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The Dalang–Morton–Willinger version of the fundamental theorem of asset pricing for the Bid-Ask financial market model.

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Abstract: We provide an equivalent characterisation of absence of arbitrage opportunity (NA^w) for the Bid and Ask financial market model analog to the Dalang–Morton–Willinger theorem formulated for discrete-time financial market models without friction. This result completes and improves the so-called Grigoriev theorem for conic models in the two dimensional case by showing that the set of all terminal liquidation values is closed under (NA^w).

Keywords and phrases: Financial market models, Absence of arbitrage opportunities, Liquidation value, Bid and Ask prices, Transaction costs.

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1. Introduction

For the discrete-time models without friction, it is well known that the (self-financing) portfolio processes are modelled by their liquidation values. An arbitrage opportunity is the terminal value V_T at time T of a portfolio process starting from a zero initial capital such that $P(V_T \geq 0) = 1$ and $V_T \neq 0$. The Dalang–Morton–Willinger (DMW) theorem [4] formulates an equivalent characterisation of absence of arbitrage opportunity (NA^w). Precisely, it states that the (NA^w) condition is equivalent to the existence of a martin-

gale probability measure, i.e. an equivalent probability measure under which the price process is a martingale. Moreover, under the (NA^w) condition, it is shown that the set of all terminal portfolio processes starting from the zero initial capital is closed in probability.

The models with friction were first considered in the pioneering paper [9] and, later, were extensively studied, e.g. in the papers [11], [7], [14], [6], [13]. With proportional transaction costs, it is classical to express the portfolio processes as stochastic vectors of \mathbf{R}^d , $d \geq 1$, whose components are the quantities of different assets held by a portfolio manager. Indeed, in presence of transaction costs, the exchanges are allowed between the assets at different transaction rates so that it is not possible to describe directly the dynamics of portfolio liquidation values. Fortunately, the self-financing property is simple when the discrete-time portfolio processes $V = (V_t)_{t=0, \dots, T}$ are expressed in physical units (see [12, Chapter 3]): it is given by $V_t - V_{t-1} \in -\mathbf{G}_t$, $t \leq T$, where $\mathbf{G}_t \subseteq \mathbf{R}^d$ is the random set of all *solvent* positions whose liquidation values are non negative at time t . It is usual to define the initial value of V as a vector $V_{0-} \in \mathbf{R}^d$ of initial investments in each asset such that $V_0 - V_{0-} \in -\mathbf{G}_0$. This means that it is possible to immediately rebalance at time 0 the initial capital V_{0-} into the new position V_0 . An arbitrage opportunity is defined as a (self-financing) portfolio process $(V_t)_{t=0, \dots, T}$ with $V_{0-} = 0$ such that the liquidation value $\mathbf{L}_T(V_T)$ at time T is non negative and $\mathbf{P}(\mathbf{L}_T(V_T) > 0) > 0$. This is clearly the definition which is also adopted for frictionless models. Absence of such arbitrage opportunity is then denoted by (NA^w) , i.e. one can not get a positive terminal liquidation value when starting from the zero initial capital. We may show that (NA^w) holds if and only if there is no vector-valued portfolio processes V starting from $V_{0-} = 0$ and ending up with a terminal value in the first orthant \mathbf{R}_+^d , see [12, Section 3.2.1]. The Grigoriev theorem [5] characterizes the condition (NA^w) for the two dimensional conic model which may be seen as a financial market model with one Bond and one risky asset defined by its Bid and Ask prices. This is a generalization of the famous result of [9], which is formulated under the No Free Lunch condition and appears to be equivalent to (NA^w) by Grigoriev's theorem. The Grigoriev theorem states that (NA^w) holds if and only if there exists a *Consistent Price System* (CPS), i.e. a martingale $Z \in \mathbf{R}^2$ such that $Z_t \in \mathbf{G}_t^* \setminus \{0\}$ a.s. for all $t = 0, 1, \dots, T$ where $\mathbf{G}_t^* := \{z \in \mathbf{R}^2 : zx \geq 0, \forall x \in \mathbf{G}_t\}$ is the positive dual of \mathbf{G}_t . When the solvency sets \mathbf{G}_t are half-planes, the Bid and Ask prices coincide with the middle price S meaning that there is no transaction cost. In that case, \mathbf{G}_t^* is the half-line of \mathbf{R}_+^2 supported by the vector $(1, S_t)$ and

a CPS is of the form $Z = (\rho, \rho S)$ such that, with $dQ/dP := \rho_T$, we obtain a risk neutral probability measure Q for S , i.e. the Grigoriev's theorem and the DMW theorem coincide.

Nevertheless, the set of all vector-valued terminal portfolio processes is not generally closed under (NA^w) , see [12, Example 1, Section 3.2.4] so that the Grigoriev theorem is not exactly the analog of the DMW theorem which also claims the closedness property under (NA^w) for frictionless models. Actually, with proportional transaction costs, closedness is only obtained in the literature under a strong absence of arbitrage opportunity, i.e. the robust no-arbitrage property (NA^r) , see [12, Lemma 3.2.8], meaning that the (NA^w) condition still holds with strictly smaller transaction costs. This is an essential property as it allows to deduce a dual characterization of super-hedging prices (see [1] and [3]) for models with strictly positive transaction costs.

The reason why closedness does not hold under (NA^w) for conic models is actually due to the fact that terminal claims are expressed in physical units contrarily to the DMW theorem where it is natural to work with the liquidation values. This is indeed confirmed by our present contribution as we propose to prove that the set of all terminal liquidation values is closed under (NA^w) for the Bid and Ask model including transaction costs. Therefore, we have generalised the DMW theorem and (NA^w) appears to be equivalent to the existence of a risk-neutral probability measure under which the expectation of all terminal liquidation values is non positive when starting from the zero initial endowment. We also deduce a dual characterization of the prices super hedging a contingent claim when they are only expressed in the first asset (the bond).

2. Model and basic properties

Notations.

If $x, y \in \mathbf{R}^2$, we denote by xy the Euclidean scalar product between x and y . We define $e_1 := (1, 0) \in \mathbf{R}^2$.

For a subset \mathbf{G} of \mathbf{R}^2 , $\partial\mathbf{G}$ is the boundary of \mathbf{G} and $\text{int } \mathbf{G}$ is its interior.

\mathbf{R}_+^2 is the set of all vectors in \mathbf{R}^2 having only non negative components.

\mathbb{E} designates the expectation of a random variable. When necessary, we denote it \mathbb{E}_Q when it is considered under $Q \sim P$.

For a set-valued random mapping E , $L^0(E, \mathcal{F})$ is the metric space of all E -valued random variables which are \mathcal{F} -measurable.

$L^p(E, \mathcal{F}, \mathbb{P})$, $p \in [1, \infty)$ (resp. $p = \infty$ or $p = 0$), is the normed space of all E -valued random variables which are \mathcal{F} -measurable and admitting a moment of order p under the probability \mathbb{P} (resp. bounded with the strong topology or the weak one $\sigma(L^\infty, L^1)$ when $p = \infty$ and endowed with the convergence in probability for $p = 0$). We use the short notations $L^0 = L^0(\mathbf{R}, \mathcal{F})$ and $L^0_+ = L^0(\mathbf{R}_+, \mathcal{F})$.

For any subset \mathcal{X} of $L^0(\mathbf{R}, \mathcal{F})$ and $p \in [1, \infty) \cup \{0\}$, we denote by $\overline{\mathcal{X}}^p$, the closure of $\mathcal{X}^p := \mathcal{X} \cap L^p(\mathbf{R}, \mathcal{F}, \mathbb{P})$ with respect to the L^p -topology (the topology of convergence in probability if $p = 0$). If $Q \sim \mathbb{P}$, we denote by $\overline{\mathcal{X}}^p(Q)$ the closure under Q .

The Bid-Ask model. Let $(\Omega, \mathbb{F} := (\mathcal{F}_t)_{t=0, \dots, T}, \mathbb{P})$ be a discrete-time complete stochastic basis. The financial market model we consider is defined by one Bond $S^0 = 1$ and one risky asset characterised by Bid and Ask price processes S^b and S^a adapted to the filtration \mathbb{F} .

Equivalently, the model may be defined by a sequence of \mathbb{F} -adapted closed and conic sets $(\mathbf{G}_t)_{t=0, \dots, T}$ of \mathbf{R}^2 which are measurable in the sense that:

$$\text{Graph } \mathbf{G}_t := \{(\omega, x) \in \Omega \times \mathbf{R}^2 : x \in \mathbf{G}_t(\omega)\} \in \mathcal{F}_t \otimes \mathcal{B}(\mathbf{R}^2), \quad t = 0, \dots, T.$$

In finance, \mathbf{G}_t , $t = 0, \dots, T$, is interpreted as the set of all financial positions it is possible to liquidate without any debt. Precisely, let us define the liquidation value process as

$$\mathbf{L}_t(x) := x^1 + (x^2)^+ S_t^b - (x^2)^- S_t^a, \quad x = (x^1, x^2) \in \mathbf{R}^2,$$

where x is the vector of physical units of assets S^0 and S respectively held in the portfolio at time t . We recall the notations $x^+ = \max(x, 0)$ and $x^- = -\min(x, 0)$. We may show that $\mathbf{L}_t(x) = \sup\{\alpha \in \mathbf{R} : x - \alpha e_1 \in \mathbf{G}_t\}$, i.e. $\mathbf{L}_t(x)$ is the maximal amount of cash the portfolio manager may obtain when liquidating the financial position x . We have $\mathbf{G}_t = \{x \in \mathbf{R}^2 : \mathbf{L}_t(x) \geq 0\}$. This is a particular case of the Kabanov model with proportional transaction costs, see [12, Chapter 3]. At any time $t = 0, \dots, T$, we easily observe that the following properties hold:

Lemma 2.1.

1. *The mapping $x \mapsto \mathbf{L}_t(x)$ is concave hence continuous.*
2. $\mathbf{L}_t(x^0, z) = x^0 + \mathbf{L}_t((0, z))$ for all $x^0, z \in \mathbf{R}$.
3. $x - \mathbf{L}_t(x)e_1 \in \partial \mathbf{G}_t$ for all $x \in \mathbf{R}^2$.

Note that the boundary $\partial\mathbf{G}_t = \{x \in \mathbf{R}^2 : \mathbf{L}_t(x) = 0\}$ is composed of two half lines respectively supported by the random vectors $(S_t^a, -1)$ and $(-S_t^b, 1)$. The positive dual of \mathbf{G}_t is $\mathbf{G}_t^* := \{z \in \mathbf{R}^2 : zx \geq 0 : \text{for all } x \in \mathbf{G}_t\}$. The latter is a random cone of \mathbf{R}_+^2 whose boundary is the union of the two half lines in \mathbf{R}_+^2 supported by the vectors $(1, S_t^b)$ and $(1, S_t^a)$. We have $\mathbf{G}_t^* \setminus \{0\} \subseteq \text{int}\mathbf{R}_+^2$.

Definition 2.2. *A self-financing portfolio process $(V_t)_{t=0, \dots, T}$ starting from the initial endowment V_{0-} is an \mathbb{F} -adapted process such that*

$$\Delta V_t := V_t - V_{t-1} \in -\mathbf{G}_t, \quad \forall t = 0, \dots, T \quad a.s. \quad (2.1)$$

The interpretation is simple: when changing the position $V_{t-1} = V_t + (-\Delta V_t)$ into a new one V_t at time t , we liquidate without any debt the residual part, i.e. $-\Delta V_t \in \mathbf{G}_t$. In the Kabanov model, we also interpret (2.1) as the paiement of proportional transaction costs to change V_{t-1} into V_t . We introduce the set of all terminal values at time $t \leq T$ of portfolio processes starting from the zero initial endowment at time $u \leq t$ i.e.

$$\mathcal{A}_u^t := \sum_{s=u}^t L^0(-\mathbf{G}_s, \mathcal{F}_s).$$

Associated to this set above, the set of terminal liquidation values is:

$$\mathcal{L}_u^t := \{\mathbf{L}_t(V) : V \in \mathcal{A}_u^t\}.$$

Remark 2.3. *Notice that for any $\gamma \in \mathcal{L}_0^T$, $\gamma e_1 = \sum_{t=0}^T (-g_t) \in \mathcal{A}_0^T$ for some $g_t \in L^0(\mathbf{G}_t, \mathcal{F}_t)$. Indeed, $\gamma = \mathbf{L}_T(V_T)$ where $V_T = \sum_{t=0}^T \Delta V_t$ with $\Delta V_t \in -L^0(\mathbf{G}_t, \mathcal{F}_t)$ for all t and $V_{0-} = 0$. Moreover, $\gamma e_1 - V_T = \mathbf{L}_T(V_T)e_1 - V_T \in -\mathbf{G}_T$. Thus, $\gamma e_1 = V_T - \hat{g}_T$ where $\hat{g}_T \in L^0(\mathbf{G}_T, \mathcal{F}_T)$ and finally $\gamma e_1 \in \mathcal{A}_0^T$. Futhermore, we may assume that $g_t \in L^0(\partial\mathbf{G}_t, \mathcal{F}_t)$ for all $0 \leq t \leq T-1$. Indeed, let us write $\gamma e_1 = -(g_0 - \mathbf{L}_0(g_0)e_1) - (g_1 + \mathbf{L}_0(g_0)e_1) + \sum_{t=2}^T (-g_t)$ where $g_0 - \mathbf{L}_0(g_0)e_1 \in \partial\mathbf{G}_0$. As $\mathbf{L}_0(g_0) \geq 0$, i.e., $\mathbf{L}_0(g_0)e_1 \in \mathbf{R}_+e_1$, then $g_1 + \mathbf{L}_0(g_0)e_1 \in \mathbf{G}_1$. So replace g_0 and g_1 respectively by $g_0 - \mathbf{L}_0(g_0)e_1$ and $g_1 + \mathbf{L}_0(g_0)e_1$ and repeat the procedure for $t \geq 1$ to obtain that $g_t \in \partial\mathbf{G}_t$ for all $0 \leq t \leq T-1$.*

In the following, we suppose a technical condition (E) when $T \geq 4$ which is not very intuitive but appears to be satisfied by the classical examples of market models with Bid and Ask prices of the literature:

Condition (E): When $T \geq 4$, for all $t \leq T - 3$, we have

$$\begin{aligned} \{S_t^a = S_{t+1}^b = E(S_{t+1}^b | \mathcal{F}_t)\} &\subseteq \bigcup_{r \geq t+1} \{(S_t^a, -1) \in \mathbf{G}_r\}, \quad \text{a.s.}, \\ \{S_t^b = S_{t+1}^a = E(S_{t+1}^a | \mathcal{F}_t)\} &\subseteq \bigcup_{r \geq t+1} \{(-S_t^b, 1) \in \mathbf{G}_r\}, \quad \text{a.s.} \end{aligned}$$

Recall that $(S_t^a, -1)$ and $(S_t^b, -1)$ are the two generator vectors of \mathbf{G}_t . Notice that Condition (E) trivially holds when the l.h.s. of the inclusion is almost surely empty. This is the case when we naturally suppose that it is not possible to know by advance at time t the value of S_{t+1}^b or S_{t+1}^a . Let us present some examples where Condition (E) holds:

Example 1: This first example is a generalization of the model proposed in [7]. Let us consider a positive stochastic process $(S_t)_{t=0, \dots, T}$ we interpret as the mid-prices and a process $(\epsilon_t)_{t=0, \dots, T}$ with values in $[0, 1)$ we interpret as proportional transaction cost rates. We suppose that $(S_t)_{t=0, \dots, T}$ and $(\epsilon_t)_{t=0, \dots, T}$ are two independent processes and for every t , either the random variables $\frac{S_{t+1}}{S_t}$ or $\frac{1+\epsilon_t}{1-\epsilon_{t+1}}$ do not admit any atom. This is the case when $\frac{S_{t+1}}{S_t}$ admits a density and the rate process ϵ is constant. The Bid and Ask prices are given by

$$S_t^b := S_t(1 - \epsilon_t), \quad S_t^a := S_t(1 + \epsilon_t).$$

Then,

$$\begin{aligned} \mathbb{P}(S_t^a = S_{t+1}^b) &= \mathbb{E}(\mathbb{E}(1_{S_t^a = S_{t+1}^b} | \sigma(\epsilon_t : t \leq T))) = \mathbb{E}(\mathbb{E}(1_{\frac{S_{t+1}}{S_t} = \frac{1+\epsilon_t}{1-\epsilon_{t+1}}} | \sigma(\epsilon_t : t \leq T))) \\ &= \mathbb{E}\left(f\left(\frac{1 + \epsilon_t}{1 - \epsilon_{t+1}}\right)\right), \end{aligned}$$

where, by the independence assumption, $f(x) = \mathbb{P}\left(\frac{S_{t+1}}{S_t} = x\right)$. Using the hypothesis on S , we get that $f = 0$. A similar reasoning holds when $\frac{1+\epsilon_t}{1-\epsilon_{t+1}}$ does not admit any atom so that Condition (E) trivially holds.

Example 2: We consider a generalization of the Cox-Ross-Rubinstein model with Bid-Ask spreads proposed in [10, Section 4]. The Bid and Ask prices are given by

$$S_t^b = (1 + \zeta_t^b)S_{t-1}^a, \quad S_t^a = (1 + \zeta_t^a)S_{t-1}^b,$$

where the vector-valued process $\zeta = (\zeta^b, \zeta^a)$ is such that $S_t^b \leq S_t^a$ a.s. for all $t \leq T$. Moreover, it is supposed in [10, Section 4] that ζ^b and ζ^a take two distinct values. With $\mathcal{F}_t = \sigma(\zeta_r^b, \zeta_r^a, S_r : r \leq t)$, this hypothesis trivially implies that the equalities $\zeta_t^b = \mathbb{E}(\zeta_{t+1}^b | \mathcal{F}_t)$ and $\zeta_t^a = \mathbb{E}(\zeta_{t+1}^a | \mathcal{F}_t)$ do not hold almost surely. Here, we only suppose that the equalities $\zeta_t^b = \mathbb{E}(\zeta_{t+1}^b | \mathcal{F}_t) = 0$ and $\zeta_t^a = \mathbb{E}(\zeta_{t+1}^a | \mathcal{F}_t) = 0$ do not hold almost surely so that we may verify that Condition (E) holds. Indeed, if for example $S_{t+1}^b = S_t^a = \mathbb{E}(S_{t+1}^b | \mathcal{F}_t)$ then $\zeta_{t+1}^b = 0 = \mathbb{E}(\zeta_{t+1}^b | \mathcal{F}_t)$. We may also verify that Condition (E) holds if we only suppose that $\zeta_t^b = 0$ if and only if $\zeta_t^a = 0$. Indeed, if for example $S_{t+1}^b = S_t^a$, then $S_{t+1}^a = S_t^b$ hence $S_{t+1}^a \leq S_{t+1}^b$. Therefore, $S_t^a = S_{t+1}^a$ and $S_t^b = S_{t+1}^b$, i.e. $\mathbf{G}_t = \mathbf{G}_{t+1}$.

Example 3: Following [7], we suppose that the Bid and Ask prices are given by

$$S_t^b = S_t - \epsilon_t, \quad S_t^a = S_t + \epsilon_t, \quad t \leq T,$$

where S and ϵ are two positive processes such $S^b > 0$. Then, Condition (E) trivially holds when S and ϵ are independent and one of them does not admit any atom since, in this case, $P(S_{t+1}^b = S_t^a) = P(S_{t+1}^a = S_t^b) = 0$ for all $u \geq t + 1$.

Remark 2.4. Consider any two dimensional conic model defined by a family $(\mathbf{G}_t)_{t=0, \dots, T}$ of random closed convex cones in \mathbf{R}^2 adapted to the filtration $(\mathcal{F}_t)_{t=0, \dots, T}$. If \mathbf{G}_t contains \mathbf{R}_+^2 and is smaller than a half-plane, it is such that $\mathbf{G}_t^* = \text{cone}(\{1\} \times [S_t^b, S_t^a])$ ¹ for some \mathcal{F}_t -measurable random variables S_t^b, S_t^a such that $0 \leq S_t^b \leq S_t^a$, i.e. such a model is a Bid-Ask model.

3. The DMW theorem for discrete-time Bid-Ask models

Definition 3.1. We say that the financial market model defined by \mathbf{G} satisfies the weak no-arbitrage property (NA^w) if $\mathcal{L}_0^T \cap L^0(\mathbf{R}_+, \mathcal{F}_T) = \{0\}$.

Lemma 3.2. (NA^w) holds if and only if $\mathcal{A}_0^T \cap L^0(\mathbf{G}_T, \mathcal{F}_T) \subseteq L^0(\partial \mathbf{G}_T, \mathcal{F}_T)$.

Proof. Suppose that (NA^w) holds and consider $V_T \in \mathcal{A}_0^T \cap L^0(\mathbf{G}_T, \mathcal{F}_T)$. Then, $\mathbf{L}_T(V_T) \in \mathcal{L}_0^T \cap L^0(\mathbf{R}_+, \mathcal{F}_T)$ hence $\mathbf{L}_T(V_T) = 0$, i.e. $V_T \in \partial \mathbf{G}_T$ a.s. Reciprocally, suppose that $\mathcal{A}_0^T \cap L^0(\mathbf{G}_T, \mathcal{F}_T) \subseteq L^0(\partial \mathbf{G}_T, \mathcal{F}_T)$. Any $\gamma_T \in \mathcal{L}_0^T$ is such that $\gamma_T e_1 \in \mathcal{A}_0^T$ since $\gamma_T = \mathbf{L}_T(V_T)$ for some $V_T \in \mathcal{A}_0^T$ such that $V_T - \mathbf{L}_T(V_T)e_1 \in \mathbf{G}_T$. \square

¹The notation $\text{cone}(A)$ designates the smallest cone containing A .

The assumption of the following lemma is clearly satisfied by the Bid-Ask model.

Lemma 3.3. *If \mathbf{G}_T strictly dominates \mathbf{R}_+^2 , i.e., $\mathbf{R}_+^2 \setminus \{0\} \subset \text{int } \mathbf{G}_T$, then we have*

$$\mathcal{L}_0^T \cap L^0(\mathbf{R}_+, \mathcal{F}_T) = \{0\} \iff \mathcal{A}_0^T \cap L^0(\mathbf{R}_+^2, \mathcal{F}_T) = \{0\}.$$

Proof. (\Leftarrow) This part is trivial since $\mathcal{L}_0^T e_1 = \mathcal{A}_0^T \cap L^0(\mathbf{R}e_1, \mathcal{F}_T) \subseteq \mathcal{A}_0^T$.

(\Rightarrow) Let $V_T \in \mathcal{A}_0^T \cap L^0(\mathbf{R}_+^2, \mathcal{F}_T)$. Since $\mathbf{R}_+^2 \subseteq \mathbf{G}_T$, then $\mathbf{L}_T(V_T) \geq 0$. So the condition $\mathcal{L}_0^T \cap L^0(\mathbf{R}_+, \mathcal{F}_T) = \{0\}$ implies $\mathbf{L}_T(V_T) = 0$, hence $V_T \in \partial \mathbf{G}_T \cap \mathbf{R}_+^2 = \{0\}$. \square

The equivalent condition to (NA^w) , as expressed in the lemma above, is studied by Grigoriev [5] and [12, Theorem 3.2.15]. The Grigoriev theorem states that Condition (NA^w) holds if and only if there exists Consistent Price Systems (CPS) evolving in the positive duals of the solvency sets, precisely martingales $(Z_t)_{t=0, \dots, T}$ satisfying $Z_t \in \mathbf{G}_t^* \setminus \{0\}$ for all $t = 0, \dots, T$. Let us recall precisely the Grigoriev theorem as we shall use it in the sequel:

Theorem 3.4 (Grigoriev theorem). *For the two-dimensional conic model, the following conditions are equivalent:*

- (A) (NA^w) : $\mathcal{A}_0^T \cap L^0(\mathbf{R}_+^d, \mathcal{F}_T) = \{0\}$;
- (B) $\overline{\mathcal{A}_0^T} \cap L^0(\mathbf{R}_+^d, \mathcal{F}_T) = \{0\}$ ²;
- (C) For any $\tilde{P} \sim P$, there exists a bounded CPS under \tilde{P} .

This result is a weaker form of the Dalang-Morton-Willinger theorem, see [4]. Without friction, the set of all terminal claims obtained from the zero initial endowment appears to be closed:

Theorem 3.5 (DMW theorem). *Let \mathcal{A}_0^T be the set of all real-valued terminal (liquidation) portfolio values starting from zero. The following conditions are equivalent:*

- (A) (NA^w) : $\mathcal{A}_0^T \cap L^0(\mathbf{R}_+, \mathcal{F}_T) = \{0\}$;
- (B) \mathcal{A}_0^T is closed in L^0 and $\mathcal{A}_0^T \cap L^0(\mathbf{R}_+, \mathcal{F}_T) = \{0\}$;
- (C) There is a bounded and strictly positive P -martingale ρ such that ρS is a P -martingale ³.

²The closure is taken in L^0 .

³This means that the probability measure Q defined by $dQ/dP = \rho_T$ is a martingale probability measure. This still holds under any $\tilde{P} \sim P$.

With proportional transaction costs, \mathcal{A}_0^T is not necessarily closed if the terminal claims are expressed in physical units, see [12, Example 1, Section 3.2.4]. The Grigoriev theorem just claims that (NA^w) is equivalent to the No Free Lunch condition (B). In this paper we show, under a weak assumption on the model, a Dalang-Morton-Willinger version of the Grigoriev theorem, precisely we prove that under Condition (NA^w) , the set of all terminal liquidation values we get from the portfolio processes starting from zero is closed. The proof is given in the next section. The main contribution of this paper is to show the implication $(1) \Rightarrow (2)$. An open problem is to study whether condition (E) is really necessary.

Theorem 3.6. *Suppose that Condition (E) holds if $T \geq 3$. The following conditions are equivalent:*

- 1 (NA^w)
- 2 \mathcal{L}_0^T is closed in probability and $\mathcal{L}_0^T \cap L^0(\mathbf{R}_+, \mathcal{F}_T) = \{0\}$.
- 3 There exists $Q \sim P$ with $dQ/dP \in L^\infty$ such that $\mathbb{E}_Q \mathbf{L}_T(V) \leq 0$ for all $\mathbf{L}_T(V) \in \mathcal{L}_0^T \cap L^1(P)$.
- 4 There exists $Q \sim P$ with $dQ/dP \in L^\infty$ such that for all $t \leq T - 1$, $\mathbb{E}_Q(S_{t+1}^a | \mathcal{F}_t) \geq S_t^b$ and $\mathbb{E}_Q(S_{t+1}^b | \mathcal{F}_t) \leq S_t^a$.
- 5 There exists $Q \sim P$ with $dQ/dP \in L^\infty$ and a Q -martingale \tilde{S} such that $\tilde{S} \in [S^b, S^a]$.

In the following, we denote by $\mathcal{M}^\infty(P)$ the set of all $Q \sim P$ such that $dQ/dP \in L^\infty$ and $\mathbb{E}_Q \mathbf{L}_T(V) \leq 0$ for all $\mathbf{L}_T(V) \in \mathcal{L}_0^T$. For any contingent claim $\xi \in L^1(\mathbf{R}, \mathcal{F}_T, P)$, we define Γ_ξ as the set of all initial endowments we need to start a portfolio process whose terminal liquidation value coincides with ξ , i.e.

$$\Gamma_\xi := \{x \in \mathbf{R} : \exists V \in \mathcal{A}_0^T : \mathbf{L}_T(xe_1 + V_T) = \xi\}.$$

Corollary 3.7. *Suppose that Condition (E) holds if $T \geq 3$. Let us consider a payoff $\xi \in L^0(\mathbf{R}, \mathcal{F}_T)$ satisfying $\mathbb{E}_P|\xi| < \infty$. Then, under Condition (NA^w) , $\Gamma_\xi = [\sup_{Q \in \mathcal{M}^\infty(P)} \mathbb{E}_Q \xi, \infty)$.*

Notice that it is possible, if necessary, to change the probability measure P into an equivalent one so that ξ becomes integrable and the result above is valid.

4. Proofs of the main results

4.1. Proof of Theorem 3.6.

Note that the implication (2) \Rightarrow (3) is immediate by [12, Theorem 2.1.4]. The implications (3) \Rightarrow (1) and (2) \Rightarrow (1) are also trivial. Notice that the implication (3) \Rightarrow (4) is easily obtained by considering the liquidation values at time $t+1$ of the positions $(S_t^b, -1)1_{F_t}$ and $(-S_t^a, 1)1_{F_t}$ for all $F_t \in \mathcal{F}_t$. The implication (4) \Rightarrow (5) is deduced from [2, Theorem 4.5]. At last, if (5) holds, take $\rho_t = \mathbb{E}(dQ/dP)$ and define $Z = (\rho, \rho\tilde{S})$. We may verify that Z is a CPS hence (NA^w) holds. It is also important for the remaining part of the proof to observe that the implication (1) \Rightarrow (4) holds. Indeed, if (NA^w) holds, we deduce by the Grigoriev theorem that $\overline{\mathcal{L}_0^T} \cap L^0(\mathbf{R}_+, \mathcal{F}_T) = \{0\}$ where $\overline{\mathcal{L}_0^T}$ designates here the closure in L^1 . We then deduce (3) by [12, Theorem 2.1.4] hence (4) also holds.

It remains to show that (1) \Rightarrow (2). Suppose that (NA^w) and let us prove that \mathcal{L}_0^T is closed in probability. Recall that, by the Grigoriev theorem, there exists a CPS Z , i.e. a martingale Z such that $Z_u \in \mathbf{G}_u^* \setminus \{0\}$ for all $u \leq T$. For the one step model there is nothing to prove since $\mathcal{L}_T^T = -L^0(\mathbf{R}_+, \mathcal{F}_T)$. Indeed, if $\mathbf{L}_T(-g_T) \geq 0$ for some $g_T \in L^0(\mathbf{G}_T, \mathcal{F}_T)$, then $g_T \in (-\mathbf{G}_T) \cap \mathbf{G}_T \subseteq \partial\mathbf{G}_T$ hence $\mathbf{L}_T(-g_T) = 0$. Let us consider the two step model.

Assume that the sequence $\gamma_T^n \in \mathcal{L}_{T-1}^T$ converges to γ_T^∞ . From Remark 2.3, we may suppose that $\gamma_T^n e_1 = -g_{T-1}^n - g_T^n$ where $g_{T-1}^n \in L^0(\partial\mathbf{G}_{T-1}, \mathcal{F}_{T-1})$ and $g_T^n \in L^0(\mathbf{G}_T, \mathcal{F}_T)$.

On the set $\Lambda_{T-1} := \{\liminf |g_{T-1}^n| = \infty\} \in \mathcal{F}_{T-1}$, we normalize the sequences by setting $\tilde{\gamma}_T^n := \frac{\gamma_T^n}{|g_{T-1}^n|}$, $\tilde{g}_{T-1}^n := \frac{g_{T-1}^n}{|g_{T-1}^n|}$ and $\tilde{g}_T^n := \frac{g_T^n}{|g_{T-1}^n|}$. Then,

$$\tilde{\gamma}_T^n e_1 = -\tilde{g}_{T-1}^n - \tilde{g}_T^n.$$

As $|\tilde{g}_{T-1}^n| = 1$, by passing to some \mathcal{F}_{T-1} -measurable random sequence we may assume that \tilde{g}_{T-1}^n converges to $\tilde{g}_{T-1}^\infty \in \mathbf{G}_{T-1}$, see [12, Lemma 2.1.2]. As $\tilde{\gamma}_T^n e_1$ converges to zero, we deduce that \tilde{g}_T^n converges to $\tilde{g}_T^\infty \in \mathbf{G}_T$. Finally, we get the following equality:

$$\tilde{g}_{T-1}^\infty + \tilde{g}_T^\infty = 0$$

where $\tilde{g}_{T-1}^\infty \in \partial\mathbf{G}_{T-1}$ and $\tilde{g}_T^\infty \in \mathbf{G}_T$. Note that, we may define $\tilde{g}_{T-1}^\infty = \tilde{g}_T^\infty = 0$ on $\Omega \setminus \Lambda_{T-1} \in \mathcal{F}_{T-1}$. Let us consider a CPS Z . From, $Z_T(\tilde{g}_{T-1}^\infty + \tilde{g}_T^\infty) = 0$

we deduce that $Z_{T-1}\tilde{g}_{T-1}^\infty + \mathbb{E}(Z_T\tilde{g}_T^\infty|\mathcal{F}_{T-1}) = 0$. As the two terms in the left side of this equality are non negative by duality, we deduce that $Z_{T-1}\tilde{g}_{T-1}^\infty = Z_T\tilde{g}_T^\infty = 0$. Moreover, $\tilde{g}_T^\infty = -\tilde{g}_{T-1}^\infty$ is \mathcal{F}_{T-1} implies that $0 = \mathbb{E}(Z_T\tilde{g}_T^\infty|\mathcal{F}_{T-1}) = Z_{T-1}\tilde{g}_{T-1}^\infty$. Then, $Z_{T-1}\tilde{g}_{T-1}^\infty = Z_T\tilde{g}_T^\infty$ implies that Z_{T-1} and Z_T belong to the same half-line of \mathbf{R}_+^2 . In particular, since $Z_T \in \mathbf{G}_T^*$, we also have $Z_{T-1} \in \mathbf{G}_T^*$. We deduce that $Z_{T-1}\gamma_T^n e_1 = -Z_{T-1}g_{T-1}^n - Z_T g_T^n \leq 0$ as $g_{T-1}^n \in \mathbf{G}_{T-1}$ and $g_T^n \in \mathbf{G}_T$. Since $Z_{T-1}e_1 > 0$, we deduce that $\gamma_T^n \leq 0$. Therefore, we may replace g_{T-1}^n by $\bar{g}_{T-1}^n = 0 \in \mathbf{G}_{T-1}$ and g_T^n by $\bar{g}_T^n = -\gamma_T^n e_1 \in \mathbf{G}_T$ so that we still have $\gamma_T^n = -\bar{g}_{T-1}^n - \bar{g}_T^n$. Finally, we may write on Ω , $\gamma_T^n = -\hat{g}_{T-1}^n - \hat{g}_T^n$, where $\hat{g}_{T-1}^n = g_{T-1}^n 1_{\Omega \setminus \Lambda_{T-1}} \in L^0(\mathbf{G}_{T-1}, \mathcal{F}_{T-1})$ and $\hat{g}_T^n = g_T^n 1_{\Omega \setminus \Lambda_{T-1}} + \bar{g}_T^n 1_{\Lambda_{T-1}} \in L^0(\mathbf{G}_T, \mathcal{F}_T)$. By construction, $\liminf_n |\hat{g}_{T-1}^n| < \infty$ hence we may suppose that $\hat{g}_{T-1}^n \rightarrow \hat{g}_{T-1}^\infty \in L^0(\mathbf{G}_{T-1}, \mathcal{F}_{T-1})$ by [12, Lemma 2.1.2]. We deduce that $\hat{g}_T^n \rightarrow \hat{g}_T^\infty \in L^0(\mathbf{G}_T, \mathcal{F}_T)$ hence $\gamma_T^\infty = -\hat{g}_{T-1}^\infty - \hat{g}_T^\infty \in \mathcal{L}_{T-1}^T$.

Let us consider the three step model. Suppose that the sequence $\gamma_T^n \in \mathcal{L}_{T-2}^T$ converges to γ_T^∞ . By Remark 2.3, we may suppose that $\gamma_T^n e_1 = -g_{T-2}^n - g_{T-1}^n - g_T^n$ where $g_t^n \in \partial \mathbf{G}_t$ for $t = T-2, T-1$ and $g_T^n \in \mathbf{G}_T$.

Let us first consider the subset $\Gamma_{T-2} := \{\liminf |g_{T-2}^n| < \infty\} \in \mathcal{F}_{T-2}$. By [12, Lemma 2.1.2], we may assume that g_{T-2}^n is convergent to $g_{T-2}^\infty \in L^0(\mathbf{G}_{T-2}, \mathcal{F}_{T-2})$. Next we consider the following two sub-cases:

(a) On the set where $\liminf |g_{T-1}^n| < \infty$, we may suppose that the sequences g_{T-1}^n and g_T^n are both pointwise convergent by [12, Theorem 5.2.3].

(b) On the set $\Lambda_{T-1} := \{\liminf_n |g_{T-1}^n| = \infty\} \in \mathcal{F}_{T-1}$, we normalize the sequences by dividing the term $|g_{T-1}^n|$ and we get

$$\tilde{\gamma}_T^n e_1 = -\tilde{g}_{T-2}^n - \tilde{g}_{T-1}^n - \tilde{g}_T^n,$$

where $\tilde{\gamma}_T^n := \frac{\gamma_T^n}{|g_{T-1}^n|}$, $\tilde{g}_t^n := \frac{g_t^n}{|g_{T-1}^n|}$ for all $t = T-2, T-1, T$. As $|\tilde{g}_{T-1}^n| = 1$, up to some \mathcal{F}_{T-1} -measurable random sequence, we may assume that \tilde{g}_{T-1}^n converges to $\tilde{g}_{T-1}^\infty \in \partial \mathbf{G}_{T-1}$. As $\tilde{\gamma}_T^n$ and \tilde{g}_{T-2}^n both converge to zero, necessarily \tilde{g}_T^n is convergent to some limit $\tilde{g}_T^\infty \in \mathbf{G}_T$ and

$$\tilde{g}_{T-1}^\infty + \tilde{g}_T^\infty = 0.$$

Since we only consider \mathcal{F}_{T-1} -measurable sets, we may claim that $\mathbf{L}_T(\tilde{g}_T^\infty) = \mathbf{L}_T(-\tilde{g}_{T-1}^\infty) \in \mathcal{L}_{T-2}^T \cap L^0(\mathbf{R}_+, \mathcal{F}_T) = \{0\}$ hence $\tilde{g}_T^\infty \in \partial \mathbf{G}_T$ by (NA^w) .

Assume $\tilde{g}_{T-1}^{\infty(1)} > 0$ and $\tilde{g}_{T-1}^{\infty(2)} < 0$, the second case where $\tilde{g}_{T-1}^{\infty(1)} < 0$ and $\tilde{g}_{T-1}^{\infty(2)} > 0$ being similar. The equality $\tilde{g}_{T-1}^\infty + \tilde{g}_T^\infty = 0$ implies that $\tilde{g}_T^{\infty(1)} < 0$

and $\tilde{g}_T^{\infty(2)} > 0$. This implies that we may replace g_{T-1}^n by $g_{T-1}^n 1_{g_{T-1}^{n(2)} < 0} + \tilde{g}_{T-1}^{\infty} 1_{g_{T-1}^{n(2)} \geq 0}$ and g_T^n by $g_T^n 1_{g_T^{n(2)} > 0} + \tilde{g}_T^{\infty} 1_{g_T^{n(2)} \leq 0}$. So, we may assume that $g_{T-1}^{n(2)} < 0$ and $g_T^{n(2)} > 0$. Let us define the \mathcal{F}_{T-1} -measurable positive real-valued random variable

$$\beta^n := \frac{g_{T-2}^{n(2)} + g_{T-1}^{n(2)}}{\tilde{g}_{T-1}^{\infty(2)}}.$$

As $\tilde{g}_{T-1}^{\infty} + \tilde{g}_T^{\infty} = 0$, we rewrite $\gamma_T^n e_1$ as $\gamma_T^n e_1 = -g_{T-2}^n - (g_{T-1}^n - \beta^n \tilde{g}_{T-1}^{\infty}) - (g_T^n - \beta^n \tilde{g}_T^{\infty})$. By definition of β^n , the second component of the term $g_{T-1}^n - \beta^n \tilde{g}_{T-1}^{\infty}$ is

$$(g_{T-1}^n - \beta^n \tilde{g}_{T-1}^{\infty})^{(2)} = -g_{T-2}^{n(2)}.$$

Notice that $\tilde{g}_{T-1}^{\infty} \in \partial \mathbf{G}_{T-1}$ and $\tilde{g}_T^{\infty} \in \partial \mathbf{G}_T$, i.e., $\mathbf{L}_{T-1}(\tilde{g}_{T-1}^{\infty}) = \tilde{g}_{T-1}^{\infty(1)} + \tilde{g}_{T-1}^{\infty(2)} S_{T-1}^a = 0$ and $\mathbf{L}_T(\tilde{g}_T^{\infty}) = \tilde{g}_T^{\infty(1)} + \tilde{g}_T^{\infty(2)} S_T^b = 0$ with $\tilde{g}_T^{\infty(2)} = -\tilde{g}_{T-1}^{\infty(2)} > 0$. So we have $\frac{\tilde{g}_{T-1}^{\infty(1)}}{\tilde{g}_{T-1}^{\infty(2)}} = -S_{T-1}^a$ and $\frac{\tilde{g}_T^{\infty(1)}}{\tilde{g}_T^{\infty(2)}} = -S_T^b$. As $\tilde{g}_{T-1}^{\infty} = -\tilde{g}_T^{\infty}$, we have $S_{T-1}^a = S_T^b$. The first component of the term $g_{T-1}^n - \beta^n \tilde{g}_{T-1}^{\infty}$ is

$$\begin{aligned} (g_{T-1}^n - \beta^n \tilde{g}_{T-1}^{\infty})^{(1)} &= g_{T-1}^{n(1)} - \beta^n \tilde{g}_{T-1}^{\infty(1)} \\ &= g_{T-1}^{n(1)} - \frac{g_{T-2}^{n(2)} + g_{T-1}^{n(2)}}{\tilde{g}_{T-1}^{\infty(2)}} \tilde{g}_{T-1}^{\infty(1)} \\ &= g_{T-1}^{n(1)} + (g_{T-2}^{n(2)} + g_{T-1}^{n(2)}) S_{T-1}^a \\ &= g_{T-1}^{n(1)} + g_{T-1}^{n(2)} S_{T-1}^a + g_{T-2}^{n(2)} S_{T-1}^a \\ &= g_{T-2}^{n(2)} S_{T-1}^a \end{aligned}$$

since $g_{T-1}^{n(1)} + g_{T-1}^{n(2)} S_{T-1}^a = \mathbf{L}_{T-1}(g_{T-1}^n) = 0$.

Then, $g_{T-1}^n - \beta^n \tilde{g}_{T-1}^{\infty} = (g_{T-2}^{n(2)} S_{T-1}^a, -g_{T-2}^{n(2)})$. Notice that it satisfies the condition $\liminf |g_{T-1}^n - \beta^n \tilde{g}_{T-1}^{\infty}| < \infty$ since $S_{T-1}^a \in L^\infty(\mathbf{R}, \mathcal{F}_{T-1})$ and $\liminf |g_{T-2}^{n(2)}| < \infty$. Liquidate this position at time $T-1$ to get

$$\mathbf{L}_{T-1}(g_{T-1}^n - \beta^n \tilde{g}_{T-1}^{\infty}) = g_{T-2}^{n(2)} S_{T-1}^a - g_{T-2}^{n(2)} S_{T-1}^a \mathbf{I}_{g_{T-2}^{n(2)} \geq 0} - g_{T-2}^{n(2)} S_{T-1}^b \mathbf{I}_{g_{T-2}^{n(2)} < 0}.$$

This implies that $\mathbf{L}_{T-1}((g_{T-1}^n - \beta^n \tilde{g}_{T-1}^{\infty}) \mathbf{I}_{g_{T-2}^{n(2)} \geq 0}) = 0$, i.e.,

$$(g_{T-1}^n - \beta^n \tilde{g}_{T-1}^{\infty}) \mathbf{I}_{g_{T-2}^{n(2)} \geq 0} \in \partial \mathbf{G}_{T-1}.$$

Otherwise, if we liquidate the position at time T , we obtain that

$$\mathbf{L}_T(g_{T-1}^n - \beta^n \tilde{g}_{T-1}^\infty) = g_{T-2}^{n(2)} S_T^b - g_{T-2}^{n(2)} S_T^a \mathbf{I}_{g_{T-2}^{n(2)} \geq 0} - g_{T-2}^{n(2)} S_T^b \mathbf{I}_{g_{T-2}^{n(2)} < 0}.$$

This implies that $\mathbf{L}_T((g_{T-1}^n - \beta^n \tilde{g}_{T-1}^\infty) \mathbf{I}_{g_{T-2}^{n(2)} < 0}) = 0$, i.e.,

$$(g_{T-1}^n - \beta^n \tilde{g}_{T-1}^\infty) \mathbf{I}_{g_{T-2}^{n(2)} < 0} \in \partial \mathbf{G}_T.$$

About the term $g_T^n - \beta^n \tilde{g}_T^\infty$, it is easy to see that the second component is zero since $\beta^n = \frac{g_T^{n(2)}}{\tilde{g}_T^{\infty(2)}}$. The first component is

$$\begin{aligned} (g_T^n - \beta^n \tilde{g}_T^\infty)^{(1)} &= g_T^{n(1)} - \beta^n \tilde{g}_T^{\infty(1)} \\ &= g_T^{n(1)} - \frac{g_T^{n(2)}}{\tilde{g}_T^{\infty(2)}} \tilde{g}_T^{\infty(1)} \\ &= g_T^{n(2)} \left(\frac{g_T^{n(1)}}{g_T^{n(2)}} - \frac{\tilde{g}_T^{\infty(1)}}{\tilde{g}_T^{\infty(2)}} \right) \end{aligned}$$

As $g_T^n \in \mathbf{G}_T$, $\mathbf{L}_T(g_T^n) = g_T^{n(1)} + g_T^{n(2)} S_T^b \geq 0$. So we have $\frac{g_T^{n(1)}}{g_T^{n(2)}} \geq -S_T^b = \frac{\tilde{g}_T^{\infty(1)}}{\tilde{g}_T^{\infty(2)}}$. Thus, $(g_T^n - \beta^n \tilde{g}_T^\infty)^{(1)} \geq 0$, i.e.,

$$g_T^n - \beta^n \tilde{g}_T^\infty \in \mathbf{R}_+ e_1.$$

On the set Γ_{T-2} , we finally obtain that $\gamma_T^n e_1 = -g_{T-2}^n - \hat{g}_{T-1}^n - \hat{g}_T^n$, where

$$\begin{aligned} \hat{g}_{T-1}^n &:= g_{T-1}^n \mathbf{1}_{\Omega \setminus \Lambda_{T-1}} + (g_{T-1}^n - \beta^n \tilde{g}_{T-1}^\infty) \mathbf{I}_{g_{T-2}^{n(2)} \geq 0} \mathbf{1}_{\Lambda_{T-1}} \in L^0(\mathbf{G}_{T-1}, \mathcal{F}_{T-1}), \\ \hat{g}_T^n &:= g_T^n \mathbf{1}_{\Omega \setminus \Lambda_{T-1}} + (g_{T-1}^n - \beta^n \tilde{g}_{T-1}^\infty) \mathbf{I}_{g_{T-2}^{n(2)} < 0} \mathbf{1}_{\Lambda_{T-1}} \\ &\quad + (g_T^n - \beta^n \tilde{g}_T^\infty) \mathbf{1}_{\Lambda_{T-1}} \in L^0(\mathbf{G}_T, \mathcal{F}_T). \end{aligned}$$

By construction, notice that $\liminf_n |\hat{g}_{T-1}^n| < \infty$ on Γ_{T-2} . By [12, Lemma 2.1.2], we may assume that \hat{g}_{T-1}^n is convergent to $\hat{g}_{T-1}^\infty \in L^0(\mathbf{G}_{T-1}, \mathcal{F}_{T-1})$ at least for some \mathcal{F}_{T-1} -measurable subsequence $n_k(\omega)$, $\omega \in \Omega$. As we already know that $g_{T-2}^n \rightarrow g_{T-2}^\infty \in L^0(\mathbf{G}_{T-2}, \mathcal{F}_{T-2})$, we still have $g_{T-2}^{n_k} \rightarrow g_{T-2}^\infty \in L^0(\mathbf{G}_{T-2}, \mathcal{F}_{T-2})$ even if $g_{T-2}^{n_k}$ is no more \mathcal{F}_{T-2} -measurable. We deduce that $\hat{g}_T^{n_k} \rightarrow \hat{g}_T^\infty \in L^0(\mathbf{G}_T, \mathcal{F}_T)$ and, finally, $\gamma_T^n \mathbf{1}_{\Gamma_{T-2}} = -g_{T-2}^\infty - \hat{g}_{T-1}^\infty - \hat{g}_T^\infty \in \mathcal{L}_{T-2}^T$.

On the set $\Lambda_{T-2} := \{\liminf |g_{T-2}^n| = \infty\}$, we use the normalization procedure to get

$$\bar{\gamma}_T^n e_1 = -\bar{g}_{T-2}^n - \bar{g}_{T-1}^n - \bar{g}_T^n,$$

where $\bar{\gamma}_T^n := \frac{\gamma_T^n}{|g_{T-2}^n|}$, $\bar{g}_t^n := \frac{g_t^n}{|g_{T-2}^n|}$ for $t \geq T-2$. As $\liminf |g_{T-2}^n| = 1$, we may argue as we did on the complementary set Γ_{T-2} and assume that $\bar{g}_t^n \rightarrow \bar{g}_t^\infty \in L^0(\mathbf{G}_t, \mathcal{F}_t)$ for $t \geq T-2$ such that

$$\bar{g}_{T-2}^\infty + \bar{g}_{T-1}^\infty + \bar{g}_T^\infty = 0$$

with $\bar{g}_t^\infty \in \partial \mathbf{G}_t$ if $t = T-2, T-1$ and $|\bar{g}_{T-2}^\infty| = 1$. Since $\mathbf{L}_T(\bar{g}_T^\infty) = \mathbf{L}_T(-\bar{g}_{T-2}^\infty - \bar{g}_{T-1}^\infty) \in \mathcal{L}_{T-2}^T \cap L^0(\mathbf{R}_+, \mathcal{F}_T) = \{0\}$, we also get that $\bar{g}_T^\infty \in \partial \mathbf{G}_T$.

For any CPS Z , taking the conditional expectation knowing \mathcal{F}_{T-2} in the equality $Z_T(\bar{g}_{T-2}^\infty + \bar{g}_{T-1}^\infty + \bar{g}_T^\infty) = 0$, we deduce that

$$Z_{T-2} \bar{g}_{T-2}^\infty + \mathbb{E}(Z_{T-1} \bar{g}_{T-1}^\infty | \mathcal{F}_{T-2}) + \mathbb{E}(Z_T \bar{g}_T^\infty | \mathcal{F}_{T-2}) = 0.$$

All the terms of the l.h.s. being non negative, we finally obtain that $Z_t \bar{g}_t^\infty = 0$ for all $t \geq T-2$. As $\bar{g}_T^\infty = -\bar{g}_{T-2}^\infty - \bar{g}_{T-1}^\infty$ is \mathcal{F}_{T-1} -measurable, taking the conditional expectation knowing \mathcal{F}_{T-1} in the equality $Z_T \bar{g}_T^\infty = 0$, we deduce that $Z_{T-1} \bar{g}_T^\infty = 0$. But $Z_{T-1}(\bar{g}_{T-2}^\infty + \bar{g}_{T-1}^\infty + \bar{g}_T^\infty) = 0$ hence $Z_{T-1} \bar{g}_{T-2}^\infty = 0$. Since $|\bar{g}_{T-2}^\infty| = 1$, the two equalities $Z_{T-2} \bar{g}_{T-2}^\infty = Z_{T-1} \bar{g}_{T-2}^\infty = 0$ implies that Z_{T-1} and Z_{T-2} belong to a same half line of $\mathbf{G}_{T-1}^* \cap \mathbf{G}_{T-2}^*$.

Assume $\bar{g}_{T-2}^{\infty(1)} > 0$ and $\bar{g}_{T-2}^{\infty(2)} < 0$, the other case where $\bar{g}_{T-2}^{\infty(1)} < 0$ and $\bar{g}_{T-2}^{\infty(2)} > 0$ is similar. Up to some \mathcal{F}_{T-2} -measurable random sequence $(\tilde{n}_k)_k$ we may assume that $g_{T-2}^{n(1)} > 0$ and $g_{T-2}^{n(2)} < 0$. We then consider the following two sub cases:

(c) When $\bar{g}_T^\infty \neq 0$. In the case where $\bar{g}_{T-1}^\infty \neq 0$, the two equalities $Z_{T-1} \bar{g}_{T-1}^\infty = Z_{T-1} \bar{g}_T^\infty = 0$ implies that \bar{g}_{T-1}^∞ and \bar{g}_T^∞ are collinear. As $Z_T(\bar{g}_{T-2}^\infty + \bar{g}_{T-1}^\infty + \bar{g}_T^\infty) = 0$, we have $Z_T(\bar{g}_{T-2}^\infty + \bar{g}_{T-1}^\infty) = 0$ where $Z_T \bar{g}_{T-1}^\infty = Z_T \bar{g}_T^\infty = 0$ since $\bar{g}_{T-1}^\infty \in \mathbf{R} \bar{g}_T^\infty$. It follows that $Z_T \bar{g}_{T-2}^\infty = 0$. In the case where $\bar{g}_{T-1}^\infty = 0$, the equality $Z_T(\bar{g}_{T-2}^\infty + \bar{g}_{T-1}^\infty) = 0$ implies that $Z_T \bar{g}_{T-2}^\infty = 0$. Finally, $Z_{T-2} \bar{g}_{T-2}^\infty = Z_{T-1} \bar{g}_{T-2}^\infty = Z_T \bar{g}_{T-2}^\infty = 0$ implies that Z_{T-2} , Z_{T-1} and Z_T are both collinear with some positive coefficients of collinearity when both \bar{g}_{T-2}^∞ and \bar{g}_T^∞ are different from zero. This implies that $Z_T \in \mathbf{G}_t^*$ for all $t \geq T-2$. Since $\gamma_T^n e_1 = -g_{T-2}^n - g_{T-1}^n - g_T^n$, we deduce that $Z_T \gamma_T^n e_1 = -(Z_T g_{T-2}^n + Z_T g_{T-1}^n + Z_T g_T^n)$ hence $Z_T \gamma_T^n e_1 \leq 0$ and $\gamma_T^n \leq 0$ on the set $\Lambda_{T-2} \cap \{\bar{g}_T^\infty \neq 0\}$.

(d) When $\bar{g}_T^\infty = 0$, $\bar{g}_{T-2}^\infty + \bar{g}_{T-1}^\infty = 0$. Let us define the \mathcal{F}_{T-2} -measurable positive real-valued random variable as

$$\bar{\beta}^n := \frac{g_{T-2}^{n(2)}}{\bar{g}_{T-2}^{\infty(2)}}.$$

Since $\bar{g}_{T-1}^\infty = -\bar{g}_{T-2}^\infty$, $\bar{g}_{T-1}^{\infty(1)} < 0$ and $\bar{g}_{T-1}^{\infty(2)} > 0$, we may replace g_{T-1}^n by $g_{T-1}^n \mathbf{1}_{\{g_{T-1}^{n(1)} < 0; g_{T-1}^{n(2)} > 0\}} + \bar{g}_{T-1}^\infty \mathbf{1}_{\Omega \setminus \{g_{T-1}^{n(1)} < 0; g_{T-1}^{n(2)} > 0\}}$ and assume that $g_{T-1}^{n(1)} < 0$ and $g_{T-1}^{n(2)} > 0$. As $\bar{g}_{T-2}^\infty + \bar{g}_{T-1}^\infty = 0$, we rewrite $\gamma_T^n e_1$ as $\gamma_T^n e_1 = -(g_{T-2}^n - \bar{\beta}^n \bar{g}_{T-2}^\infty) - (g_{T-1}^n - \bar{\beta}^n \bar{g}_{T-1}^\infty) - g_T^n$ where $g_{T-2}^n - \bar{\beta}^n \bar{g}_{T-2}^\infty = 0$ by construction. As $\bar{g}_{T-2}^\infty \in \partial \mathbf{G}_{T-2}$ and $\bar{g}_{T-1}^\infty \in \partial \mathbf{G}_{T-1}$, $\mathbf{L}_{T-2}(\bar{g}_{T-2}^\infty) = \bar{g}_{T-2}^{\infty(1)} + \bar{g}_{T-2}^{\infty(2)} S_{T-2}^a = 0$ and $\mathbf{L}_{T-1}(\bar{g}_{T-1}^\infty) = \bar{g}_{T-1}^{\infty(1)} + \bar{g}_{T-1}^{\infty(2)} S_{T-1}^b = 0$. So we have $\frac{\bar{g}_{T-2}^{\infty(1)}}{\bar{g}_{T-2}^{\infty(2)}} = -S_{T-2}^a$ and $\frac{\bar{g}_{T-1}^{\infty(1)}}{\bar{g}_{T-1}^{\infty(2)}} = -S_{T-1}^b$. As $\bar{g}_{T-1}^\infty = -\bar{g}_{T-2}^\infty$, $S_{T-2}^a = S_{T-1}^b$.

The second component of $g_{T-1}^n - \bar{\beta}^n \bar{g}_{T-1}^\infty$ is $(g_{T-1}^n - \bar{\beta}^n \bar{g}_{T-1}^\infty)^{(2)} = -g_T^{n(2)}$. Thus, $\bar{\beta}^n = \frac{g_{T-1}^{n(2)} + g_T^{n(2)}}{\bar{g}_{T-1}^{\infty(2)}}$, which is \mathcal{F}_{T-1} -measurable. Notice that $g_{T-1}^n - \bar{\beta}^n \bar{g}_{T-1}^\infty$ is also \mathcal{F}_{T-1} -measurable. Its first component is

$$\begin{aligned} (g_{T-1}^n - \bar{\beta}^n \bar{g}_{T-1}^\infty)^{(1)} &= g_{T-1}^{n(1)} - \bar{\beta}^n \bar{g}_{T-1}^{\infty(1)} \\ &= g_{T-1}^{n(1)} - \frac{g_{T-1}^{n(2)} + g_T^{n(2)}}{\bar{g}_{T-1}^{\infty(2)}} \bar{g}_{T-1}^{\infty(1)} \\ &= g_{T-1}^{n(1)} + (g_{T-1}^{n(2)} + g_T^{n(2)}) S_{T-1}^b \\ &= g_{T-1}^{n(1)} + g_{T-1}^{n(2)} S_{T-1}^b + g_T^{n(2)} S_{T-1}^b \\ &= g_T^{n(2)} S_{T-1}^b \end{aligned}$$

since $g_{T-1}^{n(1)} + g_{T-1}^{n(2)} S_{T-1}^b = \mathbf{L}_{T-1}(g_{T-1}^n) = 0$. Then

$$g_{T-1}^n - \bar{\beta}^n \bar{g}_{T-1}^\infty = (g_T^{n(2)} S_{T-1}^b, -g_T^{n(2)})$$

so that

$$\gamma_T^n = -g_T^{n(2)} S_{T-1}^b - g_T^{n(1)}.$$

Liquidate the position $g_{T-1}^n - \bar{\beta}^n \bar{g}_{T-1}^\infty$ at time $T - 1$ to get

$$\mathbf{L}_{T-1}(g_{T-1}^n - \bar{\beta}^n \bar{g}_{T-1}^\infty) = g_T^{n(2)} S_{T-1}^b - g_T^{n(2)} S_{T-1}^a \mathbf{1}_{g_T^{n(2)} > 0} - g_T^{n(2)} S_{T-1}^b \mathbf{1}_{g_T^{n(2)} \leq 0}.$$

This implies that $\mathbf{L}_{T-1}((g_{T-1}^n - \bar{\beta}^n \bar{g}_{T-1}^\infty)1_{g_T^{n(2)} \leq 0}) = 0$, i.e.,

$$(g_{T-1}^n - \bar{\beta}^n \bar{g}_{T-1}^\infty)1_{g_T^{n(2)} \leq 0} \in \partial \mathbf{G}_{T-1}.$$

Similarly we deduce by liquidating it at time $T - 2$ that

$$(g_{T-1}^n - \bar{\beta}^n \bar{g}_{T-1}^\infty)1_{g_T^{n(2)} > 0} \in \partial \mathbf{G}_{T-2}.$$

If we set $h_{T-1}^n := g_{T-2}^n + g_{T-1}^n$, then

$$h_{T-1}^n = g_{T-1}^n - \bar{\beta}^n \bar{g}_{T-1}^\infty = (g_T^{n(2)} S_{T-1}^b, -g_T^{n(2)})$$

belongs to the boundary of \mathbf{G}_{T-2} or \mathbf{G}_{T-1} .

In the case where $g_T^{n(2)} \leq 0$, it is possible to change g_{T-1}^n into $g_{T-1}^n - \bar{\beta}^n \bar{g}_{T-1}^\infty$ which still belongs to the boundary of \mathbf{G}_{T-1} .

On the set $\{g_T^{n(2)} > 0\} \in \mathcal{F}_{T-1}$, we consider two sub cases. On the set $\{\sup_n g_T^{n(2)} < \infty\} \in \mathcal{F}_{T-1}$, up to a convex combination (see [12, Theorem A.2.3]), we assume that $g_T^{n(2)}$ is convergent. As $\gamma_T^n = -g_T^{n(2)} S_{T-1}^b - g_T^{n(1)}$, $g_T^{n(1)}$ is also convergent, i.e. g_T^n converges to $g_T \in L^0(\mathbf{G}_T, \mathcal{F}_T)$. Then $\gamma_T^\infty e_1 = \lim_n (-g_{T-2}^n - g_{T-1}^n - g_T^n) = \lim_n (-g_{T-2}^n - g_{T-1}^n - g_T)$, i.e. we may replace g_T^n by g_T . By the initial normalization on Λ_{T-2} , we then get that $\gamma_T^n = -g_T^{(2)} S_{T-1}^b - g_T^{(1)}$ instead of $\gamma_T^n = -g_T^{n(2)} S_{T-1}^b - g_T^{n(1)}$, i.e. γ_T^n does not depend on n any more: we may write $\gamma_T^n e_1 = -g_{T-2}^1 - g_{T-1}^1 - g_T^1$ where g_t^1 is a constant substitution of g_t^n for each $t \geq T - 2$.

On the \mathcal{F}_{T-1} -measurable set $\{\sup_n g_T^{n(2)} = \infty\}$, first recall that for any CPS Z , Z_{T-2} and Z_{T-1} are colinear vectors of \mathbf{R}_+^2 orthogonal to the line generated by \bar{g}_{T-2}^∞ . Through the normalizing procedure which consists in dividing by $g_T^{n(2)}$ on both sides of the equality $\gamma_T^n e_1 = -h_{T-1}^n - g_T^n$, we get that

$$\tilde{h}_{T-1} + \tilde{g}_T = 0$$

where $\tilde{h}_{T-1} \in L^0(\mathbf{R} \bar{g}_{T-2}^\infty, \mathcal{F}_{T-1})$ and $\tilde{g}_T \in L^0(\mathbf{G}_T, \mathcal{F}_T)$. Taking conditional expectation knowing \mathcal{F}_{T-1} in the equality $Z_T(\tilde{h}_{T-1} + \tilde{g}_T) = 0$, we deduce that $Z_{T-1} \tilde{h}_{T-1} + \mathbb{E}(Z_T \tilde{g}_T | \mathcal{F}_{T-1}) = 0$. As $Z_{T-1} \tilde{h}_{T-1} = 0$ and $Z_T \tilde{g}_T \geq 0$, we then deduce that $Z_T \tilde{g}_T = 0$ hence $Z_T \tilde{h}_{T-1} = 0$. The equality $Z_{T-1} \tilde{h}_{T-1} = Z_T \tilde{h}_{T-1} = 0$ with $\tilde{h}_{T-1} \neq 0$ implies that Z_T and Z_{T-1} also belong to the same half line. Finally $Z_T \in \mathbf{G}_t^*$ for all $t \geq T - 2$ and, by a similar argument as above, we deduce that $\gamma_T^n \leq 0$.

Gathering with the case (c) where $\bar{g}_T^\infty \neq 0$, we can conclude that on the \mathcal{F}_{T-2} -measurable set Λ_{T-2} , we have $\gamma_T^n e_1 = -g_{T-2}^1 - \hat{g}_{T-1}^n - \hat{g}_T^n$ where

$$\hat{g}_t^n := \sum_{i=1}^4 \hat{g}_t^{ni} \in L^0(\mathbf{G}_t, \mathcal{F}_t), \quad t \geq T-1,$$

with

$$\begin{aligned} \hat{g}_{T-1}^{n1} &:= (\|g_{T-2}^1\| \bar{g}_{T-1}^\infty) \mathbf{I}_{\bar{g}_T^\infty \neq 0}, \\ \hat{g}_{T-1}^{n2} &:= (-g_{T-2}^1 + g_{T-1}^n - \bar{\beta}^n \bar{g}_{T-1}^\infty) \mathbf{I}_{\bar{g}_T^\infty = 0} \mathbf{I}_{g_T^{n(2)} \leq 0}, \\ \hat{g}_{T-1}^{n2} &:= g_{T-1}^1 \mathbf{I}_{\bar{g}_T^\infty = 0} \mathbf{I}_{g_T^{n(2)} > 0} \mathbf{I}_{\sup_n g_T^{n(2)} < \infty}, \\ \hat{g}_{T-1}^{n2} &:= (-g_{T-2}^1) \mathbf{I}_{\bar{g}_T^\infty = 0} \mathbf{I}_{g_T^{n(2)} > 0} \mathbf{I}_{\sup_n g_T^{n(2)} = \infty} \end{aligned}$$

and

$$\begin{aligned} \hat{g}_T^{n1} &:= (\|g_{T-2}^1\| \bar{g}_T^\infty - \gamma_T^n e_1) \mathbf{I}_{\bar{g}_T^\infty \neq 0}, \\ \hat{g}_T^{n2} &:= g_T^n \mathbf{I}_{\bar{g}_T^\infty = 0} \mathbf{I}_{g_T^{n(2)} \leq 0}, \\ \hat{g}_T^{n3} &:= g_T^1 \mathbf{I}_{\bar{g}_T^\infty = 0} \mathbf{I}_{g_T^{n(2)} > 0} \mathbf{I}_{\sup_n g_T^{n(2)} < \infty}, \\ \hat{g}_T^{n4} &:= (-\gamma_T^n e_1) \mathbf{I}_{\bar{g}_T^\infty = 0} \mathbf{I}_{g_T^{n(2)} > 0} \mathbf{I}_{\sup_n g_T^{n(2)} = \infty}. \end{aligned}$$

We can easily see that $\hat{g}_{T-1}^{ni} \in \partial \mathbf{G}_{T-1}$ when $i = 2, 3, 4$ and $\hat{g}_T^{ni} \in \mathbf{G}_T$ for all $i = 1, 2, 3, 4$. The expressions of \hat{g}_t^{n1} , $t = T-1, T$ are obtained from the following equality: $g_{T-2}^n = \|g_{T-2}^n\| \bar{g}_{T-2}^\infty = \|g_{T-2}^n\| \bar{g}_{T-2}^\infty = -\|g_{T-2}^n\| (\bar{g}_{T-1}^\infty + \bar{g}_T^\infty)$ with $n = 1$ and $\gamma_T^1 e_1 = -g_{T-2}^1 - (-g_{T-2}^1) - (-\gamma_T^1 e_1)$.

Since the sequence $(g_{T-2}^n)_{n \geq 1}$ is replaced by the constant g_{T-2}^1 , we may follow the reasoning we did on the complementary Γ_{T-2} so that we finally obtain $\gamma_T^\infty e_1 = -g_{T-2}^\infty - g_{T-1}^\infty - g_T^\infty$ where $g_t^\infty \in \mathbf{G}_t$ for all $t = T-2, T-1, T$ and $\gamma_T^\infty = \mathbf{L}_T(\gamma_T^\infty e_1) \in \mathcal{L}_{T-2}^T$.

We now consider the general case. Let us suppose by induction that the statement above holds for the model with dates between $t+1$ and T with $t+1 \geq T-2$ and let us prove it from t to T . To do so, consider a sequence $\delta_T^n = -g_t^n - g_{t+1}^n - \dots - g_T^n$, where we may suppose w.l.o.g. that $g_u^n \in L^0(\partial \mathbf{G}_u, \mathcal{F}_u)$ for all $u = t, \dots, T-1$ and $g_T^n \in L^0(\mathbf{G}_T, \mathcal{F}_T)$, converges to $\delta_T \in L^0(\mathbf{R}e_1, \mathcal{F}_T)$. We claim that $\delta_T^n = -\hat{g}_t^n - \hat{g}_{t+1}^n - \dots - \hat{g}_T^n + \epsilon_T^n$, where $\hat{g}_u^n \in L^0(\mathbf{G}_u, \mathcal{F}_u)$ for all $u = t, \dots, T-1$ and $\hat{g}_T^n \in L^0(\mathbf{G}_T, \mathcal{F}_T)$ are such that $\liminf_n |g_u^n| < \infty$

a.s. for all $u = t, \dots, T$ and $\lim_n \epsilon_T^n = 0$ a.s. Moreover we claim that, on the set $\{\liminf_n |g_t^n| = +\infty\}$, \hat{g}_t^n is either 0 or a constant element of the sequence $(g_u^n)_{n \geq 0}$ and $\hat{g}_t^n = g_t^n$ on the set $\{\liminf_n |\hat{g}_t^n| < \infty\}$. Notice that these properties are verified for $t = T - 1$ and $t = T - 2$ as shown above.

Let us denote by u the smallest instant such that $P(\liminf_n |g_u^n| = +\infty) > 0$. As $\liminf_n |g_u^n| < \infty$ a.s. for every $r = t, \dots, u - 1$, we may successively suppose that g_r^n is a.s. convergent to some $g_r \in L^0(\partial \mathbf{G}_r, \mathcal{F}_r)$ when $r \leq u - 1$ by [12, Lemma 2.1.2]. It is then possible to make the substitution $g_i^n = g_i$ for every $i \leq u - 1$, letting aside a residual error ϵ_T^n which tends a.s. to zero as in the claim. We suppose that $u \leq T - 1$, otherwise there is nothing to prove.

We first work on the set $\Lambda_t := \{\liminf_n |g_t^n| < \infty\} \in \mathcal{F}_t$ so that $u \geq t + 1$. We split Λ_t into $\Lambda_u := \{\liminf_n |g_u^n| < \infty\} \in \mathcal{F}_u$ and its complimentary $\Gamma_u := \{\liminf_n |g_u^n| = \infty\}$. On the latter set, dividing by $|g_u^n|$, we get the normalisation

$$\tilde{\delta}_T^n = -\tilde{g}_t^n - \tilde{g}_{t+1}^n - \dots - \tilde{g}_T^n$$

where $\tilde{\delta}_T^n = \frac{\delta_T^n}{|g_u^n|}$ and $\tilde{g}_r^n := \frac{g_r^n}{|g_u^n|}$ for $r = t, \dots, T$. As $\tilde{\delta}_T^n$ and \tilde{g}_r^n with $r = t, \dots, u - 1$ all converge to 0, we may use the induction hypothesis and suppose that $\liminf_n |\tilde{g}_r^n| < \infty$ on Γ_u if $r \geq u$. By [12, Lemma 2.1.2], we may suppose that $\tilde{g}_r^n \rightarrow \tilde{g}_r^\infty \in L^0(\mathbf{G}_r, \mathcal{F}_r)$ if $r \geq u$ and, finally,

$$\tilde{g}_u^\infty + \tilde{g}_{u+1}^\infty + \dots + \tilde{g}_T^\infty = 0. \quad (4.2)$$

We deduce that $\tilde{g}_T^\infty \in L^0(\partial \mathbf{G}_T, \mathcal{F}_T)$ under (NA^w). Let us consider the stopping time τ , being the first instant $\tau \geq u + 1$ such that $\tilde{g}_u^\infty + \tilde{g}_{u+1}^\infty + \dots + \tilde{g}_\tau^\infty = 0$. By Lemma 4.1 and Lemma 4.2, for all $r \geq u$, there exists $k_r \in L^0(\mathbf{R}, \mathcal{F}_r)$ such that $\tilde{g}_r^\infty 1_{r \leq \tau} = k_r \tilde{g}_u^\infty 1_{r \leq \tau}$. Let us introduce the first instant $\sigma \in \{t + 1, \tau\}$ such that $k_\sigma < 0$, which is possible by (4.2) since $k_u = 1$. We consider the case where $\tilde{g}_u^{\infty(1)} > 0$ and $\tilde{g}_u^{\infty(2)} < 0$, then $\tilde{g}_\sigma^{\infty(1)} < 0$ and $\tilde{g}_\sigma^{\infty(2)} > 0$. The symmetric case may be solved similarly.

Since $\mathbf{L}_u(\tilde{g}_u^\infty) = \mathbf{L}_\sigma(\tilde{g}_\sigma^\infty) = 0$, $\frac{\tilde{g}_u^{\infty(1)}}{\tilde{g}_u^{\infty(2)}} = \frac{\tilde{g}_\sigma^{\infty(1)}}{\tilde{g}_\sigma^{\infty(2)}} = -S_u^a = -S_\sigma^b$ by Lemma 4.2. As $(\tilde{g}_t^n + \dots + \tilde{g}_u^n)^{(2)} = -(\tilde{g}_{u+1}^n + \dots + \tilde{g}_T^n)^{(2)} - \delta_T^{n(2)}$ converges to $\tilde{g}_u^{\infty(2)} < 0$, up to some \mathcal{F}_u -measurable random sequences we may assume that $\tilde{g}_u^{n(2)} < 0$ and $(\tilde{g}_t^n + \dots + \tilde{g}_u^n)^{(2)} < 0$. Let

$$\beta^n := \frac{(g_t^n + \dots + g_u^n)^{(2)}}{\tilde{g}_u^{\infty(2)}}$$

which is positive and \mathcal{F}_u -measurable. Now we rewrite δ_T^n as

$$\delta_T^n = -g_t - \cdots - g_{u-1} - (g_u^n - \beta^n \tilde{g}_u^\infty) - \beta^n \tilde{g}_u^\infty - g_{u+1}^n - \cdots - g_T^n.$$

The second component of the term $g_u^n - \beta^n \tilde{g}_u^\infty$ is

$$(g_u^n - \beta^n \tilde{g}_u^\infty)^{(2)} = -(g_t + \cdots + g_{u-1})^{(2)}.$$

And the first component of this term is

$$\begin{aligned} (g_u^n - \beta^n \tilde{g}_u^\infty)^{(1)} &= g_u^{n(1)} - \beta^n \tilde{g}_u^{\infty(1)} \\ &= g_u^{n(1)} - (g_t^n + \cdots + g_u^n)^{(2)} \frac{\tilde{g}_u^{\infty(1)}}{\tilde{g}_u^{\infty(2)}} \\ &= g_u^{n(1)} + (g_t^n + \cdots + g_u^n)^{(2)} S_u^a \\ &= g_u^{n(1)} + g_u^{n(2)} S_u^a + (g_t^n + \cdots + g_{u-1}^n)^{(2)} S_u^a \\ &= (g_t + \cdots + g_{u-1})^{(2)} S_u^a \end{aligned}$$

Thus,

$$g_u^n - \beta^n \tilde{g}_u^\infty = ((g_t + \cdots + g_{u-1})^{(2)} S_u^a, -(g_t + \cdots + g_{u-1})^{(2)})$$

which is constant, and so satisfies $\liminf_n |g_u^n - \beta^n \tilde{g}_u^\infty| < +\infty$.

On the set $\Lambda_{u-1}^1 := \{(g_t + \cdots + g_{u-1})^{(2)} \geq 0\}$, it has $\mathbf{L}_u(g_u^n - \beta^n \tilde{g}_u^\infty) = (g_t + \cdots + g_{u-1})^{(2)} S_u^a - (g_t + \cdots + g_{u-1})^{(2)} S_u^a = 0$. This implies that

$$(g_u^n - \beta^n \tilde{g}_u^\infty) 1_{\Lambda_{u-1}^1} \in \partial \mathbf{G}_u.$$

On the set $\Lambda_{u-1}^2 := \{(g_t + \cdots + g_{u-1})^{(2)} < 0\}$, it has $\mathbf{L}_\sigma(g_u^n - \beta^n \tilde{g}_u^\infty) = (g_t + \cdots + g_{u-1})^{(2)} S_u^a - (g_t + \cdots + g_{u-1})^{(2)} S_\sigma^b = 0$. This implies that

$$(g_u^n - \beta^n \tilde{g}_u^\infty) 1_{\Lambda_{u-1}^2} \in \partial \mathbf{G}_\sigma.$$

As $\sigma \in \{u+1, \dots, T\}$, then $(g_u^n - \beta^n \tilde{g}_u^\infty) 1_{\Lambda_{u-1}^2} = \sum_{k=u+1}^T 1_{\sigma=k} 1_{\Lambda_{u-1}^2} (g_u^n - \beta^n \tilde{g}_u^\infty)$ where $\bar{g}_k := 1_{\sigma=k} 1_{\Lambda_{u-1}^2} (g_u^n - \beta^n \tilde{g}_u^\infty) \in L^0(\partial \mathbf{G}_k, \mathcal{F}_k)$. On the other hand, the second component of the term $(-\beta^n \tilde{g}_u^\infty - g_{u+1}^n - \cdots - g_T^n)^{(2)} = \delta_T^{n(2)} \rightarrow 0$ so that we may use the induction argument as $u \geq t+1$ to deduce that

$-\beta^n \tilde{g}_u^\infty - g_{u+1}^n - \dots - g_T^n = -\sum_{k=u}^T \check{g}_k^n + \tilde{\epsilon}_T^n$ where every $\check{g}_k^n \in L^0(\mathbf{G}_k, \mathcal{F}_k)$ admits a finite limit infimum and $\tilde{\epsilon}_T^n \rightarrow 0$ a.s. as $n \rightarrow \infty$. Finally, we may write

$$\begin{aligned}
\delta_T^n 1_{\Lambda_t} 1_{\Lambda_u} &= -\sum_{r=t}^{u-1} g_r 1_{\Lambda_u} - (g_u^n - \beta^n \tilde{g}_u^\infty) 1_{\Lambda_u} - \beta^n \tilde{g}_u^\infty 1_{\Lambda_u} - \sum_{r=u+1}^T g_r^n 1_{\Lambda_u} \\
&= -\sum_{r=t}^{u-1} g_r 1_{\Lambda_u} - (g_u^n - \beta^n \tilde{g}_u^\infty) 1_{\Lambda_{u-1}^1} 1_{\Lambda_u} - (g_u^n - \beta^n \tilde{g}_u^\infty) 1_{\Lambda_{u-1}^2} 1_{\Lambda_u} \\
&\quad - \sum_{k=u}^T \check{g}_k^n 1_{\Lambda_u} + \tilde{\epsilon}_T^n \\
&= -\sum_{r=t}^{u-1} g_r 1_{\Lambda_u} - (g_u^n - \beta^n \tilde{g}_u^\infty) 1_{\Lambda_{u-1}^1} 1_{\Lambda_u} - \sum_{k=u+1}^T \bar{g}_k 1_{\Lambda_u} - \sum_{k=u}^T \check{g}_k^n 1_{\Lambda_u} + \tilde{\epsilon}_T^n \\
&= -g_t 1_{\Lambda_u} - \dots - g_{u-1} 1_{\Lambda_u} - \sum_{k=u}^T \hat{g}_k^n 1_{\Lambda_u} + \tilde{\epsilon}_T^n
\end{aligned}$$

where $\hat{g}_u^n = (g_u^n - \beta^n \tilde{g}_u^\infty) 1_{\Lambda_{u-1}^1} 1_{\Lambda_u} + \check{g}_u^n \in L^0(\mathbf{G}_u, \mathcal{F}_u)$ and $\hat{g}_k^n = \bar{g}_k + \check{g}_k^n \in L^0(\mathbf{G}_k, \mathcal{F}_k)$ for $k \geq u+1$. Note that, for every k , $\liminf_n |\hat{g}_k^n| < \infty$ a.s. On the complementary set Γ_u of Λ_u , we may suppose that $g_u^n \rightarrow g_u \in L^0(\mathbf{G}_u, \mathcal{F}_u)$ by [12, Lemma 2.1.2] and finally assume without loss of generality that $\delta_T^n 1_{\Gamma_u} = -g_t 1_{\Gamma_u} - g_{t+1} 1_{\Gamma_u} - \dots - g_u 1_{\Gamma_u} - g_{u+1}^n 1_{\Gamma_u} - \dots - g_T^n$. We then deduce from above that $\delta_T^n 1_{\Lambda_t} = \delta_T^n 1_{\Lambda_t} 1_{\Lambda_u} + \delta_T^n 1_{\Lambda_t} 1_{\Gamma_u}$ is of the form $\delta_T^n 1_{\Lambda_t} = -g_t 1_{\Lambda_t} - \dots - g_{u-1} - g_u^n - \dots - g_T^n$ where $\liminf |g_u^n| < \infty$ a.s. This implies that we may repeat the procedure above with some date $u^1 \geq u+1$ instead of u . As the number of dates is finite, we finally conclude on Λ_t .

Next we consider the case where $\liminf_n |g_t^n| = +\infty$. Using the classical normalization procedure, we get

$$\bar{\delta}_T^n = -\bar{g}_t^n - \bar{g}_{t+1}^n - \dots - \bar{g}_T^n$$

where $\bar{\gamma}_T^n := \frac{\gamma_T^n}{|g_T^n|}$ and $\bar{g}_i^n := \frac{g_i^n}{|g_i^n|}$ for all $i = t, \dots, T$. Since $|\bar{g}_t^n| = 1$, up to the first case where $\liminf_n |g_t^n| < +\infty$ we deduce an equality of the type $\bar{g}_t^\infty + \bar{g}_{t+1}^\infty + \dots + \bar{g}_T^\infty = 0$. As $\bar{g}_t^\infty \neq 0$, let us consider the stopping time $\bar{\tau} \geq t+1$ as the first instant such that $\bar{g}_t^\infty + \bar{g}_{t+1}^\infty + \dots + \bar{g}_{\bar{\tau}}^\infty = 0$. Then, for

any CPS $(Z_r)_{r=t, \dots, T}$, Z_t, \dots, Z_T are collinear by Lemma 4.1. It follows that $Z_{t+1} \in \mathbf{G}_t^* \cap \mathbf{G}_{t+1}^*$ and $(Z_r)_{r=t+1, \dots, T}$ is a CPS for the market model from $t+1$ to T defined by the solvency sets $\tilde{\mathbf{G}}_{t+1} = \mathbf{G}_t + \mathbf{G}_{t+1} = \mathbf{G}_t \cup \mathbf{G}_{t+1} \subseteq (Z_t)^*$ and $\tilde{\mathbf{G}}_u = \mathbf{G}_u$ for $u \geq t+2$. This means that Condition (NA^w) holds for the model $(\tilde{\mathbf{G}}_r)_{r=t+1, \dots, T}$. Since $g_t^n + g_{t+1}^n \in L^0(\tilde{\mathbf{G}}_{t+1}, \mathcal{F}_{t+1})$, we may apply the induction hypothesis and deduce that $-\delta_T^n = \hat{g}_{t+1}^n + \hat{g}_{t+2}^n + \dots + \hat{g}_T^n$ where $\hat{g}_u^n \in L^0(\mathbf{G}_u, \mathcal{F}_u)$ satisfies $\liminf_n |\hat{g}_u^n| < \infty$ a.s. for $u \geq t+2$ and \hat{g}_{t+1}^n is either $g_t^n + g_{t+1}^n$ when the latter is convergent or 0 or some constant element $g_t^k + g_{t+1}^k$. In any case, we may assume that $-\delta_T^n = g_t^n + g_{t+1}^n + g_{t+2}^n + \dots + g_T^n$ where $g_u^n \in L^0(\mathbf{G}_u, \mathcal{F}_u)$, $u \leq t+1$, $g_u \in L^0(\mathbf{G}_u, \mathcal{F}_u)$, $u \geq t+2$, and $(g_t^n + g_{t+1}^n)_{n \geq 1}$ is a convergent sequence. Only the case where $\liminf_n |g_t^n| = +\infty$ is of interest as it is possible to conclude otherwise. By the normalisation procedure, we deduce the equality $\tilde{g}_t + \tilde{g}_{t+1} = 0$ where $\tilde{g}_t \in L^0(\mathbf{G}_t, \mathcal{F}_t)$ is positively collinear to g_t^n and $|\tilde{g}_t| = 1$ and $\tilde{g}_{t+1} \in L^0(\mathbf{G}_{t+1}, \mathcal{F}_{t+1})$. We deduce that $S_t^a = S_{t+1}^b$ when $\bar{g}_t^{\infty(1)} > 0$ and $\bar{g}_t^{\infty(2)} < 0$ and $S_t^b = S_{t+1}^a$ otherwise. It follows that $S_t^a = S_{t+1}^b = E(S_{t+1}^b | \mathcal{F}_t)$ or $S_t^b = S_{t+1}^a = E(S_{t+1}^a | \mathcal{F}_t)$. On the set $\{\bar{g}_t^{\infty(1)} > 0\}$, let us consider the first instant $\hat{\tau} \geq t+1$ such that $(S_{\hat{\tau}}^a, -1) \in \mathbf{G}_{\hat{\tau}}$. By Condition (E), the stopping time $\hat{\tau}$ satisfies $\hat{\tau} \leq T$. Hence, it is possible to rewrite $g_t^n = \sum_{r=t+1}^T g_n^t 1_{\tau \leq r}$ where, for all $r \geq t+1$, $g_n^t 1_{\tau \leq r} \in L^0(\mathbf{G}_r, \mathcal{F}_r)$. Similarly, we may rewrite g_t^n on the set $\{\bar{g}_t^{\infty(1)} < 0\}$ so that we may apply the induction hypothesis and conclude about the statement from t to T . The conclusion follows. \square

4.2. Proof of Corollary 3.7.

The inclusion $\Gamma_\xi \subseteq [\sup_{Q \in \mathcal{M}^\infty(P)} \mathbb{E}_Q \xi, \infty)$ is trivial. Let us now consider $x \geq \sup_{Q \in \mathcal{M}^\infty(P)} \mathbb{E}_Q \xi, \infty)$ and suppose by contradiction that $x \notin \Gamma_\xi$, i.e. $\xi - x \notin \mathcal{L}_0^T$. As $\mathcal{L}_0^T \cap L^1(\mathcal{P})$ is closed in L^1 under (NA^w), we deduce by the Hahn-Banach separation theorem the existence of $\eta \in L^\infty$ and $c \in \mathbf{R}$ such that $\mathbb{E}(\eta X) < c < \mathbb{E}(\eta(\xi - x))$ for all $X \in \mathcal{L}_0^T \cap L^1(\mathcal{P})$. Since \mathcal{L}_0^T is a cone, we deduce that $\mathbb{E}(\eta X) \leq 0$ for all $X \in \mathcal{L}_0^T \cap L^1(\mathcal{P})$. Moreover, as \mathcal{L}_0^T contains $-L^0(\mathbf{R}_+, \mathcal{F}_T)$ we deduce that $\eta \geq 0$ and, after normalization, we have $\mathbb{E}(\eta) = 1$. Moreover, if we take $\eta' = dQ/d\mathbb{P}$ such that $Q \in \mathcal{M}^\infty(P) \neq \emptyset$, then we may choose $\alpha \in (0, 1)$ sufficiently close to 1 so that $\hat{\eta} := \alpha\eta + (1 - \alpha)\eta'$ satisfies $\mathbb{E}(\hat{\eta}(\xi - x)) > 0$ since $c > 0$. Moreover, $\mathbb{E}(\hat{\eta} X) \leq 0$ for all $X \in \mathcal{L}_0^T \cap L^1(\mathcal{P})$ and $\hat{\eta} > 0$ satisfies $\mathbb{E}\hat{\eta} = 1$. Therefore, if we define \hat{Q} such that $d\hat{Q}/d\mathbb{P} = \hat{\eta}$, then $\hat{Q} \in \mathcal{M}^\infty(P)$ in contradiction with $\mathbb{E}(\hat{\eta}(\xi - x)) > 0$, i.e. $x < \mathbb{E}_{\hat{Q}} \xi$. \square

4.3. Auxiliary results

We denote by $\mathcal{T}_{t,T}$ the set of all stopping times τ with values in $\{t, \dots, T\}$. By definition, if $\tau \in \mathcal{T}_{t,T}$, then $\{\tau = u\} \in \mathcal{F}_u$ whatever $u = t, \dots, T$ hence $\{\tau \geq u\} \in \mathcal{F}_u$.

Lemma 4.1. *Suppose that for some stopping time $\tau \in \mathcal{T}_{t,T}$, we have $g_t + \dots + g_\tau = 0$ a.s. where $g_u \in L^1(\mathbf{G}_u, \mathcal{F}_u)$ are integrable selections of \mathbf{G}_u for all $u = t, \dots, T$. Moreover, assume that with probability one, $g_t + \dots + g_r \neq 0$ for all $r < \tau$. Then, for all bounded CPS Z, Z_t, \dots, Z_τ are a.s. collinear.*

Proof. By assumption, we have $\sum_{u=t}^T Z_T g_u 1_{u \leq \tau} = 0$. Taking the conditional expectation knowing \mathcal{F}_t , we deduce that

$$0 = \sum_{u=t}^T E(Z_T g_u 1_{u \leq \tau} | \mathcal{F}_t) = \sum_{u=t}^T E(Z_u g_u 1_{u \leq \tau} | \mathcal{F}_t).$$

By duality, $Z_u g_u \geq 0$ hence $Z_u g_u 1_{u \leq \tau} = 0$ a.s. for all $u = t, \dots, T$. Notice that $\{T \leq \tau\} = \{T = \tau\} = \Omega \setminus \{\tau \leq T-1\} \in \mathcal{F}_{T-1}$ and $g_T 1_{T \leq \tau} = g_T 1_{T \leq \tau} - (g_t + \dots + g_{T-1}) 1_{\tau=T} \in L^0(\mathbf{R}^2, \mathcal{F}_{T-1})$. Therefore, $Z_T g_T 1_{T \leq \tau} = 0$ implies that $Z_{T-1} g_T 1_{T \leq \tau} = 0$ by taking the conditional expectation knowing \mathcal{F}_{T-1} . It follows that $Z_{T-1} (g_{T-1} 1_{T-1 \leq \tau} + g_T 1_{T \leq \tau}) = 0$. Suppose by induction that $Z_u (g_u 1_{u \leq \tau} + \dots + g_T 1_{T \leq \tau}) = 0$ for some $u \in \{t+1, \dots, T\}$. As $g_u 1_{u \leq \tau} + \dots + g_T 1_{T \leq \tau} = -(g_t + \dots + g_{u-1}) 1_{u \leq \tau} \in L^0(\mathbf{R}^2, \mathcal{F}_{u-1})$, we deduce from $Z_u (g_u 1_{u \leq \tau} + \dots + g_T 1_{T \leq \tau}) = 0$ that $Z_{u-1} (g_u 1_{u \leq \tau} + \dots + g_T 1_{T \leq \tau}) = 0$ by taking the conditional expectation knowing \mathcal{F}_{u-1} . As $Z_{u-1} g_{u-1} 1_{u-1 \leq \tau} = 0$ a.s., we finally get that $Z_{u-1} (g_{u-1} 1_{u-1 \leq \tau} + \dots + g_T 1_{T \leq \tau}) = 0$.

As $Z_u (g_u 1_{u \leq \tau} + \dots + g_T 1_{T \leq \tau}) = 0$ and $Z_{u-1} (g_u 1_{u \leq \tau} + \dots + g_T 1_{T \leq \tau}) = 0$ where $g_u 1_{u \leq \tau} + \dots + g_T 1_{T \leq \tau} = -(g_t + \dots + g_{u-1}) 1_{u \leq \tau} \neq 0$ if $u \leq \tau$, we deduce that Z_u and Z_{u-1} are collinear if $u \leq \tau$. The conclusion follows. \square

Lemma 4.2. *Let τ be the first passage time such that $g_t + g_{t+1} + \dots + g_u = 0$ a.s. where $g_u \in L^1(\mathbf{G}_u, \mathcal{F}_u)$ are integrable selections of \mathbf{G}_u for all $u = t, \dots, T$. If $g_t \neq 0$, then (NA^w) implies that there exist some stopping times $\sigma \in \{t+1, \dots, \tau\}$ such that g_t and g_σ are collinear with negative coefficient of collinearity.*

Proof. The first passage time τ is obviously a stopping time which can be expressed as

$$\tau := \min\{u : t+1 \leq u \leq T \text{ such that } g_t + g_{t+1} + \dots + g_u = 0 \text{ a.s.}\}.$$

This stopping time τ satisfies the assumptions of Lemma 4.1 so that the condition (NA^w) implies that, for every bounded CPS Z , Z_t is collinear with Z_u a.s. if $u \leq \tau$. By taking the conditional expectation knowing \mathcal{F}_t , from the equality $\sum_{u=t}^T Z_T g_u 1_{u \leq \tau} = 0$, we deduce by the tower property and by duality that $Z_u g_u 1_{u \leq \tau} = 0$ a.s. for all $u = t, \dots, T$. Thus, $Z_t g_u 1_{u \leq \tau} = 0$. By the definition of τ , $Z_t g_t 1_{t \leq \tau} = Z_t g_t = 0$ as $\tau \geq t + 1$. Now the equality $Z_t g_u 1_{u \leq \tau} = 0$ and $Z_t g_t = 0$ implies that $g_u 1_{u \leq \tau}$ and g_t are collinear, i.e., $g_u 1_{u \leq \tau} = k_u g_t$ where $k_u \in L^0(\mathbf{R}, \mathcal{F}_u)$ satisfies $k_u = k_u 1_{u \leq \tau}$. As $\sum_{u=t}^T g_u 1_{u \leq \tau} = g_t + \sum_{u=t+1}^T k_u g_t = (1 + \sum_{u=t+1}^T k_u) g_t = 0$ and $g_t \neq 0$, there exist a.s. a first instant u such that $k_u < 0$. Define the stopping time

$$\sigma := \min\{u : t + 1 \leq u \leq \tau \text{ such that } k_u < 0\}.$$

As $1_{\sigma \leq \tau} = 1$ and $k_\sigma < 0$, then we can deduce from $g_\sigma 1_{\sigma \leq \tau} = k_\sigma g_t$ that g_t and g_σ are collinear with negative coefficient of collinearity. \square

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