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Wadge Degrees of ω-Languages of Petri Nets

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Abstract

We prove that ω-languages of (non-deterministic) Petri nets and ω-languages of (non-deterministic) Turing machines have the same topological complexity: the Borel and Wadge hierarchies of the class of ω-languages of (non-deterministic) Petri nets are equal to the Borel and Wadge hierarchies of the class of ω-languages of (non-deterministic) Turing machines which also form the class of effective analytic sets. In particular, for each non-null recursive ordinal $\alpha < \omega_1^{\mathrm{CK}}$ there exist some $\mathbf{\Sigma}^0_\alpha$-complete and some $\mathbf{\Pi}^0_\alpha$-complete ω-languages of Petri nets, and the supremum of the set of Borel ranks of ω-languages of Petri nets is the ordinal γ^1_2, which is strictly greater than the first non-recursive ordinal ω_1^{CK}. We also prove that there are some $\mathbf{\Sigma}^1_1$-complete, hence non-Borel, ω-languages of Petri nets, and that it is consistent with ZFC that there exist some ω-languages of Petri nets which are neither Borel nor $\mathbf{\Sigma}^1_1$-complete. This answers the question of the topological complexity of ω-languages of (non-deterministic) Petri nets which was left open in [9, 19].

Keywords and phrases Automata and formal languages; logic in computer science; Petri nets; infinite words; Cantor topology; Borel hierarchy; Wadge hierarchy; Wadge degrees.

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1 Introduction

In the sixties, Büchi was the first to study acceptance of infinite words by finite automata with the now called Büchi acceptance condition, in order to prove the decidability of the monadic second order theory of one successor over the integers. Since then there has been a lot of work on regular ω-languages, accepted by Büchi automata, or by some other variants of automata over infinite words, like Muller or Rabin automata, see [44, 43, 32]. The acceptance of infinite words by other finite machines, like pushdown automata, counter automata, Petri nets, Turing machines, ..., with various acceptance conditions, has also been studied, see [43, 10, 5, 45, 40, 41, 42].

The Cantor topology is a very natural topology on the set Σ^ω of infinite words over a finite alphabet Σ which is induced by the prefix metric. Then a way to study the complexity of languages of infinite words accepted by finite machines is to study their topological complexity and firstly to locate them with regard to the Borel and the projective hierarchies [44, 10, 27, 43].

Every ω-language accepted by a deterministic Büchi automaton is a $\mathbf{\Pi}^0_2$-set. On the other hand, it follows from Mac Naughton's Theorem that every regular ω-language is accepted by a deterministic Muller automaton, and thus is a boolean combination of ω-languages accepted by deterministic Büchi automata. Therefore every regular ω-language is a $\mathbf{\Delta}^0_3$-set. Moreover Landweber proved that the Borel complexity of any ω-language accepted by a Muller or Büchi automaton can be effectively computed (see [26, 32]). In a similar way, every ω-language accepted by a deterministic Muller Turing machine, and thus also by any Muller deterministic finite machine is a $\mathbf{\Delta}^0_3$-set, [10, 43].

On the other hand, the Wadge hierarchy is a great refinement of the Borel hierarchy, firstly defined by Wadge via reductions by continuous functions [46]. The trace of the Wadge hierarchy on the ω-regular languages is called the Wagner hierarchy. It has been completely described by Klaus

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Wagner in [47]. Its length is the ordinal ω^ω. Wagner gave an automaton-like characterization of this hierarchy, based on the notions of chain and superchain, together with an algorithm to compute the Wadge (Wagner) degree of any given ω-regular language, see also [2, 3, 34, 37, 36, 38].

The Wadge hierarchy of deterministic context-free ω-languages was determined by Duparc in [7, 8]. Its length is the ordinal $\omega^{(\omega^2)}$. We do not know yet whether this hierarchy is decidable or not. But the Wadge hierarchy induced by deterministic partially blind 1-counter automata was described in an effective way in [12], and other partial decidability results were obtained in [13]. Then, it was proved in [15] that the Wadge hierarchy of 1-counter or context-free ω-languages and the Wadge hierarchy of effective analytic sets (which form the class of all the ω-languages accepted by non-deterministic Turing machines) are equal. Moreover similar results hold about the Wadge hierarchy of infinitary rational relations accepted by 2-tape Büchi automata, [16]. Finally, the Wadge hierarchy of ω-languages of deterministic Turing machines was determined by Selivanov in [35].

We consider in this paper acceptance of infinite words by Petri nets. Petri nets are used for the description of distributed systems [11, 33, 21], and form a very important mathematical model in Concurrency Theory that has been developed for general concurrent computation. In the context of Automata Theory, Petri nets may be defined as (partially) blind multicounter automata, as explained in [45, 10, 20]. First, one can distinguish between the places of a given Petri net by dividing them into the bounded ones (the number of tokens in such a place at any time is uniformly bounded) and the unbounded ones. Then each unbounded place may be seen as a partially blind counter, and the tokens in the bounded places determine the state of the partially blind multicounter automaton that is equivalent to the initial Petri net. The transitions of the Petri net may then be seen as the finite control of the partially blind multicounter automaton and the labels of these transitions are then the input symbols. The infinite behavior of Petri nets was first studied by Valk [45] and by Carstensen in the case of deterministic Petri nets [1].

On one side, the topological complexity of ω-languages of *deterministic* Petri nets is completely determined. They are $\mathbf{\Delta}^0_3$-sets and their Wadge hierarchy has been determined by Duparc, Finkel and Ressayre in [9]; its length is the ordinal ω^{ω^2}. On the other side, Finkel and Skrzypczak proved in [19] that there exist $\mathbf{\Sigma}^0_3$-complete, hence non $\mathbf{\Delta}^0_3$, ω-languages accepted by *non-deterministic* one-partially-blind-counter Büchi automata. But, up to our knowledge, this was the only known result about the topological complexity of ω-languages of *non-deterministic* Petri nets. Notice that ω-languages accepted by (non-blind) one-counter Büchi automata have the same topological complexity as ω-languages of Turing machines, [15], but the non-blindness of the counter, i.e. the ability to use the zero-test of the counter, was essential in the proof of this result.

Using a simulation of a given real time 1-counter (with zero-test) Büchi automaton $\mathcal{A}$ accepting ω-words x over the alphabet Σ by a real time 4-blind-counter Büchi automaton $\mathcal{B}$ reading some special codes $h(x)$ of the words x, we prove here that ω-languages of *non-deterministic* Petri nets and effective analytic sets have the same topological complexity: the Borel and Wadge hierarchies of the class of ω-languages of Petri nets are equal to the Borel and Wadge hierarchies of the class of effective analytic sets. In particular, for each non-null recursive ordinal $\alpha < \omega_1^{\mathrm{CK}}$ there exist some $\mathbf{\Sigma}^0_\alpha$-complete and some $\mathbf{\Pi}^0_\alpha$-complete ω-languages of Petri nets, and the supremum of the set of Borel ranks of ω-languages of Petri nets is the ordinal γ^1_2, which is strictly greater than the first non-recursive ordinal ω_1^{CK}. We also prove that there are some $\mathbf{\Sigma}^1_1$-complete, hence non-Borel, ω-languages of Petri nets, and that it is consistent with ZFC that there exist some ω-languages of Petri nets which are neither Borel nor $\mathbf{\Sigma}^1_1$-complete.

The paper is organized as follows. In Section 2 we review the notions of (blind) counter automata and ω-languages. In Section 3 we recall notions of topology, and in particular the Borel and Wadge hierarchies on a Cantor space. We prove our main results in Section 4. Concluding remarks are given in Section 5.

2 Counter Automata

We assume the reader to be familiar with the theory of formal (ω-)languages [43, 32]. We recall the usual notations of formal language theory.

If Σ is a finite alphabet, a *non-empty finite word* over Σ is any sequence $x = a_1 \dots a_k$, where $a_i \in \Sigma$ for $i = 1, \dots, k$, and k is an integer ≥ 1. The *length* of x is k, denoted by $|x|$. The *empty word* is denoted by λ; its length is 0. $\Sigma^\star$ is the *set of finite words* (including the empty word) over Σ, and we denote $\Sigma^+ = \Sigma^\star \setminus \{\lambda\}$. A (finitary) *language* V over an alphabet Σ is a subset of $\Sigma^\star$.

The *first infinite ordinal* is ω. An ω-*word* over Σ is an ω-sequence $a_1 \dots a_n \dots$, where for all integers $i \geq 1$, $a_i \in \Sigma$. When $\sigma = a_1 \dots a_n \dots$ is an ω-word over Σ, we write $\sigma(n) = a_n$, $\sigma[n] = \sigma(1)\sigma(2)\dots\sigma(n)$ for all $n \geq 1$ and $\sigma[0] = \lambda$.

The usual concatenation product of two finite words u and v is denoted $u \cdot v$ (and sometimes just uv). This product is extended to the product of a finite word u and an ω-word v: the infinite word $u \cdot v$ is then the ω-word such that:

$(u \cdot v)(k) = u(k)$ if $k \leq |u|$, and $(u \cdot v)(k) = v(k - |u|)$ if $k > |u|$.

The *set of* ω-*words* over the alphabet Σ is denoted by Σ^ω. An ω-*language* V over an alphabet Σ is a subset of Σ^ω, and its complement (in Σ^ω) is $\Sigma^\omega \setminus V$, denoted V^-.

The *prefix relation* is denoted $\sqsubseteq$: a finite word u is a *prefix* of a finite word v (respectively, an infinite word v), denoted $u \sqsubseteq v$, if and only if there exists a finite word w (respectively, an infinite word w), such that $v = u \cdot w$.

Let k be an integer ≥ 1. A k-counter machine has k *counters*, each of which containing a non-negative integer. The machine can test whether the content of a given counter is zero or not, but this is not possible if the counter is a blind (sometimes called partially blind, as in [20]) counter. This means that if a transition of the machine is enabled when the content of a blind counter is zero then the same transition is also enabled when the content of the same counter is a non-zero integer. And transitions depend on the letter read by the machine, the current state of the finite control, and the tests about the values of the counters. Notice that in the sequel we shall only consider real-time automata, i.e. λ-transitions are not allowed (but the general results of this paper will be easily extrended to the case of non-real-time automata).

Formally a *real time* k-counter machine is a 4-tuple $\mathcal{M}$=(K, Σ, Δ, q_0), where K is a finite set of states, Σ is a finite input alphabet, $q_0 \in K$ is the initial state, and $\Delta \subseteq K \times \Sigma \times \{0, 1\}^k \times K \times \{0, 1, -1\}^k$ is the transition relation.

If the machine $\mathcal{M}$ is in state q and $c_i \in \mathbf{N}$ is the content of the ith counter $\mathcal{C}_i$ then the configuration (or global state) of $\mathcal{M}$ is the $(k+1)$-tuple $(q, c_1, \dots, c_k)$.

For $a \in \Sigma$, $q, q' \in K$ and $(c_1, \dots, c_k) \in \mathbf{N}^k$ such that $c_j = 0$ for $j \in E \subseteq \{1, \dots, k\}$ and $c_j > 0$ for $j \notin E$, if $(q, a, i_1, \dots, i_k, q', j_1, \dots, j_k) \in \Delta$ where $i_j = 0$ for $j \in E$ and $i_j = 1$ for $j \notin E$, then we write:

$$a : (q, c_1, \dots, c_k) \mapsto_{\mathcal{M}} (q', c_1 + j_1, \dots, c_k + j_k).$$

Thus the transition relation must obviously satisfy:
if $(q, a, i_1, \dots, i_k, q', j_1, \dots, j_k) \in \Delta$ and $i_m = 0$ for some $m \in \{1, \dots, k\}$ then $j_m = 0$ or $j_m = 1$ (but j_m may not be equal to -1).

Moreover if the counters of $\mathcal{M}$ are blind, then, if $(q, a, i_1, \dots, i_k, q', j_1, \dots, j_k) \in \Delta$ holds, and $i_m = 0$ for some $m \in \{1, \dots, k\}$ then $(q, a, i_1, \dots, i_k, q', j_1, \dots, j_k) \in \Delta$ also holds if $i_m = 1$ and the other intergers are unchanged.

An ω-sequence of configurations $r = (q_i, c_1^i, \dots c_k^i)_{i \geq 1}$ is called a run of $\mathcal{M}$ on an ω-word $\sigma = a_1 a_2 \dots a_n \dots$ over Σ iff:

(1) $(q_1, c_1^1, \dots c_k^1) = (q_0, 0, \dots, 0)$

(2) for each $i \geq 1$,

$a_i : (q_i, c_1^i, \dots c_k^i) \mapsto_{\mathcal{M}} (q_{i+1}, c_1^{i+1}, \dots c_k^{i+1})$.

For every such run r, $\mathrm{In}(r)$ is the set of all states entered infinitely often during r.

▶ **Definition 2.1.** *A Büchi k-counter automaton is a 5-tuple $\mathcal{M}$=$(K, \Sigma, \Delta, q_0, F)$, where $\mathcal{M}'$=(K, Σ, Δ, q_0) is a k-counter machine and $F \subseteq K$ is the set of accepting states. The ω-language accepted by $\mathcal{M}$ is: $L(\mathcal{M})$= $\{\sigma \in \Sigma^\omega \mid$ there exists a run r of $\mathcal{M}$ on σ such that $\mathrm{In}(r) \cap F \neq \emptyset\}$*

▶ **Definition 2.2.** *A Muller k-counter automaton is a 5-tuple $\mathcal{M}$=$(K, \Sigma, \Delta, q_0, \mathcal{F})$, where $\mathcal{M}'$=(K, Σ, Δ, q_0) is a k-counter machine and $\mathcal{F} \subseteq 2^K$ is the set of accepting sets of states. The ω-language accepted by $\mathcal{M}$ is: $L(\mathcal{M})$=$\{\sigma \in \Sigma^\omega \mid$ there exists a run r of $\mathcal{M}$ on σ such that $\exists F \in \mathcal{F}\ \mathrm{In}(r) = F\}$*

It is well known that an ω-language is accepted by a non-deterministic (real time) Büchi k-counter automaton iff it is accepted by a non-deterministic (real time) Muller k-counter automaton [10]. Notice that it cannot be shown without using the non determinism of automata and this result is no longer true in the deterministic case.

The class of ω-languages accepted by real time k-counter Büchi automata (respectively, real time k-blind-counter Büchi automata) is denoted $\mathbf{r}\text{-}\mathbf{CL}(k)_\omega$. (respectively, $\mathbf{r}\text{-}\mathbf{BCL}(k)_\omega$). (Notice that in previous papers, as in [15], the class $\mathbf{r}\text{-}\mathbf{CL}(k)_\omega$ was denoted $\mathbf{r}\text{-}\mathbf{BCL}(k)_\omega$ so we have slightly changed the notation in order to distinguish the different classes).

The class $\mathbf{CL}(1)_\omega$ is a strict subclass of the class $\mathbf{CFL}_\omega$ of context free ω-languages accepted by pushdown Büchi automata.

If we omit the counter of a real-time Büchi 1-counter automaton, then we simply get the notion of Büchi automaton. The class of ω-languages accepted by Büchi automata is the class of regular ω-languages.

3 Hierarchies in a Cantor Space

3.1 Borel hierarchy and analytic sets

We assume the reader to be familiar with basic notions of topology which may be found in [29, 27, 43, 32]. There is a natural metric on the set Σ^ω of infinite words over a finite alphabet Σ containing at least two letters which is called the *prefix metric* and is defined as follows. For $u, v \in \Sigma^\omega$ and $u \neq v$ let $\delta(u, v) = 2^{-l_{\mathrm{pref}(u,v)}}$ where $l_{\mathrm{pref}(u,v)}$ is the first integer n such that the $(n+1)^{st}$ letter of u is different from the $(n+1)^{st}$ letter of v. This metric induces on Σ^ω the usual Cantor topology in which the *open subsets* of Σ^ω are of the form $W \cdot \Sigma^\omega$, for $W \subseteq \Sigma^\star$. A set $L \subseteq \Sigma^\omega$ is a *closed set* iff its complement $\Sigma^\omega - L$ is an open set.

Define now the *Borel Hierarchy* of subsets of Σ^ω:

▶ **Definition 3.1.** *For a non-null countable ordinal α, the classes $\mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Pi}^0_\alpha$ of the Borel Hierarchy on the topological space Σ^ω are defined as follows:*
$\mathbf{\Sigma}^0_1$ is the class of open subsets of Σ^ω, $\mathbf{\Pi}^0_1$ is the class of closed subsets of Σ^ω,
and for any countable ordinal $\alpha \geq 2$:
$\mathbf{\Sigma}^0_\alpha$ is the class of countable unions of subsets of Σ^ω in $\bigcup_{\gamma<\alpha} \mathbf{\Pi}^0_\gamma$.
$\mathbf{\Pi}^0_\alpha$ is the class of countable intersections of subsets of Σ^ω in $\bigcup_{\gamma<\alpha} \mathbf{\Sigma}^0_\gamma$.

The class of *Borel sets* is $\mathbf{\Delta}^1_1 := \bigcup_{\xi<\omega_1} \mathbf{\Sigma}^0_\xi = \bigcup_{\xi<\omega_1} \mathbf{\Pi}^0_\xi$, where ω_1 is the first uncountable ordinal. There are also some subsets of Σ^ω which are not Borel. In particular the class of Borel subsets of Σ^ω is strictly included into the class $\mathbf{\Sigma}^1_1$ of *analytic sets* which are obtained by projection of Borel sets.

▶ **Definition 3.2.** *A subset A of Σ^ω is in the class $\mathbf{\Sigma}^1_1$ of analytic sets iff there exists another finite set Y and a Borel subset B of $(\Sigma \times Y)^\omega$ such that $x \in A \leftrightarrow \exists y \in Y^\omega$ such that $(x, y) \in B$, where*

(x, y) is the infinite word over the alphabet $\Sigma \times Y$ such that $(x, y)(i) = (x(i), y(i))$ for each integer $i \geq 1$.

We now define completeness with regard to reduction by continuous functions. For a countable ordinal $\alpha \geq 1$, a set $F \subseteq \Sigma^\omega$ is said to be a $\mathbf{\Sigma}^0_\alpha$ (respectively, $\mathbf{\Pi}^0_\alpha$, $\mathbf{\Sigma}^1_1$)-*complete set* iff for any set $E \subseteq Y^\omega$ (with Y a finite alphabet): $E \in \mathbf{\Sigma}^0_\alpha$ (respectively, $E \in \mathbf{\Pi}^0_\alpha$, $E \in \mathbf{\Sigma}^1_1$) iff there exists a continuous function $f : Y^\omega \to \Sigma^\omega$ such that $E = f^{-1}(F)$.

Let us now recall the definition of the arithmetical hierarchy of ω-languages, see for example [43, 29]. Let Σ be a finite alphabet. An ω-language $L \subseteq \Sigma^\omega$ belongs to the class Σ_n if and only if there exists a recursive relation $R_L \subseteq (\mathbb{N})^{n-1} \times \Sigma^\star$ such that

$$L = \{\sigma \in \Sigma^\omega \mid \exists a_1 \ldots Q_n a_n \quad (a_1, \ldots, a_{n-1}, \sigma[a_n + 1]) \in R_L\}$$

where Q_i is one of the quantifiers $\forall$ or $\exists$ (not necessarily in an alternating order). An ω-language $L \subseteq \Sigma^\omega$ belongs to the class Π_n if and only if its complement $\Sigma^\omega - L$ belongs to the class Σ_n. The inclusion relations that hold between the classes Σ_n and Π_n are the same as for the corresponding classes of the Borel hierarchy and the classes Σ_n and Π_n are strictly included in the respective classes $\mathbf{\Sigma}^0_n$ and $\mathbf{\Pi}^0_n$ of the Borel hierarchy.

As in the case of the Borel hierarchy, projections of arithmetical sets (of the second Π-class) lead beyond the arithmetical hierarchy, to the analytical hierarchy of ω-languages. The first class of the analytical hierarchy of ω-languages is the (lightface) class Σ^1_1 of effective analytic sets. An ω-language $L \subseteq \Sigma^\omega$ belongs to the class Σ^1_1 if and only if there exists a recursive relation $R_L \subseteq (\mathbb{N}) \times \{0, 1\}^\star \times \Sigma^\star$ such that:

$$L = \{\sigma \in \Sigma^\omega \mid \exists \tau (\tau \in \{0, 1\}^\omega \wedge \forall n \exists m((n, \tau[m], \sigma[m]) \in R_L))\}$$

Thus an ω-language $L \subseteq \Sigma^\omega$ is in the class Σ^1_1 iff it is the projection of an ω-language over the alphabet $\{0, 1\} \times \Sigma$ which is in the class Π_2 of the arithmetical hierarchy.

Kechris, Marker and Sami proved in [25] that the supremum of the set of Borel ranks of (lightface) Π^1_1, so also of (lightface) Σ^1_1, sets is the ordinal γ^1_2. This ordinal is precisely defined in [25]. It holds that $\omega_1^{\mathrm{CK}} < \gamma^1_2$, where ω_1^{CK} is the first non-recursive ordinal, called the Chruch-Kleene ordinal. But the exact value of the ordinal γ^1_2 may depend on axioms of set theory [25].

Notice that it seems still unknown whether *every* non null ordinal $\gamma < \gamma^1_2$ is the Borel rank of a (lightface) Π^1_1 (or Σ^1_1) set. On the other hand it is known that every ordinal $\gamma < \omega_1^{\mathrm{CK}}$ is the Borel rank of a (lightface) Δ^1_1-set, since for every ordinal $\gamma < \omega_1^{\mathrm{CK}}$ there exist some $\mathbf{\Sigma}^0_\gamma$-complete and $\mathbf{\Pi}^0_\gamma$-complete sets in the class Δ^1_1.

Recall that a Büchi Turing machine is just a Turing machine working on infinite inputs with a Büchi-like acceptance condition, and that the class of ω-languages accepted by Büchi Turing machines is the class Σ^1_1 of effective analytic sets [5, 43].

3.2 Wadge hierarchy

We now introduce the Wadge hierarchy, which is a great refinement of the Borel hierarchy defined via reductions by continuous functions, [6, 46].

▶ **Definition 3.3** (Wadge [46]). *Let X, Y be two finite alphabets. For $L \subseteq X^\omega$ and $L' \subseteq Y^\omega$, L is said to be Wadge reducible to L' ($L \leq_W L'$) iff there exists a continuous function $f : X^\omega \to Y^\omega$, such that $L = f^{-1}(L')$. l L and L' are Wadge equivalent iff $L \leq_W L'$ and $L' \leq_W L$. This will be denoted by $L \equiv_W L'$. And we shall say that $L <_W L'$ iff $L \leq_W L'$ but not $L' \leq_W L$.*
A set $L \subseteq X^\omega$ is said to be self dual iff $L \equiv_W L^-$, and otherwise it is said to be non self dual.

The relation $\leq_W$ is reflexive and transitive, and $\equiv_W$ is an equivalence relation. The *equivalence classes* of $\equiv_W$ are called *Wadge degrees*. The Wadge hierarchy WH is the class of Borel subsets of a set X^ω, where X is a finite set, equipped with $\leq_W$ and with $\equiv_W$.
For $L \subseteq X^\omega$ and $L' \subseteq Y^\omega$, if $L \leq_W L'$ and $L = f^{-1}(L')$ where f is a continuous function from X^ω into Y^ω, then f is called a continuous reduction of L to L'. Intuitively it means that L is less complicated than L' because to check whether $x \in L$ it suffices to check whether $f(x) \in L'$ where f is a continuous function. Hence the Wadge degree of an ω-language is a measure of its topological complexity.
Notice that in the above definition, we consider that a subset $L \subseteq X^\omega$ is given together with the alphabet X.
We can now define the *Wadge class* of a set L:

▶ **Definition 3.4.** *Let L be a subset of X^ω. The Wadge class of L is :*
$[L] = \{L' \mid L' \subseteq Y^\omega$ *for a finite alphabet* Y *and* $L' \leq_W L\}$.

Recall that each *Borel class* $\mathbf{\Sigma^0_\alpha}$ and $\mathbf{\Pi^0_\alpha}$ is a *Wadge class*. A set $L \subseteq X^\omega$ is a $\mathbf{\Sigma^0_\alpha}$ (respectively $\mathbf{\Pi^0_\alpha}$)*-complete set* iff for any set $L' \subseteq Y^\omega$, L' is in $\mathbf{\Sigma^0_\alpha}$ (respectively $\mathbf{\Pi^0_\alpha}$) iff $L' \leq_W L$.

There is a close relationship between Wadge reducibility and games which we now introduce.

▶ **Definition 3.5.** *Let $L \subseteq X^\omega$ and $L' \subseteq Y^\omega$. The Wadge game $W(L, L')$ is a game with perfect information between two players, player 1 who is in charge of L and player 2 who is in charge of L'. Player 1 first writes a letter $a_1 \in X$, then player 2 writes a letter $b_1 \in Y$, then player 1 writes a letter $a_2 \in X$, and so on. The two players alternatively write letters a_n of X for player 1 and b_n of Y for player 2. After ω steps, the player 1 has written an ω-word $a \in X^\omega$ and the player 2 has written an ω-word $b \in Y^\omega$. The player 2 is allowed to skip, even infinitely often, provided he really writes an ω-word in ω steps. The player 2 wins the play iff [$a \in L \leftrightarrow b \in L'$], i.e. iff :*

$$[(a \in L \text{ and } b \in L') \text{ or } (a \notin L \text{ and } b \notin L' \text{ and } b \text{ is infinite})].$$

Recall that a strategy for player 1 is a function $\sigma : (Y \cup \{s\})^\star \to X$. And a strategy for player 2 is a function $f : X^+ \to Y \cup \{s\}$.
σ is a winning stategy for player 1 iff he always wins a play when he uses the strategy σ, i.e. when the nth letter he writes is given by $a_n = \sigma(b_1 \cdots b_{n-1})$, where b_i is the letter written by player 2 at step i and $b_i = s$ if player 2 skips at step i.
A winning strategy for player 2 is defined in a similar manner.

Martin's Theorem states that every Gale-Stewart game $G(X)$ (see [24]), with X a Borel set, is determined and this implies the following :

▶ **Theorem 3.6** (Wadge). *Let $L \subseteq X^\omega$ and $L' \subseteq Y^\omega$ be two Borel sets, where X and Y are finite alphabets. Then the Wadge game $W(L, L')$ is determined: one of the two players has a winning strategy. And $L \leq_W L'$ iff the player 2 has a winning strategy in the game $W(L, L')$.*

▶ **Theorem 3.7** (Wadge). *Up to the complement and $\equiv_W$, the class of Borel subsets of X^ω, for a finite alphabet X having at least two letters, is a well ordered hierarchy. There is an ordinal $|WH|$, called the length of the hierarchy, and a map d^0_W from WH onto $|WH| - \{0\}$, such that for all $L, L' \subseteq X^\omega$:*
$d^0_W L < d^0_W L' \leftrightarrow L <_W L'$ *and*
$d^0_W L = d^0_W L' \leftrightarrow [L \equiv_W L'$ *or* $L \equiv_W L'^-]$.

The Wadge hierarchy of Borel sets of **finite rank** has length ${}^1\varepsilon_0$ where ${}^1\varepsilon_0$ is the limit of the ordinals α_n defined by $\alpha_1 = \omega_1$ and $\alpha_{n+1} = \omega_1^{\alpha_n}$ for n a non negative integer, ω_1 being the first non countable ordinal. Then ${}^1\varepsilon_0$ is the first fixed point of the ordinal exponentiation of base ω_1. The

length of the Wadge hierarchy of Borel sets in $\mathbf{\Delta^0_\omega} = \mathbf{\Sigma^0_\omega} \cap \mathbf{\Pi^0_\omega}$ is the ω_1^{th} fixed point of the ordinal exponentiation of base ω_1, which is a much larger ordinal. The length of the whole Wadge hierarchy of Borel sets is a huge ordinal, with regard to the ω_1^{th} fixed point of the ordinal exponentiation of base ω_1. It is described in [46, 6] by the use of the Veblen functions.

4 Wadge Degrees of ω-Languages of Petri Nets

We are firstly going to prove the following result.

▶ **Theorem 4.1.** *The Wadge hierarchy of the class* $\mathbf{r}$-$\mathbf{BCL}(4)_\omega$ *is equal to the Wadge hierarchy of the class* $\mathbf{r}$-$\mathbf{CL}(1)_\omega$.

In order to prove this result, we first define a coding of ω-words over a finite alphabet Σ by ω-words over the alphabet $\Sigma \cup \{A, B, 0\}$ where A, B and 0 are new letters not in Σ.

We shall code an ω-word $x \in \Sigma^\omega$ by the ω-word $h(x)$ defined by

$$h(x) = A0x(1)B0^2x(2)A \cdots B0^{2n}x(2n)A0^{2n+1}x(2n+1)B \cdots$$

This coding defines a mapping $h : \Sigma^\omega \to (\Sigma \cup \{A, B, 0\})^\omega$.
The function h is continuous because for all ω-words $x, y \in \Sigma^\omega$ and each positive integer n, it holds that $\delta(x, y) < 2^{-n} \to \delta(h(x), h(y)) < 2^{-n}$.

We now state the following lemma.

▶ **Lemma 4.2.** *Let $\mathcal{A}$ be a real time 1-counter Büchi automaton accepting ω-words over the alphabet Σ. Then one can construct a real time 4-blind-counter Büchi automaton $\mathcal{B}$ reading words over the alphabet $\Gamma = \Sigma \cup \{A, B, 0\}$, such that $L(\mathcal{A}) = h^{-1}(L(\mathcal{B}))$, i.e.*

$$\forall x \in \Sigma^\omega \quad h(x) \in L(\mathcal{B}) \longleftrightarrow x \in L(\mathcal{A}).$$

Proof. Let $\mathcal{A} = (K, \Sigma, \Delta, q_0, F)$ be a real time 1-counter Büchi automaton accepting ω-words over the alphabet Σ. We are going to explain informally the behaviour of the 4-blind-counter Büchi automaton $\mathcal{B}$ when reading an ω-word of the form $h(x)$, even if we are going to see that $\mathcal{B}$ may also accept some infinite words which do not belong to the range of h. Recall that $h(x)$ is of the form

$$h(x) = A0x(1)B0^2x(2)A \cdots B0^{2n}x(2n)A0^{2n+1}x(2n+1)B \cdots$$

Notice that in particular every ω-word in $h(\Sigma^\omega)$ is of the form:

$$y = A0^{n_1}x(1)B0^{n_2}x(2)A \cdots B0^{n_{2n}}x(2n)A0^{n_{2n+1}}x(2n+1)B \cdots$$

where for all $i \geq 1$, $n_i > 0$ is a positive integer, and $x(i) \in \Sigma$.

Moreover it is easy to see that the set of ω-words $y \in \Gamma^\omega$ which can be written in the above form is a regular ω-language $\mathcal{R} \subseteq \Gamma^\omega$, and thus we can assume, using a classical product construction (see for instance [32]), that the automaton $\mathcal{B}$ will only accept some ω-words of this form.

Now the reading by the automaton $\mathcal{B}$ of an ω-word of the above form

$$y = A0^{n_1}x(1)B0^{n_2}x(2)A \cdots B0^{n_{2n}}x(2n)A0^{n_{2n+1}}x(2n+1)B \cdots$$

will give a decomposition of the ω-word y of the following form:

$$y = Au_1v_1x(1)Bu_2v_2x(2)Au_3v_3x(3)B \cdots Bu_{2n}v_{2n}x(2n)Au_{2n+1}v_{2n+1}x(2n+1)B \cdots$$

where, for all integers $i \geq 1$, $u_i, v_i \in 0^\star$, $x(i) \in \Sigma$, $|u_1| = 0$.

The automaton $\mathcal{B}$ will use its four *blind* counters, which we denote $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4$, in the following way. Recall that the automaton $\mathcal{B}$ being non-deterministic, we do not describe the unique run of $\mathcal{B}$ on y, but the general case of a possible run.

At the beginning of the run, the value of each of the four counters is equal to zero. Then the counter $\mathcal{C}_1$ is increased of $|u_1|$ when reading u_1, i.e. the counter $\mathcal{C}_1$ is actually not increased since $|u_1| = 0$ and the finite control is here used to check this. Then the counter $\mathcal{C}_2$ is increased of 1 for each letter 0 of v_1 which is read until the automaton reads the letter $x(1)$ and then the letter B. Notice that at this time the values of the counters $\mathcal{C}_3$ and $\mathcal{C}_4$ are still equal to zero. Then the behaviour of the automaton $\mathcal{B}$ when reading the next segment $0^{n_2}x(2)A$ is as follows. The counters $\mathcal{C}_1$ is firstly decreased of 1 for each letter 0 read, when reading k_2 letters 0, where $k_2 \geq 0$ (notice that here $k_2 = 0$ because the value of the counter $\mathcal{C}_1$ being equal to zero, it cannot decrease under 0). Then the counter $\mathcal{C}_2$ is decreased of 1 for each letter 0 read, and next the automaton has to read one more letter 0, leaving unchanged the counters $\mathcal{C}_1$ and $\mathcal{C}_2$, before reading the letter $x(2)$. The end of the decreasing mode of $\mathcal{C}_1$ coincide with the beginning of the decreasing mode of $\mathcal{C}_2$, and this change may occur in a *non-deterministic way* (because the automaton $\mathcal{B}$ cannot check whether the value of $\mathcal{C}_1$ is equal to zero). Now we describe the behaviour of the counters $\mathcal{C}_3$ and $\mathcal{C}_4$ when reading the segment $0^{n_2}x(2)A$. Using its finite control, the automaton $\mathcal{B}$ has checked that $|u_1| = 0$, and then if there is a transition of the automaton $\mathcal{A}$ such that $x(1) : (q_0, |u_1|) \mapsto_{\mathcal{A}} (q_1, |u_1| + N_1)$ then the counter $\mathcal{C}_3$ is increased of 1 for each letter 0 read, during the reading of the $k_2 + N_1$ first letters 0 of 0^{n_2}, where k_2 is described above as the number of which the counter $\mathcal{C}_1$ has been decreased. This determines u_2 by $|u_2| = k_2 + N_1$ and then the counter $\mathcal{C}_4$ is increased by 1 for each letter 0 read until $\mathcal{B}$ reads $x(2)$, and this determines v_2. Notice that the automaton $\mathcal{B}$ keeps in its finite control the memory of the state q_1 of the automaton $\mathcal{A}$, and that, after having read the segment $0^{n_2} = u_2v_2$, the values of the counters $\mathcal{C}_3$ and $\mathcal{C}_4$ are respectively $|\mathcal{C}_3| = |u_2| = k_2 + N_1$ and $|\mathcal{C}_4| = |v_2| = n_2 - (|u_2|)$.

Now the run will continue. Notice that generally when reading a segment $B0^{n_{2n}}x(2n)A$ the counters $\mathcal{C}_1$ and $\mathcal{C}_2$ will successively decrease when reading the first $(n_{2n} - 1)$ letters 0 and then will remain unchanged when reading the last letter 0, and the counters $\mathcal{C}_3$ and $\mathcal{C}_4$ will successively increase, when reading the (n_{2n}) letters 0. Again the end of the decreasing mode of $\mathcal{C}_1$ coincide with the beginning of the decreasing mode of $\mathcal{C}_2$, and this change may occur in a *non-deterministic way*. But the automaton has kept in its finite control whether $|u_{2n-1}| = 0$ or not and also a state q_{2n-2} of the automaton $\mathcal{A}$. Now, if there is a transition of the automaton $\mathcal{A}$ such that $x(2n-1) : (q_{2n-2}, |u_{2n-1}|) \mapsto_{\mathcal{A}} (q_{2n-1}, |u_{2n-1}| + N_{2n-1})$ for some integer $N_{2n-1} \in \{-1; 0, 1\}$, and the counter $\mathcal{C}_1$ is decreased of 1 for each letter 0 read, when reading k_{2n} first letters 0 of $0^{n_{2n}}$, then the counter $\mathcal{C}_3$ is increased of 1 for each letter 0 read, during the reading of the $k_{2n} + N_{2n-1}$ first letters 0 of $0^{n_{2n}}$, and next the counter $\mathcal{C}_4$ is increased by 1 for each letter 0 read until $\mathcal{B}$ reads $x(2n)$, and this determines v_{2n}. Then after having read the segment $0^{n_{2n}} = u_{2n}v_{2n}$, the values of the counters $\mathcal{C}_3$ and $\mathcal{C}_4$ have respectively increased of $|u_{2n}| = k_{2n} + N_{2n-1}$ and $|v_{2n}| = n_{2n} - |u_{2n}|$. Notice that one cannot ensure that, after the reading of $0^{n_{2n}} = u_{2n}v_{2n}$, the exact values of these counters are $|\mathcal{C}_3| = |u_{2n}| = k_{2n} + N_{2n-1}$ and $|\mathcal{C}_4| = |v_{2n}| = n_{2n} - |u_{2n}|$. Actually this is due to the fact that one cannot ensure that the values of $\mathcal{C}_3$ and $\mathcal{C}_4$ are equal to zero at the beginning of the reading of the segment $B0^{n_{2n}}x(2n)A$ although we will see this is true and important in the particular case of a word of the form $y = h(x)$.

The run will continue in a similar manner during the reading of the next segment $A0^{n_{2n+1}}x(2n+1)B$, but here the role of the counters $\mathcal{C}_1$ and $\mathcal{C}_2$ on one side, and of the counters $\mathcal{C}_3$ and $\mathcal{C}_4$ on the other side, will be interchanged. More precisely the counters $\mathcal{C}_3$ and $\mathcal{C}_4$ will successively decrease when reading the first $(n_{2n+1} - 1)$ letters 0 and then will remain unchanged when reading the last

letter 0, and the counters $\mathcal{C}_1$ and $\mathcal{C}_2$ will successively increase, when reading the (n_{2n+1}) letters 0. The end of the decreasing mode of $\mathcal{C}_3$ coincide with the beginning of the decreasing mode of $\mathcal{C}_4$, and this change may occur in a *non-deterministic way*. But the automaton has kept in its finite control whether $|u_{2n}| = 0$ or not and also a state q_{2n-1} of the automaton $\mathcal{A}$. Now, if there is a transition of the automaton $\mathcal{A}$ such that $x(2n) : (q_{2n-1}, |u_{2n}|) \mapsto_{\mathcal{A}} (q_{2n}, |u_{2n}| + N_{2n})$ for some integer $N_{2n} \in \{-1; 0, 1\}$, and the counter $\mathcal{C}_3$ is decreased of 1 for each letter 0 read, when reading k_{2n+1} first letters 0 of $0^{n_{2n+1}}$, then the counter $\mathcal{C}_1$ is increased of 1 for each letter 0 read, during the reading of the $k_{2n+1} + N_{2n}$ first letters 0 of $0^{n_{2n+1}}$, and next the counter $\mathcal{C}_2$ is increased by 1 for each letter 0 read until $\mathcal{B}$ reads $x(2n+1)$, and this determines v_{2n+1}. Then after having read the segment $0^{n_{2n+1}} = u_{2n+1}v_{2n+1}$, the values of the counters $\mathcal{C}_1$ and $\mathcal{C}_2$ have respectively increased of $|u_{2n+1}| = k_{2n+1} + N_{2n}$ and $|v_{2n+1}| = n_{2n+1} - |u_{2n+1}|$. Notice that again one cannot ensure that, after the reading of $0^{n_{2n+1}} = u_{2n+1}v_{2n+1}$, the exact values of these counters are $|\mathcal{C}_1| = |u_{2n+1}| = k_{2n+1} + N_{2n}$ and $|\mathcal{C}_2| = |v_{2n+1}| = n_{2n+1} - |u_{2n+1}|$. This is due to the fact that one cannot ensure that the values of $\mathcal{C}_1$ and $\mathcal{C}_2$ are equal to zero at the beginning of the reading of the segment $A0^{n_{2n+1}}x(2n+1)B$ although we will see this is true and important in the particular case of a word of the form $y = h(x)$.

The run then continues in the same way if it is possible and in particular if there is no blocking due to the fact that one of the counters of the automaton $\mathcal{B}$ would have a negative value.

Now an ω-word $y \in \mathcal{R} \subseteq \Gamma^\omega$ of the above form will be accepted by the automaton $\mathcal{B}$ if there is such an infinite run for which a final state $q_f \in F$ of the automaton $\mathcal{A}$ has been stored infinitely often in the finite control of $\mathcal{B}$ in the way which has just been described above.

We now consider the particular case of an ω-word of the form $y = h(x)$, for some $x \in \Sigma^\omega$.

Let then $y = h(x) = A0x(1)B0^2x(2)A0^3x(3)B \cdots B0^{2n}x(2n)A0^{2n+1}x(2n+1)B \cdots$

We are going to show that, if y is accepted by the automaton $\mathcal{B}$, then $x \in L(\mathcal{A})$. Let us consider a run of the automaton $\mathcal{B}$ on y as described above and which is an accepting run. We first show by induction on $n \geq 1$, that after having read an initial segment of the form

$$A0x(1)B0^2x(2)A \cdots A0^{2n-1}x(2n-1)B$$

the values of the counters $\mathcal{C}_3$ and $\mathcal{C}_4$ are equal to zero, and the values of the counters $\mathcal{C}_1$ and $\mathcal{C}_2$ satisfy $|\mathcal{C}_1| + |\mathcal{C}_2| = 2n - 1$. And similarly after having read an initial segment of the form $A0x(1)B0^2x(2)A \cdots B0^{2n}x(2n)A$ the values of the counters $\mathcal{C}_1$ and $\mathcal{C}_2$ are equal to zero, and the values of the counters $\mathcal{C}_3$ and $\mathcal{C}_4$ satisfy $|\mathcal{C}_3| + |\mathcal{C}_4| = 2n$.

For $n = 1$, we have seen that after having read the initial segment $A0x(1)B$, the values of the counters $\mathcal{C}_1$ and $\mathcal{C}_2$ will be respectively 0 and $|v_1|$ and here $|v_1| = 1$ and thus $|\mathcal{C}_1| + |\mathcal{C}_2| = 1$. On the other hand the counters $\mathcal{C}_3$ and $\mathcal{C}_4$ have not yet increased so that the value of each of these counters is equal to zero. During the reading of the segment 0^2 of $0^2x(2)A$ the counters $\mathcal{C}_1$ and $\mathcal{C}_2$ successively decrease. But here $\mathcal{C}_1$ cannot decrease (with the above notations, it holds that $k_2 = 0$) so $\mathcal{C}_2$ must decrease of 1 because after the decreasing mode the automaton $\mathcal{B}$ must read a last letter 0 without decreasing the counters $\mathcal{C}_1$ and $\mathcal{C}_2$ and then the letter $x(2) \in \Sigma$. Thus after having read $0^2x(2)A$ the values of $\mathcal{C}_1$ and $\mathcal{C}_2$ are equal to zero. Moreover the counters $\mathcal{C}_3$ and $\mathcal{C}_4$ had their values equal to zero at the beginning of the reading of $0^2x(2)A$ and they successively increase during the reading of 0^2 and they remain unchanged during the reading of $x(2)A$ so that their values satisfy $|\mathcal{C}_3| + |\mathcal{C}_4| = 2$ after the reading of $0^2x(2)A$.

Assume now that for some integer $n > 1$ the claim is proved for all integers $k < n$ and let us prove it for the integer n. By induction hypothesis we know that at the beginning of the reading of the segment $A0^{2n-1}x(2n-1)B$ of y, the values of the counters $\mathcal{C}_1$ and $\mathcal{C}_2$ are equal to zero, and the values of the counters $\mathcal{C}_3$ and $\mathcal{C}_4$ satisfy $|\mathcal{C}_3| + |\mathcal{C}_4| = 2n - 2$. When reading the $(2n-2)$ first letters 0 of $A0^{2n-1}x(2n-1)B$ the counters $\mathcal{C}_3$ and $\mathcal{C}_4$ successively decrease and they must

decrease completely because after there must remain only one letter 0 to be read by $\mathcal{B}$ before the letter $x(2n-1)$. Therefore after the reading of $A0^{2n-1}x(2n-1)B$ the values of the counters $\mathcal{C}_3$ and $\mathcal{C}_4$ are equal to zero. And since the values of the counters $\mathcal{C}_1$ and $\mathcal{C}_2$ are equal to zero before the reading of $0^{2n-1}x(2n-1)B$ and these counters successively increase during the reading of 0^{2n-1}, their values satisfy $|\mathcal{C}_1| + |\mathcal{C}_2| = 2n-1$ after the reading of $A0^{2n-1}x(2n-1)B$. We can reason in a very similar manner for the reading of the next segment $B0^{2n}x(2n)A$, the role of the counters $\mathcal{C}_1$ and $\mathcal{C}_2$ on one side, and of the counters $\mathcal{C}_3$ and $\mathcal{C}_4$ on the other side, being simply interchanged. This ends the proof of the claim by induction on n.

It is now easy to see by induction that for each integer $n \geq 2$, it holds that $k_n = |u_{n-1}|$. Then , since with the above notations we have $|u_{n+1}| = k_{n+1} + N_n = |u_n| + N_n$, and there is a transition of the automaton $\mathcal{A}$ such that $x(n) : (q_{n-1}, |u_n|) \mapsto_{\mathcal{A}} (q_n, |u_n| + N_n)$ for $N_n \in \{-1; 0, 1\}$, it holds that $x(n) : (q_{n-1}, |u_n|) \mapsto_{\mathcal{A}} (q_n, |u_{n+1}|)$. Therefore the sequence $(q_i, |u_i|)_{i\geq 0}$ is an accepting run of the automaton $\mathcal{A}$ on the ω-word x and $x \in L(\mathcal{A})$. Notice that the state q_0 of the sequence $(q_i)_{i\geq 0}$ is also the initial state of $\mathcal{A}$.

Conversely, it is easy to see that if $x \in L(\mathcal{A})$ then there exists an accepting run of the automaton $\mathcal{B}$ on the ω-word $h(x)$ and $h(x) \in L(\mathcal{B})$. □

The above Lemma 4.2 shows that, given a real time 1-counter (with zero-test) Büchi automaton $\mathcal{A}$ accepting ω-words over the alphabet Σ, one can construct a real time 4-blind-counter Büchi automaton $\mathcal{B}$ which can simulate the 1-counter automaton $\mathcal{A}$ on the code $h(x)$ of the word x. On the other hand, we cannot describe precisely the ω-words which are accepted by $\mathcal{B}$ but are not in the set $h(\Sigma^\omega)$. However we can see that all these words have a special shape, as stated by the following lemma.

▶ **Lemma 4.3.** *Let $\mathcal{A}$ be a real time 1-counter Büchi automaton accepting ω-words over the alphabet Σ, and let $\mathcal{B}$ be the real time 4-blind-counter Büchi automaton reading words over the alphabet $\Gamma = \Sigma \cup \{A, B, 0\}$ which is constructed in the proof of Lemma 4.2. Let $y \in L(\mathcal{B}) \setminus h(\Sigma^\omega)$ being of the following form*

$$y = A0^{n_1}x(1)B0^{n_2}x(2)A0^{n_3}x(3)B \cdots B0^{n_{2n}}x(2n)A0^{n_{2n+1}}x(2n+1)B \cdots$$

and let i_0 be the smallest integer i such that $n_i \neq i$. Then it holds that either $i_0 = 1$ or $n_{i_0} < i_0$.

Let $\mathcal{L} \subseteq \Gamma^\omega$ be the ω-language containing the ω-words over Γ which belong to one of the following ω-languages.

- $\mathcal{L}_1$ is the set of ω-words over the alphabet $\Sigma \cup \{A, B, 0\}$ which have not any initial segment in $A \cdot 0 \cdot \Sigma \cdot B$.
- $\mathcal{L}_2$ is the set of ω-words over the alphabet $\Sigma \cup \{A, B, 0\}$ which contain a segment of the form $B \cdot 0^n \cdot a \cdot A \cdot 0^m \cdot b$ or of the form $A \cdot 0^n \cdot a \cdot B \cdot 0^m \cdot b$ for some letters $a, b \in \Sigma$ and some positive integers $m \leq n$.

▶ **Lemma 4.4.** *The ω-language $\mathcal{L}$ is accepted by a (non-deterministic) real-time 1-blind counter Büchi automaton.*

Proof. First, it is easy to see that $\mathcal{L}_1$ is in fact a regular ω-language, and thus it is also accepted by a real-time 1-blind counter Büchi automaton (even without active counter). On the other hand it is also easy to construct a real time 1-blind counter Büchi automaton accepting the ω-language $\mathcal{L}_2$. The class of ω-languages accepted by *non-deterministic* real time 1-blind counter Büchi automata being closed under finite union in an effective way, one can construct a real time 1-blind counter Büchi automaton accepting $\mathcal{L}$. □

▶ **Lemma 4.5.** *Let $\mathcal{A}$ be a real time 1-counter Büchi automaton accepting ω-words over the alphabet Σ. Then one can construct a real time 4-blind counter Büchi automaton $\mathcal{P}_{\mathcal{A}}$ such that*

$$L(\mathcal{P}_{\mathcal{A}}) = h(L(\mathcal{A})) \cup \mathcal{L}.$$

Proof. Let $\mathcal{A}$ be a real time 1-counter Büchi automaton accepting ω-words over Σ. We have seen in the proof of Lemma 4.2 that one can construct a real time 4-blind counter Büchi automaton $\mathcal{B}$ reading words over the alphabet $\Gamma = \Sigma \cup \{A, B, 0\}$, such that $L(\mathcal{A}) = h^{-1}(L(\mathcal{B}))$, i.e. $\forall x \in \Sigma^\omega \quad h(x) \in L(\mathcal{B}) \longleftrightarrow x \in L(\mathcal{A})$. Moreover By Lemma 4.3 it holds that $L(\mathcal{B}) \setminus h(\Sigma^\omega) \subseteq \mathcal{L}$. and thus

$$h(L(\mathcal{A})) \cup \mathcal{L} = L(\mathcal{B}) \cup \mathcal{L}$$

But By Lemma 4.4 the ω-language $\mathcal{L}$ is accepted by a (non-deterministic) real-time 1-blind counter Büchi automaton, hence also by a real-time 4-blind counter Büchi automaton. The class of ω-languages accepted by a (non-deterministic) real-time 4-blind counter Büchi automata is closed under finite union in an effective way, and thus one can construct a real time 4-blind counter Büchi automaton $\mathcal{P}_{\mathcal{A}}$ such that $\quad L(\mathcal{P}_{\mathcal{A}}) = h(L(\mathcal{A})) \cup \mathcal{L}$. □

We are now going to prove that if $L(\mathcal{A}) \subseteq \Sigma^\omega$ is accepted by a real time 1-counter automaton $\mathcal{A}$ with a Büchi acceptance condition then $L(\mathcal{P}_{\mathcal{A}}) = h(L(\mathcal{A})) \cup \mathcal{L}$ will have the same Wadge degree as the ω-language $L(\mathcal{A})$, except for some very simple cases.

We first notice that $h(\Sigma^\omega)$ is a closed subset of Γ^ω. Indeed it is the image of the compact set Σ^ω by the continuous function h, and thus it is a compact hence also closed subset of $\Gamma^\omega = (\Sigma \cup \{A, B, 0\})^\omega$. Thus its complement $h(\Sigma^\omega)^- = (\Sigma \cup \{A, B, 0\})^\omega - h(\Sigma^\omega)$ is an open subset of Γ^ω. Moreover the set $\mathcal{L}$ is an open subset of Γ^ω, as it can be easily seen from its definition and one can easily define, from the definition of the ω-language $\mathcal{L}$, a finitary language $V \subseteq \Gamma^\star$ such that $\mathcal{L} = V \cdot \Gamma^\omega$. We shall also denote $\mathcal{L}' = h(\Sigma^\omega)^- \setminus \mathcal{L}$ so that Γ^ω is the dijoint union $\Gamma^\omega = h(\Sigma^\omega) \cup \mathcal{L} \cup \mathcal{L}'$. Notice that $\mathcal{L}'$ is the difference of the two open sets $h(\Sigma^\omega)^-$ and $\mathcal{L}$.

We now wish to return to the proof of the above Theorem 4.1 stating that the Wadge hierarchy of the class $\mathbf{r}\text{-}\mathbf{BCL}(4)_\omega$ is equal to the Wadge hierarchy of the class $\mathbf{r}\text{-}\mathbf{CL}(1)_\omega$.

To prove this result we firstly consider non self dual Borel sets. We recall the definition of Wadge degrees introduced by Duparc in [6] and which is a slight modification of the previous one.

▶ **Definition 4.6.**

(a) $d_w(\emptyset) = d_w(\emptyset^-) = 1$

(b) $d_w(L) = sup\{d_w(L') + 1 \mid L' \text{ non self dual and } L' <_W L\}$
(for either L self dual or not, $L >_W \emptyset$).

Wadge and Duparc used the operation of sum of sets of infinite words which has as counterpart the ordinal addition over Wadge degrees.

▶ **Definition 4.7** (Wadge, see [46, 6]). *Assume that $X \subseteq Y$ are two finite alphabets, $Y - X$ containing at least two elements, and that $\{X_+, X_-\}$ is a partition of $Y - X$ in two non empty sets. Let $L \subseteq X^\omega$ and $L' \subseteq Y^\omega$, then*

$$L' + L =_{df} L \cup \{u \cdot a \cdot \beta \mid u \in X^\star,\ (a \in X_+ \text{ and } \beta \in L') \text{ or } (a \in X_- \text{ and } \beta \in L'^-)\}$$

This operation is closely related to the *ordinal sum* as it is stated in the following:

▶ **Theorem 4.8** (Wadge, see [46, 6]). *Let $X \subseteq Y$, $Y - X$ containing at least two elements, $L \subseteq X^\omega$ and $L' \subseteq Y^\omega$ be non self dual Borel sets. Then $(L + L')$ is a non self dual Borel set and $d_w(L' + L) = d_w(L') + d_w(L)$.*

A player in charge of a set $L' + L$ in a Wadge game is like a player in charge of the set L but who can, at any step of the play, erase his previous play and choose to be this time in charge of L' or of L'^{-}. Notice that he can do this only one time during a play.

The following lemma was proved in [15]. Notice that below the emptyset is considered as an ω-language over an alphabet Δ such that $\Delta - \Sigma$ contains at least two elements.

▶ **Lemma 4.9.** *Let $L \subseteq \Sigma^\omega$ be a non self dual Borel set such that $d_w(L) \geq \omega$. Then it holds that $L \equiv_W \emptyset + L$.*

We can now prove the following lemma.

▶ **Lemma 4.10.** *Let $L \subseteq \Sigma^\omega$ be a non self dual Borel set acccepted by a real time 1-counter Büchi automaton $\mathcal{A}$. Then there is an ω-language L' accepted by a real time 4-blind counter Büchi automaton such that $L \equiv_W L'$.*

Proof. Recall first that there are regular ω-languages of every finite Wadge degree, [43, 34]. These regular ω-languages are Boolean combinations of open sets, and they obviously belong to the class $\mathbf{r}\text{-}\mathbf{BCL}(4)_\omega$ since every regular ω-language belongs to this class.

So we have only to consider the case of non self dual Borel sets of Wadge degrees greater than or equal to ω.

Let then $L = L(\mathcal{A}) \subseteq \Sigma^\omega$ be a non self dual Borel set, acccepted by a real time 1-counter Büchi automaton $\mathcal{A}$, such that $d_w(L) \geq \omega$. By Lemma 4.5, $L(\mathcal{P}_\mathcal{A}) = h(L(\mathcal{A})) \cup \mathcal{L}$ is accepted by a a real time 4-blind counter Büchi automaton $\mathcal{P}_\mathcal{A}$, where the mapping $h : \Sigma^\omega \to (\Sigma \cup \{A, B, 0\})^\omega$ is defined, for $x \in \Sigma^\omega$, by:

$$h(x) = A0x(1)B0^2x(2)A0^3x(3)B \cdots B0^{2n}x(2n)A0^{2n+1}x(2n+1)B \cdots$$

We set $L' = L(\mathcal{P}_\mathcal{A})$ and we now prove that $L' \equiv_W L$.

Firstly, it is easy to see that $L \leq_W L'$. In order to prove this we can consider the Wadge game $W(L, L')$. It is easy to see that Player 2 has a winning strategy in this game which consists in essentially copying the play of Player 1, except that Player 2 actually writes a beginning of the code given by h of what has been written by Player 1. This is achieved in such a way that Player 2 has written the initial word $A0x(1)B0^2x(2)A \cdots B0^{2n}x(2n)$ while Player 1 has written the initial word $x(1)x(2) \cdots x(2n)$ (respectively, $A0x(1)B0^2x(2)A \cdots B0^{2n}x(2n)A0^{2n+1}x(2n+1)$ while Player 1 has written the initial word $x(1)x(2) \cdots x(2n+1)$) Notice that one can admit that a player writes a finite word at each step of the play instead of a single letter. This does not change the winner of a Wadge game. At the end of a play if Player 1 has written the ω-word x then Player 2 has written $h(x)$ and thus $x \in L(\mathcal{A}) \iff h(x) \in L'$ and Player 2 wins the play.

To prove that $L' \leq_W L$, it suffices to prove that $L' \leq_W \emptyset + (\emptyset + L)$ because Lemma 4.9 states that $\emptyset + L \equiv_W L$, and thus also $\emptyset + (\emptyset + L) \equiv_W L$. Consider the Wadge game $W(L', \emptyset + (\emptyset + L))$. Player 2 has a winning strategy in this game which we now describe.

As long as Player 1 remains in the closed set $h(\Sigma^\omega)$ (this means that the word written by Player 1 is a prefix of some infinite word in $h(\Sigma^\omega)$) Player 2 essentially copies the play of player 1 except that Player 2 skips when player 1 writes a letter not in Σ. He continues forever with this strategy if the word written by player 1 is always a prefix of some ω-word of $h(\Sigma^\omega)$. Then after ω steps Player 1 has written an ω-word $h(x)$ for some $x \in \Sigma^\omega$, and Player 2 has written x. So in that case $h(x) \in L'$ iff $x \in L(\mathcal{A})$ iff $x \in \emptyset + (\emptyset + L)$.

But if at some step of the play, Player 1 "goes out of" the closed set $h(\Sigma^\omega)$ because the word he has now written is not a prefix of any ω-word of $h(\Sigma^\omega)$, then Player 1 "enters" in the open set $h(\Sigma^\omega)^- = \mathcal{L} \cup \mathcal{L}'$ and will stay in this set. Two cases may now appear.

First case. When Player 1 "enters" in the open set $h(\Sigma^\omega)^- = \mathcal{L} \cup \mathcal{L}'$, he actually enters in the open set $\mathcal{L} = V \cdot \Gamma^\omega$ (this means that Player 1 has written an initial segment in V). Then the final word written by Player 1 will surely be inside L'. Player 2 can now write a letter of $\Delta - \Sigma$ in such a way that he is now like a player in charge of the wholeset and he can now writes an ω-word u so that his final ω-word will be inside $\emptyset + L$, and also inside $\emptyset + (\emptyset + L)$. Thus Player 2 wins this play too.

Second case. When Player 1 "enters" in the open set $h(\Sigma^\omega)^- = \mathcal{L} \cup \mathcal{L}'$, he does not enter in the open set $\mathcal{L} = V \cdot \Gamma^\omega$. Then Player 2, being first like a player in charge of the set $(\emptyset + L)$, can write a letter of $\Delta - \Sigma$ in such a way that he is now like a player in charge of the emptyset and he can now continue, writing an ω-word u. If Player 1 never enters in the open set $\mathcal{L} = V \cdot \Gamma^\omega$ then the final word written by Player 1 will be in $\mathcal{L}'$ and thus surely outside L', and the final word written by Player 2 will be outside the emptyset. So in that case Player 2 wins this play too. If at some step of the play Player 1 enters in the open set $\mathcal{L} = V \cdot \Gamma^\omega$ then his final ω-word will be surely in L'. In that case Player 1, in charge of the set $\emptyset + (\emptyset + L)$, can again write an extra letter and choose to be in charge of the wholeset and he can now write an ω-word v so that his final ω-word will be inside $\emptyset + (\emptyset + L)$. Thus Player 2 wins this play too.

Finally we have proved that $L \leq_W L' \leq_W L$ thus it holds that $L' \equiv_W L$. This ends the proof. □

End of Proof of Theorem 4.1.

Let $L \subseteq \Sigma^\omega$ be a Borel set accepted by a real time 1-counter Büchi automaton $\mathcal{A}$. If the Wadge degree of L is finite, it is well known that it is Wadge equivalent to a regular ω-language, hence also to an ω-language in the class $\mathbf{r\text{-}BCL}(4)_\omega$. If L is non self dual and its Wadge degree is greater than or equal to ω, then we know from Lemma 4.10 that there is an ω-language L' accepted by a a real time 4-blind counter Büchi automaton such that $L \equiv_W L'$.

It remains to consider the case of self dual Borel sets. The alphabet Σ being finite, a self dual Borel set L is always Wadge equivalent to a Borel set in the form $\Sigma_1 \cdot L_1 \cup \Sigma_2 \cdot L_2$, where (Σ_1, Σ_2) form a partition of Σ, and $L_1, L_2 \subseteq \Sigma^\omega$ are non self dual Borel sets such that $L_1 \equiv_W L_2^-$. Moreover L_1 and L_2 can be taken in the form $L_{(u_1)} = u_1 \cdot \Sigma^\omega \cap L$ and $L_{(u_2)} = u_2 \cdot \Sigma^\omega \cap L$ for some $u_1, u_2 \in \Sigma^\star$, see [7]. So if $L \subseteq \Sigma^\omega$ is a self dual Borel set accepted by a real time 1-counter Büchi automaton then $L \equiv_W \Sigma_1 \cdot L_1 \cup \Sigma_2 \cdot L_2$, where (Σ_1, Σ_2) form a partition of Σ, and $L_1, L_2 \subseteq \Sigma^\omega$ are non self dual Borel sets accepted by real time 1-counter Büchi automata. We have already proved that there is an ω-language L_1' in the class $\mathbf{r\text{-}BCL}(4)_\omega$ such that $L_1' \equiv_W L_1$ and an ω-language L_2' in the class $\mathbf{r\text{-}BCL}(4)_\omega$ such that $L_2'^- \equiv_W L_2$. Thus $L \equiv_W \Sigma_1 \cdot L_1 \cup \Sigma_2 \cdot L_2 \equiv_W \Sigma_1 \cdot L_1' \cup \Sigma_2 \cdot L_2'$ and $\Sigma_1 \cdot L_1' \cup \Sigma_2 \cdot L_2'$ is an ω-language in the class $\mathbf{r\text{-}BCL}(4)_\omega$.

The reverse direction is immediate: if $L \subseteq \Sigma^\omega$ is a Borel set accepted by a 4-blind counter Büchi automaton $\mathcal{A}$, then it is also accepted by a Büchi Turing machine and thus by [15, Theorem 25] there exists a real time 1-counter Büchi automaton $\mathcal{B}$ such that $L(\mathcal{A}) \equiv_W L(\mathcal{B})$. □

We have only considered in the above Theorem 4.1 the Wadge hierarchy of **Borel sets**. But we know that there exist also some non-Borel ω-languages accepted by real time 1-counter Büchi automata, and even some $\mathbf{\Sigma}^1_1$-complete ones, [14].

By Lemma 4.7 of [18] the conclusion of the above Lemma 4.9 is also true if L is assumed to be an analytic but non-Borel set.

▶ **Lemma 4.11** ([18]). *Let $L \subseteq \Sigma^\omega$ be an analytic but non-Borel set. Then $L \equiv_W \emptyset + L$.*

Next the proof of the above Lemma 4.10 can be adapted to the case of an analytic but non-Borel set, and we can state the following result.

▶ **Lemma 4.12.** *Let $L \subseteq \Sigma^\omega$ be an analytic but non-Borel set acccepted by a real time 1-counter Büchi automaton $\mathcal{A}$. Then there is an ω-language L' accepted by a real time 4-blind counter Büchi*

automaton such that $L \equiv_W L'$.

Proof. It is very similar to the proof of the above Lemma 4.10, using Lemma 4.11 instead of the above Lemma 4.9. □

It was proved in [15] that the Wadge hierarchy of the class $\mathbf{r}\text{-}\mathbf{CL}(1)_\omega$ is equal to the Wadge hierarchy of the class Σ^1_1 of effective analytic sets. Using Lemma 4.11 instead of the above Lemma 4.9, the proofs of [15] can also be adapted to the case of a non-Borel set to show that for every effective analytic but non-Borel set $L \subseteq \Sigma^\omega$, where Σ is a finite alphabet, there exists an ω-language L' in $\mathbf{r}\text{-}\mathbf{CL}(1)_\omega$ such that $L' \equiv_W L$.

We can finally summarize our results by the following theorem.

▶ **Theorem 4.13.** *The Wadge hierarchy of the class $\mathbf{r}\text{-}\mathbf{BCL}(4)_\omega$ is the Wadge hierarchy of the class $\mathbf{r}\text{-}\mathbf{CL}(1)_\omega$ and also of the class Σ^1_1 of effective analytic sets. Moreover for every effective analytic set $L \subseteq \Sigma^\omega$ there exists an ω-language L' in the class $\mathbf{r}\text{-}\mathbf{BCL}(4)_\omega$ such that $L \equiv_W L'$.*

▶ **Remark 4.14.** *Since the class $\mathbf{r}\text{-}\mathbf{PN}_\omega = \bigcup_{k\geq 1} \mathbf{r}\text{-}\mathbf{BCL}(k)_\omega$ of ω-languages of real-time non-deterministic Petri nets satisfy the following inclusions $\mathbf{r}\text{-}\mathbf{BCL}(4)_\omega \subseteq \mathbf{r}\text{-}\mathbf{PN}_\omega \subseteq \Sigma^1_1$, it holds that the Wadge hierarchy of the class $\mathbf{r}\text{-}\mathbf{PN}_\omega$ is also equal to the Wadge hierarchy of the class Σ^1_1 of effective analytic sets. Moreover the same result holds for the class $\mathbf{PN}_\omega$ of ω-languages of (possibly non-real-time) non-deterministic Petri nets.*

On the other hand, for each non-null countable ordinal α the $\mathbf{\Sigma}^0_\alpha$-complete sets (respectively, the $\mathbf{\Pi}^0_\alpha$-complete sets) form a single Wadge degree. Moreover for each non-null recursive ordinal $\alpha < \omega_1^{\mathrm{CK}}$ there are some $\mathbf{\Sigma}^0_\alpha$-complete sets and some $\mathbf{\Pi}^0_\alpha$-complete sets in the effective class Δ^1_1. Thus we can infer the following result from the above Theorem 4.13 and from the results of [25].

▶ **Corollary 4.15.** *For each non-null recursive ordinal $\alpha < \omega_1^{\mathrm{CK}}$ there exist some $\mathbf{\Sigma}^0_\alpha$-complete and some $\mathbf{\Pi}^0_\alpha$-complete ω-languages in the class $\mathbf{r}\text{-}\mathbf{BCL}(4)_\omega$. And the supremum of the set of Borel ranks of ω-languages in the class $\mathbf{r}\text{-}\mathbf{BCL}(4)_\omega$ is the ordinal γ^1_2, which is precisely defined in [25].*

Since it was proved in [14] that there is a $\mathbf{\Sigma}^1_1$-complete set accepted by a real-time 1-counter Büchi automaton, we also get the following result.

▶ **Corollary 4.16.** *There exists some $\mathbf{\Sigma}^1_1$-complete set in the class $\mathbf{r}\text{-}\mathbf{BCL}(4)_\omega$.*

Notice that if we assume the axiom of $\mathbf{\Sigma}^1_1$-determinacy, then any set which is analytic but not Borel is $\mathbf{\Sigma}^1_1$-complete, see [24], and thus there is only one more Wadge degree (beyond Borel sets) containing $\mathbf{\Sigma}^1_1$-complete sets. On the other hand, if the axiom of (effective) Σ^1_1-determinacy does not hold, then there exist some effective analytic sets which are neither Borel nor $\mathbf{\Sigma}^1_1$-complete. Recall that ZFC is the commonly accepted axiomatic framework for Set Theory in which all usual mathematics can be developed.

▶ **Corollary 4.17.** *It is consistent with ZFC that there exist some ω-languages of Petri nets in the class $\mathbf{r}\text{-}\mathbf{BCL}(4)_\omega$ which are neither Borel nor $\mathbf{\Sigma}^1_1$-complete.*

We also prove that it is highly undecidable to determine the topological complexity of a Petri net ω-language. As usual, since there is a finite description of a real time 1-counter Büchi automaton or of a 4-blind-counter Büchi automaton, we can define a Gödel numbering of all 1-counter Büchi automata or of all 4-blind-counter Büchi automata and then speak about the 1-counter Büchi automaton (or 4-blind-counter Büchi automaton) of index z. Recall first the following result, proved in [17], where we denote $\mathcal{A}_z$ the real time 1-counter Büchi automaton of index z reading words over a fixed finite alphabet Σ having at least two letters. We refer the reader to a textbook like [30, 31] for more background about the analytical hierarchy of subsets of the set $\mathbb{N}$ of natural numbers.

▶ **Theorem 4.18.** *Let α be a countable ordinal. Then*
1. $\{z \in \mathbb{N} \mid L(\mathcal{A}_z)$ *is in the Borel class* $\mathbf{\Sigma}^0_\alpha\}$ *is* Π^1_2*-hard.*
2. $\{z \in \mathbb{N} \mid L(\mathcal{A}_z)$ *is in the Borel class* $\mathbf{\Pi}^0_\alpha\}$ *is* Π^1_2*-hard.*
3. $\{z \in \mathbb{N} \mid L(\mathcal{A}_z)$ *is a Borel set* $\}$ *is* Π^1_2*-hard.*

Using the previous constructions we can now easily show the following result, where $\mathcal{P}_z$ is the real time 4-blind-counter Büchi automaton of indez z.

▶ **Theorem 4.19.** *Let $\alpha \geq 2$ be a countable ordinal. Then*
1. $\{z \in \mathbb{N} \mid L(\mathcal{P}_z)$ *is in the Borel class* $\mathbf{\Sigma}^0_\alpha\}$ *is* Π^1_2*-hard.*
2. $\{z \in \mathbb{N} \mid L(\mathcal{P}_z)$ *is in the Borel class* $\mathbf{\Pi}^0_\alpha\}$ *is* Π^1_2*-hard.*
3. $\{z \in \mathbb{N} \mid L(\mathcal{P}_z)$ *is a Borel set* $\}$ *is* Π^1_2*-hard.*

Proof. It follows from the fact that one can easily get an injective recursive function $g : \mathbb{N} \to \mathbb{N}$ such that $\mathcal{P}_{\mathcal{A}_z} = h(L(\mathcal{A}_z)) \cup \mathcal{L} = L(\mathcal{P}_{g(z)})$ and from the following equivalences which hold for each countable ordinal $\alpha \geq 2$:

1. $L(\mathcal{A}_z)$ is in the Borel class $\mathbf{\Sigma}^0_\alpha$ (resp., $\mathbf{\Pi}^0_\alpha$) $\Longleftrightarrow L(\mathcal{P}_{g(z)})$ is in the Borel class $\mathbf{\Sigma}^0_\alpha$ (resp., $\mathbf{\Pi}^0_\alpha$).
2. $L(\mathcal{A}_z)$ is a Borel set $\Longleftrightarrow L(\mathcal{P}_{g(z)})$ is a Borel set. □

5 Concluding remarks

We have proved that the Wadge hierarchy of Petri nets ω-languages, and even of ω-languages in the class $\mathbf{r}\text{-}\mathbf{BCL}(4)_\omega$, is equal to the Wadge hierarchy of effective analytic sets, and that it is highly undecidable to determine the topological complexity of a Petri net ω-language. In some sense our results show that, in contrast with the finite behavior, the infinite behavior of Petri nets is closer to the infinite behavior of Turing machines than to that of finite automata.

It remains open for further study to determine the Borel and Wadge hierarchies of ω-languages accepted by automata with less than four blind counters. Since this paper has been written, Michał Skrzypczak has informed us that he has recently proved in this direction that there exists a $\mathbf{\Sigma}^1_1$-complete ω-language accepted by a 1-blind-counter automaton, [39]. In particular, it then remains open to determine whether there exist some ω-languages accepted by 1-blind-counter automata which are Borel of rank greater than 3, or which could be neither Borel nor $\mathbf{\Sigma}^1_1$-complete.

Finally we mention that, in an extended version of this paper, we prove that the determinacy of Wadge games between two players in charge of ω-languages of Petri nets is equivalent to the (effective) analytic determinacy, which is known to be a large cardinal assumption, and thus is not provable in the axiomatic system ZFC. Based on the constructions used in the proofs of the above results, we also show that the equivalence and the inclusion problems for ω-languages of Petri nets are Π^1_2-complete, hence highly undecidable.

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ANNEXE 1

We give in this annexe some proofs which could not be included in the paper, due to lack of space.

Proof of Lemma 4.3.

Assume first that $y \in L(\mathcal{B}) \setminus h(\Sigma^\omega)$ is of the following form

$$y = A0^{n_1}x(1)B0^{n_2}x(2)A \cdots B0^{n_{2n}}x(2n)A0^{n_{2n+1}}x(2n+1)B \cdots$$

and that the smallest integer i such that $n_i \neq i$ is an even integer $i_0 > 1$. Consider an infinite accepting run of $\mathcal{B}$ on y. It follows from the proof of the above Lemma 4.2 that after the reading of the initial segment

$$A0^{n_1}x(1)B0^{n_2}x(2)A \cdots A0^{i_0-1}x(i_0-1)B$$

the values of the counters $\mathcal{C}_3$ and $\mathcal{C}_4$ are equal to zero, and the values of the counters $\mathcal{C}_1$ and $\mathcal{C}_2$ satisfy $|\mathcal{C}_1| + |\mathcal{C}_2| = i_0 - 1$. Thus since the two counters must successively decrease during the next $n_{i_0} - 1$ letters 0, it holds that $n_{i_0} - 1 \leq i_0 - 1$ because otherwise either $\mathcal{C}_1$ or $\mathcal{C}_2$ would block. Therefore $n_{i_0} < i_0$ since $n_{i_0} \neq i_0$ by definition of i_0. The reasoning is very similar in the case of an odd integer i_0, the role of the counters $\mathcal{C}_1$ and $\mathcal{C}_2$ on one side, and of the counters $\mathcal{C}_3$ and $\mathcal{C}_4$ on the other side, being simply interchanged. $\square$

Proof of Corollary 4.17.

Recall that ZFC is the commonly accepted axiomatic framework for Set Theory in which all usual mathematics can be developed. The determinacy of Gale-Stewart games $G(A)$, where A is an (effective) analytic set, denoted **Det**(Σ^1_1), is not provable in ZFC; Martin and Harrington have proved that it is a large cardinal assumption equivalent to the existence of a particular real, called the real $0^\sharp$, see [23, page 637]. It is also known that the determinacy of (effective) analytic Gale-Stewart games is equivalent to the determinacy of (effective) analytic Wadge games, denoted **W-Det**(Σ^1_1), see [28].

It is known that, if ZFC is consistent, then there is a model of ZFC in which the determinacy of (effective) analytic Gale-Stewart games, and thus also the determinacy of (effective) analytic Wadge games, do not hold. It follows from [22, Theorem 4.3] that in such a model of ZFC there exists an effective analytic set which is neither Borel nor $\mathbf{\Sigma}^1_1$-complete. The result now follows from Theorem 4.13.

ANNEXE 2

Some additional results are given here.

We proved in [18] that the determinacy of Wadge games between two players in charge of ω-languages accepted by real time 1-counter Büchi automata, denoted **W-Det**(**r-CL**$(1)_\omega$), is equivalent to the (effective) analytic Wadge determinacy.

We can now state the following result, proved within the axiomatic system ZFC.

▶ **Theorem 5.1.** *The determinacy of Wadge games between two players in charge of ω-languages in the class* **r-BCL**$(4)_\omega$ *is equivalent to the effective analytic (Wadge) determinacy, and thus is not provable in the axiomatic system ZFC.*

Proof. It was proved in [18] that the following equivalence holds: **W-Det**(**r-CL**$(1)_\omega$) $\Longleftrightarrow$ **W-Det**(Σ^1_1). The implication **W-Det**(Σ^1_1)$\Longrightarrow$ **W-Det**(**r-BCL**$(4)_\omega$) is obvious since the class $\mathbf{BCL}(4)_\omega$ is included into the class Σ^1_1. To prove the reverse implication, we assume that **W-Det**(**r-BCL**$(4)_\omega$) holds and we show that every Wadge game $W(L(\mathcal{A}), L(\mathcal{B}))$ between two players in charge of ω-languages of the class **r-CL**$(1)_\omega$ is determined (we assume without loss of generality that the two real time 1-counter Büchi automata $\mathcal{A}$ and $\mathcal{B}$ read words over the same alphabet Σ).

It is sufficient to consider the cases where at least one of two ω-languages $L(\mathcal{A})$ and $L(\mathcal{B})$ is non-Borel, since the Borel Wadge determinacy is provable in ZFC. On the other hand, we have seen how we can construct some real time 4-blind-counter Büchi automata $\mathcal{P}_\mathcal{A}$ and $\mathcal{P}_\mathcal{B}$ such that $L(\mathcal{P}_\mathcal{A}) = h(L(\mathcal{A})) \cup \mathcal{L}$ and $L(\mathcal{P}_\mathcal{B}) = h(L(\mathcal{B})) \cup \mathcal{L}$.

We can firstly consider the case where $L(\mathcal{A})$ is Borel of Wadge degree smaller than ω, and $L(\mathcal{B})$ is non-Borel. In that case $L(\mathcal{A})$ is in particular a $\mathbf{\Pi}^0_2$-set. Recall now that we can infer from Hurewicz's Theorem, see [24, page 160], that an analytic subset of Σ^ω is either $\mathbf{\Pi}^0_2$-hard or a $\mathbf{\Sigma}^0_2$-set. Thus $L(\mathcal{B})$ is $\mathbf{\Pi}^0_2$-hard and Player 2 has a winning strategy in the game $W(L(\mathcal{A}), L(\mathcal{B}))$.

Secondly we consider the case where $L(\mathcal{A})$ and $L(\mathcal{B})$ are either non-Borel or Borel of Wadge degree greater than ω. By hypothesis we know that the Wadge game $W(L(\mathcal{P}_\mathcal{A}), L(\mathcal{P}_\mathcal{B}))$ is determined, and that one of the players has a winning strategy. Using the above constructions and reasonings we used in the proofs of Lemmas 4.5 and 4.10, we can easily show that the same player has a winning strategy in the Wadge game $W(L(\mathcal{A}), L(\mathcal{B}))$.

We now consider the two following cases:

First case. Player 2 has a w.s. in the game $W(L(\mathcal{P}_\mathcal{A}), L(\mathcal{P}_\mathcal{B}))$. If $L(\mathcal{B})$ is Borel then $L(\mathcal{P}_\mathcal{B})$ is easily seen to be Borel and then $L(\mathcal{P}_\mathcal{A})$ is also Borel because $L(\mathcal{P}_\mathcal{A}) \leq_W L(\mathcal{P}_\mathcal{B})$. Thus $L(\mathcal{A})$ is also Borel and the game $W(L(\mathcal{A}), L(\mathcal{B}))$ is determined. Assume now that $L(\mathcal{B})$ is not Borel. Consider the Wadge game $W(L(\mathcal{A}), \emptyset + (\emptyset + L(\mathcal{B})))$. We claim that Player 2 has a w.s. in that game which is easily deduced from a w.s. of Player 2 in the Wadge game $W(L(\mathcal{P}_\mathcal{A}), L(\mathcal{P}_\mathcal{B}))$ $= W(h(L(\mathcal{A})) \cup \mathcal{L}, h(L(\mathcal{B})) \cup \mathcal{L})$. Consider a play in this latter game where Player 1 remains in the closed set $h(\Sigma^\omega)$: she writes a beginning of a word in the form

$$A0x(1)B0^2x(2)A \cdots B0^{2n}x(2n)A \cdots$$

Then player 2 writes a beginning of a word in the form

$$A0x'(1)B0^2x'(2)A \cdots B0^{2p}x'(2p)A \cdots$$

where $p \leq n$. Then the strategy for Player 2 in $W(L(\mathcal{A}), \emptyset + (\emptyset + L(\mathcal{B})))$ consists to write $x'(1).x'(2) \ldots x'(2p)$. when Player 1 writes $x(1).x(2) \ldots x(2n)$. (Notice that Player 2 is allowed to skip, provided he really writes an ω-word in ω steps). If the strategy for Player 2 in $W(L(\mathcal{P}_\mathcal{A}), L(\mathcal{P}_\mathcal{B}))$ was at some step to go out of the closed set $h(\Sigma^\omega)$ then this means that the word he has now written

is not a prefix of any ω-word of $h(\Sigma^\omega)$, and Player 2 "enters" in the open set $h(\Sigma^\omega)^- = \mathcal{L} \cup \mathcal{L}'$ and will stay in this set. Two subcases may now appear.

Subcase A. When Player 2 in the game $W(L(\mathcal{P}_\mathcal{A}), L(\mathcal{P}_\mathcal{B}))$ "enters" in the open set $h(\Sigma^\omega)^- = \mathcal{L} \cup \mathcal{L}'$, he actually enters in the open set $\mathcal{L}$. Then the final word written by Player 2 will surely be inside $L(\mathcal{P}_\mathcal{B})$. Player 2 in the Wadge game $W(L(\mathcal{A}), \emptyset + (\emptyset + L(\mathcal{B})))$ can now write a letter of $\Delta - \Sigma$ in such a way that he is now like a player in charge of the wholeset and he can now writes an ω-word u so that his final ω-word will be inside $\emptyset + (\emptyset + L(\mathcal{B}))$. Thus Player 2 wins this play too.

Subcase B. When Player 2 in the game $W(L(\mathcal{P}_\mathcal{A}), L(\mathcal{P}_\mathcal{B}))$ "enters" in the open set $h(\Sigma^\omega)^- = \mathcal{L} \cup \mathcal{L}'$, he does not enter in the open set $\mathcal{L}$. Then Player 2, in the Wadge game $W(L(\mathcal{A}), \emptyset + (\emptyset + L(\mathcal{B})))$, being first like a player in charge of the set $(\emptyset + L(\mathcal{B}))$, can write a letter of $\Delta - \Sigma$ in such a way that he is now like a player in charge of the emptyset and he can now continue, writing an ω-word u. If Player 2 in the game $W(L(\mathcal{P}_\mathcal{A}), L(\mathcal{P}_\mathcal{B}))$ never enters in the open set $\mathcal{L}$ then the final word written by Player 2 will be in $\mathcal{L}'$ and thus surely outside $L(\mathcal{P}_\mathcal{B})$, and the final word written by Player 2 will be outside the emptyset. So in that case Player 2 wins this play too in the Wadge game $W(L(\mathcal{A}), \emptyset + (\emptyset + L(\mathcal{B})))$. If at some step of the play, in the game $W(L(\mathcal{P}_\mathcal{A}), L(\mathcal{P}_\mathcal{B}))$, Player 2 enters in the open set $\mathcal{L}$ then his final ω-word will be surely in $L(\mathcal{P}_\mathcal{B}))$. In that case Player 2, in charge of the set $\emptyset + (\emptyset + L(\mathcal{B})))$ in the Wadge game $W(L(\mathcal{A}), \emptyset + (\emptyset + L(\mathcal{B})))$, can again write an extra letter and choose to be in charge of the wholeset and he can now write an ω-word v so that his final ω-word will be inside $\emptyset + (\emptyset + L(\mathcal{B})))$. Thus Player 2 wins this play too.

So we have proved that Player 2 has a w.s. in the Wadge game $W(L(\mathcal{A}), \emptyset + (\emptyset + L(\mathcal{B})))$ or equivalently that $L(\mathcal{A}) \leq_W \emptyset + (\emptyset + L(\mathcal{B}))$. But by Lemma 5.3 we know that $L(\mathcal{B}) \equiv_W \emptyset + (\emptyset + L(\mathcal{B}))$ and thus $L(\mathcal{A}) \leq_W L(\mathcal{B})$ which means that Player 2 has a w.s. in the Wadge game $W(L(\mathcal{A}), L(\mathcal{B}))$.

Second case. Player 1 has a w.s. in the game $W(L(\mathcal{P}_\mathcal{A}), L(\mathcal{P}_\mathcal{B}))$. Notice that this implies that $L(\mathcal{P}_\mathcal{B}) \leq_W L(\mathcal{P}_\mathcal{A})^-$. Thus if $L(\mathcal{A})$ is Borel then $L(\mathcal{P}_\mathcal{A})$ is Borel, $L(\mathcal{P}_\mathcal{A})^-$ is also Borel, and $L(\mathcal{P}_\mathcal{B})$ is Borel as the inverse image of a Borel set by a continuous function, and $L(\mathcal{B})$ is also Borel, so the Wadge game $W(L(\mathcal{A}), L(\mathcal{B}))$ is determined. We now assume that $L(\mathcal{A})$ is not Borel and we consider the Wadge game $W(L(\mathcal{A})), L(\mathcal{B}))$. Player 1 has a w.s. in this game which is easily constructed from a w.s. of the same player in the game $W(L(\mathcal{P}_\mathcal{A}), L(\mathcal{P}_\mathcal{B}))$ as follows. For this consider a play in this latter game where Player 2 does not go out of the closed set $h(\Sigma^\omega)$.

He writes a beginning of a word in the form

$$A0x(1)B0^2x(2)A \cdots B0^n x(n)A \cdots$$

Then player 1 writes a beginning of a word in the form

$$A0x'(1)B0^2x'(2)A \cdots B0^p x'(p)A \cdots$$

where $n \leq p$ (notice that here without loss of generality the notation implies that n and p are even, since the segments $B0^n x(n)A$ and $B0^p x'(p)A$ begin with a letter B but this is not essential in the proof). Then the strategy for Player 1 in $W(L(\mathcal{A})), L(\mathcal{B})$ consists to write $x'(1).x'(2)\ldots x'(p)$. when Player 2 writes $x(1).x(2)\ldots x(n)$. After ω steps, the ω-word written by Player 1 is in $L(\mathcal{A})$ iff the ω-word written by Player 2 is not in the set $L(\mathcal{B})$, and thus Player 1 wins the play.

If the strategy for Player 1 in $W(L(\mathcal{P}_\mathcal{A}), L(\mathcal{P}_\mathcal{B}))$ was at some step to go out of the closed set $h(\Sigma^\omega)$ then this means that she "enters" in the open set $h(\Sigma^\omega)^- = \mathcal{L} \cup \mathcal{L}'$ and will stay in this set. Two subcases may now appear.

Subcase A. When Player 1 in the game $W(L(\mathcal{P}_\mathcal{A}), L(\mathcal{P}_\mathcal{B}))$ "enters" in the open set $h(\Sigma^\omega)^- = \mathcal{L} \cup \mathcal{L}'$, she actually enters in the open set $\mathcal{L}$. Then the final word written by Player 1 will surely be inside $L(\mathcal{P}_\mathcal{A})$. But she wins the play since she follows a winning strategy and this leads to a

contradiction. Indeed if Player 2 decided to also enter in in the open set $\mathcal{L}$ then Player 2 would win the play. Thus this case is actually not possible.

Subcase B. When Player 1 in the game $W(L(\mathcal{P}_{\mathcal{A}}), L(\mathcal{P}_{\mathcal{B}}))$ "enters" in the open set $h(\Sigma^\omega)^- = \mathcal{L} \cup \mathcal{L}'$, she does not enter in the open set $\mathcal{L}$. But Player 2 would be able to do the same and enter in $h(\Sigma^\omega)^- = \mathcal{L} \cup \mathcal{L}'$ but not (for the moment) in the open set $\mathcal{L}$. And if at some step of the play, Player 1 would enter in the open set $\mathcal{L}$ then Player 2 could do the same, and thus Player 2 would win the play. Again this is not possible since Player 1 wins the play since she follows a winning strategy.

Finally both subcases A and B cannot occur and this shows that Player 1 has a w.s. in the Wadge game $L(\mathcal{A}) \leq_W L(\mathcal{B})$. □

We now add the following undecidability results. We first recall the following result, proved in [17].

▶ **Theorem 5.2** ([17]). *The equivalence and the inclusion problems for ω-languages accepted by real time 1-counter Büchi automata are Π^1_2-complete, i.e.:*

1. $\{(z, z') \in \mathbb{N} \mid L(\mathcal{A}_z) = L(\mathcal{A}_{z'})\}$ *is* Π^1_2*-complete*
2. $\{(z, z') \in \mathbb{N} \mid L(\mathcal{A}_z) \subseteq L(\mathcal{A}_{z'})\}$ *is* Π^1_2*-complete*

Using the previous constructions we can now easily show the following result, where $\mathcal{P}_z$ is the real time 4-blind-counter Büchi automaton of indez z.

▶ **Theorem 5.3.** *The equivalence and the inclusion problems for ω-languages of Petri nets, or even for ω-languages in the class* $\mathbf{r}\text{-}\mathbf{BCL}(4)_\omega$*, are Π^1_2-complete.*

1. $\{(z, z') \in \mathbb{N} \mid L(\mathcal{P}_z) = L(\mathcal{P}_{z'})\}$ *is* Π^1_2*-complete*
2. $\{(z, z') \in \mathbb{N} \mid L(\mathcal{P}_z) \subseteq L(\mathcal{P}_{z'})\}$ *is* Π^1_2*-complete*

Proof. Firstly, it is easy to see that each of these decision problems is in the class Π^1_2, since the equivalence and the inclusion problems for ω-languages of Turing machines are already in the class Π^1_2, see [4, 17]. The completeness part follows from the above Theorem 5.2, from the fact that there exists an injective recursive function $g : \mathbb{N} \to \mathbb{N}$ such that $\mathcal{P}_{\mathcal{A}_z} = \mathcal{P}_{g(z)}$, and then from the following equivalences:

1. $L(\mathcal{A}_z) = L(\mathcal{A}_{z'}) \Longleftrightarrow L(\mathcal{P}_{g(z)}) = L(\mathcal{P}_{g(z')})$
2. $L(\mathcal{A}_z) \subseteq L(\mathcal{A}_{z'}) \Longleftrightarrow L(\mathcal{P}_{g(z)}) \subseteq L(\mathcal{P}_{g(z')})$

which clearly imply that the equivalence (respectively, inclusion) problem for ω-languages of real-time 1-counter automata is Turing reducible to the equivalence (respectively, inclusion) problem for ω-languages of real time 4-blind-counter Büchi automata. □