Table 2. If a colouring of any edge $e \in \mathcal{U}$ is thus necessitated, the edge e is removed from \mathcal{U} and inserted into \mathcal{R} or \mathcal{B} (whichever is appropriate). In addition to this set update, the function $\text{CheckUndecidedEdges}(\mathcal{R}, \mathcal{B}, \mathcal{U})$ also updates the value of a global boolean variable `OK`, originally initialised as having the value `True`, to the value `False` if it is determined that some edge $e^* \in \mathcal{U}$ can be coloured neither red nor blue (as a result of avoiding the contradiction of some pair of properties in the bounding list). This process of calling the function $\text{CheckUndecidedEdges}(\mathcal{R}, \mathcal{B}, \mathcal{U})$ is repeated until the colouring of no further edges in \mathcal{U} is necessitated by the bounding list.

If all edges have either been coloured red or blue ($\mathcal{U} = \emptyset$ in step 6), a valid $x(m, n, p)$ colouring has been found, in which case the boolean value `True` is returned. Otherwise, an edge $e \in \mathcal{U}$ is randomly selected (step 7) and branched upon, attempting to colour it either red or blue by calling the algorithm recursively with the updated input variable triples $\mathcal{R} \cup \{e\}, \mathcal{B}, \mathcal{U} \setminus \{e\}$ or $\mathcal{R}, \mathcal{B} \cup \{e\}, \mathcal{U} \setminus \{e\}$ (step 8). If this cannot be achieved, then the boolean value `False` is returned (step 9).

Property	Description
a	No red triangle is formed
b	No blue clique of order 4 is formed
c	The red degree of each vertex is at most 3 by Corollary 1
d	The blue degree of each vertex is at most 5 by Corollary 1

Tab. 2: Bounding list of properties used to illustrate that $r(3, 4) = 9$.

For some value p and a fixed vertex $v \in V(K_p)$, the set \mathcal{R} may be initialised without loss of generality in the context of seeking an $x(m, n, p)$ colouring by taking $p - x(m, n - 1)$ arbitrary edges incident to v according to Corollary 2. The set \mathcal{B} may similarly be initialised by taking any $p - x(m - 1, n)$ of the remaining edges incident to v . This is illustrated in Figure 6(a) for $v = 1$ in the context of an $r(3, 4, 8)$ colouring, upon which Algorithm 1 returns the boolean value `True`, resulting in the $r(3, 4, 8)$ colouring shown in Figure 6(b), which shows that $r(3, 4) > 8$.

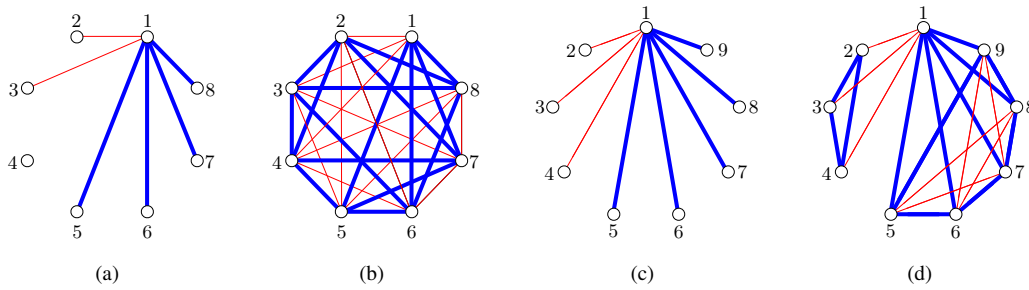


Fig. 6: (a) Initialisation of the input sets \mathcal{R} and \mathcal{B} to Algorithm 1 in pursuit of an $r(3, 4, 8)$ colouring. (b) The $r(3, 4, 8)$ colouring returned by Algorithm 1 with input sets \mathcal{R} and \mathcal{B} initialised as in (a). (c) Initialisation of the input sets \mathcal{R} and \mathcal{B} to Algorithm 1 in pursuit of an $r(3, 4, 9)$ colouring. (d) A larger initialisation of the input sets \mathcal{R} and \mathcal{B} to Algorithm 1, without loss of generality, in pursuit of an $r(3, 4, 9)$ colouring.

If, however, the same initialisation process is followed in the context of an $r(3, 4, 9)$ colouring, as shown in Figure 6(c), Algorithm 1 returns the boolean value `False` as a result of not being able to compute an $r(3, 4, 9)$ colouring, because $r(3, 4) = 9$. Although a conceptually simple algorithm, the ranges of possible values of the smallest unknown Ramsey numbers are currently so large that direct application of Algorithm 1 to determine these Ramsey numbers is not technologically possible. The rapid growth in the number of branches and associated computing times required to determine the Ramsey numbers $r(3, n)$ for $n \in \{3, 4, 5, 6\}$ via Algorithm 1 is, for example, illustrated in Table 3.

There is, however, a better way of initialising the input sets \mathcal{R} and \mathcal{B} to Algorithm 1 than the initialisations shown in Figure 6 (a) and (c). Knowledge about smaller avoidance colourings may be utilised in the initialisations. The subgraph induced in any eventual $r(3, 4, 9)$ colouring by the vertex subset $\{5, 6, 7, 8, 9\}$ in Figure 6(c), for example, is necessarily an $r(3, 3, 5)$ colouring, for if this were not the case, then the subgraph induced in the $r(3, 4, 9)$ colouring by the vertex subset $\{1, 5, 6, 7, 8, 9\}$ would contain either a red triangle or a complete graph of order 4 as subgraph, a contradiction. Similarly, the subgraph induced in any eventual $r(3, 4, 9)$ colouring by the vertex subset $\{2, 3, 4\}$ in Figure 6(c) is necessarily an $r(2, 4, 3)$ colouring, for otherwise the subgraph induced in the $r(3, 4, 9)$ colouring by the vertex

Ramsey number	Best lower bound			Best upper bound		
	Value	Branches	Time	Value	Branches	Time
$r(3, 3)$	> 5	2	$\ll 1$	≤ 6	0	$\ll 1$
$r(3, 4)$	> 8	12	$\ll 1$	≤ 9	234	< 1
$r(3, 5)$	> 13	1 034	< 1	≤ 14	0	$\ll 1$
$r(3, 6)$	> 17	7 969 625	276	≤ 18	6 944 615 486	416 233

Tab. 3: The number of branches and associated computing times required to determine the Ramsey numbers $r(3, n)$ for $n \in \{3, 4, 5, 6\}$. The times are given in seconds and were measured on an Intel core of eight 3.4GHz, 7.7Gb RAM processors running in a 64 bit Linux operating system.

subset $\{1, 2, 3, 4\}$ would contain a red triangle as subgraph, again a contradiction. The input sets \mathcal{R} and \mathcal{B} to Algorithm 1 may therefore be initialised, without loss of generality, as shown in Figure 6(d), because there is only one $r(2, 4, 3)$ colouring (a blue triangle) and only one $r(3, 3, 5)$ colouring (consisting of a red five-cycle embedded within a blue five-cycle, as shown in Figure 1(d)).

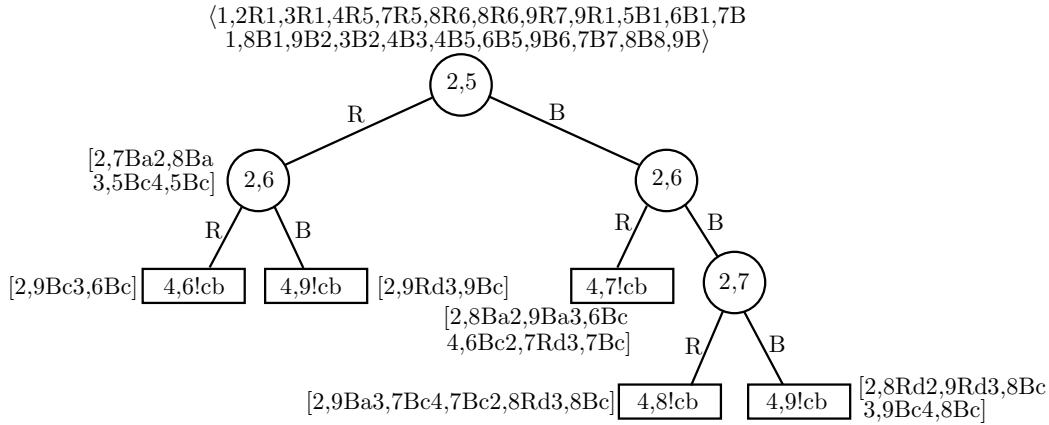


Fig. 7: The branch-and-bound process followed by Algorithm 1 to show that $r(3, 4) \leq 9$.

When initialising the input sets \mathcal{R} and \mathcal{B} to Algorithm 1 as shown in Figure 6(d), the branching process illustrated in Figure 7 results, which shows that $r(3, 4) \leq 9$. In the figure, branching nodes are denoted by circles (containing the edges that are branched on in coordinate form), while bounding nodes are denoted by rectangles (containing the reasons for contradictions preventing further growth of the tree in the form $i, j!xy$ which means that the edge ij can be coloured neither red because of reason x in Table 2 nor blue because of reason y in Table 2). Square brackets are used to present lists of edges whose colours are fixed by a branching decision, where a string of the form i, jRx means that the edge ij is necessarily red so as to avoid contradicting property x in Table 2, and similarly for blue edges. Angular brackets are used at the top of the tree to present a list of edge colours corresponding to the initialisation of the input sets \mathcal{R} and \mathcal{B} to Algorithm 1, where i, jB means that edge ij is coloured blue, and similarly for red edges.

Note that because of the extended initialisation in Figure 6(d), the tree in Figure 7 has merely eight branches instead of the 234 branches that result according to Table 3 when the sparser initialisation in Figure 6(c) is used. But even with the denser initialisation procedure described above, which utilises information about smaller avoidance colourings, the rapid growth in the number of branches and associated

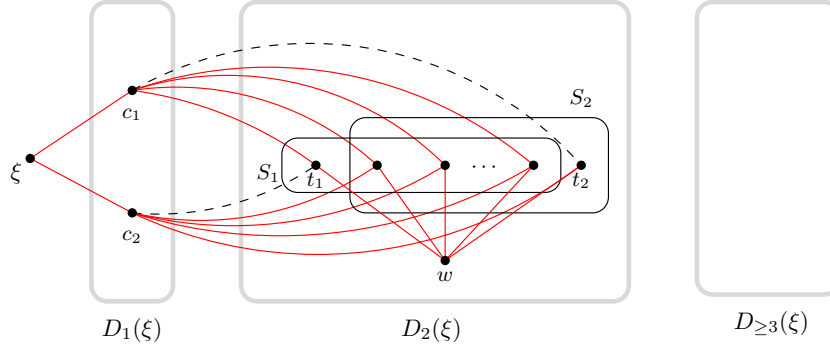


Fig. 8: The neighbours of the endpoints of a star in $D_2(\xi)$.

computing times required by Algorithm 1 places the computation of most Ramsey numbers of the form $r(3, n)$ for $n > 6$ out of the current reach of computing technology. It is, nevertheless, possible to apply the algorithm judiciously in order to determine some of these Ramsey number values, as we show in the next sections.

4 Enumeration of $t(3, 8, 21)$ colourings

We denote the subgraph of the blue [red, resp.] subgraph of an (m, n, p) -colouring induced by some set $A \subseteq V(K_p)$ by $\langle A \rangle_{\text{blue}}$ [$\langle A \rangle_{\text{red}}$, resp.]. Let $D_i(v)$ [$D_{>i}(v)$ or $D_{\geq i}(v)$, resp.] be the set of vertices at distance $i \in \mathbb{N}$ [at a distance greater than i or at a distance at least i , resp.] from some specified vertex v in the red subgraph of a $(3, n, p)$ -colouring. Since we avoid an irredundant set of cardinality 3 in the blue subgraph, we have the following useful result, adapted from Hattingh (1990).

Lemma 4 (Hattingh (1990)) *Let v be any vertex of a $t(3, n, p)$ -colouring.*

- (a) *If xyz is a path in $\langle D_{>1}(v) \rangle_{\text{red}}$ and $x, z \in D_2(v)$, then x and z are joined by red edges to a common vertex in $D_1(v)$.*
- (b) *If $A \subseteq D_{>1}(v)$ contains at most one vertex of $D_{>2}(v)$, then $\langle A \rangle_{\text{red}}$ is bipartite.*

The following generalisation of the result in Lemma 4(a) is useful.

Corollary 3 *Let $K_{1,k}$ be a star in $\langle D_{\geq 2}(\xi) \rangle_{\text{red}}$ with centre in $D_{\geq 2}(\xi)$ and endpoints in $D_2(\xi)$. Then all the endpoints of this star have a common neighbour in $D_1(\xi)$.*

Proof: By induction on the number of endpoints of the star. The result of Lemma 4(a) serves as the base case for the induction. Suppose, as induction hypothesis, that all the endpoints in $D_2(\xi)$ of any red star $K_{1,k}$ with centre in $D_{\geq 2}(\xi)$ have a common neighbour in $D_1(\xi)$. We show by contradiction that all the endpoints in $D_2(\xi)$ of any red star $K_{1,k+1}$ with centre in $D_{\geq 2}(\xi)$ have a common neighbour in $D_1(\xi)$. Let S be the set of endpoints of such a star with centre w , and let t_1 and t_2 be two vertices in S . Suppose, to the contrary, that this star does not have a common neighbour in $D_1(\xi)$. Let S_1 and S_2 be subsets of cardinality k of S such that $S = S_1 \cup \{t_2\} = S_2 \cup \{t_1\}$, as shown in Figure 8. Then it follows by the induction hypothesis that all vertices in S_1 have a common neighbour in $D_1(\xi)$, and similarly for the vertices in S_2 . Let these common neighbours be c_1 and c_2 , respectively. Then the edges $c_1 t_2$ and

c_2t_1 must both be blue (since the vertices in S do not all have a common neighbour in $D_1(\xi)$). But then $\xi c_1t_1wt_2c_2\xi$ is a red six-cycle with blue diagonals ξw , c_1t_2 and c_2t_1 — contradicting Lemma 1. \square

It follows from Figure 4.1 in Burger et al. (2014) that $t(3, 8, 21)$ colourings exist. These colourings have, however, not yet been enumerated. Let Ξ and $\bar{\Xi}$ be respectively the red and blue subgraphs of a $t(3, 8, 21)$ colouring, and denote the minimum and maximum degrees of Ξ by $\delta(\Xi)$ and $\Delta(\Xi)$, respectively. Suppose ξ is a vertex of red degree $\Delta(\Xi)$ in this colouring. Then the colourings in Figures 2–5 are relevant in the context of the $t(3, 8, 21)$ colouring $(\Xi, \bar{\Xi})$ as a result of the following observation.

Observation 1 *The colouring induced in $(\Xi, \bar{\Xi})$ by the vertices in $D_{\geq 3}(\xi)$ is a $t(3, 8 - \Delta(\Xi), 20 - |D_1(\xi) \cup D_2(\xi)|)$ colouring.*

The subgraph Ξ also has the following properties.

Lemma 5 (Burger et al. (2014), Lemmas 5–6) *$D_1(\xi)$ is an independent set of Ξ , $\Delta(\Xi) \leq 5$, $|D_2(\xi)| \leq 11$ and $|D_{>2}(\xi)| < t(3, 8 - \Delta(\Xi))$.*

By Corollary 2, we therefore have the bounds $3 \leq \delta(\Xi) \leq \Delta(\Xi) \leq 5$. Note, however, that $\Delta(\Xi) \neq 3$, for otherwise we would have a cubic graph Ξ of odd order. Therefore,

$$4 \leq \Delta(\Xi) \leq 5. \quad (3)$$

The subgraph $\langle D_2(\xi) \rangle_{\text{red}}$ of Ξ is bipartite by Lemma 4(b). Suppose $\langle D_2(\xi) \rangle_{\text{red}}$ contains c components and denote the bipartitions of these components by (X_i, Y_i) for all $i = 1, \dots, c$. We call a component of the form (X_i, Y_i) an $(|X_i|, |Y_i|)$ component, and we assume, without loss of generality, that $|X_i| \geq |Y_i|$ for all $i = 1, \dots, c$. Define $X = \cup_{i=1}^c X_i$ and $Y = \cup_{i=1}^c Y_i$. Then $|X| \geq |Y|$.

4.1 Four-step enumeration process

Our approach toward enumerating $t(3, 8, 21)$ colourings consists of the following four steps:

Step 1. For each possible pair (i, j) satisfying $0 \leq j \leq i \leq 6$, we determine all non-isomorphic (i, j) components in $D_2(\xi)$. That is, connected, bipartite subgraphs with partite sets of cardinalities i and j satisfying the above inequality chain.

Step 2. We consider each combination of values for the triple $|X|, |Y|, \Delta(\Xi)$ satisfying the inequalities in Lemma 5 and (3). These combinations are listed as the ten subcases in Table 4 where, of course, $\Delta(\Xi) = |D_1(\xi)|$. In each subcase of the table, we insert red edges in $D_2(\xi)$ according to all permissible combinations of (i, j) component combinations determined in Step 1.

Step 3. For each of the red subgraphs on $D_2(\xi)$ in Step 2 above, we consider all non-isomorphic ways in which to join vertices in $D_1(\xi)$ to vertices in $D_2(\xi)$ by means of red edges, satisfying the requirements of Corollary 3. The number of these non-isomorphic red-edge connections between $D_1(\xi)$ and $D_2(\xi)$ is given by a Stirling number of the second kind⁽ⁱ⁾. These cases are pruned by ensuring that the properties in Table 5 are satisfied.

⁽ⁱ⁾ The Stirling number $S(a, b)$ of the second kind counts the number of distinct ways of distributing a distinguishable objects among b indistinguishable containers. In the above context of red-edge connections between $D_1(\xi)$ and $D_2(\xi)$, the objects represent the endpoints of a Corollary 3 star in $D_2(\xi)$ and the containers represent the vertices of $D_1(\xi)$.

Subcase	$ D_0(\xi) $	$ D_1(\xi) $	$ D_2(\xi) $	$ X $	$ Y $	$ D_{>2}(\xi) $	Reference
I	1	4	11	6	5	5	Table 7
IIa	1	4	10	6	4	6	Table 8
IIb	1	4	10	5	5	6	Table 9
IIIa	1	4	9	6	3	7	Table 10
IIIb	1	4	9	5	4	7	Table 11
IVa	1	4	8	6	2	8	Table 12
IVb	1	4	8	5	3	8	Table 13
IVc	1	4	8	4	4	8	Table 14
V	1	5	11	6	5	4	Table 15
VIa	1	5	10	6	4	5	Table 16
VIb	1	5	10	5	5	5	Table 17

Tab. 4: Subcases considered in Step 2 of our $t(3, 8, 21)$ colouring enumeration procedure.

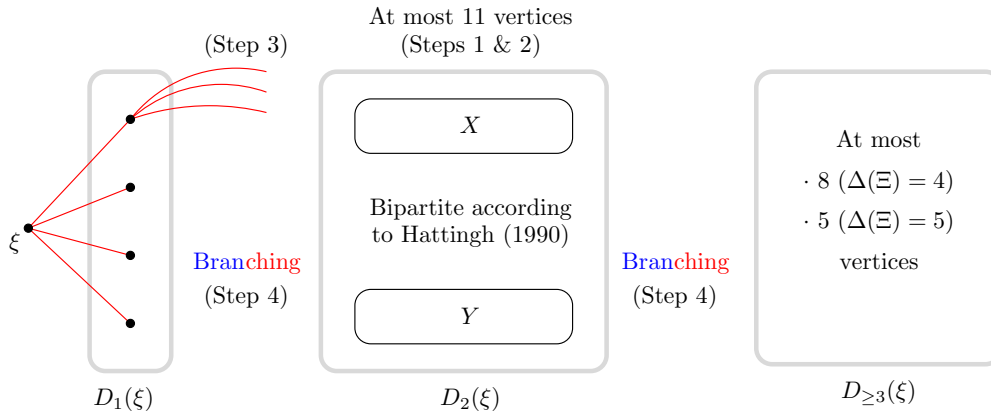


Fig. 9: Edges branched upon in Step 4 of our $t(3, 8, 21)$ colouring enumeration procedure.

Step 4. For each graph found in Step 3 above, and for each permissible avoidance graph on $D_3(\xi)$ (that is, for each $t(3, 4, p)$ colouring in Figures 2–5 in the case where $\Delta(\Xi) = 4$, or for each $t(3, 3, p)$ colouring in Figure 1 in the case where $\Delta(\Xi) = 5$), we employ our recursive branching algorithm of §3 to decide the colours of the remaining edges joining vertices in $D_1(\xi)$ to $D_2(\xi)$ as well as edges joining vertices in $D_2(\xi)$ to $D_3(\xi)$, as illustrated in Figure 9. For the branching process we use the bounding properties in Table 6.

It is possible to rule out certain types of components in $\langle D_2(\xi) \rangle_{\text{red}}$. We start, however, by noting that in order to avoid a clique of order 8 in $\langle \{\xi\} \cup X \cup D_{>3}(\xi) \rangle_{\text{blue}}$, the following result holds.

Property	Description
e	No red triangle is formed; see Lemma 1
f	No red 6-cycle with blue diagonals is formed; see Lemma 1
g	The red degree of each vertex is at most $\Delta(\Xi)$; see (3)
h	Every vertex in $D_2(\xi)$ is joined by a red edge to some vertex in $D_1(\xi)$

Tab. 5: Pruning rules applied to the Stirling edges of Step 3.

Property	Description
i	No red triangle is formed; see Lemma 1
j	No red 6-cycle with blue diagonals is formed; see Lemma 1
k	The red degree of each vertex is at most $\Delta(\Xi)$; see (3)
l	The endpoints in $D_2(\xi)$ of a star in $\langle D_{\geq 2}(\xi) \rangle_{\text{red}}$ with centre in $D_{\geq 2}(\xi)$ all have a common neighbour in $D_1(v)$; see Corollary 3
m	No odd red cycle is formed; see Lemma 4(b)
n	No blue complete graph of order 7 is formed in $D_{\geq 2}(\xi)$

Tab. 6: Bounding list of properties used for the colouring of edges between vertices in $D_1(\xi)$ and vertices in $D_2(\xi)$ as well as between vertices in $D_2(\xi)$ and vertices in $D_{>2}(\xi)$ (in Step 4).

Observation 2 *If $|X| = 6$, then $D_{>3}(\xi) = \emptyset$.*

We also have the following restriction on small components of $\langle D_2(\xi) \rangle_{\text{red}}$.

Lemma 6 *If $|X| = 6$, $|Y| = 5$ and $\langle D_2(\xi) \rangle_{\text{red}}$ has exactly one $(1, 0)$ component, then $\langle D_2(\xi) \rangle_{\text{red}}$ is not a subgraph of Ξ .*

Proof: By contradiction. Suppose, to the contrary, that $|X| = 6$ and that $\langle D_2(\xi) \rangle_{\text{red}}$ has exactly one $(1, 0)$ component in Ξ . Then any vertex $v \in D_3(\xi)$ is joined to the $(1, 0)$ component by a red edge, for otherwise $\langle \{\xi, v\} \cup D_2(\xi) \rangle_{\text{blue}}$ contains a clique of order 8. The subgraph $\langle D_3(\xi) \rangle_{\text{red}}$ is therefore edgeless (so as to avoid triangles in $\langle D_{\geq 2}(\xi) \rangle_{\text{red}}$). But then $\langle D_1(\xi) \cup D_3(\xi) \rangle_{\text{blue}}$ contains a clique of order 9 (and hence of order 8), a contradiction. \square

We similarly have the following restriction on large components of $\langle D_2(\xi) \rangle_{\text{red}}$.

Lemma 7 *$\langle D_2(\xi) \rangle_{\text{red}}$ is not a $(6, 5)$ component.*

Proof: By contradiction. Suppose, to the contrary, that $\langle D_2(\xi) \rangle_{\text{red}}$ is a $(6, 5)$ component. Since $|X| = 6$, it then follows from Observation 2 that $D_{>3}(\xi) = \emptyset$. Every vertex $v \in D_3(\xi)$ is therefore joined to X by a red edge, for otherwise $\langle \{\xi, v\} \cup X \rangle_{\text{blue}}$ contains a clique of order 8. Hence all edges from $D_3(\xi)$ to Y in $(\xi, \bar{\xi})$ are blue (so as to avoid odd cycles in $\langle D_{\geq 2}(\xi) \rangle_{\text{red}}$). But then $\langle \{\xi, v, w\} \cup Y \rangle_{\text{blue}}$ contains a clique of order 8, by Lemma 4(b), since there is at least one blue edge vw for some pair $v, w \in D_3(\xi)$, a contradiction. \square

4.2 Results for the case $\Delta(\Xi) = 4$

Suppose $\Delta(\Xi) = 4$. Then $|D_2(\xi)| \leq 11$ and $|D_{>2}(\xi)| < t(3, 4) = 9$ by Lemma 5. Furthermore, $\langle D_{>1}(\xi) \rangle_{\text{red}}$ does not contain a $(4, 1)$ component (because of the stringent red degree restriction on each vertex).

The results obtained when applying our four-step enumeration process described above to subcases I–IVc in Table 4 are presented in Tables 7–14, respectively. In these tables, $\Phi_{\leq 2}$ and $\Phi_{\geq 3}$ denote respectively the set of valid $t(3, 8, 21)$ subcolourings on the vertices in $\{\xi\} \cup D_1(\xi) \cup D_2(\xi)$ and the set of valid $t(3, 8, 21)$ subcolourings on the vertices in $D_{\geq 3}(\xi)$ found upon completion of Steps 1–3 in our four-step enumeration process of §4.1. Furthermore, Φ denotes the set of valid $t(3, 8, 21)$ subcolourings involving only edges between vertices in $D_1(\xi)$ and $D_2(\xi)$, and between vertices in $D_2(\xi)$ and $D_{\geq 3}(\xi)$, as a result of applying our branching algorithm of §3 (*i.e.* Step 4 of our four-step enumeration process of §4.1). The

No	X	Y	$ \Phi_{\leq 2} $	$ \Phi_{\geq 3} $	$ \Phi $	Time
(i)	15	05	— Ruled out by Lemma 6 —			
(ii)	141	041	— Ruled out by Lemma 6 —			
(iii)	132	032	— Ruled out by Lemma 6 —			
(iv)	1311	0311	— Ruled out by Lemma 6 —			
(v)	1221	0221	— Ruled out by Lemma 6 —			
(vi)	12111	02111	— Ruled out by Lemma 6 —			
(vii)	111111	011111	— Ruled out by Lemma 6 —			
(viii)	24	14	0	—	0	2.8
(ix)	231	131	0	—	0	0.1
(x)	222	122	0	—	0	0.1
(xi)	2211	1211	6	9	0	22.4
(xii)	21111	11111	1	9	0	9.7
(xiii)	33	23	12	9	0	22.2
(xiv)	321	221	6	9	0	11.5
(xv)	3111	2111	1	9	0	4.6
(xvi)	42	32	0	—	0	0.3
(xvii)	411	311	0	—	0	2.0
(xviii)	51	41	0	—	0	11.4
(xix)	6	5	— Ruled out by Lemma 7 —			

Tab. 7: Results obtained for Subcase I in Table 4.

No	X	Y	$ \Phi_{\leq 2} $	$ \Phi_{\geq 3} $	$ \Phi $	Time
(i)	114	004	0	—	0	2.9
(ii)	1131	0031	9	14	0	27.9
(iii)	1122	0022	7	14	0	25.2
(iv)	11211	00211	16	14	0	55.1
(v)	111111	001111	5	14	0	42.7
(vi)	123	013	8	14	0	26.6
(vii)	1221	0121	20	14	0	73.7
(viii)	12111	01111	9	14	0	37.5
(ix)	132	022	12	14	0	35.6
(x)	1311	0211	11	14	0	33.8
(xi)	141	031	1	14	0	5.1
(xii)	15	04	0	—	0	3.0
(xiii)	222	112	4	14	0	19.8
(xiv)	2211	1111	8	14	0	28.1
(xv)	231	121	13	14	0	40.8
(xvi)	24	13	0	—	0	1.6
(xvii)	33	22	6	14	0	19.3
(xviii)	33	13	4	14	0	13.7
(xix)	321	121	4	14	0	12.0
(xx)	3111	1111	1	14	0	5.4
(xxi)	42	22	0	—	0	0.1
(xxii)	411	211	1	14	0	5.4
(xxiii)	51	31	0	—	0	0.2
(xxiv)	6	4	0	—	0	1.9

Tab. 8: Results obtained for Subcase IIa in Table 4.

No	X	Y	$ \Phi_{\leq 2} $	$ \Phi_{\geq 3} $	$ \Phi $	Time
(i)	5	5	0	—	0	11.8
(ii)	41	41	0	—	0	2.8
(iii)	32	32	10	14	0	172.3
(iv)	311	311	1	14	0	15.2
(v)	221	221	3	14	0	293.5
(vi)	2111	2111	4	14	0	356.3
(vii)	11111	11111	1	14	0	35.0

Tab. 9: Results obtained for Subcase IIb in Table 4.

No	X	Y	$ \Phi_{\leq 2} $	$ \Phi_{\geq 3} $	$ \Phi $	Time
(i)	1113	0003	10	8	0	62.6
(ii)	11121	00021	20	8	0	71.3
(iii)	111111	000111	11	8	0	45.3
(iv)	1122	0012	18	8	0	304.3
(v)	11211	00111	24	8	0	66.2
(vi)	1131	0021	27	8	0	55.0
(vii)	114	003	3	8	0	6.0
(viii)	1221	0111	20	8	0	128.9
(ix)	123	012	20	8	0	49.7
(x)	222	111	4	8	0	132.0
(xi)	33	12	8	8	0	13.8
(xii)	51	21	1	8	0	2.1
(xiii)	6	3	0	—	0	0.1

Tab. 10: Results obtained for Subcase IIIa in Table 4.

No	X	Y	$ \Phi_{\leq 2} $	$ \Phi_{\geq 3} $	$ \Phi $	Time
(i)	14	04	1	8	0	3.3
(ii)	131	031	7	8	0	26.0
(iii)	122	022	4	8	0	125.1
(iv)	1211	0211	12	8	0	508.5
(v)	11111	01111	3	8	0	54.6
(vi)	23	13	5	8	0	48.8
(vii)	221	121	10	8	2	357.3
(viii)	2111	1111	4	8	0	40.4
(ix)	32	22	6	8	0	26.2
(x)	311	211	6	8	0	77.1
(xi)	41	31	1	8	0	2.1
(xii)	5	4	0	—	0	0.8

Tab. 11: Results obtained for Subcase IIIb in Table 4.

No	X	Y	$ \Phi_{\leq 2} $	$ \Phi_{\geq 3} $	$ \Phi $	Time
(i)	11112	00002	12	2	0	357.3
(ii)	111111	000011	15	2	0	302.1
(iii)	11121	00011	27	2	0	777.5
(iv)	1131	0011	9	2	0	102.1
(v)	15	02	2	2	0	0.9
(vi)	33	11	2	2	0	2.1
(vii)	123	011	8	2	0	82.1
(viii)	114	002	6	2	0	5.1
(ix)	1122	0011	15	2	0	610.2
(x)	1113	0002	18	2	0	149.2

Tab. 12: Results obtained for Subcase IVa in Table 4.

No	X	Y	$ \Phi_{\leq 2} $	$ \Phi_{\geq 3} $	$ \Phi $	Time
(i)	1 1 3	0 0 3	14	2	0	143.3
(ii)	1 1 2 1	0 0 2 1	18	2	0	659.4
(iii)	1 1 1 1 1	0 0 1 1 1	10	2	0	212.2
(iv)	1 2 2	0 1 2	12	2	0	1 405.7
(v)	1 2 1 1	0 1 1 1	15	2	0	317.5
(vi)	1 3 1	0 2 1	19	2	0	84.1
(vii)	1 4	0 3	3	2	0	1.6
(viii)	2 2 1	1 1 1	9	2	3	274.3
(ix)	2 3	1 2	8	2	1	43.0
(x)	3 2	1 2	4	2	0	10.4
(xi)	3 1 1	1 1 1	2	2	0	7.7
(xii)	4 1	2 1	3	2	0	1.7
(xiii)	5	3	0	—	0	0.1

Tab. 13: Results obtained for Subcase IVb in Table 4.

No	X	Y	$ \Phi_{\leq 2} $	$ \Phi_{\geq 3} $	$ \Phi $	Time
(i)	4	4	1	2	0	0.8
(ii)	3 1	3 1	6	2	0	20.6
(iii)	2 2	2 2	3	2	0	307.9
(iv)	2 1 1	2 1 1	6	2	0	460.2
(v)	1 1 1 1	1 1 1 1	2	2	0	213.2

Tab. 14: Results obtained for Subcase IVc in Table 4.

No	X	Y	$ \Phi_{\leq 2} $	$ \Phi_{\geq 3} $	$ \Phi $	Time
(i)	1 5	0 5	—	Ruled out by Lemma 6	—	—
(ii)	1 4 1	0 4 1	—	Ruled out by Lemma 6	—	—
(iii)	1 3 2	0 3 2	—	Ruled out by Lemma 6	—	—
(iv)	1 3 1 1	0 3 1 1	—	Ruled out by Lemma 6	—	—
(v)	1 2 2 1	0 2 2 1	—	Ruled out by Lemma 6	—	—
(vi)	1 2 1 1 1	0 2 1 1 1	—	Ruled out by Lemma 6	—	—
(vii)	1 1 1 1 1 1	0 1 1 1 1 1	—	Ruled out by Lemma 6	—	—
(viii)	2 4	1 4	86	3	0	88.8
(ix)	2 3 1	1 3 1	93	3	0	191.2
(x)	2 2 2	1 2 2	27	3	0	102.1
(xi)	2 2 1 1	1 2 2 1	80	3	0	340.2
(xii)	2 1 1 1 1	1 1 1 1 1	25	3	0	270.3
(xiii)	3 3	2 3	38	3	0	60.1
(xiv)	3 2 1	2 2 1	82	3	0	306.4
(xv)	3 1 1 1	2 1 1 1	61	3	0	329.3
(xvi)	4 2	3 2	76	3	0	205.9
(xvii)	4 1 1	3 1 1	165	3	0	263.8
(xviii)	5 1	4 1	188	3	0	187.5
(xix)	6	5	—	Ruled out by Lemma 7	—	—

Tab. 15: Results obtained for Subcase V in Table 4.

No	X	Y	$ \Phi_{\leq 2} $	$ \Phi_{\geq 3} $	$ \Phi $	Time
(i)	1 1 4	0 0 4	—	Ruled out by Lemma 8	—	
(ii)	1 1 3 1	0 0 3 1	—	Ruled out by Lemma 8	—	
(iii)	1 1 2 2	0 0 2 2	—	Ruled out by Lemma 8	—	
(iv)	1 1 2 1 1	0 0 2 1 1	—	Ruled out by Lemma 8	—	
(v)	1 1 1 1 1 1	0 0 1 1 1 1	—	Ruled out by Lemma 8	—	
(vi)	1 2 3	0 1 3	77	1	0	401.8
(vii)	1 2 2 1	0 1 2 1	118	1	0	2 309.0
(viii)	1 2 1 1 1	0 1 1 1 1	79	1	0	580.4
(ix)	1 3 2	0 2 2	66	1	0	1 361.2
(x)	1 3 1 1	0 2 1 1	132	1	0	1 393.9
(xi)	1 4 1	0 3 1	288	1	0	411.8
(xii)	1 5	0 4	200	1	0	58.2
(xiii)	2 2 2	1 1 2	32	1	0	491.6
(xiv)	2 2 1 1	1 1 1 1	48	1	0	372.3
(xv)	2 3 1	1 2 1	92	1	0	1 095.1
(xvi)	2 4	1 3	94	1	0	182.7
(xvii)	3 3	2 2	26	1	0	354.7
(xviii)	3 3	1 3	14	1	0	264.7
(xix)	3 2 1	1 2 1	20	1	0	766.5
(xx)	3 1 1 1	1 1 1 1	15	1	0	214.6
(xxi)	4 2	2 2	26	1	0	679.7
(xxii)	4 1 1	2 1 1	39	1	0	283.8
(xxiii)	5 1	3 1	83	1	0	21.9
(xxiv)	6	4	50	1	0	24.3

Tab. 16: Results obtained for Subcase VIa in Table 4.

No	X	Y	$ \Phi_{\leq 2} $	$ \Phi_{\geq 3} $	$ \Phi $	Time
(i)	5	5	45	1	0	51.6
(ii)	4 1	4 1	90	1	0	37.6
(iii)	3 2	3 2	22	1	0	902.6
(iv)	3 1 1	3 1 1	46	1	0	1 241.8
(v)	2 2 1	2 2 1	17	1	0	1 113.3
(vi)	2 1 1 1	2 1 1 1	22	1	0	1 786.5
(vii)	1 1 1 1 1	1 1 1 1 1	6	1	0	807.0

Tab. 17: Results obtained for Subcase VIb in Table 4.

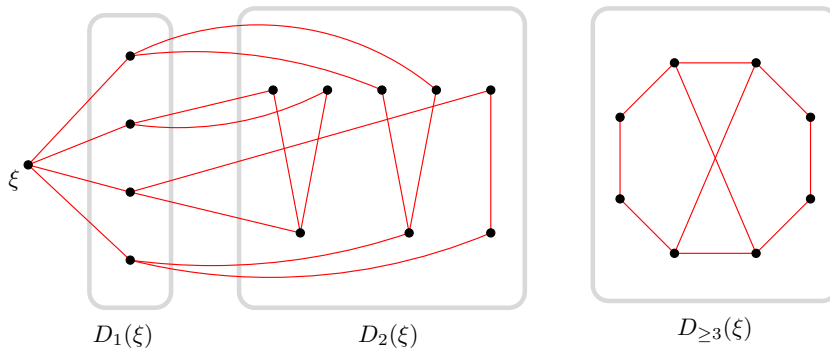


Fig. 10: An example of the result of applying the first three steps of the enumeration process to Subcase IVb(viii) of Table 13.

times reported in the tables are measured in seconds and represent the times required to carry out the four-step enumeration process on the processor described in the caption of Table 3.

Consider, for example, Subcase IVb(viii) in Table 13. An example of a result of applying the first three steps of our four-step enumeration process in this case (*i.e.* before the branching of Step 4 commences) is shown in Figure 10. Note that one of the two possible $t(3, 4, 8)$ colourings in Figure 5, namely G_{36} , appears as $\langle D_{\geq 3}(\xi) \rangle_{\text{red}}$ in Figure 10. All non-red edges entirely within the sets $D_1(\xi)$, $D_2(\xi)$ and $D_3(\xi)$ are blue, and so are the edges between ξ and $D_2(\xi)$, between ξ and $D_3(\xi)$, and between $D_1(\xi)$ and $D_3(\xi)$. Branching occurs only on the remaining edges between $D_1(\xi)$ and $D_2(\xi)$ as well as on the edges between $D_2(\xi)$ and $D_3(\xi)$, eventually resulting in the $(3, 8, 21)$ colouring H_3 shown in Figure 11.

4.3 Results for the case $\Delta(\Xi) = 5$

Suppose now $\Delta(\Xi) = 5$. Then $|D_2(\xi)| \leq 11$ and $|D_{>2}(\xi)| < t(3, 3) = 6$ by Lemma 5. Furthermore, we have the following restriction on the number of small components in $\langle D_2(\xi) \rangle_{\text{red}}$.

Lemma 8 *If $|X| = 6$, $|Y| = 4$, $\Delta(\Xi) = 5$ and $\langle D_{>2}(\xi) \rangle_{\text{red}}$ has exactly two $(1, 0)$ components, then $\langle D_2(\xi) \rangle_{\text{red}}$ is not a subgraph of Ξ .*

Proof: By contradiction. Suppose, to the contrary, that $|X| = 6$, $|Y| = 4$, $\Delta(\Xi) = 5$ and $\langle D_{>2}(\xi) \rangle_{\text{red}}$ has exactly two $(1, 0)$ components, but that $\langle D_2(\xi) \rangle_{\text{red}}$ is a subgraph of Ξ . It follows from Observation 2 that $D_{>3}(\xi) = \emptyset$ and hence from Observation 1 that $\langle D_3(\xi) \rangle_{\text{red}}$ is the red subgraph of a $t(3, 3, 5)$ colouring. Furthermore, each vertex $v \in D_3(\xi)$ is joined to a $(1, 0)$ component by a red edge, for otherwise $\{\xi, v\} \cup D_2(\xi)_{\text{blue}}$ contains a clique of order 8. Consequently one of the $(1, 0)$ components is joined to at least three vertices of $D_3(\xi)$ by red edges. These three vertices form a triangle in $\langle D_3(\xi) \rangle_{\text{blue}}$ in order to avoid forming a triangle in Ξ , but this contradicts the fact that $\langle D_3(\xi) \rangle_{\text{red}}$ is the red subgraph of a $t(3, 3, 5)$ colouring. \square

The results obtained when applying our four-step enumeration process of §4.1 to subcases V–VIb in Table 4 are presented in Tables 15–17, respectively.

4.4 Validation of results

It is known that there is only one $t(3, 3, 5)$ colouring, two $t(3, 4, 8)$ colourings, six $t(3, 5, 11)$ colourings and thirty four $t(3, 6, 14)$ colourings (Burger and van Vuuren, 2011, Table 3). We validated our computer implementation of the four-step enumeration process described in §4.1 by verifying these results independently. We also verified that the results obtained by this process are independent of the choice of the vertex ξ in each case.

4.5 The sets of $t(3, 7, 17)$ colourings and $t(3, 8, 21)$ colourings

A total of 298 $t(3, 7, 17)$ colourings were uncovered during the enumeration process described in the previous sections. These colourings have not been enumerated before and are available online at Burger (2014).

Furthermore, a total of six $t(3, 8, 21)$ colourings were similarly found, as summarised in Tables 7–17. Among these colourings there are, however, only three pairwise non-isomorphic colourings — all corresponding to the case where $\Delta(\Xi) = 4$. The red subgraphs of these colourings are shown in Figure 11.

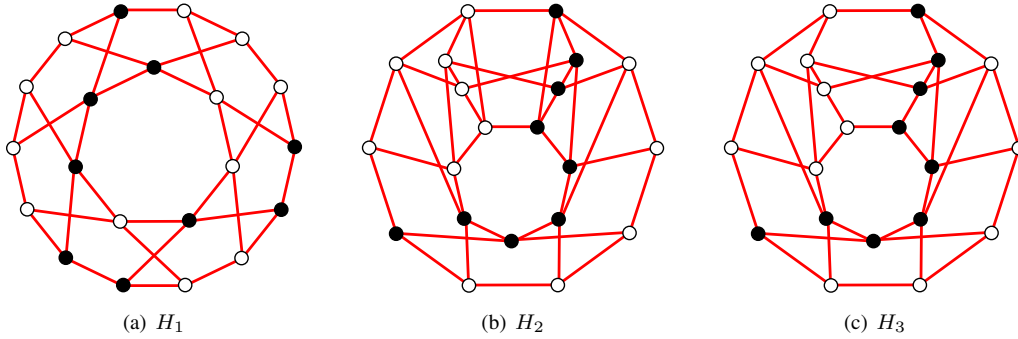


Fig. 11: The red subgraphs of the complete set of $t(3, 8, 21)$ colourings. A minimal dominating (maximal irredundant) set is indicated with solid vertices for each graph.

5 The Ramsey numbers $s(3, 8)$ and $w(3, 8)$

Since the circulant graph R in Figure 12(a) contains no irredundant set of cardinality 8 and the circulant graph B in Figure 12(b) contains no irredundant set of cardinality 3, it follows that the pair (R, B) in Figure 12 forms an $s(3, 8, 20)$ colouring.

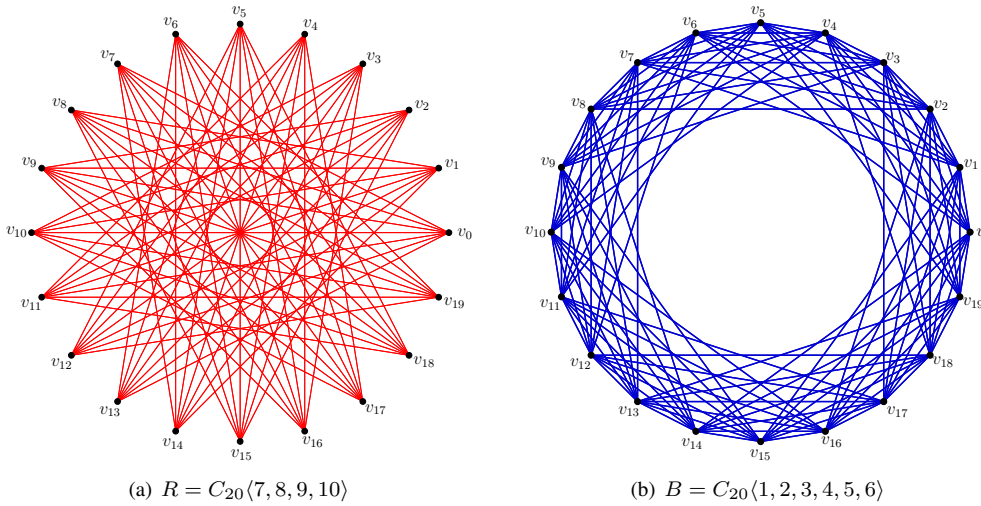


Fig. 12: An $s(3, 8, 20)$ colouring (R, B) in which both R and B are circulant graphs.

It therefore follows by (2) and Table 1 that

$$21 \leq s(3, 8) \leq w(3, 8) \leq t(3, 8) = 22. \tag{4}$$

Lemma 2 implies that sets of $s(3, 8, 21)$ and $w(3, 8, 21)$ colourings may be found from among the set of $t(3, 8, 21)$ colourings enumerated in §4. The graphs in Figure 11 each contain a minimal dominating (or

maximal irredundant) set of cardinality 9, so none of these graphs represent $w(3, 8, 21)$ colourings, or $s(3, 8, 21)$ colourings, and hence we have the following result in view of (4).

Theorem 1 $s(3, 8) = w(3, 8) = 21$.

6 Further work

In order to pave the way for the computation of the Ramsey numbers $s(3, 9)$ and $t(3, 9)$, we establish the following bounds.

Theorem 2 $24 \leq s(3, 9) \leq t(3, 9) \leq 27$.

Proof: Using the notation introduced in Figure 12, the circulant graph $R' = C_{23}\langle 8, 9, 10, 11 \rangle$ contains no irredundant set of cardinality 9 and the circulant graph $B' = C_{23}\langle 1, 2, 3, 4, 5, 6, 7 \rangle$ contains no irredundant set of cardinality 3. The pair (R', B') therefore forms an $s(3, 9, 23)$ colouring and hence $s(3, 9) \geq 24$.

In order to establish the desired upper bound on $t(3, 9)$, we show by contradiction that no $t(3, 9, 27)$ colouring exists. Suppose, to the contrary, that such a colouring indeed exists, and let Ξ' be its red subgraph. Then it follows from Corollary 2 that $\delta(\Xi') \geq 27 - t(3, 8) = 27 - 22 = 5$. It also follows from Corollary 1 that $\Delta(\Xi') < t(2, 9) = 9$, and so $\Delta(\Xi') = 5, 6, 7$ or 8 .

Let v be a vertex of maximum degree in Ξ' and suppose $\Delta(\Xi') = 8$. Then $D_{\geq 3}(v)$ is empty in order to avoid a clique of order 9 in the blue subgraph $\overline{\Xi'}$. But then $D_2(v)$ has cardinality 18 and forms a clique of order 9 in $\overline{\Xi'}$ by Lemma 4(b), a contradiction.

Suppose next that v has degree 6 or 7. Then by an argument similar to the one above, we may assume that $D_3(v)$ is not empty (so as to avoid a large set $D_2(v)$). But then it follows from Lemma 4(b) that $D_2(v)$ has fewer than 14 vertices in order to avoid a clique of order 9 in $\overline{\Xi'}$ (on the vertices of a partite set of the bipartite graph in $D_{\geq 2}(v)$ together with v). Furthermore, similar to Observation 1, we also have that $D_{\geq 3}(v)$ has fewer than $t(3, 9 - \Delta(\Xi'))$ vertices, and so Ξ' has fewer than $1 + \Delta(\Xi') + 13 + t(3, 9 - \Delta(\Xi'))$ vertices. For $\Delta(\Xi') = 6$ or 7 , this value is 25 or 24, respectively — contradicting the fact that Ξ' has order 27.

It follows that Ξ' is a 5-regular graph, which is a contradiction, because it has odd order. \square

As further work, we propose that the values of $s(3, 9)$ and $t(3, 9)$ be computed according to our enumeration strategy in §4.1. In order to render this strategy feasible for graphs of orders 24–27, however, we anticipate that our branch-and-bound procedure in §3 will have to be refined, possibly by selecting the branching order judiciously. We furthermore expect that the development of new theoretical results will be required in order to extend the bounding list of Table 6.

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