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# An efficient spectral method for the numerical solution to stochastic differential equations

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## Abstract

We consider a new approach for the numerical approximation of stochastic differential equations driven by white noise. The proposed method shares some features with the stochastic collocation techniques and, in particular, it takes advantage of the assumption of smoothness of the functional to be approximated, to achieve fast convergence. The solution to the stochastic differential equation is represented by means of Lagrange polynomials. The coefficients of the polynomial basis are functions of time and they can be computed by solving a system of deterministic ordinary differential equations. Numerical examples are presented to illustrate the accuracy and the efficiency of the proposed method.

### *Keywords:*

Stochastic differential equations, Numerical approximation, Spectral expansion, Stochastic collocation.

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## 1. Introduction

Stochastic differential equations (SDE) driven by a white noise are present in a wide variety of areas. Historically, they were first used to model the motion of Brownian particles at a microscopic level and a link between microscopic and macroscopic quantities could be made by Einstein and Langevin at the beginning of the 20th century. Later, the same ideas were used to model various physical phenomena. For example, non-Newtonian fluid models have been designed by considering the action of white noise to account for the effect of the thermal agitation on polymer chains [18]. Then, starting in the mid seventies, there has been an upsurge of interest in the study of SDE by the mathematical finance community after the work of Black and Scholes [3]. Another important aspect of stochastic problems lies in the study of the propagation of white noise into a physical system described by partial differential equations [13, 28]. All these

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topics have in common a need for fast and accurate numerical methods to solve their underlying SDE.

One of the most common scheme of approximation is the Euler-Maruyama method and its closest cousin, the Milstein scheme [15]. Both are limited to a weak order of convergence (in time) equal to 1. More important is the strong order of convergence, which is also equal to 1 for Milsten but drops to 1/2 for Euler-Maruyama. Unfortunately, designing general numerical schemes with higher order of convergence is not quite as easy as it is for its deterministic Runge-Kutta counterpart [4] and different strategies have to be thought of. One of them, called “exact sampling” consists in generating realizations of the path for the SDE [2, 1]. This method is based on an acceptance/rejection sampling algorithm. It is “exact” in the sense that the marginal distribution of the simulated values coincides with the marginal distribution of the continuous-time process at grid points. Nevertheless one should bear in mind that all the methods quoted above are plagued with a slow rate of convergence typical to Monte-Carlo simulations (proportional to  $1/\sqrt{K}$ , where  $K$  is the number of realizations of the sample paths). Although antithetic sampling or control variable techniques may be used to improve the convergence, such control variable approaches usually remain an ad hoc art [14].

In order to remedy this shortcoming, there has recently been increasing interest in considering the stochastic collocation method, or its modal counterpart the Polynomial Chaos expansion, to compute approximate solutions of SDEs. Since the work of Wiener in 1938 [21] and the seminal book of Ghanem and Spanos in the 90’s [8], Polynomial Chaos (PC) based on orthonormal polynomials have been very successful at solving a wide variety of random problems involving random variables. Later, its analogous based on Lagrange polynomials was popularized by the paper of Hesthaven and Xiu [25]. Quickly these two techniques became the methods of choices for stochastic problems involving a limited number of random variables. The interested reader might refer to the book of Xiu [24] for an overview of the possibilities offered by the PC and the stochastic collocation method. Building on the success of these polynomial-based approximations, numerical experiments were carried out in an attempt to apply the same ideas in the context of SDE driven by white noise. The least we can say is that the transition from random variables to stochastic process is not just a simple adaptation to existing techniques. The two main reasons are the infinite dimensional character of random process, leading to the so-called “curse of dimensionality” and the imperative need that the marginal distribution of the continuous-time process be properly taken into account. When this approach is considered, the Brownian motion is represented as a truncated Karhunen-Loeve series expansion and the stochastic differential equation reduces to a random differential equation [13, 28] involving a finite number of independent random variables. It has been shown by Wong and Zakai [22] that such approximation converges to the Stratanovich SDE and therefore, Itô SDE should be first converted into its corresponding Stratanovich equation. However, numerical experiments have shown that such a procedure failed to handle long-time simulations and treating non-linearities may be a problem. Discussion of these

issues and possible remedies may be found in the recent book of Zhang and Karniadakis [27].

Here, we propose a radically new approach that shares some features with the stochastic collocation method (SCM). In its simplest form, the SCM was successful in representing a random variable (r.v.)  $Y$  of unknown distribution (not necessary Gaussian) as a function of a r.v.  $X$  of known distributions (Gaussian, uniform, exponential...) under the form

$$Y = \sum_{i=1}^n y_i L_i(X), \quad (1)$$

where  $\{L_i(x)\}_{1 \leq i \leq n}$  form a Lagrange polynomial basis and  $\{y_i\}_{1 \leq i \leq n}$  are  $n$  real numbers. If a functional relationship of the form  $Y = f(X)$  exists, then the coefficients  $y_i$  are simply  $f(x_i)$  and equation (1) is the projection of  $f$  onto a Lagrange polynomial basis. If  $f$  is sufficiently regular and the collocations points  $\{x_i\}_{1 \leq i \leq n}$  are carefully chosen, one can observe spectral convergence to the exact solution [10, 11]. In this paper, following the same idea, we seek to express an unknown process  $(Y_t)_{t \in [0, T]}$  (not necessary Gaussian) as a function of a known process  $(G_t)_{t \in [0, T]}$  (Wiener, Ornstein-Uhlenbeck...) under the form

$$Y_t = \sum_{i=1}^n y_i(t) L_i(G_t), \quad \text{for } t \in [0, T],$$

where  $\{L_i(x)\}_{1 \leq i \leq n}$  still form a Lagrange polynomial basis but now,  $\{y_i(t)\}_{1 \leq i \leq n}$  are  $n$  deterministic functions of time. Similarly to the SCM, macroscopic quantities (mean, variance, autocorrelation function...) can easily be computed from the coefficients of such representation as long as the probability density function of  $G_t$  is known. In doing so, there is no need to discretize the process under the form of a finite number of independent random variables and the curse of dimensionality is no longer a problem. Another benefit of this approach is that it does not require a Monte-Carlo simulation to compute macroscopic quantities. We will use such representation for the numerical solution to SDE and the coefficients  $y_i(t)$  will be shown to be the solution of a system of ordinary differential equations. In this paper, we assume that  $(G_t)_{t \in [0, T]}$  is a Gaussian process of the form

$$G_t = \int_0^t k(s) dW_s, \quad (2)$$

where  $(W_s)_{s \in [0, T]}$  is a Wiener process and  $k : [0, T] \rightarrow \mathbb{R}$  is a real function computed in tandem with  $y_i(t)$ .

This paper is organized as follows. In section 2, we describe the basic ideas of the stochastic collocation method when applied to random variables problems. In an effort to make the paper self-contained, we also restate some classical results of polynomial approximation theory and quadrature rules. Then, we explain how the stochastic collocation method can be adapted in the context of random processes. Section 3 describes the type of SDEs addressed in this paper.

Based on Itô's lemma, in section 4 we derive an ordinary differential equation for the functions  $y_i(t)$ . Section 5 offers simple worked out examples when the coefficients  $y_i(t)$  and the function  $k(t)$  can be computed explicitly. We then move on to more complex numerical examples, followed by some conclusions.

## 2. The stochastic collocation method (SCM)

### 2.1. Polynomial approximation and quadrature rules

The two main ingredients of the SCM (interpolating polynomials and quadrature rule) are borrowed from the deterministic community and we start to recall some of their main features. Let  $P_n$  denote the linear space of polynomials of degree at most  $n$ . Given  $(n + 1)$  distinct points  $x_0, x_1, \dots, x_n$  and  $(n + 1)$  corresponding values  $y_0, y_1, \dots, y_n$ , there exists a unique polynomial  $Q \in P_n$  such that  $Q(x_i) = y_i$ . Lagrange provides an elegant way of constructing such polynomials under the form

$$Q(x) = \sum_{i=0}^n y_i L_i(x),$$

where

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^n \frac{x - x_j}{x_i - x_j}, \quad i = 0, \dots, n \quad (3)$$

form a polynomial basis of degree  $n$ .

Let  $f \in C^{n+1}(I)$  such that  $f(x_i) = y_i$  and  $I$  is the smallest interval containing all the points  $x_i$ . Following standard results in numerical analysis, the interpolating error satisfies

$$\forall x \in I, \quad f(x) - Q(x) = \frac{f^{(n+1)}(\xi_x)}{(n+1)!} \prod_{i=0}^{n+1} (x - x_i),$$

where  $\xi_x \in I$ . From this equality, we see that the quality of the approximation not only depends on the regularity of  $f$  but also on the choice of the points  $x_i$ . This can be made more precise using the following theorem, see [10, 11].

**Theorem 2.1.** *Assume that  $f$  is a continuous function on the interval  $I = [-1, 1]$  and  $Q$  is a  $n$  th-order interpolating polynomial of  $f$  based on grid points  $x_i$ , then*

$$\|f - Q\|_{\infty} \leq (1 + \Lambda_n) \|f - Q^*\|_{\infty},$$

where  $Q^*$  is the best  $n$  th-order interpolating polynomial of  $f$  and

$$\Lambda_n = \max_{x \in [-1, 1]} \left( \sum_{i=0}^n |L_i(x)| \right)$$

is called the Lebesgue constant.

This theorem shows that for a continuous function, the Lebesgue constant, and thus the quality of the approximation, only depends on the choice of the points  $x_i$ . Unfortunately, designing optimal set of points is not a trivial task and for numerical simulations, one is often led to choose points that correspond to quadrature rules since they usually exhibit good approximating properties. We recall that a quadrature rule approximates an integral as a weighted sum as follows

$$\int_I f(x)p(x)dx \simeq \sum_{i=0}^n \omega_i f(x_i), \quad (4)$$

where  $\omega_i$  are the quadrature weight and  $x_i$  are the quadrature points. The positive function  $p$  typically corresponds to a probability density of support  $I$ . For example, if  $p(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$  and  $I = \mathbb{R}$ , then the corresponding cubature rule is called Gauss-Hermite quadrature [9]. It can be shown that Gauss quadrature rules (4) are exact when  $f$  is a polynomial of degree less or equal to  $(2n - 1)$ . For the construction of general cubature rules, one can consult [7, 5, 6, 19], for example. Equipped with those two ingredients, we can now move on to the stochastic collocation method.

## 2.2. The SCM applied to random variables

Denote by  $(\Omega, \mathcal{F}, \mathbb{P})$  the probability space, where as usual  $\Omega$  is the set of all possible outcomes,  $\mathcal{F}$  is a  $\sigma$ -algebra over  $\Omega$ , and  $\mathbb{P}$  is a function  $\mathcal{F} \rightarrow [0, 1]$  that gives a probability measure on  $\mathcal{F}$ . Consider an  $\mathbb{R}$ -valued random variable  $X$  of support  $I \subset \mathbb{R}$  that describes input uncertainties. We assume that the probability law of  $X$  is known and that it is defined by a probability density function  $p(x)$ ,  $x \in I$ . Let  $Y = f(X)$  be a random model described by a function  $f : I \rightarrow \mathbb{R}$ . The beauty of the SCM resides in the interpolation points that are chosen to coincide with the quadrature points of (4). Given the set of collocation points  $\{x_i\}_{0 \leq i \leq n}$ , the function  $f$  can be evaluated at those points and approximated as

$$f(x) \simeq \sum_{i=0}^n f(x_i)L_i(x).$$

The quality of the approximation solely depends on the regularity of  $f$  and the choice of the collocation points. It is independent of the nature of the input argument  $x$ , should it be a real variable, a random variable or a stochastic process. In the case of random variables, we can write the approximation

$$Y = f(X) \simeq \sum_{i=0}^n f(x_i)L_i(X). \quad (5)$$

Macroscopic quantities such as the mean or the variance can be computed directly from (5) without having to resort to Monte-Carlo simulations. In addition, using the property of Lagrange polynomials  $L_i(x_j) = \delta_{ij}$  ( $\delta_{ij}$  is the Kronecker

delta) together with (4) and (5), it is easy to show that the formulas for the mean and the variance simplify to

$$\mathbb{E}(Y) \simeq \sum_{i=0}^n \omega_i f(x_i) \quad (6)$$

and

$$\text{var}(Y) \simeq \sum_{i=0}^n \omega_i f(x_i)^2 - \mathbb{E}(Y)^2 \quad (7)$$

In the next subsection, we will see how these ideas can be applied in a similar way in the context of stochastic processes.

### 2.3. The SCM applied to stochastic processes

Let  $(G_t)_{t \in [0, T]}$  be a Gaussian stochastic process defined on a filtered probability space  $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0, T]})$ . We further assume that for all fixed  $t \in [0, T]$ ,  $G_t$  is a zero mean r.v. with variance  $\sigma^2(t)$ . Let  $f(t, x) : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  be a real function and  $(Y_t)_{t \in [0, T]}$  be a stochastic process (not necessarily Gaussian) defined by the functional relationship

$$Y_t = f(t, G_t), \quad t \in [0, T].$$

Applying the same reasoning as before, for any fixed instant of time  $t \in [0, T]$ ,  $Y_t$  can be approximated as

$$Y_t = f(t, G_t) \simeq \sum_{i=0}^n f_i(t) L_i(G_t), \quad (8)$$

where the Lagrange polynomials are defined by (3) together with  $(n+1)$  distinct points  $\{x_i\}_{0 \leq i \leq n}$ . It can be easily seen that the coefficients  $f_i(t)$  are obtained by the evaluation of  $f(t, x_i)$ . The probability density function of  $G_t$  writes

$$p_{G_t}(t, x) = \frac{1}{\sqrt{2\pi}\sigma(t)} \exp\left(-\frac{x^2}{2\sigma(t)^2}\right)$$

and therefore the computation of the mean and the variance of  $Y_t$  will require quadrature rules for integrals of the form

$$\int_{\mathbb{R}} g(t, x) p_{G_t}(t, x) dx,$$

for some real functions  $g : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ . Denoting by  $(\omega_i, x_i)_{0 \leq i \leq n}$  the Gauss-Hermite quadrature points and weights associated to a standard normal distribution, then a simple change of variable reveals that  $(\omega_i, \sigma(t)x_i)_{0 \leq i \leq n}$  are the points and weights associated to the density  $p_{G_t}(t, x)$ . The most natural

way to transcribe the SCM to random process would then be to consider time-dependent Lagrange polynomials of the form

$$L_i(t, x) = \prod_{\substack{j=0 \\ j \neq i}}^n \frac{x - \sigma(t)x_j}{\sigma(t)x_i - \sigma(t)x_j}, \quad i = 0, \dots, n.$$

By so doing, we would obtain formulas similar to (6)-(7) for the mean and the variance of  $Y_t$  i.e.

$$\mathbb{E}(Y_t) \simeq \sum_{i=0}^n \omega_i f(t, \sigma(t)x_i),$$

and

$$\text{var}(Y_t) \simeq \sum_{i=0}^n \omega_i f(t, \sigma(t)x_i)^2 - \mathbb{E}(Y_t)^2.$$

Although this choice of collocation point seems the most natural one, numerical results for the resolution of SDE using such representation were disappointing in terms of convergence and stability. This is because we need a set of interpolating points that are also capable of accurately representing derivatives of a function and not only the function itself. Accordingly, in this paper, we have used the Chebyshev nodes defined on the interval  $[a, b]$  by

$$x_i = \frac{a+b}{2} + \frac{b-a}{2} \cos\left(\frac{2i+1}{2(n+1)}\pi\right), \quad i = 0, \dots, n \quad (9)$$

to construct the Lagrange basis. Such basis offers good convergence properties since this set of points gives a near optimal Lebesgue constant [20]. However, the formulas for the mean and the variance of  $Y_t$  do not simplify and we give their new expressions in the sequel.

Let  $(\tilde{\omega}_i, \tilde{x}_i)_{0 \leq i \leq m}$  denote cubature points and nodes for some general quadrature rule, for any fixed  $t \in [0, T]$

$$\int_{\mathbb{R}} g(t, x) dx \simeq \sum_{i=0}^m \tilde{\omega}_i g(t, \tilde{x}_i). \quad (10)$$

Then, assuming an approximation of  $Y_t$  under the form (8), we have

$$\begin{aligned} \mathbb{E}(Y_t) &= \int_{\mathbb{R}} \left( \sum_{k=0}^n f_k(t) L_k(x) \right) p_{G_t}(t, x) dx, \\ \mathbb{E}(Y_t) &\simeq \sum_{i=0}^m \sum_{k=0}^n \tilde{\omega}_i f_k(t) L_k(\tilde{x}_i) p_{G_t}(t, \tilde{x}_i), \end{aligned} \quad (11)$$

and similarly

$$\begin{aligned} \text{var}(Y_t) &= \int_{\mathbb{R}} \left( \sum_{k=0}^n f_k(t) L_k(x) \right)^2 p_{G_t}(t, x) dx - \mathbb{E}(Y_t)^2, \\ \text{var}(Y_t) &\simeq \sum_{i=0}^m \tilde{\omega}_i \left( \sum_{k=0}^n f_k(t) L_k(\tilde{x}_i) \right)^2 p_{G_t}(t, \tilde{x}_i) - \mathbb{E}(Y_t)^2. \end{aligned} \quad (12)$$

For stochastic processes, the computation of the autocorrelation function  $\mathbb{E}(Y_t Y_s)$ ,  $(t, s) \in [0, T]^2$  is also of particular importance. This requires the knowledge of the joint density of the vector  $(G_t, G_s)$  for  $t \neq s$ . It is given in the following lemma :

**Lemma 2.2.** *Let  $(G_t)_{t \in [0, T]}$  be a centered Gaussian process under the form (2). For fixed  $t, s$  into  $[0, T]$  with  $s > t$ , the vector  $(G_t, G_s)$  is a centered Gaussian vector of covariance matrix  $\Gamma(t, s) = \begin{pmatrix} \sigma(t)^2 & \sigma(t)^2 \\ \sigma(t)^2 & \sigma(s)^2 \end{pmatrix}$ , where  $\sigma(t)^2 = \int_0^t k^2(u) du$ . Furthermore, the joint density of the vector  $(G_t, G_s)$  takes the form*

$$p_{(G_t, G_s)}(t, s, x, y) = \frac{1}{2\pi \sqrt{\det(\Gamma(t, s))}} e^{-\frac{1}{2 \det(\Gamma(t, s))} (\sigma(s)^2 x^2 + \sigma(t)^2 (y^2 - 2xy))},$$

with  $\det(\Gamma(t, s)) = \sigma(t)^2(\sigma(s)^2 - \sigma(t)^2)$ .

**Proof 2.3.** *Let  $(G_t)_{t \in [0, T]}$  be the process given by (2). For  $s > t$ , we have*

$$\begin{pmatrix} G_t \\ G_s \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} G_t \\ G_s - G_t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \int_0^t k(u) dW_u \\ \int_t^s k(u) dW_u \end{pmatrix}$$

The vector  $\begin{pmatrix} \int_0^t k(u) dW_u \\ \int_t^s k(u) dW_u \end{pmatrix}$  is a centered Gaussian vector, since its component

are independent centered Gaussian random variables. Therefore  $\begin{pmatrix} G_t \\ G_s \end{pmatrix}$  is a centered Gaussian vector as a linear transformation of a centered Gaussian vector. Using Itô isometry we have

$$\text{var}(G_t) = \mathbb{E} \left( \left( \int_0^t k(u) dW_u \right)^2 \right) = \int_0^t k(u)^2 du := \sigma(t)^2.$$

For  $s > t$ , using the independence of  $G_t$  and  $G_s - G_t$  and the fact that  $\mathbb{E}(G_t) = 0$ , we obtain

$$\mathbb{E}(G_t G_s) = \mathbb{E}(G_t (G_s - G_t)) + \mathbb{E}(G_t^2) = \mathbb{E}(G_t^2) = \sigma(t)^2,$$

which gives the components of the covariance matrix. The expression of the joint density arises directly from this result.

Using the approximation (8) and the quadrature rule (10), for any fixed  $s$  and  $t$  into  $[0, T]$ , and assuming  $s > t$ , we have

$$\begin{aligned}\mathbb{E}(Y_t Y_s) &= \int_{\mathbb{R}^2} \left( \sum_{k=0}^n f_k(t) L_k(x) \right) \left( \sum_{l=0}^n f_l(s) L_l(y) \right) p_{G_t}(t, s, x, y) dx dy \\ \mathbb{E}(Y_t Y_s) &\simeq \sum_{i=0}^m \sum_{j=0}^m \sum_{k=0}^n \sum_{l=0}^n \tilde{\omega}_i \tilde{\omega}_j f_k(t) f_l(s) L_k(\tilde{x}_i) L_l(\tilde{x}_j) p_{G_t, G_s}(t, s, \tilde{x}_i, \tilde{x}_j) \quad (13)\end{aligned}$$

In the numerical section, means, variances and autocorrelation functions of considered processes will be evaluated with formulas (11), (12), and (13).

### 3. Itô stochastic differential equation

In this paper, we are interested in the numerical solution to SDEs involving a random white noise term  $dW_t$  that write under the standard form

$$\begin{cases} dX_t = a(t, X_t)dt + b(t, X_t)dW_t, & t > 0 \\ X_{t=0} = X_0 \end{cases}, \quad (14)$$

where, as usual,  $(W_t)_{t \in [0, T]}$  denotes Brownian motion on a filtered probability space  $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0, T]})$ ,  $X_0$  is  $\mathcal{F}_0$ -measurable and  $a(t, x) : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  and  $b(t, x) : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  are two deterministic functions. Equation (14) should be interpreted as its equivalent integral form

$$X_t = X_0 + \int_0^t a(s, X_s) ds + \int_0^t b(s, X_s) dW_s,$$

where the first integral on the right-hand side is a Riemann integral and the second integral is considered as an Itô integral, see [17]. We assume that standard conditions are satisfied to ensure a unique strong solution to equation (14) i.e.

- The initial condition satisfies  $\mathbb{E}(X_0^2) < \infty$  and is independent of  $(W_t)_{t \in [0, T]}$
- For all  $t \in [0, T]$  and  $x, y \in \mathbb{R}$ ,  $a(t, x)$  and  $b(t, x)$  are continuous
- For all  $t \in [0, T]$  and  $x, y \in \mathbb{R}$ ,  $a(t, x)$  and  $b(t, x)$  satisfy Lipschitz condition with respect to the second variable

$$|a(t, x) - a(t, y)| + |b(t, x) - b(t, y)| \leq K |x - y|$$

In the numerical section, we will consider SDE of the form (14) that satisfy the above conditions.

#### 4. Description of the proposed method

We now seek an approximate solution to (14) under the form

$$X_t = f(t, G_t) = \sum_{i=0}^n f_i(t) L_i(G_t), \quad (15)$$

where  $\{L_i(x)\}_{0 \leq i \leq n}$  are the Lagrange polynomials defined by (3) and  $f$  is some (unknown) smooth function.  $(G_t)_{t \in [0, T]}$  is a Gaussian process that we assume to be of the form

$$G_t = \int_0^t k(s) dW_s, \quad (16)$$

where  $k : [0, T] \rightarrow \mathbb{R}$  is a continuous real function such that  $k(0) \neq 0$ . The value of  $k(0)$  is unimportant and we take arbitrary  $k(0) = 1$ . In case  $k(0) \neq 1$ , we use the ratio  $k(s)/k(0)$  to define the stochastic process  $G_t$  in (16) instead of  $k(s)$ .

Such representation covers a large variety of Gaussian processes: for example, the Wiener process for  $k(s) = 1$ , the Ornstein-Uhlenbeck process for  $k(s) = e^{\theta s}$  with  $\theta$  a real parameter, the Brownian bridge process for  $k(s) = \frac{\alpha}{\alpha - s}$  ( $0 \leq s < \alpha$ ) with  $\alpha$  a real positive parameter...

Using Itô isometry, the standard deviation  $\sigma(t)$  of  $G_t$  can be shown to be

$$\sigma(t) = \sqrt{\int_0^t k(s)^2 ds}. \quad (17)$$

The unknown of the problem are the  $(n+1)$  coefficients  $f_i(t)$  of equation (15) together with the function  $k(s)$  of equation (16). Once those quantities have been computed, the zero mean Gaussian process  $(G_t)_{t \in [0, T]}$  is fully specified, and the process  $(X_t)_{t \in [0, T]}$  is also completely represented by its approximation (15). It is then possible to compute macroscopic quantities with equations (11), (12) and (13). The following theorem specifies the expression of  $f_i(t)$  and  $k(t)$ .

**Theorem 4.1.** *Let consider the Itô stochastic differential equation (14) and let assume that its exact solution can be expressed as an explicit function  $f(t, G_t)$  with  $(G_t)_{t \in [0, T]}$  a Gaussian process defined by (16) and  $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  a function whose second order derivatives are continuous. Then  $f$  satisfies following system of partial differential equations for all  $(t, x)$  in  $[0, T] \times \mathbb{R}$  :*

$$\begin{cases} \frac{\partial f}{\partial t}(t, x) = a(t, f(t, x)) - \frac{1}{2} \frac{\partial b(t, f(t, x))}{\partial x} \times b(t, f(t, x)) \\ \frac{\partial f}{\partial x}(t, x) k(t) = b(t, f(t, x)) \end{cases} \quad (18)$$

**Proof 4.2.**  *$f$  being assumed to have second order derivatives continuous, Itô lemma can be applied to obtain*

$$dX_t = df(t, G_t) = \frac{\partial f}{\partial t}(t, G_t) dt + \frac{\partial f}{\partial x}(t, G_t) dG_t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(t, G_t) dG_t dG_t. \quad (19)$$

In addition, from equation (16), we have

$$dG_t = k(t)dW_t$$

and

$$dG_t dG_t = k(t)^2 dW_t dW_t = k(t)^2 dt$$

Replacing those expressions into (19), we get

$$dX_t = \left( \frac{\partial f}{\partial t}(t, G_t) + \frac{k(t)^2}{2} \frac{\partial^2 f}{\partial x^2}(t, G_t) \right) dt + k(t) \frac{\partial f}{\partial x}(t, G_t) dW_t.$$

Identifying this equation with (14) after substituting  $X_t$  by  $f(t, G_t)$ , the following functional relationship should hold for all  $(t, x) \in [0, T] \times \mathbb{R}$  :

$$\begin{cases} a(t, f(t, x)) = \frac{\partial f}{\partial t}(t, x) + \frac{k(t)^2}{2} \frac{\partial^2 f}{\partial x^2}(t, x) \\ b(t, f(t, x)) = k(t) \frac{\partial f}{\partial x}(t, x) \end{cases} \quad (20)$$

Differentiating the second equation in (20) with respect to  $x$  and replacing  $k(t) \frac{\partial f}{\partial x}(t, x)$  by  $b(t, f(t, x))$ , we have

$$\frac{\partial b(t, f(t, x))}{\partial x} \times b(t, f(t, x)) = k(t)^2 \frac{\partial^2 f}{\partial x^2}(t, x).$$

And replacing anew the above second order derivative into the first equation of the system (20), we obtain the equivalent system (18).

We now seek an approximate solution to the system (18) together with given initial conditions that will be specified a bit further in the sequel. The approximate solution is assumed to take the form

$$f(t, x) = \sum_{i=0}^n f_i(t) L_i(x) \quad (21)$$

and the expression of  $f_i(t) = f(t, x_i)$  is obtained as the solution to an ordinary differential equation, as stated in the following theorem.

**Theorem 4.3.** *Assume that an approximation of  $f$  writes under the form (21). Then the coefficients  $f_i(t)$  with  $i = 0, \dots, n$  satisfy the following system of decoupled ordinary differential equations*

$$\begin{cases} f_i'(t) = -\frac{1}{2} \frac{\partial b}{\partial x}(t, f_i(t)) \times b(t, f_i(t)) + a(t, f_i(t)) \\ f_i(0) = f(0, x_i) \end{cases} \quad (22)$$

We obtain the initial conditions  $f(0, x_i)$  by solving the following differential equation :

$$\begin{cases} \frac{df}{dx}(0, x) = b(0, f(0, x)) \\ f(0, 0) = X_0 \end{cases} \quad (23)$$

Furthermore, assuming that for all  $(t, x)$  in  $[0, T] \times \mathbb{R} : \frac{\partial f}{\partial x}(t, x) \neq 0$ , then the function  $k(t)$  of equation (16) is given by

$$k(t) = \frac{b(t, f(t, x))}{\frac{\partial f}{\partial x}(t, x)} \simeq \frac{b\left(t, \sum_{i=0}^n f_i(t)L_i(x)\right)}{\sum_{i=0}^n f_i(t)L'_i(x)} \quad (24)$$

**Proof 4.4.** Following the standard collocation method, we replace the approximation of  $f$  (21) into equation (18)

$$\begin{cases} \sum_{i=0}^n f'_i(t)L_i(x) = a\left(t, \sum_{i=0}^n f_i(t)L_i(x)\right) - \frac{1}{2} \frac{\partial b}{\partial x}\left(t, \sum_{i=0}^n f_i(t)L_i(x)\right) \times b\left(t, \sum_{i=0}^n f_i(t)L_i(x)\right) \\ k(t) \times \sum_{i=0}^n f_i(t)L'_i(x) = b\left(t, \sum_{i=0}^n f_i(t)L_i(x)\right) \end{cases}$$

We then impose the equations to be satisfied at collocations points  $x_j$  and use the property of Lagrange polynomial  $L_i(x_j) = \delta_{ij}$  to end up with :

$$\begin{cases} f'_i(t) = a(t, f_i(t)) - \frac{1}{2} \frac{\partial b}{\partial x}(t, f_i(t)) \times b(t, f_i(t)) \\ k(t) = \frac{b\left(t, \sum_{i=0}^n f_i(t)L_i(x)\right)}{\sum_{i=0}^n f_i(t)L'_i(x)} \end{cases} \quad (25)$$

The differential equation for  $f_i(t)$  should be provided with initial conditions  $f_i(0) = f(0, x_i)$ . The initial conditions to the SDE (14) cannot be directly used since it only provides  $X_0 = f(0, 0)$ . However, the value of  $f(0, x_i)$  can be obtained by considering the second equation of the system (20) and setting  $t = 0$ . Since  $k(0) = 1$ , this leads to the differential equation (23) that can be solved to get the value of  $f(0, x_i)$  for  $i = 0, \dots, n$ .

**Remark 4.5.** The second equation of (18) gives an a-posteriori criterion to check whether the hypothesis (16) is valid. Indeed, if

$$k(t) = \frac{b(t, f(t, x))}{\frac{\partial f}{\partial x}(t, x)}$$

is a function of  $x$ , then the considered SDE has a solution that cannot be expressed as on explicit function of  $G_t$  defined by (16).

## 5. Worked out examples

Before moving on to more complex numerical examples, we apply the proposed method to some classical SDE for which explicit calculations can be performed by hand.

### 5.1. The geometric Brownian motion

As a first example, we consider the geometric Brownian motion process which satisfies the following SDE :

$$\begin{cases} dX_t = \mu X_t dt + \rho X_t dW_t, & t > 0 \\ X_{t=0} = X_0 \end{cases}, \quad (26)$$

where  $\mu$  and  $\rho$  are some real parameters. An analytic solutions can be found for this classical SDE and it writes, for  $t \geq 0$  :

$$X_t = X_0 \exp\left(\left(\mu - \frac{\rho^2}{2}\right)t + \rho W_t\right). \quad (27)$$

The real functions  $a(t, x)$  and  $b(t, x)$  are define on  $\mathbb{R}^+ \times \mathbb{R}$  by

$$\begin{cases} a(t, x) = \mu x \\ b(t, x) = \rho x \end{cases}.$$

The differential equation (23) to be solved for the initial conditions of (22) is

$$\begin{cases} \frac{df}{dx}(0, x) = \rho f(0, x) \\ f(0, 0) = X_0 \end{cases}.$$

Its solution is  $f(0, x) = X_0 \exp(\rho x)$  and therefore, the initial conditions of the differential equation for the coefficients  $f_i(t)$  will be

$$f_i(0) = f(0, x_i) = X_0 \exp(\rho x_i).$$

Since  $\frac{\partial b(t, x)}{\partial x} = \rho$ , the differential equation for  $f_i(t)$  with  $i = 0, \dots, n$  simplifies to

$$\begin{cases} f_i'(t) = \left(\mu - \frac{\rho^2}{2}\right) f_i(t) \\ f_i(0) = X_0 \exp(\rho x_i) \end{cases}$$

and its solution is  $f_i(t) = X_0 \exp(\rho x_i) \exp\left(\left(\mu - \frac{\rho^2}{2}\right)t\right)$ .

We now have all the elements available to compute the function  $k(t)$  that will specify the process  $(G_t)_{t \in [0, T]}$ . Replacing  $f_i(t)$  by its expression in the second equation of (25), we get after simplifications

$$k(t) = \frac{\rho \sum_{i=0}^n f_i(t) L_i(x)}{\sum_{i=0}^n f_i(t) L_i'(x)} = \frac{\rho \sum_{i=0}^n \exp(\rho x_i) L_i(x)}{\sum_{i=0}^n \exp(\rho x_i) L_i'(x)}.$$

At this point, this fraction cannot be further simplified. However, we can recognize the projection of the real function  $x \mapsto \exp(\rho x)$  onto a Lagrange basis at the numerator and the projection of the derivative of the same function at the denominator. Therefore, we have

$$k(t) = \frac{\rho \sum_{i=0}^n \exp(\rho x_i) L_i(x)}{\sum_{i=0}^n \exp(\rho x_i) L'_i(x)} \simeq \frac{\rho \exp(\rho x)}{(\exp(\rho x))'} = \frac{\rho \exp(\rho x)}{\rho \exp(\rho x)} = 1.$$

As a conclusion, the expression of the process  $(G_t)$  simplifies to

$$G_t = \int_0^t k(s) dW_s = \int_0^t dW_s = W_t,$$

which is nothing other than a Wiener process. Therefore, an approximate solution to (26) can write

$$X_t = X_0 \exp\left(\left(\mu - \frac{\rho^2}{2}\right)t\right) \sum_{i=0}^n \exp(\rho x_i) L_i(W_t). \quad (28)$$

Comparing the exact solution (27) with the approximate one (28), we notice that the term  $\sum_{i=0}^n \exp(\rho x_i) L_i(W_t)$  is the projection of the real function  $x \mapsto \exp(\rho x)$  onto a Lagrange basis evaluated at  $x = W_t$ .

### 5.2. The Ornstein-Uhlenbeck process

As a second example, we consider the Ornstein-Uhlenbeck process which satisfies the following SDE:

$$\begin{cases} dX_t = \theta(\mu - X_t)dt + \rho dW_t, & t > 0 \\ X_{t=0} = X_0 \end{cases},$$

where  $\theta$ ,  $\mu$  and  $\rho$  are some real parameters. The analytic solutions for this SDE writes, for  $t \geq 0$

$$X_t = X_0 e^{-\theta t} + \mu(1 - e^{-\theta t}) + \rho e^{-\theta t} \int_0^t e^{\theta s} dW_s \quad (29)$$

The real functions  $a(t, x)$  and  $b(t, x)$  are define on  $\mathbb{R}^+ \times \mathbb{R}$  by

$$\begin{cases} a(t, x) = \theta(\mu - x) \\ b(t, x) = \rho \end{cases},$$

and the differential equation (23) to be solved for the initial conditions of (22) reduces to

$$\begin{cases} \frac{df}{dx}(0, x) = \rho \\ f(0, 0) = X_0 \end{cases}.$$

Its solution writes  $f(0, x) = \rho x + X_0$ , which gives initial conditions

$$f_i(0) = f(0, x_i) = \rho x_i + X_0.$$

Since for all  $(t, x)$  in  $\mathbb{R}^+ \times \mathbb{R}$  we have  $\frac{\partial b(t, x)}{\partial x} = 0$ , the differential equation for  $f_i(t)$  with  $i = 0, \dots, n$  comes down to

$$\begin{cases} f_i'(t) = \theta(\mu - f_i(t)) \\ f_i(0) = \rho x_i + X_0 \end{cases}$$

and its solution is  $f_i(t) = \mu + (X_0 + \rho x_i - \mu) \exp(-\theta t)$ . As before, we compute the function  $k(t)$  by replacing  $f_i(t)$  by its expression in the second equation of (25)

$$\begin{aligned} k(t) &= \frac{\rho}{\sum_{i=0}^n (\mu + (X_0 + \rho x_i - \mu) e^{-\theta t}) L_i'(x)} \\ &= \frac{\rho}{(\mu + (X_0 - \mu) e^{-\theta t}) \sum_{i=0}^n L_i'(x) + \rho e^{-\theta t} \sum_{i=0}^n x_i L_i'(x)}. \end{aligned}$$

The denominator of the above expression can be simplified by using two properties of Lagrange's polynomials, i.e. for all  $x$  in  $\mathbb{R}$ ,

$$\sum_{i=0}^n L_i'(x) = 0 \quad \text{and} \quad \sum_{i=0}^n x_i L_i'(x) = 1;$$

leading to

$$k(t) = \frac{\rho}{\rho e^{-\theta t}} = e^{\theta t}.$$

The same properties can also be used to simplify the approximate solution of the SDE as follows :

$$\begin{aligned} X_t &= \sum_{i=0}^n (\mu + (X_0 + \rho x_i - \mu) e^{-\theta t}) L_i(G_t) \\ &= X_0 e^{-\theta t} + \mu(1 - e^{-\theta t}) + \rho e^{-\theta t} \sum_{i=0}^n x_i L_i(G_t), \\ &= X_0 e^{-\theta t} + \mu(1 - e^{-\theta t}) + \rho e^{-\theta t} G_t, \end{aligned}$$

where in the last equation, we have used the additional property: for all  $x$  in  $\mathbb{R}$ ,  $\sum_{i=0}^n x_i L_i(x) = x$ . Then, the proposed numerical method applied to the Ornstein-Uhlenbeck SDE gives the solution

$$X_t = X_0 e^{-\theta t} + \mu(1 - e^{-\theta t}) + \rho e^{-\theta t} G_t$$

with

$$G_t = \int_0^t k(s) dW_s = \int_0^t e^{\theta s} dW_s.$$

This is precisely the exact solution of the SDE (see equation (29)). This comes as no surprise, since the exact solution can be expressed as a degree one polynomial function of  $G_t$  when  $G_t$  takes the form (16) with  $k(s) = e^{\theta s}$ .

## 6. Numerical examples

In this section, we tackle two SDEs and we explain how each step of the proposed method can be numerically addressed.

### 6.1. SDE with known analytical solution

As a first numerical example, we consider the following SDE:

$$\begin{cases} dX_t = \frac{1}{2}(\tanh(X_t))^3 dt + \frac{1}{\cosh(X_t)} dW_t, & t > 0 \\ X_{t=0} = \operatorname{arcsinh}(1) \end{cases}$$

of known analytical solution (see [15, 16])

$$X_t = \operatorname{arcsinh} \left( e^{t/2} \left( 1 + \int_0^t e^{-s/2} dW_s \right) \right). \quad (30)$$

We start by computing  $(n+1)$  Chebyshev collocation points (see (9))  $\{x_i\}_{0 \leq i \leq n}$  in the interval  $[a, b] = [-6, 6]$  (the choice of this interval will be justified in the sequel). The next step is to solve the ordinary differential equation (23) twice: once in the interval  $[0, 6]$  to get the values of  $f(0, x_i)$  when  $x_i \in [0, 6]$  and once in the interval  $[-6, 0]$  to get the values of  $f(0, x_i)$  when  $x_i \in [-6, 0]$ . We have to proceed that way since the initial condition to this differential equation is given at  $x = 0$ . For the equations considered in this paper, the choice of the numerical solver was found to be of little importance (4th order Kunge-Kutta for example) since high accuracies could be obtained with mild computing efforts. The  $(n + 1)$  ordinary differentials equations (22) are then solved numerically to get an approximate solution of  $f_i(t)$  at points  $t_l \in [0, T]$ . Notice that they all take the same form but have different initial conditions. Here again, a standard ODE solver may be used. The function of time  $k(t)$  that serves to define the process  $(G_t)_{t \in [0, T]}$  can be evaluated with equation (24) at all discrete points  $t_l$  in time and at all collocation points  $x_j$  as follows:

$$k(t_l) = \frac{b(t_l, f_i(t_l))}{\sum_{i=0}^n f_i(t_l) L'_i(x_j)}$$

If the exact solution to the SDE can be expressed as an explicit function of the Gaussian process defined by (16) this expression should be independent of

$x_j$ . This gives an a-posteriori criterion for establishing whether the considered SDE can be processed with our numerical method.

The computation of the mean and the variance of  $X_t$  requires an evaluation of the standard deviation  $\sigma(t)$  of  $G_t$ . According to (17) and after a simple change of variable, this is done by evaluating the function (for all points  $t_l \in [0, T]$ )

$$\sigma(t_l) = \sqrt{t_l \int_0^1 k(t_l s)^2 ds}.$$

The above integral is computed with Fejér quadrature rule [5] based on the roots of Chebychev polynomials of first kind. The probability density function of  $G_t$  at instants of time  $t_l \in [0, T]$  is then given by

$$p_{G_t}(t_l, x) = \frac{1}{\sqrt{2\pi}\sigma(t_l)} e^{-\frac{x^2}{2\sigma(t_l)^2}},$$

and it is possible to compute the mean, the variance and the autocorrelation function of the process  $(X_t)_{t \in [0, T]}$  with formulas (11), (12) and (13), respectively. The integrals on  $\mathbb{R}$  are truncated to the interval  $[-6, 6]$  and they are also evaluated with Fejér quadrature rules. The truncation of the integrals do not affect the accuracy of the method due to the fast decay of  $p_{G_t}(t_l, x)$  with respect to  $|x|$ . In this example we can make this assertion more precise, since  $\sigma(t)$  can be computed explicitly:

$$\sigma(t) = \sqrt{\int_0^t (e^{-s/2})^2 ds} = \sqrt{1 - e^{-t}}. \quad (31)$$

For all  $t \geq 0$ , the maximum value of  $\sigma(t)$  is one, so for all  $t \geq 0$ , if  $|x| \geq 6$ ,

$$0 \leq \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma(t)^2}} \leq \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \leq \frac{1}{\sqrt{2\pi}} e^{-\frac{6^2}{2}} \simeq 6 \times 10^{-9}.$$

The interval of time for the simulations is chosen to be  $[0, T] = [0, 4]$  and the time step for representing the results are  $\Delta t = t_{l+1} - t_l = 10^{-2}$ . Notice that for this method, it is irrelevant to study the weak or strong convergence in time. Indeed, the differential equations (22) and (23) can be solved up to machine's precision with standard ODE solvers and the major part of the error comes from the interpolating error of Lagrange's polynomials. What we consider as an "exact" mean or an "exact" variance of  $X_t$  can be numerically computed from the solution (30) using high precision quadrature rules. We start by rewriting (30) as

$$X_t = \operatorname{arcsinh} \left( e^{t/2} (1 + G_t) \right),$$

with  $(G_t)_{t \in [0, T]}$  a Gaussian process defined by

$$G_t = \int_0^t e^{-s/2} dW_s.$$

Then for any  $t \in [0, T]$  the probability density function of  $G_t$  is

$$p_{G_t}(x) = \frac{1}{\sqrt{2\pi}\sigma(t)} e^{-\frac{x^2}{2\sigma(t)^2}}, \quad x \in \mathbb{R},$$

with  $\sigma(t)$  given by (31). Denoting by  $\mu_{X_t}(t)$  and  $\sigma_{X_t}^2(t)$  the exact mean and variance of  $X_t$ , we have

$$\mu_{X_t}(t) = \int_{\mathbb{R}} \operatorname{arcsinh}\left(e^{t/2}(1+x)\right) p_{G_t}(x) dx \quad (32)$$

$$\sigma_{X_t}^2(t) = \int_{\mathbb{R}} \left(\operatorname{arcsinh}\left(e^{t/2}(1+x)\right)\right)^2 p_{G_t}(x) dx - (\mu_{X_t}(t))^2 \quad (33)$$

Such integrals are evaluated after truncating their support to  $[-10, 10]$  and using Fejér quadrature rule with  $10^3$  quadrature points. The values obtained will be considered as “exact” values for  $\mu_{X_t}(t)$  and  $\sigma_{X_t}(t)$  in the sequel.

Let  $\mathbf{mean}^{exact}$  and  $\mathbf{mean}^{approx}$  be two vectors containing the values of the exact mean and the approximated mean at times  $\{t_l\}_{1 \leq l \leq 400}$ . The relative error on the mean is defined as

$$\varepsilon_{mean} = \frac{\|\mathbf{mean}^{exact} - \mathbf{mean}^{approx}\|}{\|\mathbf{mean}^{exact}\|}, \quad (34)$$

where the norms above are the Euclidean norms. The relative error on the variance is defined in a similar way. Table 1 shows such errors for different levels of discretization when the number of collocations points vary from  $n + 1 = 40$  up to  $n + 1 = 180$ .

$n + 1$	40	60	80	100
$\varepsilon_{mean}$	$6.82 \times 10^{-3}$	$3.24 \times 10^{-3}$	$6.96 \times 10^{-4}$	$3.17 \times 10^{-4}$
$\varepsilon_{var}$	$2.82 \times 10^{-2}$	$1.54 \times 10^{-2}$	$3.40 \times 10^{-3}$	$1.46 \times 10^{-3}$
$n + 1$	120	140	160	180
$\varepsilon_{mean}$	$9.01 \times 10^{-5}$	$3.05 \times 10^{-5}$	$6.26 \times 10^{-6}$	$1.13 \times 10^{-6}$
$\varepsilon_{var}$	$4.50 \times 10^{-4}$	$1.40 \times 10^{-4}$	$3.17 \times 10^{-5}$	$5.88 \times 10^{-6}$

Table 1: Errors on the mean and the variance of  $X_t$  defined by equation (34)

We can see the high rate of convergence of the proposed method. We do not provide CPU times since it only takes a few second to run for the coarse level of discretization and up to a few minutes for the finest level of discretisation. A sample trajectory is represented on Figure 1 for the exact process  $X_t$  given by

$$X_t = \operatorname{arcsinh}\left(e^{t/2}(1 + G_t)\right), \quad (35)$$

and its approximation given by

$$X_t = \sum_{i=0}^n f_i(t) L_i(G_t). \quad (36)$$

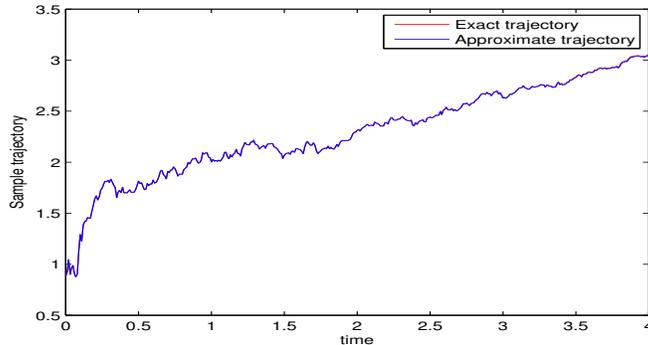


Figure 1: Sample trajectory for  $X_t$  computed from equation (35) (exact) and from its approximation equation (36)

For that, a sample path for the process  $(G_t)_{t \in [0, T]}$  needs to be generated from one trajectory of a Wiener process  $(W_t)_{t \in [0, T]}$  [12] as follows:

$$G_{t_{i+1}} = G_{t_i} + e^{-t_i/2} (W_{t_{i+1}} - W_{t_i}),$$

for the path of the exact solution; and

$$G_{t_{i+1}} = G_{t_i} + k(t_i) (W_{t_{i+1}} - W_{t_i}),$$

for the path of the approximated solution. We see that the two trajectories are almost indistinguishable (here  $n + 1 = 80$ ). For the same level of discretization, we also represent the mean and the variance (exact and approximate) on Figures 2 and 3, respectively. Here again, the two curves match perfectly well. Finally, Figure 4 shows the autocorrelation function of  $(X_t)_{t \in [0, T]}$ , as computed by (13) for  $(s, t) \in [0, 4] \times [0, 4]$ .

### 6.2. SDE without known analytical solution

As a second numerical example, we consider the following SDE

$$\begin{cases} dX_t = \frac{1}{4} (X_t + \sqrt{1 + X_t^2}) \left( 3 + \frac{2X_t}{\sqrt{1 + X_t^2}} \right) dt + (X_t + \sqrt{1 + X_t^2}) dW_t, & t > 0 \\ X_{t=0} = 0 \end{cases}$$

for which no solution can be derived analytically. This SDE is particularly challenging for Euler-like numerical methods, due to the variance of the solution that grows extremely fast with respect to  $t$ . As a consequence, the Monte-Carlo simulations to compute the mean and the variance of  $X_t$  requires a very large number of trajectories to get a converged solution. At first, we limit the interval of study to  $[0, T] = [0, 1]$  to be able to compare our numerical method with the Milstein scheme with a time step  $\Delta t = 10^{-2}$ .

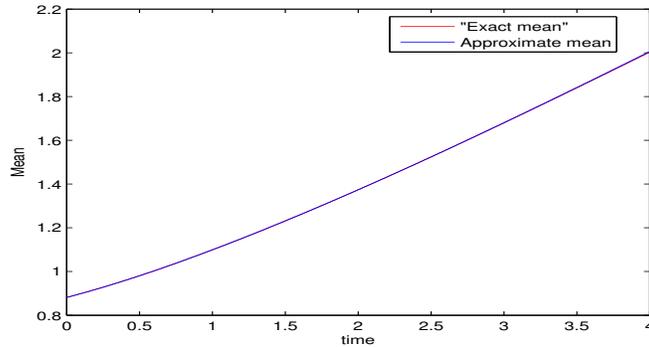


Figure 2: “Exact” and approximate mean of  $X_t$  computed with equation (32) and equation (11) respectively

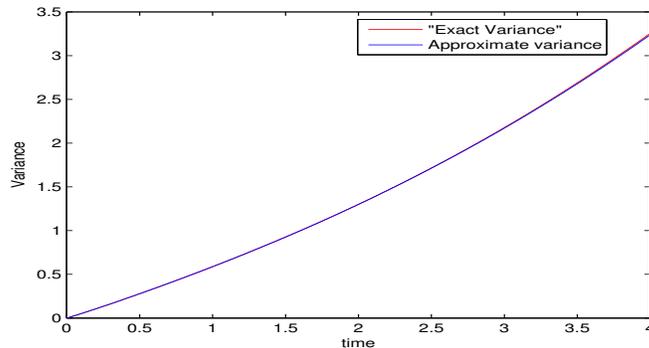


Figure 3: “Exact” and approximate variance of  $X_t$  computed with equation (33) and equation (12) respectively

The mean and the variance are then obtained with a  $10^6$  samples Monte-Carlo simulation. The Milstein scheme has a weak and a strong order of convergence in time equal to 1 and it writes :

$$\begin{cases} X_{t_{l+1}} = X_{t_l} + a(t_l, X_{t_l})\Delta t + b(t_l, X_{t_l})\Delta W + \frac{1}{2}b(t_l, X_{t_l})\frac{\partial b}{\partial x}(t_l, X_{t_l})(\Delta W^2 - \Delta t) \\ X_{t_0} = 0 \end{cases} ,$$

with  $\Delta W = W_{t_{l+1}} - W_{t_l}$ . Figure 5 and Figure 6 show the mean and the variance of  $X_t$  for our method with  $n + 1 = 60$  collocation points and for the Milstein/Monte-Carlo simulation. We can see a good match of the two curves for times  $t \leq 0.6$  but it quickly deteriorates for the Milstein scheme as we advance in time.

We now study the convergence of our scheme for long time simulations by

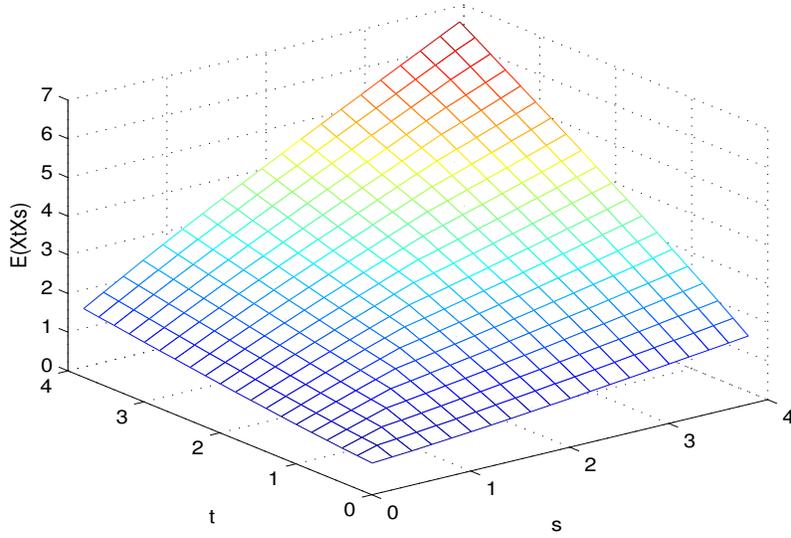


Figure 4: Autocorrelation function of  $X_t$  computed with equation (13)

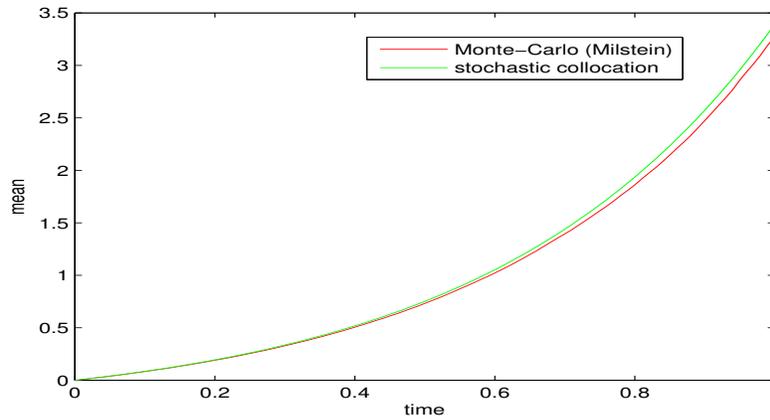


Figure 5: Comparison of the mean of  $X_t$  computed with our stochastic collocation method and with a Milstein scheme

taking  $[0, T] = [0, 8]$  and for three levels of discretization:  $n + 1 = 40$ ;  $n + 1 = 80$  and  $n + 1 = 120$ . The plots of Figures 7 and 8 show the good convergence of the mean and the variance as we increase the degree of Lagrange's polynomials. For this example, we've observed numerically that  $k(t) = 1$  and therefore the

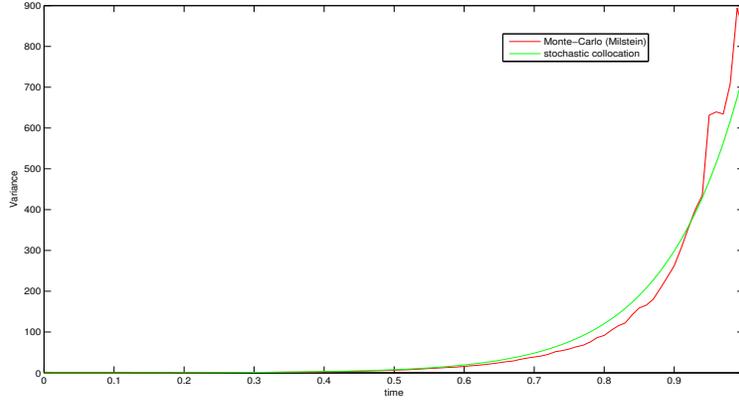


Figure 6: Comparison of the variance of  $X_t$  computed with our stochastic collocation method and with a Milstein scheme

solution is a function of  $G_t = W_t$ . However, this function cannot be analytically computed. Finally, Figure 9 shows the autocorrelation function of  $X_t$  for  $(s, t) \in [0, 1] \times [0, 1]$ . We have limited the interval for  $(s, t)$  to  $[0, 1] \times [0, 1]$  otherwise, the plot would be a flat land with an upsurge for the highest values of  $s$  and  $t$ .

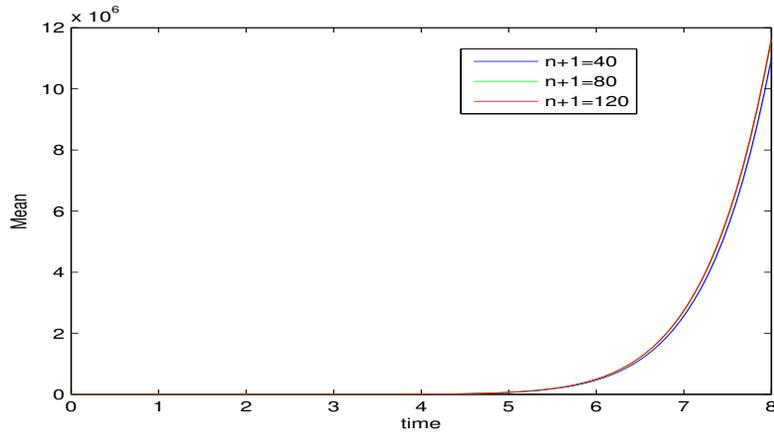


Figure 7: Mean of  $X_t$  for different degrees of Lagrange's polynomials

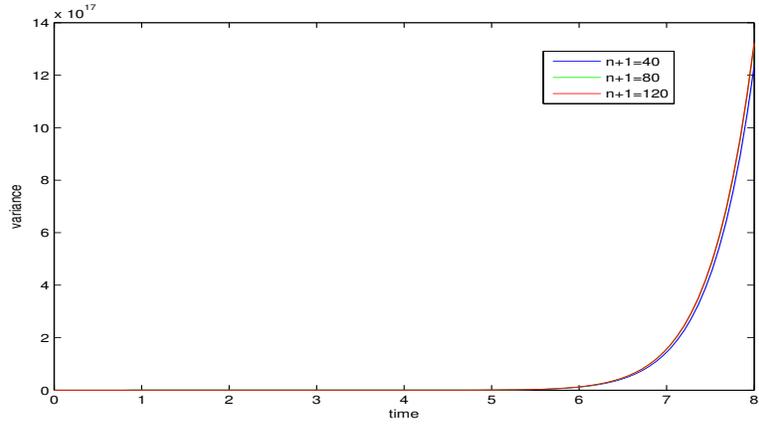


Figure 8: Variance of  $X_t$  for different degrees of Lagrange's polynomials

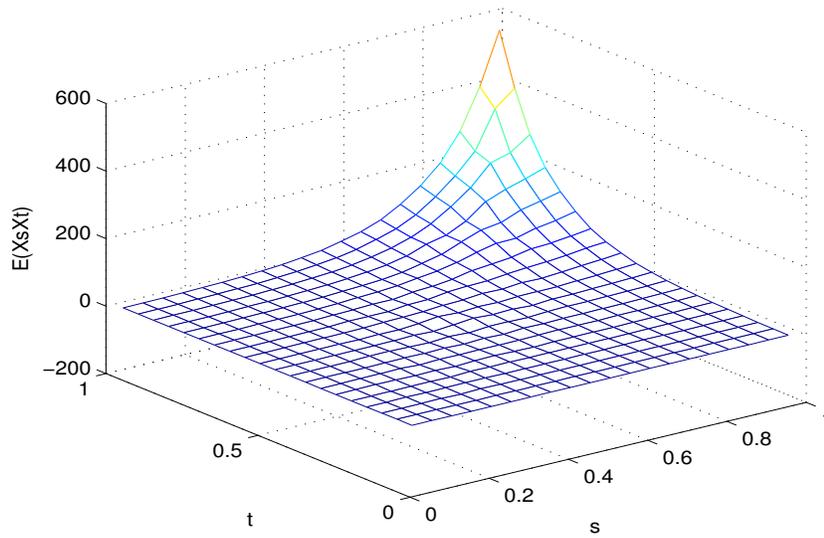


Figure 9: Autocorrelation function of  $X_t$  computed with equation (13)

## 7. Conclusion

A new class of high order numerical method for the solution to SDEs is proposed in this paper. It mainly relies on two standard mathematical and numerical ingredients: polynomial interpolation and cubature rules. However, it

is based on a new paradigm: the white noise is no longer discretized under the form of a truncated Karhunen-Loeve series, as in Polynomial chaos expansions, and the stochastic process  $(X_t)_{t \in [0, T]}$  is no longer expressed as a finite number of independent random variables. Instead,  $(X_t)_{t \in [0, T]}$  (not necessarily Gaussian) is directly sought as a polynomial function of a Gaussian process  $(G_t)_{t \in [0, T]}$  and the coefficients of the expansion are obtained by solving ordinary differential equations (ODE). The macroscopic quantities (mean, variance,...) can be computed directly from the coefficients of the decomposition. Thus, two major drawbacks of competing methods can be avoided: the curse of dimensionality for Polynomial chaos expansions and the resort to Monte-Carlo simulations, with slow convergence, for Euler-like or exact sampling techniques. Assuming that the ODEs can be solved with high degree of accuracy, then the error of our method mainly comes from the interpolation error.

Several examples were studied to illustrate the efficiency and the accuracy of the proposed method. First considering the Ornstein-Uhlenbeck and the geometric Brownian motion SDEs, hands calculations have shown that our method leads to an approximate solution that is the projection of the exact solution onto a Lagrange polynomial basis. Then, more complex SDEs were numerically solved, showing the high rate of convergence and the efficiency of our approach.

Extension of the method to stochastic processes  $(G_t)_{t \in [0, T]}$  that are not necessarily Gaussian would allow to tackle a wider class of SDEs. This is the subject of our ongoing research.

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