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# The Dalang–Morton–Willinger version of the fundamental theorem of asset pricing for the Bid-Ask financial market model.

Zhao Jun,<sup>1</sup> Emmanuel LEPINETTE<sup>1</sup>

<sup>1</sup> *Department of Applied Mathematics, Nanjing University of Science and Technology, Nanjing 210094, Jiangsu, P.R. China.  
Email: zhaojun.njust@hotmail.com*

<sup>2</sup> *Ceremade, UMR CNRS 7534, Paris Dauphine University, PSL National Research, Place du Maréchal De Lattre De Tassigny, 75775 Paris cedex 16, France.  
Email: emmanuel.lepinette@ceremade.dauphine.fr*

**Abstract:** We provide an equivalent characterisation of absence of arbitrage opportunity  $NA^w$  for the Bid and Ask financial market model analog to the Dalang–Morton–Willinger theorem formulated for discrete-time financial market models without friction. This result completes the Grigoriev theorem for conic models in the two dimensional case by showing that the set of all terminal liquidation values is closed under  $NA^w$ .

**Keywords and phrases:** Financial market models, Absence of arbitrage opportunities, Liquidation value, Bid and Ask prices, Transaction costs.

2000 MSC: 60G44, G11-G13.

## 1. Introduction

In the discrete-time models without friction, it is well known that the self-financing portfolio processes are modelled by their liquidation values. An arbitrage opportunity is the terminal value  $V_T$  at time  $T$  of such a portfolio process, starting from a zero initial capital, and such that  $P(V_T \geq 0) = 1$  and  $V_T \neq 0$ . The Dalang–Morton–Willinger (DMW) theorem [2] formulates an equivalent characterisation of absence of such an arbitrage opportunity  $NA^w$ . Precisely, it states that  $NA^w$  is equivalent to the existence of a martingale probability measure for the price process and, moreover, under  $NA^w$  the set

of all terminal portfolio processes starting from the zero initial capital is closed in probability.

The models with friction was first considered in the pioneering paper [7] and, later, were extensively studied, e.g. in the papers [9], [5], [12], [4], [11]. With proportional transaction costs, it is classical to express the portfolio processes as stochastic vectors of the invested physical units because, paying transaction costs, the exchanges are allowed between the assets. Actually, while the analog of the self-financing condition property is simple when the portfolio processes are expressed in physical units (see [10, Chapter 3]), there is no simple dynamics for their liquidation values. This is why, most of the characterisations of absence of arbitrage opportunities are formulated by expressing the portfolio processes in physical units. In particular, the Grigoriev theorem [3] provides such a characterization for the two dimensional conic model that may be seen as a financial market model with a Bond and a risky asset defined by its Bid and Ask prices. We may see it as a generalization of the famous result of [7], which is formulated under the No Free Lunch condition and appears to be equivalent to the  $NA^w$  by Grigoriev's theorem. The condition  $NA^w$  is the same than in the frictionless models, i.e. there is no positive terminal liquidation value when starting from the zero initial capital. Equivalently,  $NA^w$  holds if and only if there is no vector-valued portfolio processes starting from zero and ending up with a terminal value in the first orthant  $\mathbf{R}_+^2$ , see [10, Section 3.2.1]. It appears that the set of all vector-valued terminal portfolio processes is not necessarily closed under  $NA^w$ , see [10, Example 1, Section 3.2.4 ] so that the Grigoriev theorem is not exactly the analog of the DMW theorem. Actually, with proportional transaction costs, closedness is only obtained under a strong absence of arbitrage opportunities, i.e. the robust no-arbitrage property (see [10, Lemma 3.2.8]) meaning that the  $NA^w$  condition still holds with strictly smaller transaction costs.

In this paper, our main contribution is a version of the DMW theorem we obtain for the Bid and Ask model by proving the closedness of the set of all terminal liquidation values.

## 2. Model and basic properties

### Notations.

$e_1 = (1, 0) \in \mathbf{R}^2$ .

For a subset  $G$  of  $\mathbf{R}^2$ ,  $\partial G$  is the boundary of  $G$  and  $\text{int } G$  is its interior.

$\mathbf{R}_+^2$  is the set of all vectors in  $\mathbf{R}^2$  having only non negative components.

$\mathbb{E}$  designates the expectation of a random variable. When necessary, we denote it  $\mathbb{E}_Q$  when it is considered under  $Q \sim P$ .

For a set-valued random mapping  $E$ ,  $L^0(E, \mathcal{F})$  is the metric space of all  $E$ -valued random variables which are  $\mathcal{F}$ -measurable.

$L^p(E, \mathcal{F}, P)$ ,  $p \in [1, \infty)$  (resp.  $p = \infty$ ), is the normed space of all  $E$ -valued random variables which are  $\mathcal{F}$ -measurable and admitting a moment of order  $p$  under the probability  $P$  (resp. bounded).

For any subset  $\mathcal{X}$  of  $L^0(\mathbf{R}, \mathcal{F})$  containing  $-L_+^0$  and  $p \in [1, \infty) \cup \{0\}$ , we denote by  $\overline{\mathcal{X}}^p$ , the closure of  $\mathcal{X}^p := \mathcal{X} \cap L^p(\mathbf{R}, \mathcal{F}, P)$  with respect to the  $L^p$ -topology (the topology of convergence in probability if  $p = 0$ ). If  $Q \sim P$ , we denote by  $\overline{\mathcal{X}}^p(Q)$  the closure under  $Q$ .

**The Bid-Ask model.** We consider a discrete-time complete stochastic basis  $(\Omega, \mathbb{F} := (\mathcal{F}_t)_{t=0, \dots, T}, P)$  to construct a financial market model defined by a Bond  $S^0 = 1$  and one risky asset characterised by Bid and Ask price processes  $S^b$  and  $S^a$  adapted to the filtration  $\mathbb{F}$ .

Equivalently, the model may be defined by a sequence of  $\mathbb{F}$ -adapted closed and conic sets  $(\mathbf{G}_t)_{t=0, \dots, T}$  of  $\mathbf{R}^d$ , i.e. such that:

$$\text{Graph } \mathbf{G}_t := \{(\omega, x) \in \Omega \times \mathbf{R}^d : x \in \mathbf{G}_t(\omega)\} \in \mathcal{F}_t \otimes \mathcal{B}(\mathbf{R}^d), \quad t = 0, \dots, T.$$

where  $\mathbf{G}_t$ ,  $t = 0, \dots, T$ , is interpreted as the set of all financial positions it is possible to liquidate without any debt. Precisely, let us define the liquidation value process as

$$\mathbf{L}_t(x) := x^1 + (x^2)^+ S_t^b - (x^2)^- S_t^a, \quad x = (x^1, x^2) \in \mathbf{R}^2,$$

where  $x$  is the vector of physical units of assets  $S^0$  and  $S$  respectively held in the portfolio at time  $t$ . We recall the notation  $x^+ = \max(x, 0)$  and  $x^- = -\min(x, 0)$ . We may show that  $\mathbf{L}_t(x) = \sup\{\alpha \in \mathbf{R} : x - \alpha e_1 \in \mathbf{G}_t\}$ , i.e.  $\mathbf{L}_t(x)$  is the maximal amount of cash the agent may obtain when liquidating the financial position  $x$ , and  $\mathbf{G}_t = \{x \in \mathbf{R}^2 : \mathbf{L}_t(x) \geq 0\}$ . This is a particular case of the the Kabanov model with proportional transaction costs, see [10, Chapter 3]. At any time  $t = 0, \dots, T$ , we easily observe that the following properties hold:

**Lemma 2.1.**

1. *The mapping  $x \mapsto \mathbf{L}_t(x)$  is concave hence continuous.*
2.  *$\mathbf{L}_t(x^0, z) = x^0 + \mathbf{L}_t((0, z))$  for all  $x^0 \in \mathbf{R}$  and  $z \in \mathbf{R}^d$ .*

3.  $x - \mathbf{L}_t(x)e_1 \in \partial G_t$  for all  $x \in \mathbf{R}^2$ .

Note that the boundary  $\partial \mathbf{G}_t = \{x \in \mathbf{R}^2 : \mathbf{L}_t(x) = 0\}$  is composed of two half lines respectively generated by the random vectors  $(S_t^a, -1)$ . and  $(-S_t^b, 1)$ . The positive dual of  $\mathbf{G}_t$  is  $\mathbf{G}_t^* := \{z \in \mathbf{R}^2 : zx \geq 0 : \text{for all } x \in \mathbf{G}_t\}$ . The latter is a random cone of  $\mathbf{R}_+^2$  whose boundary is the union of the two half lines generated by the vectors  $(1, S_t^b)$  and  $(1, S_t^a)$ . We have  $\mathbf{G}_t^* \setminus \{0\} \subseteq \text{int} \mathbf{R}_+^2$ .

**Definition 2.2.** A self-financing portfolio process  $(V_t)_{t=0, \dots, T}$  starting from the initial endowment  $V_{0-}$  is an  $\mathbb{F}$ -adapted process such that

$$\Delta V_t := V_t - V_{t-1} \in -\mathbf{G}_t, \quad \forall t = 0, \dots, T \quad a.s. \quad (2.1)$$

The interpretation is simple; when changing the position  $V_{t-1}$  into a new one  $V_t$  at time  $t$ , we liquidate without any debt the remaining part, i.e.  $-\Delta V_t \in \mathbf{G}_t$ . In the Kabanov model, we also interpret (2.1) as the paiement of proportional transaction costs to change  $V_{t-1}$  into  $V_t$ . We introduce the set of all terminal values at time  $t \leq T$  of portfolio processes starting from the zero initial endowment at time  $u \leq t$  i.e.

$$\mathcal{A}_u^t := \sum_{s=u}^t L^0(-\mathbf{G}_s, \mathcal{F}_s).$$

Associated to this set above, the terminal liquidation values are

$$\mathcal{L}_u^t := \{\mathbf{L}_t(V) : V \in \mathcal{A}_u^t\}.$$

**Remark 2.3.** Notice that for any  $\gamma \in \mathcal{L}_0^T$ ,  $\gamma e_1 = \sum_{t=0}^T (-g_t) \in \mathcal{A}_0^T$  for some  $g_t \in L^0(\mathbf{G}_t, \mathcal{F}_t)$ . Indeed,  $\gamma = \mathbf{L}_T(V_T)$  where  $V_T = \sum_{t=0}^T \Delta V_t$  with  $\Delta V_t \in -L^0(\mathbf{G}_t, \mathcal{F}_t)$  for all  $t$  and  $V_{0-} = 0$ . Moreover,  $\gamma e_1 - V_T = \mathbf{L}_T(V_T)e_1 - V_T \in -\mathbf{G}_T$ . Thus,  $\gamma e_1 = V_T - \hat{g}_T$  where  $\hat{g}_T \in L^0(\mathbf{G}_T, \mathcal{F}_T)$  and finally  $\gamma e_1 \in \mathcal{A}_0^T$ . Futhermore, we may assume that  $g_t \in L^0(\partial \mathbf{G}_t, \mathcal{F}_t)$  for all  $0 \leq t \leq T-1$ . Indeed, let us write  $\gamma e_1 = -(g_0 - \mathbf{L}_0(g_0)e_1) - (g_1 + \mathbf{L}_0(g_0)e_1) + \sum_{t=2}^T (-g_t)$  where  $g_0 - \mathbf{L}_0(g_0)e_1 \in \partial \mathbf{G}_0$ . As  $\mathbf{L}_0(g_0) \geq 0$ , i.e.,  $\mathbf{L}_0(g_0)e_1 \in \mathbf{R}_+ e_1$ , then  $g_1 + \mathbf{L}_0(g_0)e_1 \in \mathbf{G}_1$ . So replace  $g_0$  and  $g_1$  respectively by  $g_0 - \mathbf{L}_0(g_0)e_1$  and  $g_1 + \mathbf{L}_0(g_0)e_1$  and repeat the above procedure for  $t \geq 1$  to obtain that  $g_t \in \partial \mathbf{G}_t$  for all  $0 \leq t \leq T-1$ .

In the following, we suppose a technical condition (E) when  $T \geq 4$  which is satisfied by the classical examples of market models with Bid and Ask prices of the literature:

**Condition (E):** When  $T \geq 4$ , for all  $t \leq T - 3$ , we have

$$\begin{aligned} \{S_t^a = S_{t+1}^b = E(S_{t+1}^b | \mathcal{F}_t)\} &\subseteq \bigcup_{r \geq t+1} \{(S_t^a, -1) \in \mathbf{G}_r\}, \quad \text{a.s.}, \\ \{S_t^b = S_{t+1}^a = E(S_{t+1}^a | \mathcal{F}_t)\} &\subseteq \bigcup_{r \geq t+1} \{(-S_t^b, 1) \in \mathbf{G}_r\}, \quad \text{a.s.} \end{aligned}$$

Recall that  $(S_t^a, -1)$  and  $(S_t^b, -1)$  are the two generator vectors of  $\mathbf{G}_t$ . Notice that Condition (E) trivially holds when the l.h.s. of the inclusion is almost surely empty. This is the case when we naturally suppose that it is not possible to know by advance at time  $t$  the value of  $S_{t+1}^b$  or  $S_{t+1}^a$ . Let us present some examples where Condition (E) holds:

**Example 1:** This first example is a generalization of the model proposed in [5]. Let us consider a positive stochastic process  $(S_t)_{t=0, \dots, T}$  we interpret as the mid-prices and a process  $(\epsilon_t)_{t=0, \dots, T}$  with values in  $[0, 1)$  we interpret as proportional transaction cost rates. We suppose that  $(S_t)_{t=0, \dots, T}$  and  $(\epsilon_t)_{t=0, \dots, T}$  are two independent processes and for every  $t$ , either the random variables  $\frac{S_{t+1}}{S_t}$  or  $\frac{1+\epsilon_t}{1-\epsilon_{t+1}}$  do not admit any atom. This is the case when  $\frac{S_{t+1}}{S_t}$  admits a density and the rate process  $\epsilon$  is constant. The Bid and Ask prices are given by

$$S_t^b := S_t(1 - \epsilon_t), \quad S_t^a := S_t(1 + \epsilon_t).$$

Then,

$$\begin{aligned} P(S_t^a = S_{t+1}^b) &= \mathbb{E}(\mathbb{E}(1_{S_t^a = S_{t+1}^b} | \sigma(\epsilon_t : t \leq T))) = \mathbb{E}(\mathbb{E}(1_{\frac{S_{t+1}}{S_t} = \frac{1+\epsilon_t}{1-\epsilon_{t+1}}} | \sigma(\epsilon_t : t \leq T))) \\ &= \mathbb{E}\left(f\left(\frac{1 + \epsilon_t}{1 - \epsilon_{t+1}}\right)\right), \end{aligned}$$

where, by the independence assumption,  $f(x) = P\left(\frac{S_{t+1}}{S_t} = x\right)$ . Using the hypothesis on  $S$ , we get that  $f = 0$ . A similar reasoning holds when  $\frac{1+\epsilon_t}{1-\epsilon_{t+1}}$  does not admit any atom so that Condition (E) trivially holds.

**Example 2:** We consider a generalization of the Cox-Ross-Rubinstein model with bid-ask spreads proposed in [8, Section 4]. The Bid and Ask prices are given by

$$S_t^b = (1 + \zeta_t^b)S_{t-1}^a, \quad S_t^a = (1 + \zeta_t^a)S_{t-1}^b,$$

where the vector-valued process  $\zeta = (\zeta^b, \zeta^a)$  is such that  $S_t^b \leq S_t^a$  a.s. for all  $t \leq T$ . Moreover, it is supposed in [8, Section 4] that  $\zeta^b$  and  $\zeta^a$  take two distinct values. With  $\mathcal{F}_t = \sigma(\zeta_r^b, \zeta_r^a, S_r : r \leq t)$ , this hypothesis trivially implies that the equalities  $\zeta_t^b = \mathbb{E}(\zeta_{t+1}^b | \mathcal{F}_t)$  and  $\zeta_t^a = \mathbb{E}(\zeta_{t+1}^a | \mathcal{F}_t)$  do not hold almost surely. Here, we only suppose that the equalities  $\zeta_t^b = \mathbb{E}(\zeta_{t+1}^b | \mathcal{F}_t) = 0$  and  $\zeta_t^a = \mathbb{E}(\zeta_{t+1}^a | \mathcal{F}_t) = 0$  do not hold almost surely so that we may verify that Condition (E) holds. Indeed, if for example  $S_{t+1}^b = S_t^a = \mathbb{E}(S_{t+1}^b | \mathcal{F}_t)$  then  $\zeta_{t+1}^b = 0 = \mathbb{E}(\zeta_{t+1}^b | \mathcal{F}_t)$ . We may also verify that Condition (E) holds if we only suppose that  $\zeta_t^b = 0$  if and only if  $\zeta_t^a = 0$ . Indeed, if for example  $S_{t+1}^b = S_t^a$ , then  $S_{t+1}^a = S_t^b$  hence  $S_{t+1}^a \leq S_{t+1}^b$ . Therefore,  $S_t^a = S_{t+1}^a$  and  $S_t^b = S_{t+1}^b$ , i.e.  $\mathbf{G}_t = \mathbf{G}_{t+1}$ .

**Example 3:** Following [5], we suppose that the Bid and Ask prices are given by

$$S_t^b = S_t - \epsilon_t, \quad S_t^a = S_t + \epsilon_t, \quad t \leq T,$$

where  $S$  and  $\epsilon$  are two positive processes such  $S^b > 0$ . Then, Condition (E) trivially holds when  $S$  and  $\epsilon$  are independent and one of them does not admit any atom since, in this case,  $P(S_{t+1}^b = S_t^a) = P(S_{t+1}^a = S_t^b) = 0$  for all  $u \geq t + 1$ .

### 3. The DMW theorem for discrete-time Bid-Ask models

**Definition 3.1.** We say that the financial market model defined by  $\mathbf{G}$  satisfies the weak no-arbitrage property ( $NA^w$ ) if  $\mathcal{L}_0^T \cap L^0(\mathbf{R}_+, \mathcal{F}_T) = \{0\}$ .

**Lemma 3.2.** ( $NA^w$ ) holds if and only if  $\mathcal{A}_0^T \cap L^0(\mathbf{G}_T, \mathcal{F}_T) \subseteq L^0(\partial \mathbf{G}_T, \mathcal{F}_T)$ .

*Proof.* Suppose that ( $NA^w$ ) holds and consider  $V_T \in \mathcal{A}_0^T \cap L^0(\mathbf{G}_T, \mathcal{F}_T)$ . Then,  $\mathbf{L}(V_T) \in \mathcal{L}_0^T \cap L^0(\mathbf{R}_+, \mathcal{F}_T)$  hence  $\mathbf{L}(V_T) = 0$ , i.e.  $V_T \in \partial \mathbf{G}_T$  a.s. Reciprocally, suppose that  $\mathcal{A}_0^T \cap L^0(\mathbf{G}_T, \mathcal{F}_T) \subseteq L^0(\partial \mathbf{G}_T, \mathcal{F}_T)$ . Any  $\gamma_T \in \mathcal{L}_0^T$  is such that  $\gamma_T e_1 \in \mathcal{A}_0^T$  since  $\gamma_T = \mathbf{L}(V_T)$  for some  $V_T \in \mathcal{A}_0^T$  such that  $V_T - \mathbf{L}(V_T)e_1 \in \mathbf{G}_T$ .  $\square$

The assumption of the following lemma is clearly satisfied for the Bid-Ask model.

**Lemma 3.3.** If  $\mathbf{G}_T$  strictly dominates  $\mathbf{R}_+^d$ , i.e.,  $\mathbf{R}_+^d \setminus \{0\} \subset \text{int } \mathbf{G}_T$ , then we have

$$\mathcal{L}_0^T \cap L^0(\mathbf{R}_+, \mathcal{F}_T) = \{0\} \iff \mathcal{A}_0^T \cap L^0(\mathbf{R}_+^d, \mathcal{F}_T) = \{0\}.$$

*Proof.* ( $\Leftarrow$ ) This part is trivial since  $\mathcal{L}_0^T e_1 = \mathcal{A}_0^T \cap L^0(\mathbf{R}e_1, \mathcal{F}_T) \subseteq \mathcal{A}_0^T$ .

( $\Rightarrow$ ) Let  $V_T \in \mathcal{A}_0^T \cap L^0(\mathbf{R}_+^d, \mathcal{F}_T)$ . Since  $\mathbf{R}_+^d \subseteq \mathbf{G}_T$ , then  $\mathbf{L}_T(V_T) \geq 0$ . So the condition  $\mathcal{L}_0^T \cap L^0(\mathbf{R}_+, \mathcal{F}_T) = \{0\}$  implies  $\mathbf{L}_T(V_T) = 0$ , hence  $V_T \in \partial \mathbf{G}_T \cap \mathbf{R}_+^d = \{0\}$ .  $\square$

The equivalent condition to  $(NA^w)$ , as expressed in the lemma above, is studied by Grigoriev [3] and [10, Theorem 3.2.15]. The Grigoriev theorem states that Condition  $(NA^w)$  holds if and only if there exists Consistent Price Systems (CPS) evolving in the positive duals of the solvency sets, precisely martingales  $(Z_t)_{t=0, \dots, T}$  satisfying  $Z_t \in \mathbf{G}_t^* \setminus \{0\}$  for all  $t = 0, \dots, T$ . This result is a weaker form of the Dalang-Morton-Willinger theorem, see [2]. Without friction, the set of all terminal claims obtained from the zero initial endowment appears to be closed. With proportional transaction costs, this is no more the case if the terminal claims are expressed in physical units, see [10, Example 1, Section 3.2.4]. In this paper, we show a Dalang-Morton-Willinger version of the Grigoriev theorem, i.e. we show that under Condition  $(NA^w)$ , the set of all terminal liquidation values we get from portfolio processes starting from zero is closed. To do so, we use the technical condition (E) when  $T \geq 4$ . This is an open problem whether our result still holds without (E) for  $T \geq 4$ .

**Theorem 3.4.** *Suppose that Condition (E) holds for  $T \geq 4$ . The following conditions are equivalent:*

- 1  $(NA^w)$
- 2  $\mathcal{L}_0^T$  is closed in probability and  $\mathcal{L}_0^T \cap L^0(\mathbf{R}_+, \mathcal{F}_T) = \{0\}$ .
- 3 There exists  $Q \sim P$  with  $dQ/dP \in L^\infty$  such that  $\mathbb{E}_Q \mathbf{L}_T(V) \leq 0$  for all  $\mathbf{L}_T(V) \in \mathcal{L}_0^T \cap L^1(P)$ .
- 4 There exists  $Q \sim P$  with  $dQ/dP \in L^\infty$  such that for all  $t \leq T - 1$ ,  $\mathbb{E}_Q(S_{t+1}^a | \mathcal{F}_t) \leq S_t^b$  and  $\mathbb{E}_Q(S_{t+1}^b | \mathcal{F}_t) \leq S_t^a$ .
- 5 There exists  $Q \sim P$  with  $dQ/dP \in L^\infty$  and a  $Q$ -martingale  $\tilde{S}$  such that  $\tilde{S} \in [S^b, S^a]$ .

In the following, we denote by  $\mathcal{M}(P)$  the set of all  $Q \sim P$  such that  $\mathbb{E}_Q \mathbf{L}_T(V) \leq 0$  for all  $\mathbf{L}_T(V) \in \mathcal{L}_0^T$ . For any contingent claim  $\xi \in L^0(\mathbf{R}, \mathcal{F}_T)$ , we define  $\Gamma_\xi$  as the set of all initial endowments we need to start a portfolio process whose terminal liquidation value coincides with  $\xi$ , i.e.

$$\Gamma_\xi := \{x \in \mathbf{R} : \exists V \in \mathcal{A}_0^T : \mathbf{L}_T(xe_1 + V) = \xi\}.$$



**Corollary 3.5.** *Suppose that Condition (E) holds for  $T \geq 4$ . Let us consider a payoff  $\xi \in L^0(\mathbf{R}, \mathcal{F}_T)$  satisfying  $\sup_{Q \in \mathcal{M}(P)} \mathbb{E}_Q |\xi| < \infty$ . Then, under Condition (NA<sup>w</sup>),  $\Gamma_\xi = [\sup_{Q \in \mathcal{M}(P)} \mathbb{E}_Q \xi, \infty)$ .*

## 4. Proofs of the main results

### 4.1. Proof of Theorem 3.4.

Note that the implication (2)  $\Rightarrow$  (3) is immediate by [10, Theorem 2.1.4]. The implications (3)  $\Rightarrow$  (1) and (2)  $\Rightarrow$  (1) are also trivial. It remains to show that (1)  $\Rightarrow$  (2). Suppose that (NA<sup>w</sup>) and let us prove that  $\mathcal{L}_0^T$  is closed in probability. Recall that, by the Grigoriev theorem, there exists a CPS  $Z$ , i.e. a martingale  $Z$  such that  $Z_u \in \mathbf{G}_u^* \setminus \{0\}$  for all  $u \leq T$ . For the one step model there is nothing to prove since  $\mathcal{L}_T^T = -L^0(\mathbf{R}_+, \mathcal{F}_T)$ . Indeed, if  $\mathbf{L}_T(-g_T) \geq 0$  for some  $g_T \in L^0(\mathbf{G}_T, \mathcal{F}_T)$ , then  $g_T \in (-\mathbf{G}_T) \cap \mathbf{G}_T \subseteq \partial \mathbf{G}_T$  hence  $\mathbf{L}_T(-g_T) = 0$ . Let us consider the two step model.

Assume that the sequence  $\gamma_T^n \in \mathcal{L}_{T-1}^T$  converges to  $\gamma_T^\infty$ . From Remark 2.3, we may suppose that  $\gamma_T^n e_1 = -g_{T-1}^n - \tilde{g}_T^n$  where  $g_{T-1}^n \in L^0(\partial \mathbf{G}_{T-1}, \mathcal{F}_{T-1})$  and  $\tilde{g}_T^n \in L^0(\mathbf{G}_T, \mathcal{F}_T)$ .

On the set  $\Lambda_{T-1} := \{\liminf |g_{T-1}^n| = \infty\} \in \mathcal{F}_{T-1}$ , we normalize the sequences by setting  $\tilde{\gamma}_T^n := \frac{\gamma_T^n}{|g_{T-1}^n|}$ ,  $\tilde{g}_{T-1}^n := \frac{g_{T-1}^n}{|g_{T-1}^n|}$  and  $\tilde{g}_T^n := \frac{\tilde{g}_T^n}{|g_{T-1}^n|}$ . Then,

$$\tilde{\gamma}_T^n e_1 = -\tilde{g}_{T-1}^n - \tilde{g}_T^n.$$

As  $|\tilde{g}_{T-1}^n| = 1$ , by passing to some  $\mathcal{F}_{T-1}$ -measurable random sequence we may assume that  $\tilde{g}_{T-1}^n$  converges to  $\tilde{g}_{T-1}^\infty \in \mathbf{G}_{T-1}$ , see [10, Lemma 2.1.2]. As  $\tilde{\gamma}_T^n e_1$  converges to zero, we deduce that  $\tilde{g}_T^n$  converges to  $\tilde{g}_T^\infty \in \mathbf{G}_T$ . Finally, we get the following equality:

$$\tilde{g}_{T-1}^\infty + \tilde{g}_T^\infty = 0$$

where  $\tilde{g}_{T-1}^\infty \in \partial \mathbf{G}_{T-1}$  and  $\tilde{g}_T^\infty \in \mathbf{G}_T$ . Note that, we may define  $\tilde{g}_{T-1}^\infty = \tilde{g}_T^\infty = 0$  on  $\Omega \setminus \Lambda_{T-1} \in \mathcal{F}_{T-1}$ . Let us consider a CPS  $Z$ . From,  $Z_T(\tilde{g}_{T-1}^\infty + \tilde{g}_T^\infty) = 0$  we deduce that  $Z_{T-1}\tilde{g}_{T-1}^\infty + \mathbb{E}(Z_T\tilde{g}_T^\infty | \mathcal{F}_{T-1}) = 0$ . As the two terms in the right side of this equality are non negative by duality, we deduce that  $Z_{T-1}\tilde{g}_{T-1}^\infty = Z_T\tilde{g}_T^\infty = 0$ . Moreover,  $\tilde{g}_T = -\tilde{g}_{T-1}$  is  $\mathcal{F}_{T-1}$  implies that  $0 = \mathbb{E}(Z_T\tilde{g}_T^\infty | \mathcal{F}_{T-1}) = Z_{T-1}\tilde{g}_T^\infty$ . Then,  $Z_{T-1}\tilde{g}_T^\infty = Z_T\tilde{g}_T^\infty$  implies that  $Z_{T-1}$  and  $Z_T$  belongs to the same half-line of  $\mathbf{R}_+^2$ . In particular, since  $Z_T \in \mathbf{G}_T^*$ , we

also have  $Z_{T-1} \in \mathbf{G}_T^*$ . We deduce that  $Z_{T-1}\gamma_T^n e_1 = -Z_{T-1}g_{T-1}^n - Z_T g_T^n \leq 0$  as  $g_{T-1}^n \in \mathbf{G}_{T-1}$  and  $g_T^n \in \mathbf{G}_T$ . Since  $Z_{T-1}e_1 > 0$ , we deduce that  $\gamma_T^n \leq 0$ . Therefore, we may replace  $g_{T-1}^n$  by  $\bar{g}_{T-1}^n = 0 \in \mathbf{G}_{T-1}$  and  $g_T^n$  by  $\bar{g}_T^n = -\gamma_T^n e_1 \in \mathbf{G}_T$  so that we still have  $\gamma_T^n = -\bar{g}_{T-1}^n - \bar{g}_T^n$ . Finally, we may write on  $\Omega$ ,  $\gamma_T^n = -\hat{g}_{T-1}^n - \hat{g}_T^n$ , where  $\hat{g}_{T-1}^n = g_{T-1}^n 1_{\Omega \setminus \Lambda_{T-1}} \in L^0(\mathbf{G}_{T-1}, \mathcal{F}_{T-1})$  and  $\hat{g}_T^n = g_T^n 1_{\Omega \setminus \Lambda_{T-1}} + \bar{g}_T^n 1_{\Lambda_{T-1}} \in L^0(\mathbf{G}_T, \mathcal{F}_T)$ . By construction,  $\liminf_n |\hat{g}_{T-1}^n| < \infty$  hence we may suppose that  $\hat{g}_{T-1}^n \rightarrow \hat{g}_{T-1}^\infty \in L^0(\mathbf{G}_{T-1}, \mathcal{F}_{T-1})$  by [10, Lemma 2.1.2]. We deduce that  $\hat{g}_T^n \rightarrow \hat{g}_T^\infty \in L^0(\mathbf{G}_T, \mathcal{F}_T)$  hence  $\gamma_T^\infty = -\hat{g}_{T-1}^\infty - \hat{g}_T^\infty \in \mathcal{L}_{T-1}^T$ .

Let us consider the three step model. Suppose that the sequence  $\gamma_T^n \in \mathcal{L}_{T-2}^T$  converges to  $\gamma_T^\infty$ . By Remark 2.3, we may suppose that  $\gamma_T^n e_1 = -g_{T-2}^n - g_{T-1}^n - g_T^n$  where  $g_t^n \in \partial \mathbf{G}_t$  for  $t = T-2, T-1$  and  $g_T^n \in \mathbf{G}_T$ .

Let us first consider the subset  $\Gamma_{T-2} := \{\liminf |g_{T-2}^n| < \infty\} \in \mathcal{F}_{T-2}$ . By [10, Lemma 2.1.2], we may assume that  $g_{T-2}^n$  is convergent to  $g_{T-2}^\infty \in L^0(\mathbf{G}_{T-2}, \mathcal{F}_{T-2})$ . Next we consider the following two sub-cases:

(a) On the set where  $\liminf |g_{T-1}^n| < \infty$ , we may suppose that the sequences  $g_{T-1}^n$  and  $g_T^n$  are both pointwise convergent by [10, Theorem 5.2.3].

(b) On the set  $\Lambda_{T-1} := \{\liminf_n |g_{T-1}^n| = \infty\} \in \mathcal{F}_{T-1}$ , we normalize the sequences by dividing the term  $|g_{T-1}^n|$  and we get

$$\tilde{\gamma}_T^n e_1 = -\tilde{g}_{T-2}^n - \tilde{g}_{T-1}^n - \tilde{g}_T^n$$

where  $\tilde{\gamma}_T^n := \frac{\gamma_T^n}{|g_{T-1}^n|}$ ,  $\tilde{g}_t^n := \frac{g_t^n}{|g_{T-1}^n|}$  for all  $t = T-2, T-1, T$ . As  $|\tilde{g}_{T-1}^n| = 1$ , up to some  $\mathcal{F}_{T-1}$ -measurable random sequence, we may assume that  $\tilde{g}_{T-1}^n$  converges to  $\tilde{g}_{T-1}^\infty \in \partial \mathbf{G}_{T-1}$ . As  $\tilde{\gamma}_T^n$  and  $\tilde{g}_{T-2}^n$  both converge to zero, necessarily  $\tilde{g}_T^n$  is convergent to some limit  $\tilde{g}_T^\infty \in \mathbf{G}_T$  and

$$\tilde{g}_{T-1}^\infty + \tilde{g}_T^\infty = 0.$$

Since we only consider  $\mathcal{F}_{T-1}$ -measurable sets, we may claim that  $\mathbf{L}_T(\tilde{g}_T^\infty) = \mathbf{L}_T(-\tilde{g}_{T-1}^\infty) \in \mathcal{L}_{T-2}^T \cap L^0(\mathbf{R}_+, \mathcal{F}_T) = \{0\}$  hence  $\tilde{g}_T^\infty \in \partial \mathbf{G}_T$  by  $NA^w$ .

Assume  $\tilde{g}_{T-1}^{\infty(1)} > 0$  and  $\tilde{g}_{T-1}^{\infty(2)} < 0$ , the second case where  $\tilde{g}_{T-1}^{\infty(1)} < 0$  and  $\tilde{g}_{T-1}^{\infty(2)} > 0$  being similar. The equality  $\tilde{g}_{T-1}^\infty + \tilde{g}_T^\infty = 0$  implies that  $\tilde{g}_T^{\infty(1)} < 0$  and  $\tilde{g}_T^{\infty(2)} > 0$ . This implies that we may replace  $g_{T-1}^n$  by  $g_{T-1}^n 1_{g_{T-1}^{n(2)} < 0} + \tilde{g}_{T-1}^\infty 1_{g_{T-1}^{n(2)} \geq 0}$  and  $g_T^n$  by  $g_T^n 1_{g_T^{n(2)} > 0} + \tilde{g}_T^\infty 1_{g_T^{n(2)} \leq 0}$ . So, we may assume that  $g_{T-1}^{n(2)} < 0$  and  $g_T^{n(2)} > 0$ . Let us define the  $\mathcal{F}_{T-1}$ -measurable positive real-valued random variable

$$\beta^n := \frac{g_{T-2}^{n(2)} + g_{T-1}^{n(2)}}{\tilde{g}_{T-1}^{\infty(2)}}.$$

As  $\tilde{g}_{T-1}^\infty + \tilde{g}_T^\infty = 0$ , we rewrite  $\gamma_T^n e_1$  as  $\gamma_T^n e_1 = -g_{T-2}^n - (g_{T-1}^n - \beta^n \tilde{g}_{T-1}^\infty) - (g_T^n - \beta^n \tilde{g}_T^\infty)$ . By definition of  $\beta^n$ , the second component of the term  $g_{T-1}^n - \beta^n \tilde{g}_{T-1}^\infty$  is

$$(g_{T-1}^n - \beta^n \tilde{g}_{T-1}^\infty)^{(2)} = -g_{T-2}^{n(2)}.$$

Notice that  $\tilde{g}_{T-1}^\infty \in \partial \mathbf{G}_{T-1}$  and  $\tilde{g}_T^\infty \in \partial \mathbf{G}_T$ , i.e.,  $\mathbf{L}_{T-1}(\tilde{g}_{T-1}^\infty) = \tilde{g}_{T-1}^{\infty(1)} + \tilde{g}_{T-1}^{\infty(2)} S_{T-1}^a = 0$  and  $\mathbf{L}_T(\tilde{g}_T^\infty) = \tilde{g}_T^{\infty(1)} + \tilde{g}_T^{\infty(2)} S_T^b = 0$  with  $\tilde{g}_T^{\infty(2)} = -\tilde{g}_{T-1}^{\infty(2)} > 0$ . So we have  $\frac{\tilde{g}_{T-1}^{\infty(1)}}{\tilde{g}_{T-1}^{\infty(2)}} = -S_{T-1}^a$  and  $\frac{\tilde{g}_T^{\infty(1)}}{\tilde{g}_T^{\infty(2)}} = -S_T^b$ . As  $\tilde{g}_{T-1}^\infty = -\tilde{g}_T^\infty$ , we have  $S_{T-1}^a = S_T^b$ . The first component of the term  $g_{T-1}^n - \beta^n \tilde{g}_{T-1}^\infty$  is

$$\begin{aligned} (g_{T-1}^n - \beta^n \tilde{g}_{T-1}^\infty)^{(1)} &= g_{T-1}^{n(1)} - \beta^n \tilde{g}_{T-1}^{\infty(1)} \\ &= g_{T-1}^{n(1)} - \frac{g_{T-2}^{n(2)} + g_{T-1}^{n(2)}}{\tilde{g}_{T-1}^{\infty(2)}} \tilde{g}_{T-1}^{\infty(1)} \\ &= g_{T-1}^{n(1)} + (g_{T-2}^{n(2)} + g_{T-1}^{n(2)}) S_{T-1}^a \\ &= g_{T-1}^{n(1)} + g_{T-1}^{n(2)} S_{T-1}^a + g_{T-2}^{n(2)} S_{T-1}^a \\ &= g_{T-2}^{n(2)} S_{T-1}^a \end{aligned}$$

since  $g_{T-1}^{n(1)} + g_{T-1}^{n(2)} S_{T-1}^a = \mathbf{L}_{T-1}(g_{T-1}^n) = 0$ .

Then,  $g_{T-1}^n - \beta^n \tilde{g}_{T-1}^\infty = (g_{T-2}^{n(2)} S_{T-1}^a, -g_{T-2}^{n(2)})$ . Notice that it satisfies the condition  $\liminf |g_{T-1}^n - \beta^n \tilde{g}_{T-1}^\infty| < \infty$  since  $S_{T-1}^a \in L^\infty(\mathbf{R}, \mathcal{F}_{T-1})$  and  $\liminf |g_{T-2}^{n(2)}| < \infty$ . Liquidate this position at time  $T-1$  to get

$$\mathbf{L}_{T-1}(g_{T-1}^n - \beta^n \tilde{g}_{T-1}^\infty) = g_{T-2}^{n(2)} S_{T-1}^a - g_{T-2}^{n(2)} S_{T-1}^a \mathbf{I}_{g_{T-2}^{n(2)} \geq 0} - g_{T-2}^{n(2)} S_{T-1}^b \mathbf{I}_{g_{T-2}^{n(2)} < 0}.$$

This implies that  $\mathbf{L}_{T-1}((g_{T-1}^n - \beta^n \tilde{g}_{T-1}^\infty) \mathbf{I}_{g_{T-2}^{n(2)} \geq 0}) = 0$ , i.e.,

$$(g_{T-1}^n - \beta^n \tilde{g}_{T-1}^\infty) \mathbf{I}_{g_{T-2}^{n(2)} \geq 0} \in \partial \mathbf{G}_{T-1}.$$

Otherwise, if we liquidate the position at time  $T$ , we obtain that

$$\mathbf{L}_T(g_{T-1}^n - \beta^n \tilde{g}_{T-1}^\infty) = g_{T-2}^{n(2)} S_T^b - g_{T-2}^{n(2)} S_T^a \mathbf{I}_{g_{T-2}^{n(2)} \geq 0} - g_{T-2}^{n(2)} S_T^b \mathbf{I}_{g_{T-2}^{n(2)} < 0}.$$

This implies that  $\mathbf{L}_T((g_{T-1}^n - \beta^n \tilde{g}_{T-1}^\infty) \mathbf{I}_{g_{T-2}^{n(2)} < 0}) = 0$ , i.e.,

$$(g_{T-1}^n - \beta^n \tilde{g}_{T-1}^\infty) \mathbf{I}_{g_{T-2}^{n(2)} < 0} \in \partial \mathbf{G}_T.$$

About the term  $g_T^n - \beta^n \tilde{g}_T^\infty$ , it is easy to see that the second component is zero since  $\beta^n = \frac{g_T^{n(2)}}{\tilde{g}_T^{\infty(2)}}$ . The first component is

$$\begin{aligned} (g_T^n - \beta^n \tilde{g}_T^\infty)^{(1)} &= g_T^{n(1)} - \beta^n \tilde{g}_T^{\infty(1)} \\ &= g_T^{n(1)} - \frac{g_T^{n(2)}}{\tilde{g}_T^{\infty(2)}} \tilde{g}_T^{\infty(1)} \\ &= g_T^{n(2)} \left( \frac{g_T^{n(1)}}{g_T^{n(2)}} - \frac{\tilde{g}_T^{\infty(1)}}{\tilde{g}_T^{\infty(2)}} \right) \end{aligned}$$

As  $g_T^n \in \mathbf{G}_T$ ,  $\mathbf{L}_T(g_T^n) = g_T^{n(1)} + g_T^{n(2)} S_T^b \geq 0$ . So we have  $\frac{g_T^{n(1)}}{g_T^{n(2)}} \geq -S_T^b = \frac{\tilde{g}_T^{\infty(1)}}{\tilde{g}_T^{\infty(2)}}$ . Thus,  $(g_T^n - \beta^n \tilde{g}_T^\infty)^{(1)} \geq 0$ , i.e.,

$$g_T^n - \beta^n \tilde{g}_T^\infty \in \mathbf{R}_+ e_1.$$

On the set  $\Gamma_{T-2}$ , we finally obtain that  $\gamma_T^n e_1 = -g_{T-2}^n - \hat{g}_{T-1}^n - \hat{g}_T^n$ , where

$$\begin{aligned} \hat{g}_{T-1}^n &:= g_{T-1}^n 1_{\Omega \setminus \Lambda_{T-1}} + (g_{T-1}^n - \beta^n \tilde{g}_{T-1}^\infty) \mathbf{I}_{g_{T-2}^{n(2)} \geq 0} 1_{\Lambda_{T-1}} \in L^0(\mathbf{G}_{T-1}, \mathcal{F}_{T-1}), \\ \hat{g}_T^n &:= g_T^n 1_{\Omega \setminus \Lambda_{T-1}} + (g_T^n - \beta^n \tilde{g}_T^\infty) \mathbf{I}_{g_{T-2}^{n(2)} < 0} 1_{\Lambda_{T-1}} \\ &\quad + (g_T^n - \beta^n \tilde{g}_T^\infty) 1_{\Lambda_{T-1}} \in L^0(\mathbf{G}_T, \mathcal{F}_T). \end{aligned}$$

By construction, notice that  $\liminf_n |\hat{g}_{T-1}^n| < \infty$  on  $\Gamma_{T-2}$ . By [10, Lemma 2.1.2], we may assume that  $\hat{g}_{T-1}^n$  is convergent to  $\hat{g}_{T-1}^\infty \in L^0(\mathbf{G}_{T-1}, \mathcal{F}_{T-1})$  at least for some  $\mathcal{F}_{T-1}$ -measurable subsequence  $n_k(\omega)$ ,  $\omega \in \Omega$ . As we already know that  $g_{T-2}^n \rightarrow g_{T-2}^\infty \in L^0(\mathbf{G}_{T-2}, \mathcal{F}_{T-2})$ , we still have  $g_{T-2}^{n_k} \rightarrow g_{T-2}^\infty \in L^0(\mathbf{G}_{T-2}, \mathcal{F}_{T-2})$  even if  $g_{T-2}^{n_k}$  is no more  $\mathcal{F}_{T-2}$ -measurable. We deduce that  $g_T^{n_k} \rightarrow g_T^\infty \in L^0(\mathbf{G}_T, \mathcal{F}_T)$  and, finally,  $\gamma_T^\infty 1_{\Gamma_{T-2}} = -g_{T-2}^\infty - \hat{g}_{T-1}^\infty - \hat{g}_T^\infty \in \mathcal{L}_{T-2}^T$ .

On the set  $\Lambda_{T-2} := \{\liminf |g_{T-2}^n| = \infty\}$ , we use the normalization procedure to get

$$\bar{\gamma}_T^n e_1 = -\bar{g}_{T-2}^n - \bar{g}_{T-1}^n - \bar{g}_T^n,$$

where  $\bar{\gamma}_T^n := \frac{\gamma_T^n}{|g_{T-2}^n|}$ ,  $\bar{g}_t^n := \frac{g_t^n}{|g_{T-2}^n|}$  for  $t \geq T-2$ . As  $\liminf |\bar{g}_{T-2}^n| = 1$ , we may argue as we did on the complementary set  $\Gamma_{T-2}$  and assume that  $\bar{g}_t^n \rightarrow \bar{g}_t^\infty \in L^0(\mathbf{G}_t, \mathcal{F}_t)$  for  $t \geq T-2$  such that

$$\bar{g}_{T-2}^\infty + \bar{g}_{T-1}^\infty + \bar{g}_T^\infty = 0$$

with  $\bar{g}_t^\infty \in \partial \mathbf{G}_t$  if  $t = T-2, T-1$  and  $|\bar{g}_{T-2}^\infty| = 1$ . Since  $\mathbf{L}_T(\bar{g}_T^\infty) = \mathbf{L}_T(-\bar{g}_{T-2}^\infty - \bar{g}_{T-1}^\infty) \in \mathcal{L}_{T-2}^T \cap L^0(\mathbf{R}_+, \mathcal{F}_T) = \{0\}$ , we also get that  $\bar{g}_T^\infty \in \partial \mathbf{G}_T$ .

For any CPS  $Z$ , taking the conditional expectation knowing  $\mathcal{F}_{T-2}$  in the equality  $Z_T(\bar{g}_{T-2}^\infty + \bar{g}_{T-1}^\infty + \bar{g}_T^\infty) = 0$ , we deduce that

$$Z_{T-2}\bar{g}_{T-2}^\infty + \mathbb{E}(Z_{T-1}\bar{g}_{T-1}^\infty | \mathcal{F}_{T-2}) + \mathbb{E}(Z_T\bar{g}_T^\infty | \mathcal{F}_{T-2}) = 0.$$

All the terms of the l.h.s. being non negative, we finally obtain that  $Z_t\bar{g}_t^\infty = 0$  for all  $t \geq T-2$ . As  $\bar{g}_T^\infty = -\bar{g}_{T-2}^\infty - \bar{g}_{T-1}^\infty$  is  $\mathcal{F}_{T-1}$ -measurable, taking the conditional expectation knowing  $\mathcal{F}_{T-1}$  in the equality  $Z_T\bar{g}_T^\infty = 0$ , we deduce that  $Z_{T-1}\bar{g}_T^\infty = 0$ . But  $Z_{T-1}(\bar{g}_{T-2}^\infty + \bar{g}_{T-1}^\infty + \bar{g}_T^\infty) = 0$  hence  $Z_{T-1}\bar{g}_{T-2}^\infty = 0$ . Since  $|\bar{g}_{T-2}^\infty| = 1$ , the two equalities  $Z_{T-2}\bar{g}_{T-2}^\infty = Z_{T-1}\bar{g}_{T-2}^\infty = 0$  implies that  $Z_{T-1}$  and  $Z_{T-2}$  belong to a same half line of  $\mathbf{G}_{T-1}^* \cap \mathbf{G}_{T-2}^*$ .

Assume  $\bar{g}_{T-2}^{\infty(1)} > 0$  and  $\bar{g}_{T-2}^{\infty(2)} < 0$ , the other case where  $\bar{g}_{T-2}^{\infty(1)} < 0$  and  $\bar{g}_{T-2}^{\infty(2)} > 0$  is similar. Up to some  $\mathcal{F}_{T-2}$ -measurable random sequence  $(\tilde{n}_k)_k$  we may assume that  $g_{T-2}^{n(1)} > 0$  and  $g_{T-2}^{n(2)} < 0$ . We then consider the following two sub cases:

(c) When  $\bar{g}_T^\infty \neq 0$ . In the case where  $\bar{g}_{T-1}^\infty \neq 0$ , the two equalities  $Z_{T-1}\bar{g}_{T-1}^\infty = Z_{T-1}\bar{g}_T^\infty = 0$  implies that  $\bar{g}_{T-1}^\infty$  and  $\bar{g}_T^\infty$  are collinear. As  $Z_T(\bar{g}_{T-2}^\infty + \bar{g}_{T-1}^\infty + \bar{g}_T^\infty) = 0$ , we have  $Z_T(\bar{g}_{T-2}^\infty + \bar{g}_{T-1}^\infty) = 0$  where  $Z_T\bar{g}_{T-1}^\infty = Z_T\bar{g}_T^\infty = 0$  since  $\bar{g}_{T-1}^\infty \in R_+\bar{g}_T^\infty$ . It follows that  $Z_T\bar{g}_{T-2}^\infty = 0$ . In the case where  $\bar{g}_{T-1}^\infty = 0$ , the equality  $Z_T(\bar{g}_{T-2}^\infty + \bar{g}_{T-1}^\infty) = 0$  implies that  $Z_T\bar{g}_{T-2}^\infty = 0$ . Finally,  $Z_{T-2}\bar{g}_{T-2}^\infty = Z_{T-1}\bar{g}_{T-2}^\infty = Z_T\bar{g}_{T-2}^\infty = 0$  implies that  $Z_{T-2}$ ,  $Z_{T-1}$  and  $Z_T$  are both collinear with some positive coefficients of collinearity when both  $\bar{g}_{T-2}^\infty$  and  $\bar{g}_T^\infty$  are different from zero. This implies that  $Z_t \in \mathbf{G}_t^*$  for all  $t \geq T-2$ . Since  $\gamma_T^n e_1 = -g_{T-2}^n - g_{T-1}^n - g_T^n$ , we deduce that  $Z_T\gamma_T^n e_1 = -(Z_Tg_{T-2}^n + Z_Tg_{T-1}^n + Z_Tg_T^n)$   $Z_T\gamma_T^n e_1 \leq 0$  hence  $\gamma_T^n \leq 0$  on the set  $\Lambda_{T-2} \cap \{\bar{g}_T^\infty \neq 0\}$ .

(d) When  $\bar{g}_T^\infty = 0$ ,  $\bar{g}_{T-2}^\infty + \bar{g}_{T-1}^\infty = 0$ . Let us define the  $\mathcal{F}_{T-2}$ -measurable positive real-valued random variable as

$$\bar{\beta}^n := \frac{g_{T-2}^{n(2)}}{\bar{g}_{T-2}^{\infty(2)}}.$$

Since  $\bar{g}_{T-1}^\infty = -\bar{g}_{T-2}^\infty$ ,  $\bar{g}_{T-1}^{\infty(1)} < 0$  and  $\bar{g}_{T-1}^{\infty(2)} > 0$ , we may replace  $g_{T-1}^n$  by  $g_{T-1}^n \mathbf{1}_{\{g_{T-1}^{n(1)} < 0; g_{T-1}^{n(2)} > 0\}} + \bar{g}_{T-1}^\infty \mathbf{1}_{\Omega \setminus \{g_{T-1}^{n(1)} < 0; g_{T-1}^{n(2)} > 0\}}$  and assume that  $g_{T-1}^{n(1)} < 0$  and  $g_{T-1}^{n(2)} > 0$ . As  $\bar{g}_{T-2}^\infty + \bar{g}_{T-1}^\infty = 0$ , we rewrite  $\gamma_T^n e_1$  as  $\gamma_T^n e_1 = -(g_{T-2}^n - \bar{\beta}^n \bar{g}_{T-2}^\infty) - (g_{T-1}^n - \bar{\beta}^n \bar{g}_{T-1}^\infty) - g_T^n$  where  $g_{T-2}^n - \bar{\beta}^n \bar{g}_{T-2}^\infty = 0$  by construction.

As  $\bar{g}_{T-2}^\infty \in \partial \mathbf{G}_{T-2}$  and  $\bar{g}_{T-1}^\infty \in \partial \mathbf{G}_{T-1}$ ,  $\mathbf{L}_{T-2}(\bar{g}_{T-2}^\infty) = \bar{g}_{T-2}^{\infty(1)} + \bar{g}_{T-2}^{\infty(2)} S_{T-2}^a = 0$  and  $\mathbf{L}_{T-1}(\bar{g}_{T-1}^\infty) = \bar{g}_{T-1}^{\infty(1)} + \bar{g}_{T-1}^{\infty(2)} S_{T-1}^b = 0$ . So we have  $\frac{\bar{g}_{T-2}^{\infty(1)}}{\bar{g}_{T-2}^{\infty(2)}} = -S_{T-2}^a$  and  $\frac{\bar{g}_{T-1}^{\infty(1)}}{\bar{g}_{T-1}^{\infty(2)}} = -S_{T-1}^b$ . As  $\bar{g}_{T-1}^\infty = -\bar{g}_{T-2}^\infty$ ,  $S_{T-2}^a = S_{T-1}^b$ .

The second component of  $g_{T-1}^n - \bar{\beta}^n \bar{g}_{T-1}^\infty$  is  $(g_{T-1}^n - \bar{\beta}^n \bar{g}_{T-1}^\infty)^{(2)} = -g_T^{n(2)}$ . Thus,  $\bar{\beta}^n = \frac{g_{T-1}^{n(2)} + g_T^{n(2)}}{\bar{g}_{T-1}^{\infty(2)}}$ , which is  $\mathcal{F}_{T-1}$ -measurable. Notice that  $g_{T-1}^n - \bar{\beta}^n \bar{g}_{T-1}^\infty$  is also  $\mathcal{F}_{T-1}$ -measurable. Its first component is

$$\begin{aligned} (g_{T-1}^n - \bar{\beta}^n \bar{g}_{T-1}^\infty)^{(1)} &= g_{T-1}^{n(1)} - \bar{\beta}^n \bar{g}_{T-1}^{\infty(1)} \\ &= g_{T-1}^{n(1)} - \frac{g_{T-1}^{n(2)} + g_T^{n(2)}}{\bar{g}_{T-1}^{\infty(2)}} \bar{g}_{T-1}^{\infty(1)} \\ &= g_{T-1}^{n(1)} + (g_{T-1}^{n(2)} + g_T^{n(2)}) S_{T-1}^b \\ &= g_{T-1}^{n(1)} + g_{T-1}^{n(2)} S_{T-1}^b + g_T^{n(2)} S_{T-1}^b \\ &= g_T^{n(2)} S_{T-1}^b \end{aligned}$$

since  $g_{T-1}^{n(1)} + g_{T-1}^{n(2)} S_{T-1}^b = \mathbf{L}_{T-1}(g_{T-1}^n) = 0$ . Then

$$g_{T-1}^n - \bar{\beta}^n \bar{g}_{T-1}^\infty = (g_T^{n(2)} S_{T-1}^b, -g_T^{n(2)})$$

so that

$$\gamma_T^n = -g_T^{n(2)} S_{T-1}^b - g_T^{n(1)}.$$

Liquidate the position  $g_{T-1}^n - \bar{\beta}^n \bar{g}_{T-1}^\infty$  at time  $T - 1$  to get

$$\mathbf{L}_{T-1}(g_{T-1}^n - \bar{\beta}^n \bar{g}_{T-1}^\infty) = g_T^{n(2)} S_{T-1}^b - g_T^{n(2)} S_{T-1}^a \mathbf{I}_{g_T^{n(2)} > 0} - g_T^{n(2)} S_{T-1}^b \mathbf{I}_{g_T^{n(2)} \leq 0}.$$

This implies that  $\mathbf{L}_{T-1}((g_{T-1}^n - \bar{\beta}^n \bar{g}_{T-1}^\infty) \mathbf{I}_{g_T^{n(2)} \leq 0}) = 0$ , i.e.,

$$(g_{T-1}^n - \bar{\beta}^n \bar{g}_{T-1}^\infty) \mathbf{I}_{g_T^{n(2)} \leq 0} \in \partial \mathbf{G}_{T-1}.$$

Similarly we deduce by liquidating it at time  $T - 2$  that

$$(g_{T-1}^n - \bar{\beta}^n \bar{g}_{T-1}^\infty) \mathbf{I}_{g_T^{n(2)} > 0} \in \partial \mathbf{G}_{T-2}.$$

If we set  $h_{T-1}^n := g_{T-2}^n + g_{T-1}^n$ , then

$$h_{T-1}^n = g_{T-1}^n - \bar{\beta}^n \bar{g}_{T-1}^\infty = (g_T^{n(2)} S_{T-1}^b, -g_T^{n(2)})$$

belongs to the boundary of  $\mathbf{G}_{T-2}$  or  $\mathbf{G}_{T-1}$ .

In the case where  $g_T^{n(2)} \leq 0$ , it is possible to change  $g_{T-1}^n$  into  $g_{T-1}^n - \bar{\beta}^n \bar{g}_{T-1}^\infty$  which still belongs to the boundary of  $\mathbf{G}_{T-1}$ .

On the set  $\{g_T^{n(2)} > 0\} \in \mathcal{F}_{T-1}$ , we consider two sub cases. On the set  $\{\sup_n g_T^{n(2)} < \infty\} \in \mathcal{F}_{T-1}$ , up to a convex combination (see [10, Theorem A.2.3]), we assume that  $g_T^{n(2)}$  is convergent. As  $\gamma_T^n = -g_T^{n(2)} S_{T-1}^b - g_T^{n(1)}$ ,  $g_T^{n(1)}$  is also convergent, i.e.  $g_T^n$  converges to  $g_T \in L^0(\mathbf{G}_T, \mathcal{F}_T)$ . Then  $\gamma_T^\infty e_1 = \lim_n (-g_{T-2}^n - g_{T-1}^n - g_T^n) = \lim_n (-g_{T-2}^n - g_{T-1}^n - g_T)$ , i.e. we may replace  $g_T^n$  by  $g_T$ . By the initial normalization on  $\Lambda_{T-2}$ , we then get that  $\gamma_T^n = -g_T^{(2)} S_{T-1}^b - g_T^{(1)}$  instead of  $\gamma_T^n = -g_T^{n(2)} S_{T-1}^b - g_T^{n(1)}$ , i.e.  $\gamma_T^n$  does not depend on  $n$  any more: we may write  $\gamma_T^n e_1 = -g_{T-2}^1 - g_{T-1}^1 - g_T^1$  where  $g_t^1$  is a constant substitution of  $g_t^n$  for each  $t \geq T-2$ .

On the  $\mathcal{F}_{T-1}$ -measurable set  $\{\sup_n g_T^{n(2)} = \infty\}$ , first recall that for any CPS  $Z$ ,  $Z_{T-2}$  and  $Z_{T-1}$  are colinear vectors of  $\mathbf{R}_+^2$  orthogonal to the line generated by  $\bar{g}_{T-2}^\infty$ . Through the normalizing procedure which consists in dividing by  $g_T^{n(2)}$  on both sides of the equality  $\gamma_T^n e_1 = -h_{T-1}^n - g_T^n$ , we get that

$$\tilde{h}_{T-1} + \tilde{g}_T = 0$$

where  $\tilde{h}_{T-1} \in L^0(\mathbf{R} \bar{g}_{T-2}^\infty, \mathcal{F}_{T-1})$  and  $\tilde{g}_T \in L^0(\mathbf{G}_T, \mathcal{F}_T)$ . Taking conditional expectation knowing  $\mathcal{F}_{T-1}$  in the equality  $Z_T(\tilde{h}_{T-1} + \tilde{g}_T) = 0$ , we deduce that  $Z_{T-1} \tilde{h}_{T-1} + \mathbb{E}(Z_T \tilde{g}_T | \mathcal{F}_{T-1}) = 0$ . As  $Z_{T-1} \tilde{h}_{T-1} = 0$  and  $Z_T \tilde{g}_T \geq 0$ , we then deduce that  $Z_T \tilde{g}_T = 0$  hence  $Z_T \tilde{h}_{T-1} = 0$ . The equality  $Z_{T-1} \tilde{h}_{T-1} = Z_T \tilde{h}_{T-1} = 0$  with  $\tilde{h}_{T-1} \neq 0$  implies that  $Z_T$  and  $Z_{T-1}$  also belongs to the same half line. Finally  $Z_T \in \mathbf{G}_t^*$  for all  $t \geq T-2$  and, by a similar argument as above, we deduce that  $\gamma_T^n \leq 0$ .

Gathering with the case (c) where  $\bar{g}_T^\infty \neq 0$ , we can conclude that on the  $\mathcal{F}_{T-2}$ -measurable set  $\Lambda_{T-2}$ , we have  $\gamma_T^n e_1 = -g_{T-2}^1 - \hat{g}_{T-1}^n - \hat{g}_T^n$  where

$$\hat{g}_t^n := \sum_{i=1}^4 \hat{g}_t^{ni} \in L^0(\mathbf{G}_t, \mathcal{F}_t), \quad t \geq T-1,$$

with

$$\begin{aligned} \hat{g}_{T-1}^{n1} &:= (\|g_{T-2}^1\| \bar{g}_{T-1}^\infty) \mathbf{I}_{\bar{g}_T^\infty \neq 0}, \\ \hat{g}_{T-1}^{n2} &:= (-g_{T-2}^1 + g_{T-1}^n - \bar{\beta}^n \bar{g}_{T-1}^\infty) \mathbf{I}_{\bar{g}_T^\infty = 0} \mathbf{I}_{g_T^{n(2)} \leq 0}, \\ \hat{g}_{T-1}^{n2} &:= g_{T-1}^1 \mathbf{I}_{\bar{g}_T^\infty = 0} \mathbf{I}_{g_T^{n(2)} > 0} \mathbf{I}_{\sup_n g_T^{n(2)} < \infty}, \\ \hat{g}_{T-1}^{n2} &:= (-g_{T-2}^1) \mathbf{I}_{\bar{g}_T^\infty = 0} \mathbf{I}_{g_T^{n(2)} > 0} \mathbf{I}_{\sup_n g_T^{n(2)} = \infty} \end{aligned}$$

and

$$\begin{aligned}
\hat{g}_T^{n1} &:= (\|g_{T-2}^1\| \bar{g}_T^\infty - \gamma_T^n e_1) \mathbf{I}_{\bar{g}_T^\infty \neq 0}, \\
\hat{g}_T^{n2} &:= g_T^n \mathbf{I}_{\bar{g}_T^\infty = 0} \mathbf{I}_{g_T^{n(2)} \leq 0}, \\
\hat{g}_T^{n3} &:= g_T^1 \mathbf{I}_{\bar{g}_T^\infty = 0} \mathbf{I}_{g_T^{n(2)} > 0} \mathbf{I}_{\sup_n g_T^{n(2)} < \infty}, \\
\hat{g}_T^{n4} &:= (-\gamma_T^n e_1) \mathbf{I}_{\bar{g}_T^\infty = 0} \mathbf{I}_{g_T^{n(2)} > 0} \mathbf{I}_{\sup_n g_T^{n(2)} = \infty}.
\end{aligned}$$

We can easily see that  $\hat{g}_{T-1}^{ni} \in \partial \mathbf{G}_{T-1}$  when  $i = 2, 3, 4$  and  $\hat{g}_T^{ni} \in \mathbf{G}_T$  for all  $i = 1, 2, 3, 4$ . The expressions of  $\hat{g}_t^{n1}$ ,  $t = T-1, T$  are obtained from the following equality:  $g_{T-2}^n = \|g_{T-2}^n\| \bar{g}_{T-2}^n = \|g_{T-2}^n\| \bar{g}_{T-2}^\infty = -\|g_{T-2}^n\| (\bar{g}_{T-1}^\infty + \bar{g}_T^\infty)$  with  $n = 1$  and  $\gamma_T^1 e_1 = -g_{T-2}^1 - (-g_{T-2}^1) - (-\gamma_T^1 e_1)$ .

Since the sequence  $(g_{T-2}^n)_{n \geq 1}$  is replaced by the constant  $g_{T-2}^1$ , we may follow the reasoning we did on the complementary  $\Gamma_{T-2}$  so that we finally obtain  $\gamma_T^\infty e_1 = -g_{T-2}^\infty - g_{T-1}^\infty - g_T^\infty$  where  $g_t^\infty \in \mathbf{G}_t$  for all  $t = T-2, T-1, T$  and  $\gamma_T^\infty = \mathbf{L}_T(\gamma_T^\infty e_1) \in \mathcal{L}_{T-2}^T$ .

We now consider the general case. Let us suppose by induction that the statement above holds for the model with dates between  $t+1$  and  $T$  with  $t+1 \geq T-2$  and let us prove it from  $t$  to  $T$ . To do so, consider a sequence  $\delta_T^n = -g_t^n - g_{t+1}^n - \dots - g_T^n$ , where we may suppose w.l.o.g. that  $g_u^n \in L^0(\partial \mathbf{G}_u, \mathcal{F}_u)$  for all  $u = t, \dots, T-1$  and  $g_T^n \in L^0(\mathbf{G}_T, \mathcal{F}_T)$ , converges to  $\delta_T \in L^0(\mathbf{R}e_1, \mathcal{F}_T)$ . We claim that  $\delta_T^n = -\hat{g}_t^n - \hat{g}_{t+1}^n - \dots - \hat{g}_T^n + \epsilon_T^n$ , where  $\hat{g}_u^n \in L^0(\mathbf{G}_u, \mathcal{F}_u)$  for all  $u = t, \dots, T-1$  and  $\hat{g}_T^n \in L^0(\mathbf{G}_T, \mathcal{F}_T)$  are such that  $\liminf_n |g_u^n| < \infty$  a.s. for all  $u = t, \dots, T$  and  $\lim_n \epsilon_T^n = 0$  a.s. Moreover we claim that, on the set  $\{\liminf_n |g_t^n| = +\infty\}$ ,  $\hat{g}_t^n$  is either 0 or a constant element of the sequence  $(g_u^n)_{n \geq 0}$  and  $\hat{g}_t^n = g_t^n$  on the set  $\{\liminf_n |g_t^n| < \infty\}$ . Notice that these properties are verified for  $t = T-1$  and  $t = T-2$  as shown above.

Let us denote by  $u$  the smallest instant such that  $P(\liminf_n |g_u^n| = +\infty) > 0$ . As  $\liminf_n |g_u^n| < \infty$  a.s. for every  $r = t, \dots, u-1$ , we may successively suppose that  $g_r^n$  is a.s. convergent to some  $g_r \in L^0(\partial \mathbf{G}_r, \mathcal{F}_r)$  when  $r \leq u-1$  by [10, Lemma 2.1.2]. It is then possible to make the substitution  $g_i^n = g_i$  for every  $i \leq u-1$ , letting aside a residual error  $\tilde{\epsilon}_T^n$  which tends a.s. to zero as in the claim. We suppose that  $u \leq T$ , otherwise there is nothing to prove.

We first work on the set  $\Lambda_t := \{\liminf_n |g_t^n| < \infty\}$  so that  $u \geq t+1$ . We split  $\Lambda_t$  into  $\Lambda_u := \{\liminf_n |g_u^n| < \infty\} \in \mathcal{F}_u$  and its complimentary  $\Gamma_u := \{\liminf_n |g_u^n| = \infty\}$ . On the latter set, dividing by  $|g_u^n|$ , we get the normalisation

$$\tilde{\delta}_T^n = -\tilde{g}_t^n - \tilde{g}_{t+1}^n - \dots - \tilde{g}_T^n$$



where  $\tilde{\delta}_T^n = \frac{\delta_T^n}{|g_u^n|}$  and  $\tilde{g}_r^n := \frac{g_r^n}{|g_u^n|}$  for  $r = t, \dots, T$ . As  $\tilde{\delta}_T^n$  and  $\tilde{g}_r^n$  with  $r = t, \dots, u-1$  all converge to 0, we may use the induction hypothesis and suppose that  $\liminf_n |\tilde{g}_r^n| < \infty$  on  $\Gamma_u$  if  $r \geq u$ . By [10, Lemma 2.1.2], we may suppose that  $\tilde{g}_r^n \rightarrow \tilde{g}_r \in L^0(\mathbf{G}_r, \mathcal{F}_r)$  if  $r \geq u$  and, finally,

$$\tilde{g}_u^\infty + \tilde{g}_{u+1}^\infty + \dots + \tilde{g}_T^\infty = 0. \quad (4.2)$$

We deduce that  $\tilde{g}_T^\infty \in L^0(\partial \mathbf{G}_T, \mathcal{F}_T)$  under (NA<sup>w</sup>). Let us consider the stopping time  $\tau$ , being the first instant  $\tau \geq u+1$  such that  $\tilde{g}_u^\infty + \tilde{g}_{u+1}^\infty + \dots + \tilde{g}_\tau^\infty = 0$ . By Lemma 4.1 and Lemma 4.2, for all  $r \geq u$ , there exists  $k_r \in L^0(\mathbf{R}, \mathcal{F}_r)$  such that  $\tilde{g}_r 1_{r \leq \tau} = k_r \tilde{g}_u 1_{r \leq \tau}$ . Let us introduce the first instant  $\sigma \in \{t+1, \tau\}$  such that  $k_\sigma < 0$ , which is possible by (4.2) since  $k_u = 1$ . We consider the case where  $\tilde{g}_u^{\infty(1)} > 0$  and  $\tilde{g}_u^{\infty(2)} < 0$ , then  $\tilde{g}_\sigma^{\infty(1)} < 0$  and  $\tilde{g}_\sigma^{\infty(2)} > 0$ . The symmetric case may be solved similarly.

Since  $\mathbf{L}_u(\tilde{g}_u^\infty) = \mathbf{L}_\sigma(\tilde{g}_\sigma^\infty) = 0$ ,  $\frac{\tilde{g}_u^{\infty(1)}}{\tilde{g}_u^{\infty(2)}} = \frac{\tilde{g}_\sigma^{\infty(1)}}{\tilde{g}_\sigma^{\infty(2)}} = -S_u^a = -S_\sigma^b$  by Lemma 4.2. As  $(\tilde{g}_t^n + \dots + \tilde{g}_u^n)^{(2)} = -(\tilde{g}_{u+1}^n + \dots + \tilde{g}_T^n)^{(2)} - \delta_T^{n(2)}$  converges to  $\tilde{g}_u^{\infty(2)} < 0$ , up to some  $\mathcal{F}_u$ -measurable random sequences we may assume that  $\tilde{g}_u^{n(2)} < 0$  and  $(\tilde{g}_t^n + \dots + \tilde{g}_u^n)^{(2)} < 0$ . Let

$$\beta^n := \frac{(g_t^n + \dots + g_u^n)^{(2)}}{\tilde{g}_u^{\infty(2)}}$$

which is positive and  $\mathcal{F}_u$ -measurable. Now we rewrite  $\delta_T^n$  as

$$\delta_T^n = -g_t - \dots - g_{u-1} - (g_u^n - \beta^n \tilde{g}_u^\infty) - \beta^n \tilde{g}_u^\infty - g_{u+1}^n - \dots - g_T^n.$$

The second component of the term  $g_u^n - \beta^n \tilde{g}_u^\infty$  is

$$(g_u^n - \beta^n \tilde{g}_u^\infty)^{(2)} = -(g_t + \dots + g_{u-1})^{(2)}.$$

And the first component of this term is

$$\begin{aligned} (g_u^n - \beta^n \tilde{g}_u^\infty)^{(1)} &= g_u^{n(1)} - \beta^n \tilde{g}_u^{\infty(1)} \\ &= g_u^{n(1)} - (g_t^n + \dots + g_u^n)^{(2)} \frac{\tilde{g}_u^{\infty(1)}}{\tilde{g}_u^{\infty(2)}} \\ &= g_u^{n(1)} + (g_t^n + \dots + g_u^n)^{(2)} S_u^a \\ &= g_u^{n(1)} + g_u^{n(2)} S_u^a + (g_t^n + \dots + g_{u-1}^n)^{(2)} S_u^a \\ &= (g_t + \dots + g_{u-1})^{(2)} S_u^a \end{aligned}$$

Thus,

$$g_u^n - \beta^n \tilde{g}_u^\infty = ((g_t + \cdots + g_{u-1})^{(2)}) S_u^a - (g_t + \cdots + g_{u-1})^{(2)}$$

which is constant, and so satisfies  $\liminf_n |g_u^n - \beta^n \tilde{g}_u^\infty| < +\infty$ .

On the set  $\Lambda_{u-1}^1 := \{(g_t + \cdots + g_{u-1})^{(2)} \geq 0\}$ , it has  $\mathbf{L}_u(g_u^n - \beta^n \tilde{g}_u^\infty) = (g_t + \cdots + g_{u-1})^{(2)} S_u^a - (g_t + \cdots + g_{u-1})^{(2)} S_u^a = 0$ . This implies that

$$(g_u^n - \beta^n \tilde{g}_u^\infty) 1_{\Lambda_{u-1}^1} \in \partial \mathbf{G}_u.$$

On the set  $\Lambda_{u-1}^2 := \{(g_t + \cdots + g_{u-1})^{(2)} < 0\}$ , it has  $\mathbf{L}_\sigma(g_u^n - \beta^n \tilde{g}_u^\infty) = (g_t + \cdots + g_{u-1})^{(2)} S_u^a - (g_t + \cdots + g_{u-1})^{(2)} S_\sigma^b = 0$ . This implies that

$$(g_u^n - \beta^n \tilde{g}_u^\infty) 1_{\Lambda_{u-1}^2} \in \partial \mathbf{G}_\sigma.$$

As  $\sigma \in \{u+1, \dots, T\}$ , then  $(g_u^n - \beta^n \tilde{g}_u^\infty) 1_{\Lambda_{u-1}^2} = \sum_{k=u+1}^T 1_{\sigma=k} 1_{\Lambda_{u-1}^2} (g_u^n - \beta^n \tilde{g}_u^\infty)$

where  $\bar{g}_k := 1_{\sigma=k} 1_{\Lambda_{u-1}^2} (g_u^n - \beta^n \tilde{g}_u^\infty) \in L^0(\partial \mathbf{G}_k, \mathcal{F}_k)$ . On the other hand, the second component of the term  $(-\beta^n \tilde{g}_u^\infty - g_{u+1}^n - \dots - g_T^n)^{(2)} = \delta_T^{n(2)} \rightarrow 0$  so that we may use the induction argument as  $u \geq t+1$  to deduce that  $-\beta^n \tilde{g}_u^\infty - g_{u+1}^n - \dots - g_T^n = -\sum_{k=u}^T \check{g}_k^n + \tilde{\epsilon}_T^n$  where every  $\check{g}_k^n \in L^0(\mathbf{G}_k, \mathcal{F}_k)$  admits a finite limit infimum and  $\tilde{\epsilon}_T^n \rightarrow 0$  a.s. as  $n \rightarrow \infty$ . Finally, we may write

$$\begin{aligned} \delta_T^n 1_{\Lambda_t} 1_{\Lambda_u} &= -\sum_{r=t}^{u-1} g_r 1_{\Lambda_u} - (g_u^n - \beta^n \tilde{g}_u^\infty) 1_{\Lambda_u} - \beta^n \tilde{g}_u^\infty 1_{\Lambda_u} - \sum_{r=u+1}^T g_r^n 1_{\Lambda_u} \\ &= -\sum_{r=t}^{u-1} g_r 1_{\Lambda_u} - (g_u^n - \beta^n \tilde{g}_u^\infty) 1_{\Lambda_{u-1}^1} - (g_u^n - \beta^n \tilde{g}_u^\infty) 1_{\Lambda_{u-1}^2} - \sum_{k=u}^T \check{g}_k^n 1_{\Lambda_u} + \tilde{\epsilon}_T^n \\ &= -\sum_{r=t}^{u-1} g_r 1_{\Lambda_u} - (g_u^n - \beta^n \tilde{g}_u^\infty) 1_{\Lambda_{u-1}^1} - \sum_{k=u+1}^T \bar{g}_k 1_{\Lambda_u} - \sum_{k=u}^T \check{g}_k^n 1_{\Lambda_u} + \tilde{\epsilon}_T^n \\ &= -g_t 1_{\Lambda_u} - \cdots - g_{u-1} 1_{\Lambda_u} - \sum_{k=u}^T \hat{g}_k^n 1_{\Lambda_u} + \tilde{\epsilon}_T^n \end{aligned}$$

where  $\hat{g}_u^n = (g_u^n - \beta^n \tilde{g}_u^\infty) 1_{\Lambda_{u-1}^1} + \check{g}_u^n \in L^0(\mathbf{G}_u, \mathcal{F}_u)$  and  $\hat{g}_k^n = \bar{g}_k + \check{g}_k^n \in L^0(\mathbf{G}_k, \mathcal{F}_k)$  for  $k \geq u+1$ . Note that, for every  $k$ ,  $\liminf_n |\hat{g}_k^n| < \infty$  a.s. On the complementary set  $\Gamma_u$  of  $\Lambda_u$ , we may suppose that  $g_u^n \rightarrow g_u \in$

$L^0(\mathbf{G}_u, \mathcal{F}_u)$  by [10, Lemma 2.1.2] and finally assume without loss of generality that  $\delta_T^n 1_{\Gamma_u} = -g_t 1_{\Gamma_u} - g_{t+1} 1_{\Gamma_u} - \cdots - g_u 1_{\Gamma_u} - g_{u+1}^n 1_{\Gamma_u} - \cdots - g_T^n$ . We then deduce from above that  $\delta_T^n 1_{\Lambda_t} = \delta_T^n 1_{\Lambda_t} 1_{\Lambda_u} + \delta_T^n 1_{\Lambda_t} 1_{\Gamma_u}$  is of the form  $\delta_T^n 1_{\Lambda_t} = -g_t 1_{\Lambda_t} - \cdots - g_{u-1} - g_u^n - \cdots - g_T^n$  where  $\liminf |g_u^n| < \infty$  a.s. This implies that we may repeat the procedure above with some date  $u^1 \geq u + 1$  instead of  $u$ . As the number of dates is finite, we finally conclude on  $\Lambda_t$ .

Next we consider the case where  $\liminf_n |g_t^n| = +\infty$ . Using the classical normalization procedure, we get

$$\bar{\delta}_T^n = -\bar{g}_t^n - \bar{g}_{t+1}^n - \cdots - \bar{g}_T^n$$

where  $\bar{\gamma}_T^n := \frac{\gamma_T^n}{|g_T^n|}$  and  $\bar{g}_i^n := \frac{g_i^n}{|g_i^n|}$  for all  $i = t, \dots, T$ . Since  $|\bar{g}_t^n| = 1$ , up to the first case where  $\liminf_n |g_t^n| < +\infty$  we deduce an equality of the type  $\bar{g}_t^\infty + \bar{g}_{t+1}^\infty + \cdots + \bar{g}_T^\infty = 0$ . As  $\bar{g}_t^\infty \neq 0$ , let us consider the stopping time  $\bar{\tau} \geq t + 1$  as the first instant such that  $\bar{g}_t^\infty + \bar{g}_{t+1}^\infty + \cdots + \bar{g}_\tau^\infty = 0$ . Then, for any CPS  $(Z_r)_{r=t, \dots, T}$ ,  $Z_t, \dots, Z_\tau$  are collinear by Lemma 4.1. It follows that  $Z_{t+1} \in \mathbf{G}_t^* \cap \mathbf{G}_{t+1}^*$  and  $(Z_r)_{r=t+1, \dots, T}$  is a CPS for the market model from  $t + 1$  to  $T$  defined by the solvency sets  $\tilde{\mathbf{G}}_t = \mathbf{G}_t + \mathbf{G}_{t+1} = \mathbf{G}_t \cup \mathbf{G}_{t+1} \subseteq (Z_t)^*$  and  $\tilde{\mathbf{G}}_u = \mathbf{G}_u$  for  $u \geq t + 2$ . This means that Condition (NA<sup>w</sup>) holds for the model  $(\tilde{\mathbf{G}}_r)_{r=t+1, \dots, T}$ . Since  $g_t^n + g_{t+1}^n \in L^0(\tilde{\mathbf{G}}_t, \mathcal{F}_{t+1})$ , we may apply the induction hypothesis and deduce that  $-\delta_T^n = \hat{g}_{t+1}^n + \hat{g}_{t+2}^n + \cdots + \hat{g}_T^n$  where  $\hat{g}_u^n \in L^0(\mathbf{G}_u, \mathcal{F}_u)$  satisfies  $\liminf_n |\hat{g}_u^n| < \infty$  a.s. for  $u \geq t + 2$  and  $\hat{g}_{t+1}^n$  is either  $g_t^n + g_{t+1}^n$  when the latter is convergent or 0 or some constant element  $g_t^k + g_{t+1}^k$ . In any case, we may assume that  $-\delta_T^n = g_t^n + g_{t+1}^n + g_{t+2}^n + \cdots + g_T^n$  where  $g_u^n \in L^0(\mathbf{G}_u, \mathcal{F}_u)$ ,  $u \leq t + 1$ ,  $g_u \in L^0(\mathbf{G}_u, \mathcal{F}_u)$ ,  $u \geq t + 2$ , and  $(g_t^n + \hat{g}_{t+1}^n)_{n \geq 1}$  is a convergent sequence. Only the case where  $\liminf_n |g_t^n| = +\infty$  is of interest. By the normalisation procedure, we deduce the equality  $\tilde{g}_t + \tilde{g}_{t+1} = 0$  where  $\tilde{g}_t \in L^0(\mathbf{G}_t, \mathcal{F}_t)$  is positively collinear to  $g_t^n$  and  $|\tilde{g}_t| = 1$  and  $\tilde{g}_{t+1} \in L^0(\mathbf{G}_{t+1}, \mathcal{F}_{t+1})$ . We deduce that  $S_t^a = S_{t+1}^b$  when  $\bar{g}_t^{\infty(1)} > 0$  and  $\bar{g}_t^{\infty(2)} < 0$  and  $S_t^b = S_{t+1}^a$  otherwise. It follows that  $S_t^a = S_{t+1}^b = E(S_{t+1}^b | \mathcal{F}_t)$  or  $S_t^b = S_{t+1}^a = E(S_{t+1}^a | \mathcal{F}_t)$ . On the set  $\{\bar{g}_t^{\infty(1)} > 0\}$ , let us consider the first instant  $\hat{\tau} \geq t + 1$  such that  $(S_{\hat{\tau}}^a, -1) \in \mathbf{G}_{\hat{\tau}}$ . By Condition (E), the stopping time  $\hat{\tau}$  satisfies  $\hat{\tau} \leq T$ . Hence, it is possible to rewrite  $g_t^n = \sum_{r=t+1}^T g_n^t 1_{\hat{t}au=r}$  where, for all  $r \geq t + 1$ ,  $g_n^t 1_{\hat{t}au} \in L^0(\mathbf{G}_r, \mathcal{F}_r)$ . Similarly, we may rewrite  $g_t^n$  on the set  $\{\bar{g}_t^{\infty(1)} < 0\}$  so that we may apply the induction hypothesis and conclude about the statement from  $t$  to  $T$ . The conclusion follows.

Notice that the implication (3)  $\Rightarrow$  (4) is easily obtained by considering the liquidation values at time  $t + 1$  of the positions  $(S_t^b, -1)1_{F_t}$  and  $(-S_t^a, 1)1_{F_t}$  for all  $F_t \in \mathcal{F}_t$ . The implication (4)  $\Rightarrow$  (5) is deduced from [1, Theorem 4.5]. At last, if (5) holds, take  $\rho_t = \mathbb{E}(dQ/dP)$  and define  $Z = (\rho, \rho\tilde{S})$ . We may verify that  $Z$  is a CPS hence  $(NA^w)$  holds.  $\square$

#### 4.2. Proof of Theorem 3.5.

The inclusion  $\Gamma_\xi \subseteq [\sup_{Q \in \mathcal{M}(P)} \mathbb{E}_Q \xi, \infty)$  is trivial. Let us now consider  $x \geq \sup_{Q \in \mathcal{M}(P)} \mathbb{E}_Q \xi, \infty)$  and suppose by contradiction that  $x \notin \Gamma_\xi$ , i.e.  $\xi - x \notin \mathcal{L}_0^T$ . As  $\mathcal{L}_0^T$  is closed in  $L^1$  under  $(NA^w)$ , we deduce by the Hahn-Banach separation theorem the existence of  $\eta \in L^\infty$  and  $c \in \mathbf{R}$  such that  $\mathbb{E}(\eta X) < c < \mathbb{E}(\eta(\xi - x))$  for all  $X \in \mathcal{L}_0^T \cap L^1(P)$ . Since  $\mathcal{L}_0^T$  is a cone, we deduce that  $\mathbb{E}(\eta X) \leq 0$  for all  $X \in \mathcal{L}_0^T \cap L^1(P)$ . Moreover, as  $\mathcal{L}_0^T$  contains  $-L^0(\mathbf{R}_+, \mathcal{F}_T)$  we deduce that  $\eta \geq 0$  and, after normalization, we have  $\mathbb{E}(\eta) = 1$ . Moreover, if we take  $\eta' = dQ/dP$  such that  $Q \in \mathcal{M}(P) \neq \emptyset$ , then we may choose  $\alpha \in (0, 1)$  sufficiently close to 1 so that  $\hat{\eta} := \alpha\eta + (1 - \alpha)\eta'$  satisfies  $\mathbb{E}(\hat{\eta}(\xi - x)) > 0$  since  $c > 0$ . Moreover,  $\mathbb{E}(\hat{\eta}X) \leq 0$  for all  $X \in \mathcal{L}_0^T \cap L^1(P)$  and  $\hat{\eta} > 0$  satisfies  $\mathbb{E}\hat{\eta} = 1$ . Therefore, if we define  $\hat{Q}$  such that  $d\hat{Q}/dP = \hat{\eta}$ , then  $\hat{Q} \in \mathcal{M}(P)$  in contradiction with  $\mathbb{E}(\hat{\eta}(\xi - x)) > 0$ , i.e.  $x < \mathbb{E}_{\hat{Q}}\xi$ .  $\square$

#### 4.3. Auxiliary results

We denote by  $\mathcal{T}_{t,T}$  the set of all stopping times  $\tau$  with values in  $\{t, \dots, T\}$ . By definition, if  $\tau \in \mathcal{T}_{t,T}$ , then  $\{\tau = u\} \in \mathcal{F}_u$  whatever  $u = t, \dots, T$  hence  $\{\tau \geq u\} \in \mathcal{F}_u$ .

**Lemma 4.1.** *Suppose that for some stopping time  $\tau \in \mathcal{T}_{t,T}$ , we have  $g_t + \dots + g_\tau = 0$  a.s. where  $g_u \in L^1(\mathbf{G}_u, \mathcal{F}_u)$  are integrable selections of  $\mathbf{G}_u$  for all  $u = t, \dots, T$ . Moreover, assume that with probability one,  $g_t + \dots + g_r \neq 0$  for all  $r < \tau$ . Then, for all bounded CPS  $Z, Z_t, \dots, Z_\tau$  are a.s. collinear.*

*Proof.* By assumption, we have  $\sum_{u=t}^T Z_T g_u 1_{u \leq \tau} = 0$ . Taking the conditional expectation knowing  $\mathcal{F}_t$ , we deduce that by the tower property that

$$0 = \sum_{u=t}^T E(Z_T g_u 1_{u \leq \tau} | \mathcal{F}_t) = \sum_{u=t}^T E(Z_u g_u 1_{u \leq \tau} | \mathcal{F}_t).$$

By duality,  $Z_u g_u \geq 0$  hence  $Z_u g_u 1_{u \leq \tau} = 0$  a.s. for all  $u = t, \dots, T$ . Notice that  $\{T \leq \tau\} = \{T = \tau\} = \Omega \setminus \{\tau \leq T - 1\} \in \mathcal{F}_{T-1}$  and  $g_T 1_{T \leq \tau} =$

$g_\tau 1_{T \leq \tau} = -(g_t + \dots + g_{T-1}) 1_{\tau=T} \in L^0(\mathbf{R}^d, \mathcal{F}_{T-1})$ . Therefore,  $Z_T g_T 1_{T \leq \tau} = 0$  implies that  $Z_{T-1} g_T 1_{T \leq \tau} = 0$  by taking the conditional expectation knowing  $\mathcal{F}_{T-1}$ . It follows that  $Z_{T-1} (g_{T-1} 1_{T-1 \leq \tau} + g_T 1_{T \leq \tau}) = 0$ . Suppose by induction that  $Z_u (g_u 1_{u \leq \tau} + \dots + g_T 1_{T \leq \tau}) = 0$  for some  $u \in \{t+1, \dots, T\}$ . As  $g_u 1_{u \leq \tau} + \dots + g_T 1_{T \leq \tau} = -(g_t + \dots + g_{u-1}) 1_{u \leq \tau} \in L^0(\mathbf{R}^d, \mathcal{F}_{u-1})$ , we deduce from  $Z_u (g_u 1_{u \leq \tau} + \dots + g_T 1_{T \leq \tau}) = 0$  by taking the conditional expectation knowing  $\mathcal{F}_{u-1}$  that  $Z_{u-1} (g_u 1_{u \leq \tau} + \dots + g_T 1_{T \leq \tau}) = 0$ . As  $Z_{u-1} g_{u-1} 1_{u-1 \leq \tau} = 0$  a.s., we finally get that  $Z_{u-1} (g_{u-1} 1_{u-1 \leq \tau} + \dots + g_T 1_{T \leq \tau}) = 0$ .

As  $Z_u (g_u 1_{u \leq \tau} + \dots + g_T 1_{T \leq \tau}) = 0$  and  $Z_{u-1} (g_u 1_{u \leq \tau} + \dots + g_T 1_{T \leq \tau}) = 0$  where  $g_u 1_{u \leq \tau} + \dots + g_T 1_{T \leq \tau} = -(g_t + \dots + g_{u-1}) 1_{u \leq \tau} \neq 0$  if  $u \leq \tau$ , we deduce that  $Z_u$  and  $Z_{u-1}$  are colinear if  $u \leq \tau$ . The conclusion follows.  $\square$

**Lemma 4.2.** *Let  $\tau$  be the first passage time such that  $g_t + g_{t+1} + \dots + g_u = 0$  a.s. where  $g_u \in L^1(\mathbf{G}_u, \mathcal{F}_u)$  are integrable selections of  $\mathbf{G}_u$  for all  $u = t, \dots, T$ . If  $g_t \neq 0$ , then  $(NA^w)$  implies that there exist some stopping times  $\sigma \in \{t+1, \dots, \tau\}$  such that  $g_t$  and  $g_\sigma$  are collinear with negative coefficient of collinearity.*

*Proof.* The first passage time  $\tau$  is obviously a stopping time which can be expressed as

$$\tau := \min\{u : t+1 \leq u \leq T \text{ such that } g_t + g_{t+1} + \dots + g_u = 0 \text{ a.s.}\}.$$

This stopping time  $\tau$  satisfies the assumptions of Lemma 4.1 so that the condition  $(NA^w)$  implies that, for every bounded CPS  $Z$ ,  $Z_t$  is collinear with  $Z_u$  a.s. if  $u \leq \tau$ . By taking the conditional expectation knowing  $\mathcal{F}_t$ , from the equality  $\sum_{u=t}^T Z_T g_u 1_{u \leq \tau} = 0$ , we deduce by the tower property and by duality that  $Z_u g_u 1_{u \leq \tau} = 0$  a.s. for all  $u = t, \dots, T$ . Thus,  $Z_t g_u 1_{u \leq \tau} = 0$ . By the definition of  $\tau$ ,  $Z_t g_t 1_{t \leq \tau} = Z_t g_t = 0$  as  $\tau \geq t+1$ . Now the equality  $Z_t g_u 1_{u \leq \tau} = 0$  and  $Z_t g_t = 0$  implies that  $g_u 1_{u \leq \tau}$  and  $g_t$  are collinear, i.e.,  $g_u 1_{u \leq \tau} = k_u g_t$  where  $k_u \in L^0(\mathbf{R}, \mathcal{F}_u)$  satisfies  $k_u = k_u 1_{u \leq \tau}$ . As  $\sum_{u=t}^T g_u 1_{u \leq \tau} = g_t + \sum_{u=t+1}^T k_u g_t = (1 + \sum_{u=t+1}^T k_u) g_t = 0$  and  $g_t \neq 0$ , there exist a.s. a first instant  $u$  such that  $k_u < 0$ . Define the stopping time

$$\sigma := \min\{u : t+1 \leq u \leq \tau \text{ such that } k_u < 0\}.$$

As  $1_{\sigma \leq \tau} = 1$  and  $k_\sigma < 0$ , then we can deduce from  $g_\sigma 1_{\sigma \leq \tau} = k_\sigma g_t$  that  $g_t$  and  $g_\sigma$  are collinear with negative coefficient of collinearity.  $\square$

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