



HAL
open science

Wave-structure interaction for long wave models with a freely moving bottom

Krisztian Benyo

► **To cite this version:**

Krisztian Benyo. Wave-structure interaction for long wave models with a freely moving bottom. 2017.
hal-01665775v1

HAL Id: hal-01665775

<https://hal.science/hal-01665775v1>

Preprint submitted on 16 Dec 2017 (v1), last revised 30 Apr 2018 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Wave-structure interaction for long wave models with a freely moving bottom

Krisztian Benyo[†]

Abstract

In this paper we address a particular fluid-solid interaction problem in which the solid object is lying at the bottom of a layer of fluid and moves under the forces created by waves travelling on the surface of this layer. More precisely, we consider the water waves problem in a fluid of fixed depth with a flat bottom topography and with an object lying on the bottom, allowed to move horizontally under the pressure forces created by the waves. After establishing the physical setting of the problem, namely the dynamics of the fluid and the mechanics of the solid motion, as well as analyzing the nature of the coupling inbetween these two, we examine in detail two particular shallow water regimes: the case of the (nonlinear) Saint-Venant system, and the (weakly nonlinear) Boussinesq system. We prove an existence and uniqueness theorem for the coupled system in both cases. Using the particular structure of the coupling terms we are able to go beyond the standard scale for the existence time of solutions to the Boussinesq system with a moving bottom.

1 Introduction

The water waves problem, which consists of describing the motion of waves at the surface of an inviscid, incompressible, and irrotational fluid of constant density under the action of gravity, has attracted a lot of attention in the last decades. The local well-posedness theory is now well-understood following the works of Wu [Wu97, Wu99] establishing the relevance of the Taylor sign condition. In the case of finite depth, which is of interest here, we refer for instance to [Lan05, Igu09, ABZ14]; the case where the bottom is also allowed to depend on time has also been treated in [ABZ11, Igu11, Mel15]. In this paper, we are interested in a particular configuration where the bottom depends on time, but instead of being in forced motion as in the above references, it evolves under the action of the hydrodynamic forces created by the surface waves. Finding its evolution is therefore a free boundary problem, which is coupled to the standard water waves problem, itself being a free boundary problem. The mathematical theory for such a configuration has not been considered yet; we refer however to [Lan17] for a related problem where the moving object is floating instead of lying on the bottom as in the present paper.

Here, our goal is not to address the local well-posedness theory for this double free boundary problem, but to give some qualitative insight on its behavior by deriving and analysing simpler asymptotic models. We shall focus on a regime which is particularly interesting for applications, namely, the shallow water regime, where the typical horizontal scale of the flow is much larger than the depth at rest. For a fixed bottom, several models arise in this setting such as the Korteweg–de Vries (KdV) equation (justified in [Cra85, KN86, SW00]), the nonlinear shallow water equations (justified in [Ovs74, KN86, ASL08a, Igu09]), the Boussinesq systems (justified in [Cra85, KN86, BCS04, BCL05]) – see also [Per67, PR83, GKSW95, Cha07, CLS12] for particular focus on topography effects – the Green-Naghdi equations [Li06, ASL08b, HI15], etc. We refer to [Lan13] for more exhaustive references.

[†]Institut de Mathématiques de Bordeaux, Université de Bordeaux, France. (krisztian.benyo@math.u-bordeaux.fr)

The author was partially supported by the Conseil Régional Nouvelle Aquitaine, the ANR-13-BS01-0003-01 DYFICOLTI and the Fondation Del Duca.

For a bottom with prescribed motion in general, the problem has already been considered, local well-posedness results ([ABZ11]) and long time existence results ([Mel15]) have been proven recently. Numerical experiments and tentatives to adapt existing and known shallow water models for a moving bottom regime have been present for a while in literature, however lacking rigorous justifications. After observing successively generated solitary waves due to a disturbance in the bottom topography advancing at critical speed ([Wu87]) they formally derived a set of generalized channel type Boussinesq systems ([TW92]), their work was extended later on in a formal study on more general long wave regimes ([Che03]). Tsunami research has also proved to be a main motivating factor with the consideration of water waves type problems with a moving bottom (see for example [Mit09] for an extensive numerical study). The mathematical justification of these models as approximations of the full problem was carried out not too long ago ([Igu11] for Saint-Venant type systems, or [HI15] for the precise Green–Naghdi system).

Here, we present a new class of problems where the bottom is still moving, but its movement is not prescribed, instead it is generated by the wave motion. A good approach to this is to place a freely moving object on the bottom of the fluid domain. The main physical motivation of this study lies in the recent development of submerged wave energy converters (submerged pressure differential devices, [AELS14] and references therein) and oscillating wave surge converters (WaveRollers and Submerged plate devices, [GIL⁺14]), as well as reef-evolution and submarine landslide modelling problems. Bibliography in the more theoretical approach is rather lacking, existing studies are heavily oriented to physical experiments ([ACDNn17] to investigate a submerged spring-block system and its numerical simulation through an adapted level set method, for further details, see for example [CM06]), as well as numerical applications ([DNZ15] for instance).

The structure of the article is as follows. This first introductory section starts with the presentation of the basic free surface fluid dynamics system and possible reformulations in the water waves setting. This is done first in the case of a fixed bottom, then in the more general case when the bottom has a prescribed motion. After that, the equations governing the motion of an object lying on the bottom are established, they derive from Newton’s equation and take into account the hydrodynamic force exerted by the fluid and a dynamic friction force. Then we introduce the characteristic scales of the variables of the system in order to derive the nondimensionalised equivalents of the different equations and formulae, preparing for the study of the asymptotic models.

In Section 2, we detail the first order asymptotic regime with respect to the shallowness parameter μ ; the resulting approximation is the well-known (nonlinear) Saint-Venant equations, in the presence of a solid moving on the bottom of the fluid domain. A key step is to derive an asymptotic approximation of the hydrodynamic force exerted on the solid. Then we establish a local in time well-posedness result for the coupled system.

In the third section, we elaborate our study on a second order asymptotic regime with respect to the shallowness parameter μ . This study concerns the so called long wave regime where the size of the waves and of the solid are assumed to be small compared to the mean fluid height. The resulting approximation is the so called (weakly nonlinear) Boussinesq system. A local in time well-posedness is shown for this coupled system as well. The standard existence time for a Boussinesq system with a moving bottom is $\mathcal{O}(1)$ with respect to the nonlinearity parameter ε , due to the presence of a source term involving time derivatives of the topography, which can potentially become large (as remarked in [Mel15]). By a precise analysis of the wave-structure coupling we are able to extend the existence time to the $\mathcal{O}\left(\frac{1}{\sqrt{\varepsilon}}\right)$ time scale. This time scale is therefore intermediate between the aforementioned $\mathcal{O}(1)$ scale, and the $\mathcal{O}\left(\frac{1}{\varepsilon}\right)$ scale that can be achieved for fixed bottoms ([ASL08a, Bur16]).

1.1 The case of a fixed bottom

As a basis for our model and our computations, we shall consider a fluid moving under the influence of gravity. The fluid domain Ω_t (depending on the time t) is delimited from below by a fixed bottom and from above by a free surface. In our case the fluid is homogeneous with a constant density ρ , moreover it is non-viscous, incompressible, and irrotational.

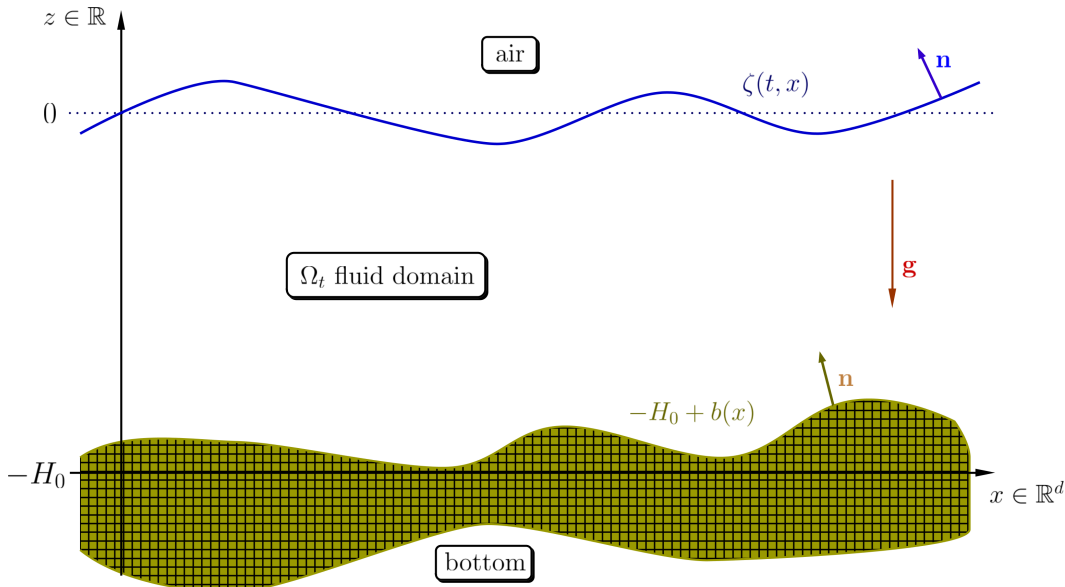


Figure 1: The water waves setting for a fixed bottom

To clarify the upcoming notations, the spatial coordinates take the form $(x, z) \in \mathbb{R}^d \times \mathbb{R}$ with x denoting the horizontal component and z the vertical one. Regarding differential operators, x or z as a subscript refers to the operator with respect to that particular variable, the absence of subscript for a space dependent operator means that it is to be taken for the whole space $(x, z) \in \mathbb{R}^{d+1}$. In general we shall work in arbitrary horizontal dimension $d \in \mathbb{N}^+$, even though the physically relevant cases are $d = 1$ and 2 only.

In what follows, we shall denote by $\zeta(t, x)$ the free surface elevation function and $b(x)$ describes the bottom topography variation at a base depth of H_0 . We remark that by supposing that the bottom is flat, we may infer that $b(x) = 0$. With this notation at our disposal we may establish that the fluid domain is

$$\Omega_t = \left\{ (x, z) \in \mathbb{R}^d \times \mathbb{R} : -H_0 + b(x) < z < \zeta(t, x) \right\}.$$

Let us also introduce the height function $h(t, x) = H_0 + \zeta(t, x) - b(x)$ that describes the total depth of the fluid at a given horizontal coordinate x and at a given time t .

In order to avoid special physical cases arising from the fluid domain Ω_t (such as islands or beaches), throughout our analysis we will often make use of the following (or similar) minimal water height condition

$$\exists h_{min} > 0, \forall (t, x) \in [0, T) \times \mathbb{R}^d, h(t, x) \geq h_{min}, \quad (1.1)$$

we refer to [dP16] for an analysis of the water waves equation allowing vanishing depth, and to [LM17] where the evolution of the shoreline is considered for the one dimensional nonlinear Saint-Venant and Serre–Green–Naghdi equations.

1.1.1 The free surface Euler equations

To describe the fluid motion under these assumptions, the free surface Euler equations are to be considered. Denoting by \mathbf{U} the velocity field and by P the fluid pressure, the homogeneous, incompressible, irrotational Euler equations take the following form:

$$\begin{cases} \partial_t \mathbf{U} + \mathbf{U} \cdot \nabla \mathbf{U} = -\frac{\nabla P}{\varrho} + \mathbf{g}, \\ \nabla \cdot \mathbf{U} = 0, \\ \nabla \times \mathbf{U} = 0, \end{cases} \quad (1.2)$$

valid in the entire fluid domain Ω_t .

Here, $\mathbf{g} = (0, -g)^\top$ in the equations denotes the gravitational acceleration as a downwards pointing vector. Moreover ϱ denotes the density of the fluid (constant due to the homogeneity assumption). It is useful to introduce the horizontal and vertical components of the velocity field, that is $\mathbf{U} = (V, w)^\top$.

The boundary conditions can be resumed as follows:

- the kinematic (or no-penetration) boundary conditions (that is, the fluid particles do not cross neither the bottom nor the free surface);
- there is no surface tension along the free surface, so the pressure at the surface is given by the atmospheric pressure, and assumed to be constant.

A mathematical restatement of the aforementioned conditions is the following:

- denoting by \mathbf{n} the unit normal vector of the fluid domain pointing upward, we have the following reformulation for the no-penetration condition for the bottom

$$\mathbf{U} \cdot \mathbf{n} = 0 \quad \text{on } \{z = -H_0 + b(x)\}, \quad (1.2)'$$

and for the free surface

$$\partial_t \zeta - \sqrt{1 + |\nabla_x \zeta|^2} \mathbf{U} \cdot \mathbf{n} = 0 \quad \text{on } \{z = \zeta(t, x)\}; \quad (1.2)''$$

- denoting by P_{atm} the atmospheric pressure, we have that

$$P = P_{atm} \quad \text{on } \{z = \zeta(t, x)\}. \quad (1.3)$$

The system of equations (1.2)-(1.2)'' and (1.3) together form the free surface Euler equations for the fluid domain Ω_t .

1.1.2 The free surface Bernoulli equations

Due to the incompressibility and irrotationality conditions (second and third equations in (1.2)), one may propose an alternative form of the equations by utilising the velocity potential Φ , since with the knowledge of this potential one may recover the velocity field as the gradient, that is

$$\mathbf{U} = \nabla \Phi, \quad \text{with } \Delta \Phi = 0 \quad \text{on } \Omega_t. \quad (1.4)$$

With this velocity potential, denoting by $\partial_{\mathbf{n}}$ the upwards normal derivative, the boundary conditions take the form

$$\partial_{\mathbf{n}}\Phi = 0 \quad \text{on } \{z = -H_0 + b(x)\}; \quad (1.4)'$$

$$\partial_t\zeta - \sqrt{1 + |\nabla_x\zeta|^2}\partial_{\mathbf{n}}\Phi = 0 \quad \text{on } \{z = \zeta(t, x)\}. \quad (1.4)''$$

So from the Euler equations (1.2), with the aforementioned boundary conditions, the velocity potential is recovered as a solution of the following Laplace equation

$$\begin{cases} \Delta\Phi = 0 & \text{in } \Omega_t, \\ \Phi|_{z=\zeta} = \psi, \quad \partial_{\mathbf{n}}\Phi|_{z=-H_0+b(x)} = 0. \end{cases} \quad (1.5)$$

Finally, by the momentum conservation part of the Euler system (first equation of (1.2)), we get that

$$\partial_t\Phi + \frac{1}{2}|\nabla\Phi|^2 + gz = -\frac{1}{\rho}(P - P_{atm}) \quad (1.6)$$

in the domain Ω_t . So the free surface Bernoulli equations are the system of equations (1.4)-(1.4)'' and (1.6).

Based on equation (1.6), we can recover the pressure in terms of the velocity potential:

$$P = -\rho\left(\partial_t\Phi + \frac{1}{2}|\nabla\Phi|^2 + gz\right) + P_{atm}. \quad (1.7)$$

This relation allows to compute the hydrodynamical force exerted on the solid by the fluid (derived from Newton's second law in Section 1.3).

1.1.3 The Zakharov / Craig–Sulem formulation

Following a somewhat historical introduction, we present another formulation of the equations (also referred to as the water waves problem). This formulation is attributed to Zakharov in his studies regarding gravitational waves [Zak68] and is based on the fact that the variables ζ and $\psi = \Phi|_{z=\zeta}$ fully determine the flow. More precisely, the water waves problem reduces to a set of two evolution equations in ζ and ψ ,

$$\begin{cases} \partial_t\zeta - \sqrt{1 + |\nabla_x\zeta|^2}\partial_{\mathbf{n}}\Phi|_{z=\zeta} = 0, \\ \partial_t\psi + g\zeta + \frac{1}{2}|\nabla_x\psi|^2 - \frac{(\sqrt{1 + |\nabla_x\zeta|^2}\partial_{\mathbf{n}}\Phi|_{z=\zeta} + \nabla_x\zeta \cdot \nabla_x\psi)^2}{2(1 + |\nabla_x\zeta|^2)} = 0, \end{cases} \quad (1.8)$$

where Φ solves the boundary value problem (1.5).

In more general terms, one can introduce the so-called Dirichlet-Neumann operator associated to the Laplace problem (1.5), but we will not pursue further this path, for more details we refer to the works of Craig and Sulem [CS93, CSS92]. For a comprehensive and detailed analysis as well as the well-posedness of the water waves problem under this formulation, we refer to [Lan13] and references therein.

1.2 The case of a moving bottom with prescribed motion

Now we consider the more general case where $b(t, x)$ is time dependent, although a given function of t (and of course x), that is the bottom is changing in a given fashion. This indicates the evolution of both the upper and lower boundaries of the fluid domain Ω_t , although in a different way, since as opposed to

the free surface, the evolution of the bottom is given. For the sake of clarity we shall precise that the fluid domain now is

$$\Omega_t = \left\{ (x, z) \in \mathbb{R}^d \times \mathbb{R} : -H_0 + b(t, x) < z < \zeta(t, x) \right\},$$

which also indicates that the height function has the form $h(t, x) = H_0 + \zeta(t, x) - b(t, x)$.

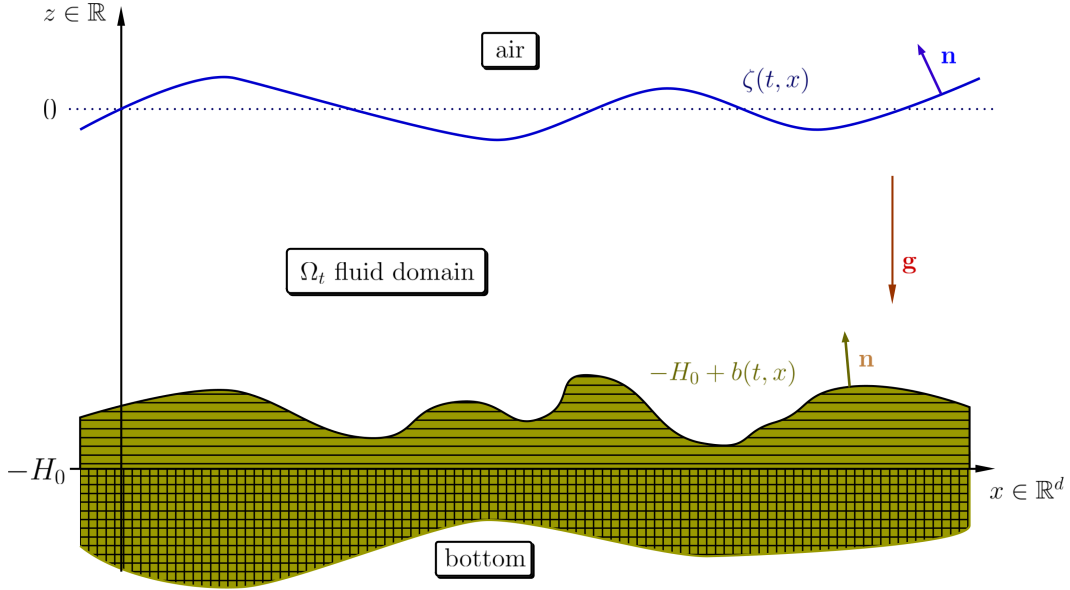


Figure 2: The water waves setting for a moving bottom

1.2.1 The Euler / Bernoulli frameworks

One may notice that the fact that b now depends on the time doesn't change anything for the equations themselves in the Euler framework, thus the free surface Euler equations (1.2) and (1.3) hold just like before. However some adjustments have to be made for the boundary conditions, namely the no-penetration condition for the bottom becomes a kinematic condition similar to the one imposed on the free surface, so we replace (1.2)' by

$$\partial_t b - \sqrt{1 + |\nabla_x b|^2} \mathbf{U} \cdot \mathbf{n} = 0 \quad \text{on } \{z = -H_0 + b(t, x)\}. \quad (1.2)^*$$

For the free surface Bernoulli equations, (1.4) and (1.6) are still valid. On the other hand the Neumann boundary condition for the bottom becomes nonhomogeneous in the Laplace equation used to recover the velocity potential. That is, we are left with the following Laplace equation instead of (1.5):

$$\begin{cases} \Delta \Phi = 0 & \text{in } \Omega_t, \\ \Phi|_{z=\zeta} = \psi, \quad \sqrt{1 + |\nabla_x b|^2} \partial_{\mathbf{n}} \Phi|_{z=-H_0+b} = \partial_t b. \end{cases} \quad (1.9)$$

Similarly to the free surface Euler equations, the boundary condition on the bottom (1.4)' is replaced by

$$\partial_t b - \sqrt{1 + |\nabla_x b|^2} \partial_{\mathbf{n}} \Phi = 0 \quad \text{on } \{z = -H_0 + b(t, x)\}. \quad (1.4)^*$$

1.2.2 The Zakharov / Craig–Sulem framework

As for the reformulation due to Zakharov, Craig, and Sulem, the recovery of the potential is based on the Laplace problem (1.9) now with non-homogeneous Neumann condition at the bottom. This can give

rise to a natural decomposition of Φ into a "fixed bottom" and a "moving bottom" component which could be used to define the Dirichlet-Neumann and Neumann-Neumann operators for this problem assuming sufficient regularity for the limiting functions. Thus one can formulate the water waves problem for moving bottom, which has already been studied; for details we refer to the article of Alazard, Burq, and Zuily [ABZ11] for the local well-posedness theory or to [Igu11] for specific studies motivated by earthquake generated tsunami research.

We remark that the formulation (1.8) of the water waves problem still holds, only now we have to use the velocity potential obtained from (1.9) instead of (1.5). Since our study focuses on shallow water regimes, it is convenient to bypass the aforementioned technicalities by introducing the following variable:

Definition 1.1. *The vertically averaged horizontal component of the velocity is given by*

$$\bar{V} = \frac{1}{h} \int_{-H_0+b}^{\zeta} \nabla_x \Phi(\cdot, z) dz, \quad (1.10)$$

where Φ solves (1.9).

The interest of this new variable \bar{V} is that a closed formulation of the water waves problem in terms of ζ and \bar{V} (instead of ζ and ψ) can be obtained, see for example [Lan17]. For our case, it is sufficient to observe that

Proposition 1.1. *If Φ solves (1.9) and \bar{V} is defined as in (1.10), then*

$$\sqrt{1 + |\nabla_x \zeta|^2} \partial_{\mathbf{n}} \Phi|_{z=\zeta} = \partial_t b - \nabla \cdot (h\bar{V}), \quad (1.11)$$

assuming sufficient regularity on the data concerning ζ , ψ , and b as well as the minimal water depth condition (1.1).

Remark 1.1. *Let $\zeta, b \in W^{1,\infty}(\mathbb{R}^d)$ such that they satisfy the minimal water depth condition (1.1). Moreover, let $\psi \in \dot{H}^{3/2}(\mathbb{R}^d) = \{f \in L^2_{loc} : \nabla_x f \in H^{1/2}(\mathbb{R}^d)\}$. Then the Laplace equation (1.9) can be solved with $\Phi \in \dot{H}^2(\Omega) = \{f \in L^2_{loc} : \nabla_x f \in H^1(\Omega)\}$ and relation (1.11) holds true, where*

$$\Omega = \{(X, z) \in \mathbb{R}^d \times \mathbb{R}, -1 + b(X) < z < \zeta(X)\} \quad (1.12)$$

is a known fluid domain. For more details, we refer to Chapter 2 of [Lan13].

With this, the water waves problem with a moving bottom takes the following form

$$\begin{cases} \partial_t \zeta + \nabla \cdot (h\bar{V}) = \partial_t b, \\ \partial_t \psi + g\zeta + \frac{1}{2} |\nabla_x \psi|^2 - \frac{(-\nabla \cdot (h\bar{V}) + \partial_t b + \nabla_x \zeta \cdot \nabla_x \psi)^2}{2(1 + |\nabla_x \zeta|^2)} = 0. \end{cases} \quad (1.13)$$

This system seemingly depends on three variables, namely ζ , \bar{V} and ψ , but in fact the Laplace-equation provides a connection between the latter two. Exploiting this connection to express (asymptotically) one variable with the other gives rise to various well-known asymptotic equations under the shallow water assumption. In Section 1.4 detailing the nondimensionalisation of the system we shall provide the necessary tools as well as some references concerning this asymptotic expansion.

1.3 The coupled problem with a freely moving object

The aim of this paper is to understand a particular case in which the bottom of the domain contains a freely moving object, the movement of which is determined by the gravity driven fluid motion. We will

work with a flat bottom with the presence of a freely moving solid object on it (see Figure 3).

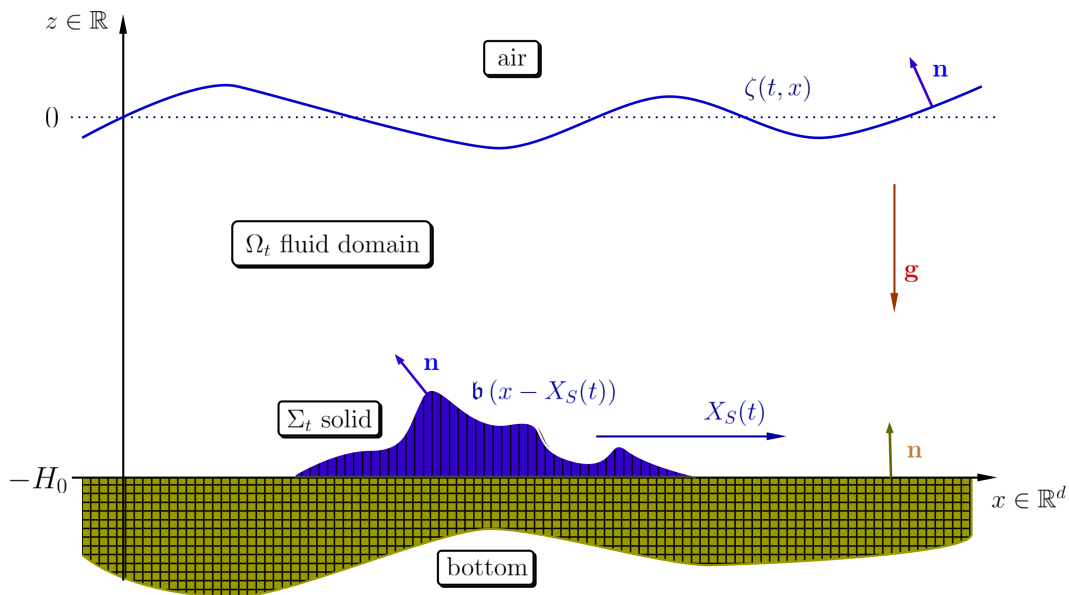


Figure 3: The coupled water waves setting in the presence of a solid

For the solid we suppose it to be rigid and homogeneous with a given mass M . The surface of the object can be characterised by two components: the part of the surface in direct contact with the fluid, denoted by Σ_t and the rest, that is the part in direct contact with the flat bottom, denoted by $I(t)$. For convenience reasons we shall suppose that Σ_t is a graph of a \mathcal{C}^∞ function with compact support $I(t)$ for any instance of t .

The solid moves horizontally in its entirety, we denote by $X_S(t)$ the displacement vector, and $v_S(t)$ the velocity (with $\dot{X}_S = v_S$). We make the additional hypothesis that the object is neither overturning, nor rotating so its movement is completely described by its displacement vector, which will be restrained to horizontal movement only. In particular, this means that the object is not allowed to start floating, the domain $I(t)$ has a constant (nonzero) area.

Under these assumptions a simplified characterisation of the function describing the bottom variation is possible:

$$b(t, x) = \mathbf{b}(x - X_S(t)), \quad (1.14)$$

where \mathbf{b} corresponds to the initial state of the solid at $t = 0$ (so that we have $X_S(0) = 0$).

Taking into account all the external forces acting on the object, Newton's second law provides us with the correct equation for the movement of the solid. The total force acting on the solid is

$$\begin{aligned} \mathbf{F}_{total} &= \mathbf{F}_{gravity} + \mathbf{F}_{solid-bottom\ interaction} + \mathbf{F}_{solid-fluid\ interaction} \\ &= M\mathbf{g} + [\mathbf{F}_{normal} + \mathbf{F}_{friction}] + \mathbf{F}_{pressure}. \end{aligned}$$

Here we made use of the fact that the force emerging from the contact of the solid with the bottom may be decomposed in two components: the normal force, perpendicular to the surface of the bottom, expressing the fact that the bottom is supporting the solid, and the (kinetic or dynamic) friction force, the tangential component, hindering the sliding of the solid. By making use of the three empirical laws of friction [Ber06], most notably the third law often attributed to Coulomb regarding the existence of a coefficient $c_{fric} > 0$ of kinetic friction (describing the material properties of the contact materia), we

may reformulate the tangential contact force as follows

$$\mathbf{F}_{friction} = \mathbf{F}_{sliding\ friction} = -c_{fric}|F_{normal}|\frac{v_S(t)}{|v_S(t)| + \delta}, \quad (1.15)$$

where $\delta \ll 1$ is a purely mathematical, small constant serving as a regularising term in order to avoid a singularity in the equation when the solid stops, that is when $v_S(t)$ is equal to 0. Normally, when the solid stops, the kinetic friction detailed just before turns into static friction, a tangential force component preventing the solid from restarting its movement. The static friction has its own coefficient, which is usually greater than c_{fric} , and its direction is determined by the horizontal force components rather than the velocity.

In order to prevent the numerous complications that would arise by implementing the physically more relevant threshold for $v_S(t) = 0$ and the associated jump in friction force, we simplify the system by regularizing the friction force, thus neglecting static effects. A more specific modelling and analysis of the transition between static and dynamic friction will be addressed in future works.

Treating the horizontal and vertical component of $\mathbf{F}_{total} = (\mathbf{F}_{total}^h, \mathbf{F}_{total}^v)^\top$ separately and using the fact that the solid is constrained to horizontal motion, we have that the vertical components are in equilibrium, thus

$$0 = -Mg + F_{normal} + \mathbf{F}_{pressure}^v, \quad (1.16)$$

and we obtain that the horizontal movement of the solid is given by

$$M\ddot{X}_S(t) = \mathbf{F}_{sliding\ friction} + \mathbf{F}_{pressure}^h. \quad (1.17)$$

Finally, by making use of the fact that

$$\mathbf{F}_{pressure} = \int_{\Sigma_t} P \mathbf{n}_{solid} d\Sigma = \int_{I(t)} P|_{z=-H_0+b(t,x)} \begin{pmatrix} \nabla_x b \\ -1 \end{pmatrix} dx,$$

due to the fact that the outwards normal vector for the surface of the solid $\mathbf{n}_{solid} = \mathbf{n}$ can be easily expressed by the bottom variation $b(t, x)$, since

$$\mathbf{n}_{solid} = \frac{1}{\sqrt{1 + |\nabla_x b|^2}} \begin{pmatrix} \nabla_x b \\ -1 \end{pmatrix}.$$

Therefore we obtain from (1.16) that

$$F_{normal} = Mg + \int_{I(t)} P|_{z=-H_0+b(t,x)} dx, \quad (1.18)$$

thus, by (1.15), (1.17) writes as

$$M\ddot{X}_S(t) = -c_{fric}|F_{normal}|\frac{\dot{X}_S(t)}{|\dot{X}_S(t)| + \delta} + \int_{I(t)} P|_{z=-H_0+b(t,x)} \nabla_x b dx. \quad (1.19)$$

So we have that Newton's equation characterising the motion of the solid takes the following form

$$M\ddot{X}_S(t) = -c_{fric} \left| Mg + \int_{I(t)} P|_{z=-H_0+b(t,x)} dx \right| \frac{\dot{X}_S(t)}{|\dot{X}_S(t)| + \delta} + \int_{I(t)} P|_{z=-H_0+b(t,x)} \nabla_x b dx. \quad (1.20)$$

A key step in our study is to handle the force term exerted by the fluid, which requires the computation of the integral of the pressure on the bottom over the solid domain. For this we will establish an appropriate formula for the pressure to be used in the integral.

In both the case of the freely moving bottom (due to the moving object) and the free surface, the kinematic no-penetration condition still applies, most notably we still have that

$$\partial_t b - \sqrt{1 + |\nabla_x b|^2} \mathbf{U} \cdot \mathbf{n} = 0 \quad \text{for } \{z = -H_0 + b(t, x)\},$$

or equivalently, on the part of the surface of the solid in contact with the fluid (Σ_t), the normal component of the fluid velocity field coincides with the normal component of the velocity of the solid, that is

$$\mathbf{U} \cdot \mathbf{n}_{solid} = v_S \cdot \mathbf{n}_{solid}^h \quad \text{for } \{z = -H_0 + b(t, x)\}. \quad (1.21)$$

To sum up, the water waves problem in the presence of a solid on the bottom is given by equations (1.13) and (1.9), where in the Neumann boundary condition, the bottom function b and its time derivative are given by (1.14), with X_S arising from (1.20) and the pressure P derived from (1.7).

1.4 Dimensionless form of the equations

The main part of the analysis consists of establishing and analysing the wave-structure interaction system for shallow water regimes, and for that we need first of all the correct parameters involving the characteristic orders of magnitude of our variables as well as the dimensionless equations obtained with the help of these quantities.

1.4.1 The different scales of the problem

First of all we present the proper dimensionless parameters relevant to the system, bearing in mind that our aim is to derive simpler asymptotic models. For that we need to introduce the various characteristic scales of the problem:

- H_0 , the base water depth,
- L , the characteristic horizontal scale of the wave motion (both for longitudinal and transversal directions),
- a_{surf} , the order of the free surface amplitude,
- a_{bott} , the characteristic height of the solid (order of the bottom topography variation in general).

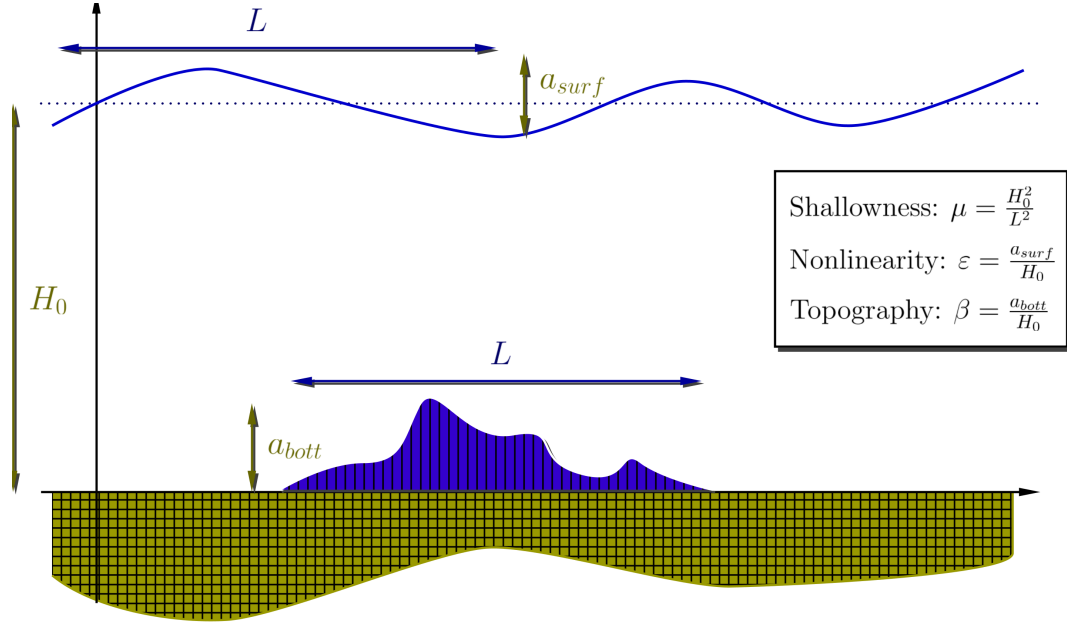


Figure 4: The characteristic scales of the coupled water waves problem

Using these quantities, we can introduce several dimensionless numbers:

- shallowness parameter $\mu = \frac{H_0^2}{L^2}$,
- nonlinearity (or amplitude) parameter $\varepsilon = \frac{a_{surf}}{H_0}$,
- bottom topography parameter $\beta = \frac{a_{bott}}{H_0}$.

Our goal in this paper is to examine asymptotic models when μ is small (shallow water regime), and under various assumptions on the characteristic size of ε and β .

With these parameters in our hand, we may remark that the natural scaling for the horizontal space variable x is L , and for its vertical counterpart z it is H_0 . Moreover the natural order of magnitude for the function characterising the free surface ζ is a_{surf} , equivalently, for the bottom b it is a_{bott} . Thus the nondimensionalised form for the water depth is

$$h = 1 + \varepsilon\zeta - \beta b.$$

Furthermore, one can establish the correct scale of the velocity potential/field through linear wave analysis, which gives rise to

$$\Phi_0 = \frac{a_{surf}}{H_0} L \sqrt{gH_0}.$$

As for the pressure, we choose the typical order of the hydrostatic pressure, that is $P_0 = \rho g H_0$. For the time parameter, from linear wave theory one can deduce the scaling as

$$t_0 = \frac{L}{\sqrt{gH_0}}.$$

Finally, for the parameters concerning the solid, we impose that the characteristic horizontal dimension of the solid is comparable to L . Following this, by taking into account the volume integral of the

density, the natural scaling for the mass of the solid is given by

$$M = M_0 \tilde{M} = \rho L^d a_{\text{bott}} \tilde{M}.$$

Our main interest will be to express the equations principally with the different orders of magnitude of μ (the shallowness parameter) to pass on to the different asymptotic regimes.

Thus the proper nondimensionalised parameters are obtained by

$$x' = \frac{x}{L}, \quad z' = \frac{z}{H_0}, \quad \zeta' = \frac{\zeta}{a_{\text{surf}}}, \quad \Phi' = \frac{\Phi}{\Phi_0}, \quad t' = \frac{t}{t_0}, \quad \text{etc.}$$

For the sake of clarity we shall omit the primes on the variables from here on.

Given the particular structure of the asymptotic regimes we are going to examine we shall make an a priori hypothesis concerning certain parameters.

Remark 1.2. *Since all the regimes handled in this article involve the hypothesis that ε and β are of the same order of magnitude we shall, for the sake of simplicity, assume that $\beta = \varepsilon$.*

An additional precision shall be made concerning the quantities involving the bottom. The explicit form of the nondimensionalised form for the water depth is

$$h(t, x) = 1 + \varepsilon(\zeta(t, x) - b(t, x)) \quad (1.22)$$

with

$$b(t, x) = \mathbf{b}(x - X_S(t)). \quad (1.23)$$

1.4.2 Nondimensionalised equations

Using the previous section and in particular taking $\varepsilon = \beta$ as in Remark 1.2, one easily derives the dimensionless version of (1.13), namely

$$\begin{cases} \partial_t \zeta + \nabla \cdot (h \bar{V}) = \partial_t b, \\ \partial_t \psi + \zeta + \frac{\varepsilon}{2} |\nabla_x \psi|^2 - \varepsilon \mu \frac{(-\nabla \cdot (h \bar{V}) + \partial_t b + \nabla_x(\varepsilon \zeta) \cdot \nabla_x \psi)^2}{2(1 + \varepsilon^2 \mu |\nabla_x \zeta|^2)} = 0, \end{cases} \quad (1.24)$$

where \bar{V} is now defined as

$$\bar{V} = \frac{1}{h} \int_{-1+\varepsilon b}^{\varepsilon \zeta} \nabla_x \Phi(\cdot, z) dz, \quad (1.25)$$

with $h = 1 + \varepsilon \zeta - \varepsilon b$, furthermore Φ solves

$$\begin{cases} \Delta^\mu \Phi = \mu \Delta_x \Phi + \partial_z^2 \Phi = 0, & \text{on } -1 + \varepsilon b \leq z \leq \varepsilon \zeta, \\ \Phi|_{z=\varepsilon \zeta} = \psi, & (\partial_z \Phi - \mu \nabla_x(\varepsilon b) \cdot \nabla_x \Phi)|_{z=-1+\varepsilon b} = \mu \partial_t b, \end{cases} \quad (1.26)$$

the nondimensionalised equivalent of the Laplace problem (1.9).

It is also necessary to nondimensionalize the formula describing the pressure (1.7), thus

$$P = \frac{P_{\text{atm}}}{\rho g H_0} - z - \varepsilon \partial_t \Phi - \frac{\varepsilon^2}{2} |\nabla_x \Phi|^2 - \frac{\varepsilon^2}{2\mu} |\partial_z \Phi|^2. \quad (1.27)$$

Here we had to separate the horizontal and the vertical part of the gradient due to the different scaling

parameters for the different directions.

We remark that the outwards normal derivative is given by

$$\mathbf{n}_{solid} = \frac{1}{\sqrt{1 + \varepsilon^2 \mu |\nabla_x b|^2}} \begin{pmatrix} \sqrt{\mu} \varepsilon \nabla_x b \\ -1 \end{pmatrix}.$$

Thus we may reformulate Newton's equation (1.20) in the following way

$$\ddot{X}_S(t) = -\frac{c_{fric}}{\sqrt{\mu}} \left| 1 + \frac{1}{\varepsilon \tilde{M}} \int_{I(t)} P|_{z=-1+\varepsilon b} dx \right| \frac{\dot{X}_S(t)}{|\dot{X}_S(t)| + \tilde{\delta}} + \frac{1}{\tilde{M}} \int_{\mathbb{R}^d} P_{z=-1+\varepsilon b} \nabla_x b dx, \quad (1.28)$$

taking into consideration the characteristic scales of the variables.

2 The $\mathcal{O}(\mu)$ asymptotic regime: The nonlinear Saint-Venant equations

We shall now start our analysis for shallow water regimes, that is an asymptotic analysis with respect to the shallowness parameter μ for the nondimensionalised water waves problem (1.24) coupled with Newton's equation (1.28) for the solid. With our notations, this means that we would like to consider systems that are valid for $\mu \ll 1$.

In this section we treat the general first order approximate system, more specifically a model with $\mathcal{O}(\mu)$ approximation that allows large wave amplitudes and large bottom variations ($\varepsilon = \mathcal{O}(1)$). So, the asymptotic regime writes as follows

$$0 \leq \mu \leq \mu_{max} \ll 1, \quad \varepsilon = 1. \quad (SV)$$

2.1 Asymptotics for the fluid model

As mentioned before, the important step in deducing asymptotic models relies on how we establish the connection between the variables \bar{V} and ψ . More precisely, it is possible to construct an asymptotic expansion of \bar{V} (depending on ζ , b and ψ). For details, we refer to chapter 3 of [Lan13]. One can equally obtain an asymptotic expansion of Φ with respect to μ , depending on the aforementioned variables. Quite obviously, the equation $\Delta^\mu \Phi = 0$ in (1.26) reduces to $\partial_z^2 \Phi = 0$ at leading order in z ; since the Neumann boundary condition in (1.26) is $\mathcal{O}(\mu)$, it follows that Φ does not depend on z at leading order, and therefore

$$\bar{V} = \nabla_x \psi + \mathcal{O}(\mu),$$

see Proposition 3.37. in [Lan13] for a rigorous proof.

So the system (1.24) for the (ζ, \bar{V}) variables simplifies as follows

$$\begin{cases} \partial_t \zeta + \nabla_x \cdot (h \bar{V}) = \partial_t b, \\ \partial_t \bar{V} + \nabla_x \zeta + (\bar{V} \cdot \nabla_x) \bar{V} = 0, \end{cases} \quad (2.1)$$

where we considered the gradient of the second equation in (1.24), and then neglected terms of order $\mathcal{O}(\mu)$. This system is known as the (nonlinear) Saint-Venant or nonlinear shallow water system.

2.2 Formal derivation of a first order asymptotic equation for the solid motion

Our strategy is as follows: we establish an asymptotic formula of order 1 for the pressure P based on (1.27). With this at our disposal, we shall establish Newton's equation (1.28) at order μ describing the displacement of the solid.

For an $\mathcal{O}(\mu)$ approximation, we shall start with the corresponding development for the velocity potential, that is

$$\Phi = \psi + \mathcal{O}(\mu), \quad (2.2)$$

where $\psi = \Phi|_{z=\varepsilon\zeta}$ as before, the restriction of the velocity potential on the free surface. Knowing this we recover the following for the time derivative of ψ (based on the second equation of the water waves problem (1.24))

$$\partial_t \psi = -\zeta - \frac{1}{2} |\nabla_x \psi|^2 + \mathcal{O}(\mu).$$

So by substituting the first order asymptotic expansion of the velocity potential described in (2.2) into the general nondimensionalised formula of the pressure (1.27) the corresponding $\mathcal{O}(\mu)$ approximation for the pressure takes the form

$$P = \frac{P_{atm}}{\rho g H_0} + (\zeta - z) + \mathcal{O}(\mu),$$

using the fact that ψ does not depend on the variable z .

So in particular, at the bottom, we find that the pressure is given by the hydrostatic formula

$$P|_{z=-1+b} = \frac{P_{atm}}{\rho g H_0} + h + \mathcal{O}(\mu). \quad (2.3)$$

Thus for Newton's equation (1.28),

$$\begin{aligned} \ddot{X}_S = & -\frac{c_{fric}}{\sqrt{\mu}} \left| 1 + \frac{1}{\tilde{M}} \int_{I(t)} \left(\frac{P_{atm}}{\rho g H_0} + h \right) dx \right| \frac{\dot{X}_S}{|\dot{X}_S| + \tilde{\delta}} \\ & + \frac{P_{atm}}{\rho g H_0 \tilde{M}} \int_{\mathbb{R}^d} \nabla_x b \, dx + \frac{1}{\tilde{M}} \int_{\mathbb{R}^d} h \nabla_x b \, dx + \mathcal{O} \left(\frac{c_{fric}}{\tilde{M}} \sqrt{\mu} \right). \end{aligned}$$

Using the fact that b is of compact support, the integral of its (and b^2 's) gradient on the whole horizontal space is 0, and the equation simplifies into

$$\begin{aligned} \ddot{X}_S = & -\frac{c_{fric}}{\sqrt{\mu}} \left| 1 + \frac{|\text{supp}(\mathbf{b})|}{\tilde{M}} \left(\frac{P_{atm}}{\rho g H_0} + 1 \right) - \frac{|\text{Volume}_{Solid}|}{\tilde{M}} + \frac{1}{\tilde{M}} \int_{I(t)} \zeta \, dx \right| \frac{\dot{X}_S}{|\dot{X}_S| + \tilde{\delta}} \\ & + \frac{1}{\tilde{M}} \int_{\mathbb{R}^d} \zeta \nabla_x b \, dx + \mathcal{O} \left(\frac{c_{fric}}{\tilde{M}} \sqrt{\mu} \right). \end{aligned}$$

Notice the presence of the friction term (the first term on the right hand side). Even though it is of order $\frac{1}{\sqrt{\mu}}$, it will not pose a problem when controlling the solid velocity, as we shall see in Lemma 2.5. later on, since it acts as a damping force.

So recalling that b is given by (1.23) the corresponding approximative equation characterising the

motion of the body is

$$\ddot{X}_S = -\frac{c_{fric}}{\sqrt{\mu}} \left| c_{solid} + \frac{1}{\tilde{M}} \int_{I(t)} \zeta dx \right| \frac{\dot{X}_S}{|\dot{X}_S| + \tilde{\delta}} + \frac{1}{\tilde{M}} \int_{\mathbb{R}^d} \zeta \nabla_x \mathbf{b}(x - X_S) dx, \quad (2.4)$$

where we made use of the following abbreviation:

$$c_{solid} = 1 + \frac{|\text{supp}(\mathbf{b})|}{\tilde{M}} \left(\frac{P_{atm}}{\rho g H_0} + 1 \right) - \frac{|\text{Volume}_{Solid}|}{\tilde{M}}. \quad (2.5)$$

2.3 The wave-structure interaction problem at first order

With (2.4) in our hand, we have all three equations for our coupled system. Indeed, notice that for the first equation in the nonlinear Saint–Venant system (2.1), the right hand side depends on X_S , since $b(t, x)$ depends on it. Thus by the chain rule the right hand side is

$$\partial_t b(t, x) = \nabla_x \mathbf{b}(x - X_S(t)) \cdot \dot{X}_S(t).$$

Our remark concerning the friction term present in the acceleration equation (2.4) becomes even more pertinent now, since we can observe a direct influence of the solid velocity (and thus an order $\frac{1}{\sqrt{\mu}}$ term) in the first equation of the fluid system (2.1). This implies that a careful attention has to be paid on the velocity estimate for the solid.

To sum it up, the free surface equations with a solid moving at the bottom in the case of the nonlinear Saint-Venant approximation take the following form

$$\begin{cases} \partial_t \zeta + \nabla_x \cdot (h \bar{V}) = \nabla_x \mathbf{b}(x - X_S) \cdot \dot{X}_S, \\ \partial_t \bar{V} + \nabla_x \zeta + (\bar{V} \cdot \nabla_x) \bar{V} = 0, \end{cases} \quad (2.6a)$$

$$\begin{cases} \ddot{X}_S = -\frac{c_{fric}}{\sqrt{\mu}} \left| c_{solid} + \frac{1}{\tilde{M}} \int_{I(t)} \zeta dx \right| \frac{\dot{X}_S}{|\dot{X}_S| + \tilde{\delta}} + \frac{1}{\tilde{M}} \int_{\mathbb{R}^d} \zeta \nabla_x \mathbf{b}(x - X_S) dx. \end{cases} \quad (2.6b)$$

In what follows, we proceed to the mathematical analysis of this system. We shall establish a local in time existence result for the coupled equations.

2.4 Local in time existence of the solution

The main result on the local well-posedness of the wave-structure interaction problem (2.6) is the following:

Theorem 2.1. *Suppose that $\varepsilon = 1$, and that μ is sufficiently small. Let us suppose that for the initial value ζ_{in} and \mathbf{b} the lower bound condition (1.1) is satisfied. If the initial values ζ_{in} and \bar{V}_{in} are in $H^s(\mathbb{R}^d)$ with $s \in \mathbb{R}$, $s > d/2 + 1$, and $X_S(0) = 0$, $\dot{X}_S(0) = v_{S_0} \in \mathbb{R}^d$ is an arbitrary initial condition for the solid motion, then there exists a solution*

$$\begin{aligned} (\zeta, \bar{V}) &\in L^\infty([0, T]; H^s(\mathbb{R}^d)) \cap Lip([0, T]; H^{s-1}(\mathbb{R}^d)), \\ X_S &\in \mathcal{C}^1([0, T]), \end{aligned}$$

to (2.6) for a sufficiently small time $T > 0$.

Proof: The demonstration is based on the fixed point theorem applied to an iterative scheme presented

in the following subsections. The brief outline of our proof is as follows:

1. Reformulation of the system,
2. Construction of the iterative scheme,
3. Existence and a priori estimates for the iterative scheme,
4. Convergence of the iterative scheme solutions.

2.4.1 Reformulation of the coupled system

Let us remark the following: the nonlinear Saint-Venant equations (2.1) admit a quasilinear hyperbolic structure. More precisely, we have the following classical reformulation:

Lemma 2.1. *The system (2.6a) may be reformulated using the variable $\mathcal{U} = (\zeta, \bar{V})^\top \in \mathbb{R}^{d+1}$ into a single equation, that is*

$$\partial_t \mathcal{U} + \sum_{j=1}^d A_j(\mathcal{U}, X_S) \partial_j \mathcal{U} + B(\mathcal{U}, X_S) = 0. \quad (2.7)$$

Proof: Let us take the following real valued $(d+1) \times (d+1)$ matrices

$$A_j(\mathcal{U}, X_S) = \left(\begin{array}{c|c} \bar{V}_j & hI_j \\ \hline I_j^\top & \bar{V}_j \text{Id}_{d \times d} \end{array} \right) \text{ for } 1 \leq j \leq d, \quad (2.8)$$

where for every $1 \leq j \leq d$ we have $I_j = e_j \in \mathbb{R}^d$ the j^{th} coordinate vector with respect to the standard Euclidean basis of \mathbb{R}^d .

We recall that $h = 1 + \zeta - b$ thus implying that the matrices $A_j(\mathcal{U}, X_S)$ indeed depend on X_S , however only through the bottom variation (1.23).

Following the notation in (2.7), after an easy computation, the additional term $B(\mathcal{U}, X_S)$ is the vector

$$B(\mathcal{U}, X_S) = \left(-\bar{V} \cdot \nabla_x \mathbf{b}(x - X_S) - \nabla_x \mathbf{b}(x - X_S) \cdot \dot{X}_S, 0, \dots, 0 \right)^\top.$$

□

From here on, we shall use the following uniform notation for the coordinate functions of \mathcal{U} :

$$\mathcal{U}_0 = \zeta, \quad \mathcal{U}_j = \bar{V}_j \text{ for } 1 \leq j \leq d.$$

As for the initial values, we have $\mathcal{U}(0, \cdot) = \mathcal{U}_{in} = (\zeta_{in}, \bar{V}_{in})$ and $X_S(0) = 0$, $\dot{X}_S(0) = v_{S_0}$. There is no restriction necessary on the initial values concerning the solid motion.

There exists a symmetrizer $S(\mathcal{U}, X_S)$ defined by

$$S(\mathcal{U}, X_S) = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & h \text{Id}_{d \times d} \end{array} \right), \quad (2.9)$$

such that the matrices $S(\mathcal{U}, X_S)A_j(\mathcal{U}, X_S)$ are symmetric. Moreover, based on our imposed lower boundary condition on h_{in} , we shall establish that $S(\mathcal{U}_{in}, 0) \geq h_{min} \text{Id}_{(d+1) \times (d+1)}$, which guarantees that the

matrix is positive definite.

Owing to the existence of such a symmetrizer S , the local well-posedness for a bottom with a prescribed motion follows from classical results [Tay97]. In our case an additional step is needed due to the presence of the coupling with the equation describing the solid motion.

Let us make one further remark, concerning the second order (nonlinear) ordinary differential equation characterising the displacement of the solid X_S in (2.6b). More precisely, let us define the functional $\mathcal{F}[\mathcal{U}](t, Y, Z)$ as

$$\mathcal{F}[\mathcal{U}](t, Y, Z) = -\frac{c_{fric}}{\sqrt{\mu}} \left| c_{solid} + \frac{1}{\bar{M}} \int_{\text{supp}(\mathbf{b})+Y} \mathcal{U}_0 dx \right| \frac{Z}{|Z| + \bar{\delta}} + \frac{1}{\bar{M}} \int_{\mathbb{R}^d} \mathcal{U}_0 \nabla_x \mathbf{b}(x - Y) dx.$$

The coupled system (2.6) has the following equivalent form

$$\begin{cases} \partial_t \mathcal{U} + \sum_{j=1}^d A_j(\mathcal{U}, X_S) \partial_j \mathcal{U} + B(\mathcal{U}, X_S) = 0, & (2.10a) \\ \ddot{X}_S = \mathcal{F}[\mathcal{U}](t, X_S, \dot{X}_S). & (2.10b) \end{cases}$$

2.4.2 The iterative scheme

To solve the coupled system (2.10) we construct a sequence $(\{\mathcal{U}^k(t, x)\}, \{X^k(t)\})_{k \in \mathbb{N}}$ of approximate solutions through the scheme

$$\begin{cases} S(\mathcal{U}^k, X^k) \partial_t \mathcal{U}^{k+1} + \sum_{j=1}^d S(\mathcal{U}^k, X^k) A_j(\mathcal{U}^k, X^k) \partial_j \mathcal{U}^{k+1} + S(\mathcal{U}^k, X^k) B(\mathcal{U}^k, X^k) = 0, & (2.11a) \\ \dot{X}^{k+1} = \mathcal{F}[\mathcal{U}^{k+1}](t, X^{k+1}, \dot{X}^{k+1}); & (2.11b) \\ \mathcal{U}^{k+1}(0, \cdot) = \mathcal{U}_{in}, \quad X^{k+1}(0) = 0, \quad \dot{X}^{k+1}(0) = v_{S_0}. \end{cases}$$

Here the matrices A_j and S are the matrices defined in (2.8) and (2.9). In what follows we will make use of the following abbreviations

$$S^k = S(\mathcal{U}^k, X^k), \quad A_j^k = A_j(\mathcal{U}^k, X^k), \quad \text{and} \quad B^k = B(\mathcal{U}^k, X^k).$$

The main goal is to prove the existence and convergence of this sequence. We will follow the footsteps of a classical method, presented by Alinhac and Gérard in [AG07] for instance, detailing only the parts where additional estimates are necessary due to our coupled system.

The iterative scheme works as follows: we choose the initial $k = 0$ elements to be $(\mathcal{U}^0, X^0) = (\mathcal{U}_{in}, 0)$. From then on, at each step k ($k \in \mathbb{N}$) we have to solve

- a linear symmetric hyperbolic PDE system (2.11a) to recover \mathcal{U}^{k+1} ,
- and then a second order nonlinear ODE (2.11b) to obtain X^{k+1} .

2.4.3 Existence and a priori estimates

Now, the aim is to establish the existence of solutions $(\mathcal{U}^{k+1}, X^{k+1})$ ($k \geq 0$) for the iterative scheme to justify their definition in (2.11). Furthermore we shall also obtain a control of the velocity fields for our

coupled system. In particular an upper bound on \mathcal{U}^k in a "large norm", partially in order to guarantee the boundedness conditions required for the existence result presented, as well as to introduce certain inequalities which will be useful for the convergence of the series.

This so called large norm shall be the following:

Definition 2.1. For an $f(t, x)$ function let us define

$$\|f\|_{T, H^s} = \sup_{t \in [0, T]} \|f(t, \cdot)\|_{H^s}. \quad (2.12)$$

With this definition at our disposal, we can state the induction hypothesis (H_k) for boundedness of solutions $(\mathcal{U}^l, X^l)_{l \leq k}$ of (2.11):

$$\text{for } 0 \leq l \leq k, \quad \|\mathcal{U}^l - \mathcal{U}_{in}\|_{T, H^s} \leq \delta_0, \quad \sup_{t \in [0, T]} |\dot{X}^l - v_{S_0}| \leq C_v T, \quad (H_k)$$

for some constant $C_v = C(\tilde{M}^{-1}, \delta_0, \|\mathbf{b}\|_{H^s})$, where $\delta_0 > 0$ a small independent constant to be defined.

Proposition 2.1. For $k \geq 0$, with the knowledge of (H_k) , there exists a solution $\mathcal{U}^{k+1} \in \mathcal{C}([0, T]; L^2(\mathbb{R}^d))$, $X^{k+1} \in \mathcal{C}^1([0, T])$ of (2.11), by an adequate choice of δ_0 and T (independent of k), moreover

$$(H_k) \Rightarrow (H_{k+1}).$$

Proof: The proof goes by induction. For $k = 0$, (H_0) is clearly verified. For the induction step, we shall treat separately the case of the PDE (part A) and the case of the ODE (part B), for the sake of clarity.

Part A: existence and energy estimate for \mathcal{U}^{k+1} : The initial values $\mathcal{U}^{k+1}(0, \cdot)$ are bounded since they are equal to the original initial values \mathcal{U}_{in} . Since we are operating by induction with respect to k , for the respective \mathcal{U}^k term we already have existence, moreover we also have the large norm estimates (H_k) at hand, which in particular guarantees the uniform boundedness (independently of the index k) for small time T and δ_0 . Also, given the simple structure of S^k and $S^k A_j^k$, they are bounded as well in Lipschitz norm.

Lemma 2.2. For $k \geq 0$, with the initial condition $\mathcal{U}^{k+1}(0, \cdot) = \mathcal{U}_{in}$ and the hypothesis (H_k) there exists a $\mathcal{C}([0, T]; H^s(\mathbb{R}^d))$ solution \mathcal{U}^{k+1} for the linear symmetric hyperbolic PDE system defined in (2.11a).

Proof: Notice that (2.11a) has a particular symmetric structure which may be exploited based on the following proposition:

Proposition 2.2. Let us consider the symmetric hyperbolic differential operator

$$L = S\partial_t + \sum_{j=1}^d SA_j \partial_j$$

with S and SA_j symmetric real valued and bounded in Lipschitz norm, with $S \geq S_0 \text{Id}$, where $S_0 > 0$. Furthermore let us consider $s \in \mathbb{R}$, $s > d/2 + 1$ and $\lambda \in \mathbb{R}$, and suppose that $2S_0\lambda \geq \lambda_0$, where

$$\lambda_0 = C(\|A_j\|_{H^s}, \|\text{div}(SA_j)\|_{L^\infty}, \|\partial_t S\|_{L^\infty}).$$

Then, for any $f \in L^1([0, T]; H^s(\mathbb{R}^d))$ and $\varphi \in H^s(\mathbb{R}^d)$ the Cauchy problem

$$\begin{aligned} Lu &= f, \quad 0 < t < T \\ u(0, \cdot) &= \varphi, \end{aligned} \quad (2.13)$$

admits a unique solution $u \in \mathcal{C}([0, T]; H^s(\mathbb{R}^d))$ that verifies the energy estimate

$$S_0 \sup_{t \in [0, T]} \left\{ e^{-\lambda t} \|u(t, \cdot)\|_{H^s} \right\} \leq \|\varphi\|_{H^s} + 2 \int_0^T e^{-\lambda t'} \|f(t', \cdot)\|_{H^s} dt'. \quad (2.14)$$

For more details as well as a complete proof, please refer to [AG07].

We want to apply Proposition 2.2. to solve the linear PDE (2.11a) for \mathcal{U}^{k+1} in $\mathcal{C}([0, T]; L^2(\mathbb{R}^d))$. For this, we only need to verify that λ_0 is bounded. By making use of the fact that S^k and A_j^k depend on X^k in a very simplistic way, throughout the function h , thus it is present as a translation for the function \mathbf{b} , which obviously does not affect the L^∞ norm, we are only left with the estimation of $\|\partial_t S\|_{L^\infty}$.

Based on the corresponding equation for \mathcal{U}^k (from the system (2.11a)), we have that

$$\partial_t \mathcal{U}^k + \sum_{j=1}^d A_j^{k-1} \partial_j \mathcal{U}^k + B^{k-1} = 0,$$

which implies that

$$\begin{aligned} \|\partial_t \mathcal{U}^k\|_{L^\infty} &\leq \sum_{j=1}^d \|A_j^{k-1} \partial_j \mathcal{U}^k\|_{L^\infty} + \left\| \overline{V}^{k-1} \cdot \nabla_x \mathbf{b}(\cdot, -X^{k-1}) + \nabla_x \mathbf{b}(\cdot, -X^{k-1}) \cdot \dot{X}^{k-1} \right\|_{L^\infty} \\ &\lesssim \sum_{j=1}^d \left(1 + \|\mathcal{U}^{k-1}\|_{L^\infty} + \|\mathbf{b}\|_{L^\infty} \right) \cdot \|\mathcal{U}^k\|_{H^s} + \|\mathbf{b}\|_{H^s} \left(\|\mathcal{U}^{k-1}\|_{H^s} + |\dot{X}^{k-1}| \right) \\ &\lesssim c(\delta_0, \|\mathbf{b}\|_{H^s}; C_v)(1 + T), \end{aligned}$$

by (H_k) and the regularity of \mathbf{b} .

Therefore, λ_0 is a constant independent of k , linear in T , and properly bounded. \square

For the large norm estimate, let us introduce the variable $R^{k+1} = \mathcal{U}^{k+1} - \mathcal{U}_{in}$, the remainder term to be examined for the induction step. From the definition of the iterative scheme (2.11a) it satisfies the system

$$S^k \partial_t R^{k+1} + \sum_{j=1}^d S^k A_j^k \partial_j R^{k+1} = -S^k B^k - \sum_{j=1}^d S^k A_j^k \partial_j \mathcal{U}_{in} =: F(t, x). \quad (2.15)$$

The initial condition for this system writes as

$$R^{k+1}(0, \cdot) = 0.$$

The estimations concerning λ_0 still hold, so, since equation (2.15) has the exact same linear structure as (2.11a), Proposition 2.2. can be applied. Thus the H^s energy estimate from Proposition 2.2. for the system (2.15) of R^{k+1} can be stated to obtain

$$S_0 \sup_{t \in [0, T]} \left\{ e^{-\lambda_0 t} \|R^{k+1}(t, \cdot)\|_{H^s} \right\} \leq 2 \int_0^T e^{-\lambda_0 t} \|F(t, \cdot)\|_{H^s} dt. \quad (2.16)$$

To conclude, we shall provide a good estimate for the right hand side of (2.15), namely

Lemma 2.3. *The function F given by (2.15) satisfies the following linear-in-time estimate*

$$\|F(t, \cdot)\|_{H^s} \leq C_F(1 + T), \quad (2.17)$$

with the constant $C_F = C(\delta_0, \|\mathbf{b}\|_{H^s}; C_v)$, independent of k and of T .

Proof: We shall estimate each term in H^s -norm.

For the first term, we have

$$\begin{aligned} \|S^k B^k\|_{H^s} &\lesssim \left(1 + \|\mathcal{U}_0^k\|_{L^\infty} + \|\mathbf{b}\|_{L^\infty}\right) \cdot \|\mathbf{b}\|_{H^s} \left(\|\bar{V}^k\|_{L^\infty} + |\dot{X}^k|\right) \\ &\quad + \left(1 + \|\mathcal{U}_0^k\|_{H^s} + \|\mathbf{b}\|_{H^s}\right) \cdot \|\nabla_x \mathbf{b}\|_{L^\infty} \|\bar{V}^k\|_{L^\infty} \\ &\lesssim C(\delta_0, \|\mathbf{b}\|_{H^s}; C_v)(1 + T) \end{aligned}$$

using the special structure of the matrix S^k as well as the fact that the Sobolev norm is translation invariant. After, we made use of standard injection inequalities, finally we deduce that it is bounded by a constant since the H^s -norm is uniformly bounded by (H_k) for \mathcal{U}^k , and \mathbf{b} is still regular.

We handle the second term similarly, thus deducing

$$\left\| S^k \left(\sum_{j=1}^d A_j^k \partial_j \mathcal{U}_{in} \right) \right\|_{L^2} \lesssim C(\delta_0, \|\mathbf{b}\|_{H^s}).$$

With this all the terms are in order for our lemma concerning F . □

So we return to (2.16) to obtain

$$S_0 \sup_{t \in [0, T]} \left\{ e^{-\lambda_0 t} \|R^{k+1}(t, \cdot)\|_{H^s} \right\} \lesssim T(1 + T).$$

By taking T small enough, we obtain the first half of (H_{k+1}) .

Part B: existence and velocity estimate for X^{k+1} : For the existence of the solution X^{k+1} of the ODE (2.11b), we shall apply the Picard–Lindelöf theorem.

Lemma 2.4. *For $k \geq 0$, with the initial conditions $X^{k+1}(0) = 0$, and $\dot{X}^{k+1}(0) = v_{S_0}$ and the hypothesis (H_k) , there exists a continuously differentiable solution X^{k+1} for the nonlinear second order nonhomogeneous ODE defined in (2.11b) for $t \in [0, T_S]$, where $T_S = C(\delta_0, \|\mathbf{b}\|_{H^s})$.*

Proof: For the Picard–Lindelöf theorem, we have to show that the nonlinear functional on the right hand side of (2.11b) is continuous in time and uniformly Lipschitz in the spatial variable.

We recall that the functional $\mathcal{F}[\mathcal{U}^{k+1}](t, Y, Z)$ has the form of

$$\mathcal{F}[\mathcal{U}^{k+1}](t, Y, Z) = -\frac{c_{fric}}{\sqrt{\mu}} \left| c_{solid} + \frac{1}{\tilde{M}} \int_{\text{supp}(\mathbf{b})+Y} \mathcal{U}_0^{k+1} dx \right| \frac{Z}{|Z| + \tilde{\delta}} + \frac{1}{\tilde{M}} \int_{\mathbb{R}^d} \mathcal{U}_0^{k+1} \nabla_x \mathbf{b}(x - Y) dx.$$

We already know that \mathcal{U}_0^{k+1} is of class $\mathcal{C}([0, T]; H^s(\mathbb{R}^d))$, so it is continuous in the time variable, regular in the space variable, moreover the function \mathbf{b} is regular and with a compact support, thus the integrals indeed exist and are bounded, furthermore based on the well known theorem concerning the continuity of a parametric integral, it will be continuous with respect to t .

All we need to show is that it is (locally) uniformly Lipschitz with respect to its second variable (Y, Z) . Examining $\mathcal{F}[\mathcal{U}^{k+1}](t, Y, Z)$, it is clear that the second term is clearly Lipschitz continuous due to the fact that \mathbf{b} is regular. As for the first term, since it contains a product of multiple terms with the variables Y and Z , by adding and subtracting intermediate terms they can be separated.

Let us take a closer look on these two separate terms. The integral term can be estimated due to

$$\int_{\text{supp}(\mathbf{b})+Y} \mathcal{U}_0^{k+1}(t, x) dx = \int_{\text{supp}(\mathbf{b})} \mathcal{U}_0^{k+1}(t, x - Y) dx,$$

and the regularity of \mathcal{U}_0^{k+1} . Since we chose $\tilde{\delta} > 0$, the function

$$Z \mapsto \frac{Z}{|Z| + \tilde{\delta}}$$

is clearly Lipschitz continuous. So, putting all the estimates together, we obtain that

$$\begin{aligned} \left| \mathcal{F}[\mathcal{U}^{k+1}](t, Y_1, Z_1) - \mathcal{F}[\mathcal{U}^{k+1}](t, Y_2, Z_2) \right| &\leq \frac{3c_{fric}}{\tilde{\delta}\sqrt{\mu}} \left(c_{solid} + \frac{\delta_0 |\{\text{supp}(\mathbf{b})\}|}{\tilde{M}} \right) |Z_1 - Z_2| \\ &\quad + 2\delta_0 L_{\nabla_x \mathbf{b}} \frac{|\{\text{supp}(\mathbf{b})\}|}{\tilde{M}} \cdot |Y_1 - Y_2|; \end{aligned}$$

where L denotes the Lipschitz constant of the corresponding function in the subscript, moreover

$$\mathfrak{B} = \{\text{supp}(\mathbf{b}) + Y_1\} \cup \{\text{supp}(\mathbf{b}) + Y_2\}.$$

□

One of the most important parts of the proof is the control on the solid velocity, since it contains an order $\frac{1}{\sqrt{\mu}}$ term which could potentially become huge, making the system explode. However,

Lemma 2.5. *A control on the solid velocity is assured by*

$$\left| \dot{X}^{k+1}(t) \right| \leq |v_{S_0}| + C_v t, \quad (2.18)$$

with a constant $C_v = C(\tilde{M}^{-1}, \delta_0, \|\mathbf{b}\|_{H^s})$ independent of k and t .

Proof: By definition X^{k+1} satisfies the corresponding second order nonlinear nonhomogeneous equation in (2.11b) so we have that

$$\ddot{X}^{k+1} = \mathcal{F}[\mathcal{U}^{k+1}](t, X^{k+1}, \dot{X}^{k+1}),$$

thus, multiplying by $\dot{X}^{k+1}(t)$, we get that

$$\begin{aligned} \ddot{X}^{k+1} \cdot \dot{X}^{k+1} &= -\frac{c_{fric}}{\sqrt{\mu}} \left| c_{solid} + \frac{1}{\tilde{M}} \int_{\text{supp}(\mathbf{b})+\dot{X}^{k+1}} \mathcal{U}_0^{k+1} dx \right| \frac{|\dot{X}^{k+1}|^2}{|\dot{X}^{k+1}| + \tilde{\delta}} + \frac{1}{\tilde{M}} \int_{\mathbb{R}^d} \mathcal{U}_0^{k+1} \nabla_x \mathbf{b}(x - X^{k+1}) dx \cdot \dot{X}^{k+1} \\ &\leq 0 + \frac{1}{\tilde{M}} \int_{\mathbb{R}^d} \mathcal{U}_0^{k+1} \nabla_x \mathbf{b}(x - X^{k+1}) \cdot \dot{X}^{k+1} dx, \end{aligned}$$

here the key remark is that the first, negative term disappeared from the equations. Thus, we are left with

$$\frac{1}{2} \frac{d}{dt} \left[|\dot{X}^{k+1}|^2 \right] \leq \frac{\delta_0}{\tilde{M}} \int_{\mathbb{R}^d} \nabla_x \mathbf{b}(x - X^{k+1}) \cdot \dot{X}^{k+1} dx \leq \frac{\delta_0}{\tilde{M}} \|\mathbf{b}\|_{H^s} |\dot{X}^{k+1}|,$$

so by a Grönwall type lemma for \dot{X}^k , we may conclude that

$$\left| \dot{X}^k(t) \right| \leq |v_{S_0}| + \frac{\delta_0}{\tilde{M}} \|\mathbf{b}\|_{H^s} t.$$

□

This concludes the velocity estimate for the object. So we proved the existence of solutions $(\mathcal{U}^{k+1}, X^{k+1})$ for the system (2.11), moreover we established the necessary elements for the upper bounds concerning the velocities in (H_{k+1}) . \square

2.4.4 Convergence

We want to establish the convergence of the series from (2.11), for that we need the L^2 -norm estimates for the difference between two subsequent elements for \mathcal{U}^k , for X^k we shall simply estimate in \mathbb{R}^d -norm.

We shall start by subtracting the equations corresponding to the k^{th} element from the equations corresponding to the $(k+1)^{\text{th}}$ element. After the subtraction we have

$$\left\{ \begin{array}{l} S^k \partial_t (\mathcal{U}^{k+1} - \mathcal{U}^k) + \sum_{j=1}^d S^k A_j^k \partial_j (\mathcal{U}^{k+1} - \mathcal{U}^k) = \\ \quad = - (S^k - S^{k-1}) \partial_t \mathcal{U}^k - \sum_{j=1}^d (S^k A_j^k - S^{k-1} A_j^{k-1}) \partial_j \mathcal{U}^k - (S^k B^k - S^{k-1} B^{k-1}), \end{array} \right. \quad (2.19a)$$

$$\left\{ \begin{array}{l} (\mathcal{U}^{k+1} - \mathcal{U}^k)(0, \cdot) = 0; \\ \frac{d^2}{dt^2} (X^{k+1} - X^k) = \mathcal{F}[\mathcal{U}^{k+1}] (t, X^{k+1}, \dot{X}^{k+1}) - \mathcal{F}[\mathcal{U}^k] (t, X^k, \dot{X}^k) \\ (X^{k+1} - X^k)(0) = 0, \quad \frac{d}{dt} (X^{k+1} - X^k)(0) = 0. \end{array} \right. \quad (2.19b)$$

We provide separately an appropriate estimate in this small norm for the solutions $(\mathcal{U}^{k+1} - \mathcal{U}^k)$ and $(X^{k+1} - X^k)$ of the system. The estimation for the ODE part (2.19b) is given by the following lemma.

Lemma 2.6. *For a solution $X^{k+1} - X^k$ of (2.19b), we have that*

$$\sup_{t \in [0, T]} |X^{k+1}(t) - X^k(t)| \lesssim C_1 T \sup_{t' \in [0, T]} \left(\left\| \mathcal{U}^{k+1}(t', \cdot) - \mathcal{U}^k(t', \cdot) \right\|_{L^2} + |X^{k+1}(t') - X^k(t')| \right), \quad (2.20)$$

with a constant $C_1 = C(\tilde{M}, \delta_0, \|\mathbf{b}\|_{H^s})$.

Notice that the right hand side of the estimate contains exactly the same differences as the ones we would like to establish an upper bound for, but due to the presence of the factor T , with T sufficiently small, it will be completely absorbed by the left hand side.

Proof: To treat the difference of products that arise multiple times, we introduce intermediary terms, just as we did for the verification of the Lipschitz-property. So following standard computations, we get that

$$\begin{aligned} |X^{k+1}(t) - X^k(t)| &= \left| \int_0^t \frac{d}{ds} (X^{k+1} - X^k)(s) ds \right| = \left| \int_0^t \int_0^s \frac{d^2}{dt^2} (X^{k+1} - X^k)(\tau) d\tau ds \right| \\ &\lesssim \int_0^t \int_0^s (1+T) \|\mathcal{U}^{k+1}\|_{L^2} |X^{k+1} - X^k|(\tau) d\tau ds + \int_0^t \left| \int_0^s \frac{d}{d\tau} (X^{k+1} - X^k)(\tau) d\tau \right| ds \\ &\quad + \int_0^t \int_0^s \sup_{t' \in [0, T]} \left\| \mathcal{U}_0^{k+1}(t', \cdot) - \mathcal{U}_0^k(t', \cdot) \right\|_{L^2} \|\nabla_x \mathbf{b}\|_{L^2} d\tau ds + \int_0^t \int_0^s |X^{k+1} - X^k|(\tau) d\tau ds \end{aligned}$$

Here we estimated each term by the controls of (H_k) and the Lipschitz properties, in order to obtain

$$|X^{k+1}(t) - X^k(t)| \lesssim T \left(\sup_{t' \in [0, T]} \left\| \mathcal{U}^{k+1}(t', \cdot) - \mathcal{U}^k(t', \cdot) \right\|_{L^2} + \sup_{t' \in [0, T]} |X^{k+1}(t') - X^k(t')| \right).$$

□

Then, for the estimation for the PDE part we have the following.

Lemma 2.7. *For a solution of (2.19a), we obtain*

$$\sup_{t \in [0, T]} \|\mathcal{U}^{k+1}(t, \cdot) - \mathcal{U}^k(t, \cdot)\|_{L^2} \lesssim C_2 T \sup_{t' \in [0, T]} \left(\|\mathcal{U}^k(t', \cdot) - \mathcal{U}^{k-1}(t', \cdot)\|_{L^2} + |X^k(t') - X^{k-1}(t')| \right), \quad (2.21)$$

with a constant $C_2 = C(\tilde{M}, \delta_0, \|\mathbf{b}\|_{H^s})$.

Proof: Once again we aim to use the energy estimate, since we have the linear system (2.19a) of the type of Proposition 2.2. for the variable $\mathcal{U}^{k+1} - \mathcal{U}^k$.

The same reasoning applies here as for the energy estimates section concerning the applicability of the proposition, so we only need a sufficient upper bound for the right hand side of (2.19a), denoted for consistency reasons by \overline{F} . We shall examine it term by term.

The first two terms are handled with standard techniques to deduce

$$\begin{aligned} \|(S^k - S^{k-1})\partial_t \mathcal{U}^k\|_{L^2} &\lesssim \|\mathcal{U}^k - \mathcal{U}^{k-1}\|_{L^2} + |X^k - X^{k-1}|, \\ \|(S^k A_j^k - S^{k-1} A_j^{k-1})\partial_j \mathcal{U}^k\|_{L^2} &\lesssim \|\mathcal{U}^k - \mathcal{U}^{k-1}\|_{L^2} + |X^k - X^{k-1}|. \end{aligned}$$

And for the third term, we deal with the product via an intermediary term, so

$$\|S^k B^k - S^{k-1} B^{k-1}\|_{L^2} \leq \|S^k - S^{k-1}\|_{L^2} \|B^k\|_{L^2} + \|S^{k-1}\|_{L^2} \|B^k - B^{k-1}\|_{L^2},$$

then, by using the definition of the source term, we get that

$$\begin{aligned} \|S^k B^k - S^{k-1} B^{k-1}\|_{L^2} &\lesssim (\|\mathcal{U}^k - \mathcal{U}^{k-1}\|_{L^2} + |X^k - X^{k-1}|) \|\overline{V}^k \cdot \nabla_x \mathbf{b}(\cdot - X^k)\|_{L^2} \\ &\quad + (\|\mathcal{U}^k - \mathcal{U}^{k-1}\|_{L^2} + |X^k - X^{k-1}|) \left\| \nabla_x \mathbf{b}(\cdot - X^k) \cdot \dot{X}^k \right\|_{L^2} \\ &\quad + (1 + \|\mathcal{U}^{k-1}\|_{L^2}) \left\| \overline{V}^k \cdot \nabla_x \mathbf{b}(\cdot - X^k) - \overline{V}^{k-1} \cdot \nabla_x \mathbf{b}(\cdot - X^{k-1}) \right\|_{L^2} \\ &\quad + (1 + \|\mathcal{U}^{k-1}\|_{L^2}) \left\| \nabla_x \mathbf{b}(\cdot - X^k) \cdot \dot{X}^k - \nabla_x \mathbf{b}(\cdot - X^{k-1}) \cdot \dot{X}^{k-1} \right\|_{L^2}. \end{aligned}$$

Again, with the apparition of intermediary terms for each product, by utilising Lemma 2.5. as well as its direct adaptation to the difference $|\dot{X}^k - \dot{X}^{k-1}|$, we get that

$$\|S^k B^k - S^{k-1} B^{k-1}\|_{L^2} \lesssim (1 + T) \sup_{t' \in [0, T]} \left(\|\mathcal{U}^k(t', \cdot) - \mathcal{U}^{k-1}(t', \cdot)\|_{L^2} + |X^k(t') - X^{k-1}(t')| \right).$$

Applying the energy estimate to (2.19a), we obtain that

$$\begin{aligned} S_0 \sup_{t \in [0, T]} \left\{ e^{-\lambda t} \|\mathcal{U}^{k+1}(t, \cdot) - \mathcal{U}^k(t, \cdot)\|_{L^2} \right\} &\leq 2 \int_0^T e^{-\lambda t} \|\overline{F}(t, \cdot)\|_{L^2} dt \\ &\lesssim T(1 + T) \sup_{t' \in [0, T]} \left\{ e^{-\lambda t'} \left(\|\mathcal{U}^k(t', \cdot) - \mathcal{U}^{k-1}(t', \cdot)\|_{L^2} + |X^k(t') - X^{k-1}(t')| \right) \right\}. \end{aligned}$$

□

To sum up the results from the two previous lemmas, we obtained that for T sufficiently small, we

have that

$$\begin{aligned} & \sup_{t \in [0, T]} \left\| \mathcal{U}^{k+1}(t, \cdot) - \mathcal{U}^k(t, \cdot) \right\|_{L^2} + \sup_{t \in [0, T]} |X^{k+1}(t) - X^k(t)| \\ & \leq \mathfrak{c} \left(\sup_{t \in [0, T]} \left\| \mathcal{U}^k(t, \cdot) - \mathcal{U}^{k-1}(t, \cdot) \right\|_{L^2} + \sup_{t \in [0, T]} |X^k(t) - X^{k-1}(t)| \right), \end{aligned} \quad (2.22)$$

with a constant $\mathfrak{c} < 1$. This assures the convergence in $L^\infty([0, T]; L^2(\mathbb{R}^d))$.

Since we have that the H^s -norm is bounded, we may extract a weakly convergent subsequence from the series, and since the limit in the sense of distributions is unique, we have convergence for the whole series in H^s too. \square

This concludes the proof of the theorem, since, to deduce the regularity implied in the statement, we only have to use the convexity of the norm, following classical regularity arguments. Thus we obtained a classical solution of the coupled system (2.6) for a sufficiently small time T .

3 The $\mathcal{O}(\mu^2)$ asymptotic regime: The Boussinesq system

In this section, we move on to the next order regarding the asymptotic regime, that is the approximations of order μ^2 . In order to simplify the computations, we consider here a weakly nonlinear regime, i.e. we assume that $\varepsilon = \mathcal{O}(\mu)$. The fluid is then governed by a Boussinesq system. Thus, the asymptotic regime writes as follows

$$0 \leq \mu \leq \mu_{max} \ll 1, \quad \varepsilon = \mathcal{O}(\mu). \quad (\text{BOUS})$$

At second order, the asymptotic expansion of \bar{V} in terms of ζ and \bar{V} (based on Proposition 3.37. of [Lan13]) is given by

$$\bar{V} = \nabla_x \psi + \frac{\mu}{3h} \nabla_x (h^3 \nabla_x \cdot \nabla_x \psi) - \frac{\mu}{2} h \nabla_x \partial_t b + \mathcal{O}(\mu^2),$$

so by making use of the definition of h , once again taking the gradient of the second equation in (1.24), and neglecting terms of order $\mathcal{O}(\mu^2)$, equations (1.24) under the Boussinesq regime (BOUS) take the form of

$$\begin{cases} \partial_t \zeta + \nabla_x \cdot (h \bar{V}) = \partial_t b, \\ \left(1 - \frac{\mu}{3} \Delta_x\right) \partial_t \bar{V} + \nabla_x \zeta + \varepsilon (\bar{V} \cdot \nabla_x) \bar{V} = -\frac{\mu}{2} \nabla_x \partial_t^2 b. \end{cases} \quad (3.1)$$

For the well-posedness of this Boussinesq system, see for instance [Lan13].

Remark 3.1. *Without the smallness assumption on $\varepsilon = \mathcal{O}(\mu)$, it is still possible to perform an asymptotic expansion at $\mathcal{O}(\mu^2)$. The resulting system is more general than the Boussinesq system (3.1) but also more complicated, it is known as the Serre–Green–Naghdi equations. For the justification of this general system in the fixed bottom case, please refer to [ASL08a], and [HI15] for a moving bottom under a forced motion.*

It is well-known for the fixed bottom case that the good timescale of Boussinesq-type systems is of order $\frac{1}{\varepsilon}$ in order to be able to properly observe the nonlinear and dispersive effects of equations (3.1) (see for instance [BCL05, SX12, Bur16]). However, for a time dependent bottom (as it is in our case), one can only infer an existence time of $\mathcal{O}(1)$, due to the source term $\partial_t b$ on the right hand side of the first equation in (3.1). Throughout this chapter, we will show that, with the presence of the solid in the system as well as with better estimates, a time of existence in $\frac{1}{\sqrt{\varepsilon}}$ is achievable.

3.1 Formal derivation of the corresponding solid motion equation

As we may observe from the Boussinesq system (3.1) the bottom related source terms are, respectively of order $\mathcal{O}(1)$ and $\mathcal{O}(\mu)$ for the first and second equations. To ensure the consistency of the whole coupled system at order $\mathcal{O}(\mu^2)$, we have to impose the same precision for the equation dealing with the solid; the surface integral present in (1.28) must therefore be approximated at order $\mathcal{O}(\mu^2)$.

Our strategy is exactly the same as for the first order approximation case in the previous section, but it is carried out to the next order of approximation. In fact, it turns out that due to the additional hypothesis on ε , the pressure formula (2.3) derived in Section 2 holds exactly in this regime as well.

Lemma 3.1. *Under the Boussinesq hypotheses (BOUS), the pressure takes the following form*

$$P = \frac{P_{atm}}{\rho g H_0} + (\varepsilon \zeta - z) + \mathcal{O}(\mu^2). \quad (3.2)$$

Proof: The residual in (2.3) is of size $\mathcal{O}(\mu)$ in the SV Saint-Venant regime; however the parameter ε was set to 1 in Section 2.4.1, and the same computations show that the residual is actually of size $\mathcal{O}(\varepsilon\mu)$, and therefore of $\mathcal{O}(\mu^2)$ with the Boussinesq scaling regime BOUS. \square

Remark 3.2. *We remark that for the Serre–Green–Naghdi system, that is without the smallness hypothesis on ε , the situation would be completely different, the expression for the pressure would take a more complex form, incorporating nonlinear effects which would lead to an eventual added mass effect for the equation of motion characterising the solid.*

Therefore, following the same computations as in Section 2.2, we obtain the same ODE for the solid displacement as (2.4), but with a dependence on ε as well,

$$\ddot{X}_S = -\frac{c_{fric}}{\sqrt{\mu}} \left| 1 + \frac{c_{solid} - 1}{\varepsilon} + \frac{1}{\tilde{M}\varepsilon} \int_{\text{supp}(\mathbf{b}) + \dot{X}_S} \zeta dx \right| \frac{\dot{X}_S}{|\dot{X}_S| + \tilde{\delta}} + \frac{\varepsilon}{\tilde{M}} \int_{\mathbb{R}^d} \zeta \nabla_x \mathbf{b}(x - X_S) dx. \quad (3.3)$$

Here, we made use of the constant of the solid c_{solid} , defined by (2.5). The additional $\frac{1}{\varepsilon}$ terms in the equation are due to the general nondimensionalised Newton's equation (1.28). It was of no importance in the previous section for the Saint-Venant regime (SV), but in the Boussinesq regime (BOUS) it has to be taken into consideration.

Once again, notice the presence of the friction terms in the solid equation, which is potentially of order $\frac{1}{\varepsilon\sqrt{\mu}}$, so we will have to reason carefully why this doesn't pose a problem for our system.

3.2 The coupled Boussinesq system

Here we present some remarks on the right hand side of the Boussinesq system (3.1). Again, we have that

$$\partial_t b(t, x) = -\nabla_x \mathbf{b}(x - X_S(t)) \cdot \dot{X}_S(t),$$

however we also have that

$$\begin{aligned} \nabla_x \partial_t^2 b(t, x) &= \nabla_x \partial_t \left(-\nabla_x \mathbf{b}(x - X_S(t)) \cdot \dot{X}_S(t) \right) \\ &= \nabla_x \left(\nabla_x^2 \mathbf{b}(x - X_S(t)) \dot{X}_S(t) \cdot \dot{X}_S(t) - \nabla_x \mathbf{b}(x - X_S(t)) \cdot \ddot{X}_S(t) \right). \end{aligned}$$

To sum it up, the free surface equations with a solid moving at the bottom in the case of the Boussinesq approximation take the following form

$$\begin{cases} \partial_t \zeta + \nabla_x \cdot (h\bar{V}) = \partial_t b, \\ \left(1 - \frac{\mu}{3} \Delta_x\right) \partial_t \bar{V} + \nabla_x \zeta + \varepsilon (\bar{V} \cdot \nabla_x) \bar{V} = -\frac{\mu}{2} \nabla_x \partial_t^2 b, \end{cases} \quad (3.4a)$$

$$\begin{cases} \ddot{X}_S = -\frac{c_{fric}}{\sqrt{\mu}} \left| 1 + \frac{c_{solid} - 1}{\varepsilon} + \frac{1}{\tilde{M}\varepsilon} \int_{\text{supp}(\mathbf{b}) + \dot{X}_S} \zeta dx \right| \frac{\dot{X}_S}{|\dot{X}_S| + \tilde{\delta}} + \frac{\varepsilon}{\tilde{M}} \int_{\mathbb{R}^d} \zeta \nabla_x \mathbf{b}(x - X_S) dx. \end{cases} \quad (3.4b)$$

3.3 A reformulation of the coupled system

Following the observations of Section 3.2, we may elaborate the source term of the coupled system. The free surface equations with a solid moving at the bottom in the case of the Boussinesq approximation can be written as

$$\begin{cases} \partial_t \zeta + \nabla_x \cdot (h\bar{V}) = -\nabla_x \mathbf{b}(x - X_S) \cdot \dot{X}_S, \\ \left(1 - \frac{\mu}{3} \Delta_x\right) \partial_t \bar{V} + \nabla_x \zeta + \varepsilon (\bar{V} \cdot \nabla_x) \bar{V} = \\ -\frac{\mu}{2} \nabla_x \left(\nabla_x^2 \mathbf{b}(x - X_S) \dot{X}_S \cdot \dot{X}_S - \nabla_x \mathbf{b}(x - X_S) \cdot \dot{X}_S \right), \end{cases} \quad (3.5a)$$

$$\begin{cases} \ddot{X}_S = -\frac{c_{fric}}{\sqrt{\mu}} \left| 1 + \frac{c_{solid} - 1}{\varepsilon} + \frac{1}{\tilde{M}\varepsilon} \int_{\text{supp}(\mathbf{b}) + \dot{X}_S} \zeta dx \right| \frac{\dot{X}_S}{|\dot{X}_S| + \tilde{\delta}} + \frac{\varepsilon}{\tilde{M}} \int_{\mathbb{R}^d} \zeta \nabla_x \mathbf{b}(x - X_S) dx. \end{cases} \quad (3.5b)$$

First of all, let us remark that a more compact formulation can be derived, just like for the coupled nonlinear Saint-Venant equations (2.6) in Section 2.4.1. This formula is obtained through the same means as in the previous section, so we will apply similar notations as well. We have the following: the fluid equations (3.5a) for the variable $\mathcal{U} = (\zeta, \bar{V})$ can be written as

$$D_\mu \partial_t \mathcal{U} + \sum_{j=1}^d A_j(\mathcal{U}, X_S) \partial_j \mathcal{U} + B(\mathcal{U}, X_S) = 0, \quad (3.6)$$

where the matrix $A_j(\mathcal{U}, X_S)$ is the same as the one defined in Lemma 2.1, that is

$$A_j(\mathcal{U}, X_S) = \left(\begin{array}{c|c} \varepsilon \bar{V}_j & h I_j \\ \hline I_j^\top & \varepsilon \bar{V}_j \text{Id}_{d \times d} \end{array} \right) \text{ for } 1 \leq j \leq d.$$

We remark that we have the following simple decomposition

$$A_j(\mathcal{U}, X_S) = \bar{I}_j + \varepsilon \bar{A}_j(\mathcal{U}, X_S) = \left(\begin{array}{c|c} 0 & I_j \\ \hline I_j^\top & 0 \end{array} \right) + \varepsilon \left(\begin{array}{c|c} \bar{V}_j & (\zeta - b) I_j \\ \hline 0 & \bar{V}_j \text{Id}_{d \times d} \end{array} \right). \quad (3.7)$$

Additionally, we have that

$$D_\mu = \left(\begin{array}{c|c} 1 & \\ \hline & \left(1 - \frac{\mu}{3}\Delta_x\right) \text{Id}_{d \times d} \end{array} \right),$$

and the source term vector takes the following form

$$B(\mathcal{U}, X_S) = \left(\begin{array}{c} -\varepsilon \bar{V} \cdot \nabla_x \mathbf{b}(x - X_S) + \nabla_x \mathbf{b}(x - X_S) \cdot \dot{X}_S \\ \frac{\mu}{2} \nabla_x \left(\nabla_x^2 \mathbf{b}(x - X_S) \dot{X}_S \cdot \dot{X}_S - \nabla_x \mathbf{b}(x - X_S) \cdot \ddot{X}_S \right) \end{array} \right).$$

Remark 3.3. *Once again, we can symmetrize equation (3.6) with the use of the matrix*

$$S(\mathcal{U}, X_S) = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & h \text{Id}_{d \times d} \end{array} \right),$$

remarking that

$$S(\mathcal{U}, X_S) = \text{Id}_{(d+1) \times (d+1)} + \varepsilon \bar{S}(\mathcal{U}, X_S) = \text{Id}_{(d+1) \times (d+1)} + \varepsilon \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & (\zeta - b) \text{Id}_{d \times d} \end{array} \right). \quad (3.8)$$

Let us make one further remark, concerning the second order (nonlinear) ordinary differential equation characterising the displacement of the solid X_S in (3.5b). Let us recall the definition of the functional $\mathcal{F}[\mathcal{U}](t, Y, Z)$ introduced in Section 2.4.1.

$$\mathcal{F}[\mathcal{U}](t, Y, Z) = -\frac{c_{fric}}{\sqrt{\mu}} \left| 1 + \frac{c_{solid} - 1}{\varepsilon} + \frac{1}{\tilde{M}\varepsilon} \int_{\text{supp}(b)+Y} \mathcal{U}_0 dx \right| \frac{Z}{|Z| + \bar{\delta}} + \frac{\varepsilon}{\tilde{M}} \int_{\mathbb{R}^d} \mathcal{U}_0 \nabla_x \mathbf{b}(x - Y) dx.$$

The coupled system (3.5) has the following equivalent form

$$\begin{cases} D_\mu \partial_t \mathcal{U} + \sum_{j=1}^d A_j(\mathcal{U}, X_S) \partial_j \mathcal{U} + B(\mathcal{U}, X_S) = 0, & (3.9a) \\ \ddot{X}_S = \mathcal{F}[\mathcal{U}](t, X_S, \dot{X}_S). & (3.9b) \end{cases}$$

3.4 A priori estimate for the coupled Boussinesq system

In this part we present the energy estimate in a Sobolev-type function space for the coupled system (3.5). This estimate is based on classical methods (Grönwall type inequalities), but for an energy functional adapted to the fluid-solid system. In the nonlinear Saint-Venant regime, we constructed an iterative scheme for the system which provided the necessary tools to deduce a local in time existence theorem. The heart of the proof was the energy estimate established on the linearised PDE system (Proposition 2.2) and a separate velocity estimate (Lemma 2.5.) for the solid system. Due to the additional dispersive

term as well as a more complicated source term on the right hand side of system (3.5a), a refined analysis of the coupling terms is necessary.

One additional remark concerns the time of existence of the system. We aim for a long time existence result, which involves the parameter ε . This scale was not present in the previous section since for the Saint-Venant regime (SV), we made use of the additional hypothesis of $\varepsilon = 1$. However this implies that in the Boussinesq regime (BOUS) more careful estimates are needed; we establish an existence time over a large $\mathcal{O}\left(\frac{1}{\sqrt{\varepsilon}}\right)$ scale, while standard methods only provide an $\mathcal{O}(1)$ existence time when the bottom is moving, because of the $\mathcal{O}(1)$ source term $\partial_t b$ in the first equation of (3.1). It is however smaller than the $\mathcal{O}\left(\frac{1}{\varepsilon}\right)$ scale for a fixed bottom ([SX12, Bur16]).

By introducing the wave-structure energy functional

$$E_B(t) = \frac{1}{2} \int_{\mathbb{R}^d} \zeta^2 dx + \frac{1}{2} \int_{\mathbb{R}^d} h(\bar{V} \cdot \bar{V}) dx + \frac{1}{2} \int_{\mathbb{R}^d} \frac{\mu}{3} h(\nabla_x \cdot \bar{V})^2 dx + \frac{1}{2\varepsilon} |\dot{X}_S|^2, \quad (3.10)$$

we can establish first of all an L^2 type energy estimate for the coupled system (3.5), from which we will be able to deduce a certain control on the velocity of the solid.

So, we have the following:

Proposition 3.1. *Let $\mu \ll 1$ sufficiently small. Then any $\mathcal{U} \in \mathcal{C}^1([0, T] \times \mathbb{R}^d)$, $X_S \in \mathcal{C}^1([0, T])$ satisfying the coupled system (3.9) (or equivalently (3.5)), with initial data $\mathcal{U}(0, \cdot) = \mathcal{U}_{in} \in \mathcal{X}^0(\mathbb{R}^d)$ and $(X_S(0), \dot{X}_S(0)) = (0, v_{S_0}) \in \mathbb{R}^d \times \mathbb{R}^d$ verifies the energy estimate*

$$\sup_{t \in [0, T]} \left\{ e^{-\sqrt{\varepsilon} c_0 t} E_B(t) \right\} \leq E_B(0), \quad (3.11)$$

where

$$c_0 = c(\|\mathcal{U}\|_{T, W^{1, \infty}}, \|b\|_{W^{3, \infty}}).$$

Proof: We follow the standard steps of a general energy estimate, adapted for the Boussinesq system with moving bottom, paying close attention to the parameters. We start by multiplying the first equation of (3.5a) by ζ , and the second equation by $h\bar{V}$, after which we integrate on \mathbb{R}^d with respect to the space variable x . This yields the following system

$$\begin{cases} \int_{\mathbb{R}^d} \partial_t \zeta \zeta dx + \int_{\mathbb{R}^d} \nabla_x \cdot (h\bar{V}) \zeta dx = - \int_{\mathbb{R}^d} \zeta \nabla_x \mathbf{b}(x - X_S) dx \cdot \dot{X}_S, \\ \int_{\mathbb{R}^d} h \left(1 - \frac{\mu}{3} \Delta_x\right) \partial_t \bar{V} \cdot \bar{V} dx + \int_{\mathbb{R}^d} h \nabla_x \zeta \bar{V} dx + \varepsilon \int_{\mathbb{R}^d} h(\bar{V} \cdot \nabla_x) \bar{V} \cdot \bar{V} dx = \\ - \frac{\mu}{2} \int_{\mathbb{R}^d} h \nabla_x \left(\nabla_x^2 \mathbf{b}(x - X_S) \dot{X}_S \cdot \dot{X}_S\right) \cdot \bar{V} dx + \frac{\mu}{2} \int_{\mathbb{R}^d} h \nabla_x^2 \mathbf{b}(x - X_S) \dot{X}_S \cdot \bar{V} dx. \end{cases}$$

Our main interest is the terms on the right hand side that represent the coupling in the source term, for the rest we shall reason briefly, since those estimates are part of the classical analysis.

The time derivative term of the second equation can be reformulated by integration by parts in the following way

$$\begin{aligned} \int_{\mathbb{R}^d} h \left(1 - \frac{\mu}{3} \Delta_x\right) \partial_t \bar{V} \cdot \bar{V} dx &= \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} h(\bar{V} \cdot \bar{V}) dx + \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} \frac{\mu}{3} h(\nabla_x \cdot \bar{V})^2 dx \\ &\quad - \frac{1}{2} \int_{\mathbb{R}^d} \partial_t h \left(\bar{V} \cdot \bar{V} + \frac{\mu}{3} (\nabla_x \cdot \bar{V})^2\right) dx + \frac{\mu}{3} \int_{\mathbb{R}^d} (\nabla_x \cdot \partial_t \bar{V}) \nabla_x h \cdot \bar{V} dx. \end{aligned}$$

For the first equation, by making use of an integration by parts as well as equation (3.5b) on the

right hand side , we get

$$\left\{ \begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} \zeta^2 dx + \frac{\tilde{M}}{2} \frac{1}{\varepsilon} \frac{d}{dt} |\dot{X}_S|^2 + \int_{\mathbb{R}^d} \varepsilon \nabla_x \zeta \cdot \bar{V} \zeta dx + \int_{\mathbb{R}^d} h(\nabla_x \cdot \bar{V}) \zeta dx = \\ & \varepsilon \int_{\mathbb{R}^d} \nabla_x \mathbf{b}(x - X_S) \cdot \bar{V} \zeta dx - \frac{\tilde{M} c_{fric}}{\varepsilon \sqrt{\mu}} \left| 1 + \frac{c_{solid} - 1}{\varepsilon} + \frac{1}{\tilde{M} \varepsilon} \int_{I(t)} \zeta dx \right| \frac{|\dot{X}_S|^2}{|\dot{X}_S| + \tilde{\delta}}, \\ & \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} h(\bar{V} \cdot \bar{V}) dx + \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} \frac{\mu}{3} h(\nabla_x \cdot \bar{V})^2 dx \\ & - \frac{1}{2} \int_{\mathbb{R}^d} \partial_t h \left(\bar{V} \cdot \bar{V} + \frac{\mu}{3} (\nabla_x \cdot \bar{V})^2 \right) dx + \frac{\mu}{3} \int_{\mathbb{R}^d} (\nabla_x \cdot \partial_t \bar{V}) \nabla_x h \cdot \bar{V} dx \\ & + \int_{\mathbb{R}^d} h \nabla_x \zeta \bar{V} dx + \varepsilon \int_{\mathbb{R}^d} h(\bar{V} \cdot \nabla_x) \bar{V} \cdot \bar{V} dx = \\ & - \frac{\mu}{2} \int_{\mathbb{R}^d} h \nabla_x \left(\nabla_x^2 \mathbf{b}(x - X_S) \dot{X}_S \cdot \dot{X}_S \right) \cdot \bar{V} dx + \frac{\mu}{2} \int_{\mathbb{R}^d} h \nabla_x^2 \mathbf{b}(x - X_S) \ddot{X}_S \cdot \bar{V} dx. \end{aligned} \right.$$

Notice that by equation (3.5b), we have been able to put part of the contribution associated to the source term $\partial_t b$ as a component of the energy $E_B(t)$ on the left hand side of the first equation. This is crucial to get an extended existence time. Moreover, on the right hand side, a now nonpositive friction term appeared that can be easily controled.

Now we add together these two equations and in what follows, by making use of term by term estimates, we arrive to a Grönwall-type inequality concerning the energy functional $E_B(t)$ (for $0 \leq t \leq T$) which then allows us to properly conclude the demonstration. Hence we are left with

$$\frac{d}{dt} E_B(t) = A_B + B_B + C_B + D_B + F_B + G_B, \quad (3.12)$$

where

$$A_B := \frac{1}{2} \int_{\mathbb{R}^d} \partial_t h \left(\bar{V} \cdot \bar{V} + \frac{\mu}{3} (\nabla_x \cdot \bar{V})^2 \right) dx, \quad (3.13)$$

$$B_B := -\frac{\mu}{3} \int_{\mathbb{R}^d} (\nabla_x \cdot \partial_t \bar{V}) \nabla_x h \cdot \bar{V} dx, \quad (3.14)$$

$$C_B := -\int_{\mathbb{R}^d} \varepsilon \nabla_x \zeta \cdot \bar{V} \zeta dx - \int_{\mathbb{R}^d} h(\nabla_x \cdot \bar{V}) \zeta dx - \int_{\mathbb{R}^d} h \nabla_x \zeta \bar{V} dx - \varepsilon \int_{\mathbb{R}^d} h(\bar{V} \cdot \nabla_x) \bar{V} \cdot \bar{V} dx, \quad (3.15)$$

$$D_B := \varepsilon \int_{\mathbb{R}^d} \nabla_x \mathbf{b}(x - X_S) \cdot \bar{V} \zeta dx - \frac{\tilde{M} c_{fric}}{\varepsilon \sqrt{\mu}} \left| 1 + \frac{c_{solid} - 1}{\varepsilon} + \frac{1}{\tilde{M} \varepsilon} \int_{I(t)} \zeta dx \right| \frac{|\dot{X}_S|^2}{|\dot{X}_S| + \tilde{\delta}}, \quad (3.16)$$

$$F_B := -\frac{\mu}{2} \int_{\mathbb{R}^d} h \nabla_x \left(\nabla_x^2 \mathbf{b}(x - X_S) \dot{X}_S \cdot \dot{X}_S \right) \cdot \bar{V} dx, \quad (3.17)$$

$$G_B := \frac{\mu}{2} \int_{\mathbb{R}^d} h \nabla_x^2 \mathbf{b}(x - X_S) \ddot{X}_S \cdot \bar{V} dx. \quad (3.18)$$

Now we proceed to estimate each term on the right hand side. By making use of the first equation of the Boussinesq system (3.5), namely that

$$\partial_t h = \varepsilon \nabla_x \cdot (h \bar{V}),$$

we can establish that

$$A_B \leq \varepsilon c(\|\mathcal{U}\|_{W^{1,\infty}}, \|\mathbf{b}\|_{W^{1,\infty}}) \left(\|\bar{V}\|_{L^2}^2 + \frac{\mu}{3} \|\nabla_x \cdot \bar{V}\|_{L^2}^2 \right).$$

As for the term B_B , we make use of the second equation of the system (3.5), which gives

$$\begin{aligned} \frac{\mu}{3} \nabla_x \cdot \partial_t \bar{V} &= \left(1 - \frac{\mu}{3} \Delta_x\right)^{-1} \left(-\frac{\mu}{3} \Delta_x\right) \zeta + \left(1 - \frac{\mu}{3} \Delta_x\right)^{-1} \varepsilon \left(-\frac{\mu}{3} \nabla_x \cdot ((\bar{V} \cdot \nabla_x) \bar{V})\right) \\ &\quad + \frac{\mu}{2} \left(1 - \frac{\mu}{3} \Delta_x\right)^{-1} \left(-\frac{\mu}{3} \Delta_x\right) \partial_t^2 b. \end{aligned}$$

Given the fact that $(1 - \frac{\mu}{3} \Delta_x)^{-1} (-\frac{\mu}{3} \Delta_x)$ is a zeroth order differential operator whose symbol is uniformly bounded with respect to μ , this gives rise to a norm estimate of the following form

$$\left\| \frac{\mu}{3} \nabla_x \cdot \partial_t \bar{V} \right\|_{L^2} \lesssim \|\zeta\|_{L^2} + \varepsilon c(\|\mathcal{U}\|_{L^\infty}) \|\bar{V}\|_{L^2} + \frac{\mu}{2} \|\partial_t^2 b\|_{L^2}. \quad (3.19)$$

Thus, by making use of the chain rule for $\partial_t^2 b$, we obtain

$$\begin{aligned} B_B &\leq c(\|\mathcal{U}\|_{W^{1,\infty}}, \|\mathbf{b}\|_{W^{1,\infty}}) \left[\varepsilon \|\zeta\|_{L^2} \|\bar{V}\|_{L^2} + \varepsilon^2 c(\|\mathcal{U}\|_{L^\infty}) \|\bar{V}\|_{L^2}^2 + \left| \dot{X}_S \right|^2 \right] \\ &\quad + \mu c(\|\mathcal{U}\|_{W^{1,\infty}}) \int_{\mathbb{R}^d} \nabla_x^2 \mathbf{b}(x - X_S) \ddot{X}_S \cdot \bar{V} dx \end{aligned}$$

The integrals incorporating the nonlinear space derivative terms correspond to

$$C_B = \sum_{j=1}^d \int_{\mathbb{R}^d} S A_j(\mathcal{U}) \partial_j \mathcal{U} \cdot \mathcal{U} dx = -\frac{1}{2} \sum_{j=1}^d \int_{\mathbb{R}^d} \partial_j (S A_j)(\mathcal{U}) \mathcal{U} \cdot \mathcal{U} dx,$$

to which we can easily find an upper bound, giving

$$C_B \leq \varepsilon c(\|\mathcal{U}\|_{W^{1,\infty}}, \|\mathbf{b}\|_{W^{1,\infty}}) \|\mathcal{U}\|_{L^2}^2.$$

For the first two source terms for the system, basic L^∞ -norm estimates and Cauchy-Schwartz inequalities provide the necessary means to conclude

$$\begin{aligned} D_B &\leq \varepsilon c(\|\mathbf{b}\|_{W^{1,\infty}}) \|\zeta\|_{L^2} \|\bar{V}\|_{L^2} + 0, \\ F_B &\leq \mu c(\|\mathcal{U}\|_{W^{1,\infty}}, \|\mathbf{b}\|_{W^{3,\infty}}) \left| \dot{X}_S \right|^2. \end{aligned}$$

We remark that the friction term can be straightforwardly bounded above by 0 in the estimate for D_B .

We leave the last source term, G_B as it is due to the presence of $\ddot{X}_S(t)$, according to equation (3.5b), it requires some attention to avoid problems arising from the friction part.

So, to sum up the previous estimates, we get that

$$\begin{aligned} \frac{d}{dt} E_B(t) &\leq \varepsilon c(\|\mathcal{U}\|_{W^{1,\infty}}, \|\mathbf{b}\|_{W^{1,\infty}}) \|\mathcal{U}\|_{L^2}^2 \\ &\quad + \varepsilon c(\|\mathcal{U}\|_{W^{1,\infty}}, \|\mathbf{b}\|_{W^{1,\infty}}) \left(\|\bar{V}\|_{L^2}^2 + \frac{\mu}{3} \|\nabla_x \cdot \bar{V}\|_{L^2}^2 \right) \\ &\quad + \varepsilon c(\|\mathcal{U}\|_{W^{1,\infty}}, \|\mathbf{b}\|_{W^{1,\infty}}) \left[\|\zeta\|_{L^2} \|\bar{V}\|_{L^2} + \varepsilon c(\|\mathcal{U}\|_{L^\infty}) \|\bar{V}\|_{L^2}^2 \right] \\ &\quad + \varepsilon c(\|\mathbf{b}\|_{W^{1,\infty}}) \|\zeta\|_{L^2} \|\bar{V}\|_{L^2} + 0 + \mu c(\|\mathcal{U}\|_{W^{1,\infty}}, \|\mathbf{b}\|_{W^{3,\infty}}) \left| \dot{X}_S \right|^2 \\ &\quad + \mu c(\|\mathcal{U}\|_{W^{1,\infty}}) \int_{\mathbb{R}^d} \nabla_x^2 \mathbf{b}(x - X_S) \ddot{X}_S \cdot \bar{V} dx. \end{aligned}$$

Here we keep intact the last source term since by simply plugging equation (3.5b) in it would yield to asymptotically singular terms.

So we may deduce that

$$\frac{d}{dt}E_B(t) \leq \varepsilon c(\|\mathcal{U}\|_{W^{1,\infty}}, \|\mathbf{b}\|_{W^{3,\infty}})E_B(t) + \mu c(\|\mathcal{U}\|_{W^{1,\infty}}) \int_{\mathbb{R}^d} \nabla_x^2 \mathbf{b}(x - X_S) \ddot{X}_S \cdot \bar{V} dx \quad (3.20)$$

from which by integrating with respect to the time variable t (keeping in mind that $0 \leq t \leq T$), we obtain

$$E_B(t) - E_B(0) \leq \varepsilon c_0 \int_0^t E_B(\tau) d\tau + \mu c(\|\mathcal{U}\|_{T,W^{1,\infty}}) \int_{\mathbb{R}^d} \int_0^t \nabla_x^2 \mathbf{b}(x - X_S(\tau)) \ddot{X}_S(\tau) \cdot \bar{V} d\tau dx, \quad (3.21)$$

where we made use of the additional notation for the constant

$$c_0 = c(\|\mathcal{U}\|_{T,W^{1,\infty}}, \|\mathbf{b}\|_{W^{3,\infty}}).$$

Lemma 3.2. *The remaining source term satisfies the following estimate for all $0 \leq t \leq T$,*

$$\begin{aligned} \mu \int_{\mathbb{R}^d} \int_0^t \nabla_x^2 \mathbf{b}(x - X_S(\tau)) \ddot{X}_S(\tau) \cdot \bar{V} d\tau dx &\leq \mu c(\|\mathbf{b}\|_{W^{2,\infty}})(E_B(t) + E_B(0)) \\ &+ \sqrt{\varepsilon} c(\|\mathbf{b}\|_{W^{3,\infty}}) \int_0^t E_B(\tau) d\tau. \end{aligned} \quad (3.22)$$

Proof: To handle the remaining source term, we apply an integration by parts in the time variable. This yields

$$\begin{aligned} \mu \int_{\mathbb{R}^d} \int_0^t \nabla_x^2 \mathbf{b}(x - X_S) \ddot{X}_S \cdot \bar{V} d\tau dx &= \\ &+ \mu \int_{\mathbb{R}^d} \nabla_x^2 \mathbf{b}(x - X_S) \dot{X}_S \cdot \bar{V}(t, x) dx - \mu \int_{\mathbb{R}^d} \nabla_x^2 \mathbf{b}(x - 0) v_{S_0} \cdot \bar{V}_{in}(x) dx \\ &- \mu \int_0^t \int_{\mathbb{R}^d} \nabla_x^2 \partial_t \mathbf{b}(x - X_S(\tau)) \dot{X}_S(\tau) \cdot \bar{V} dx d\tau - \mu \int_0^t \int_{\mathbb{R}^d} \nabla_x^2 \mathbf{b}(x - X_S(\tau)) \dot{X}_S(\tau) \cdot \partial_t \bar{V} dx d\tau. \end{aligned}$$

The first two boundary terms can be estimated similarly, so we simply obtain

$$\mu \nabla_x^2 \mathbf{b}(x - X_S) \dot{X}_S \cdot \bar{V}(t, x) dx \leq \mu c(\|\mathbf{b}\|_{W^{2,\infty}}) E_B(t),$$

and we can deduce an identical estimation for the initial data. Since we assume that μ is sufficiently small, this estimate with the energy term will be absorbed by the energy term on the left hand side of (3.21).

Based on the chain rule evoked for $\partial_t \mathbf{b}$ in Section 3.2, we have that

$$\mu \int_0^t \int_{\mathbb{R}^d} \nabla_x^2 \partial_t \mathbf{b}(x - X_S(\tau)) \dot{X}_S(\tau) \cdot \bar{V} dx d\tau \leq \mu \sqrt{\varepsilon} c(\|\mathbf{b}\|_{W^{3,\infty}}) \int_0^t E_B(\tau) d\tau.$$

Finally, by an integration by parts with respect to the spatial variable, we get that

$$\int_0^t \int_{\mathbb{R}^d} \nabla_x^2 \mathbf{b}(x - X_S(\tau)) \dot{X}_S(\tau) \cdot \partial_t \bar{V} dx d\tau = \int_0^t \int_{\mathbb{R}^d} \nabla_x \mathbf{b}(x - X_S(\tau)) \cdot \dot{X}_S(\tau) (\nabla_x \cdot \partial_t \bar{V}) dx d\tau,$$

from which, using the definition of $E_B(t)$ we deduce that

$$\begin{aligned} \mu \int_0^t \int_{\mathbb{R}^d} \nabla_x^2 \mathbf{b}(x - X_S(\tau)) \dot{X}_S(\tau) \cdot \partial_t \bar{V} \, dx \, d\tau &\leq \sqrt{\varepsilon} c(\|\mathbf{b}\|_{W^{2,\infty}}) \int_0^t \int_{\mathbb{R}^d} E_B^{1/2}(\tau) \left(\frac{\mu}{3} \nabla_x \cdot \partial_t \bar{V} \right) \, dx \, d\tau \\ &\leq \sqrt{\varepsilon} c(\|\mathbf{b}\|_{W^{2,\infty}}) \int_0^t E_B^{1/2}(\tau) \left(\|\zeta\|_{L^2} + \varepsilon c(\|\mathcal{U}\|_{L^\infty}) \|\bar{V}\|_{L^2} + \frac{\mu}{2} \|\partial_t^2 b\|_{L^2} \right) \, d\tau \\ &\leq \sqrt{\varepsilon} c(\|\mathbf{b}\|_{W^{2,\infty}}) \int_0^t E_B(\tau) \, d\tau, \end{aligned}$$

where we made use of our previous observation on $\frac{\mu}{3} \nabla_x \cdot \partial_t \bar{V}$ (inequality (3.19)), which in turn allows us to conclude this lemma. \square

So by Lemma 3.2 and inequality (3.21), we obtain that for μ sufficiently small

$$E_B(t) \leq 2E_B(0) + \sqrt{\varepsilon} \tilde{c}_0 \int_0^t E_B(\tau) \, d\tau, \quad (3.23)$$

with the constant

$$\tilde{c}_0 = c(\|\mathcal{U}\|_{T, W^{1,\infty}}, \|\mathbf{b}\|_{W^{3,\infty}}).$$

Thus, by Grönwall's inequality, we can conclude the energy estimate. \square

This concludes the L^2 -estimates (case $s = 0$). Let us mention some consequences concerning the velocity of the solid.

Corollary 3.1. *This energy estimate provides us with a natural control on the solid velocity, namely*

$$\sup_{t \in [0, T]} \left\{ e^{-\sqrt{\varepsilon} c_0 t} \left| \dot{X}_S(t) \right|^2 \right\} \leq \varepsilon \|\mathcal{U}_{in}\|_{\mathcal{X}^0}^2 + |v_{S_0}|^2, \quad (3.24)$$

where

$$c_0 = c(\|\mathcal{U}\|_{T, W^{1,\infty}}, \|b\|_{W^{3,\infty}}).$$

This implies that the solid velocity stays bounded on a $\mathcal{O}\left(\frac{1}{\sqrt{\varepsilon}}\right)$ timescale as long as c_0 stays bounded.

Remark 3.4. *Following the steps of Lemma 2.5. we would have obtained the velocity estimate*

$$\left| \dot{X}(t) \right| \leq |v_{S_0}| + \varepsilon \frac{\|\mathcal{U}\|_{L^\infty} \|b\|_{W^{1,\infty}}}{\tilde{M}} t, \quad (3.25)$$

which is a worse estimate than the one presented in the previous corollary and it cannot be used to obtain an extended existence time.

Remark 3.5. *By the identity*

$$\dot{X}_S(t) = \sqrt{\varepsilon} \frac{\dot{X}_S(t)}{\sqrt{\varepsilon}},$$

following the same reasoning as the one presented in the demonstration of Proposition 3.1. up until Corollary 3.1, it is easy to see that if the initial velocity is of order $\sqrt{\varepsilon}$, that is $\frac{v_{S_0}}{\sqrt{\varepsilon}}$ is uniformly bounded in μ and ε , then the scaled solid velocity $\frac{1}{\sqrt{\varepsilon}} \dot{X}_S(t)$ stays uniformly bounded. Moreover, as it can be seen from the velocity estimate (3.24), this uniform bound is valid up until a time of order $\mathcal{O}\left(\frac{1}{\sqrt{\varepsilon}}\right)$ as long as c_0 remains bounded.

For higher order energy estimates we are going to make use of this estimate and the differential operator $\Lambda^s = (1 - \Delta_x)^{s/2}$. The energy functional associated to these estimates writes as

$$E_B^s(t) = \frac{1}{2} \int_{\mathbb{R}^d} (\Lambda^s \zeta)^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^d} h(\Lambda^s \bar{V} \cdot \Lambda^s \bar{V}) \, dx + \frac{1}{2} \int_{\mathbb{R}^d} \frac{\mu}{3} h(\nabla_x \cdot \Lambda^s \bar{V})^2 \, dx + \frac{1}{2\varepsilon} \left| \dot{X}_S \right|^2.$$

Due to the special structure of our system, let us define the following adapted Banach-space to provide a uniformly formulated energy estimate.

Definition 3.1. *The Sobolev-type space \mathcal{X}^s is given by*

$$\mathcal{X}^s(\mathbb{R}^d) = \left\{ \mathcal{U} = (\zeta, \bar{V}) \in L^2(\mathbb{R}^d) \text{ such that } \|\mathcal{U}\|_{\mathcal{X}^s} < \infty \right\},$$

where

$$\|\mathcal{U}\|_{\mathcal{X}^s} = \|\zeta\|_{H^s} + \|\bar{V}\|_{H^s} + \sqrt{\mu} \|\nabla_x \cdot \bar{V}\|_{H^s}.$$

The last term in the \mathcal{X}^s norm appeared due to the necessity to control the dispersive smoothing through $\sqrt{\mu}$ times the divergence.

We have to modify certain parts of the proof, due to the fact that some of the cancellations used above cease to work anymore. More precisely, we have that

Proposition 3.2. *Let $\mu \ll 1$ sufficiently small and let us take $s \in \mathbb{R}$ with $s > d/2 + 1$. Let us take $\mathcal{U} \in \mathcal{C}([0, T]; \mathcal{X}^s(\mathbb{R}^d)) \cap C^1([0, T]; \mathcal{X}^{s-1}(\mathbb{R}^d))$, $X_S \in \mathcal{C}^1([0, T])$ satisfying the coupled system (3.9) (or equivalently (3.5)), with initial data $\mathcal{U}(0, \cdot) \in \mathcal{X}^s(\mathbb{R}^d)$ and $(X_S(0), \dot{X}_S(0)) = (0, \sqrt{\varepsilon} V_{S_0}) \in \mathbb{R}^d$. Then \mathcal{U} , X_S verifies the energy estimate*

$$\sup_{t \in [0, T]} \left\{ e^{-\sqrt{\varepsilon} c_s t} \left(\|\mathcal{U}\|_{\mathcal{X}^s}^2 + \frac{1}{\varepsilon} |\dot{X}_S|^2 \right) \right\} \leq \|\mathcal{U}(0, \cdot)\|_{\mathcal{X}^s}^2 + |V_{S_0}|^2 + \sqrt{\varepsilon} T \|\nabla_x b\|_{H^s}^2, \quad (3.26)$$

where

$$c_s = c(\|\mathcal{U}\|_{T, H^s}, \|b\|_{H^{s+3}}, \|\mathcal{U}_{in}\|_{\mathcal{X}^0}, |V_{S_0}|).$$

Remark 3.6. *Notice that taking into account the coupling effect for the a priori estimate assured that the constant in the exponential stays of order $\sqrt{\varepsilon}$, which guarantees a proper control on the fluid velocity over a time $\mathcal{O}(\frac{1}{\sqrt{\varepsilon}})$, which is better than what the general theory would imply for a time dependent bottom variation.*

Proof: We start by applying the operator Λ^s on the symmetrized equation ((3.9) multiplied by $S(\mathcal{U}, X_S)$), and we would like to use the techniques presented for the case of $s = 0$, treating $\Lambda^s \mathcal{U}$ as our new unknown. Thus we are left with

$$\begin{aligned} S(\mathcal{U}, X_S) D_\mu \partial_t \Lambda^s \mathcal{U} + \sum_{j=1}^d S A_j(\mathcal{U}, X_S) \partial_j \Lambda^s \mathcal{U} + \Lambda^s B(\mathcal{U}, X_S) \\ + [\Lambda^s, S(\mathcal{U}, X_S)] D_\mu \partial_t \mathcal{U} + \sum_{j=1}^d [\Lambda^s, S A_j(\mathcal{U}, X_S)] \partial_j \mathcal{U} = 0. \end{aligned} \quad (3.27)$$

Notice the presence of the additional commutator terms in the equation.

Our main idea is the same as before, after multiplying the equation by $\Lambda^s \mathcal{U}$ and intergating over \mathbb{R}^d , we make use of similar estimates as in the first part for the L^2 estimate to obtain a Grönwall type inequality for the corresponding modified energy functional $E_B^s(t)$.

For the first two terms of our new equation, which correspond to the time derivative and nonlinear terms of the original equation (3.32a), they may be treated similarly as before, obtaining the same estimates with the same constants, only for H^s -norm instead of L^2 -norm.

The main difference is the presence of the commutators in equation (3.27) due to Λ^s , and the treatment of the source term since the cancellation obtained by using the ODE (3.5b) does not work anymore. We

will make use of the well-known Kato-Ponce inequality (for this we have $s > 0$) as well as Sobolev-embedding results (for these, the condition $s > d/2 + 1$ is necessary) to establish commutator estimates. Namely, we have that for $f \in H^s$, $g \in H^{s-1}$

$$\|[\Lambda_s, f]g\|_{L^2} \lesssim \|f\|_{H^s} \|g\|_{L^\infty} + \|f\|_{W^{1,\infty}} \|g\|_{H^{s-1}} \lesssim \|f\|_{H^s} \|g\|_{H^{s-1}},$$

the latter inequality coming from the embedding $H^s(\mathbb{R}^d) \hookrightarrow W^{1,\infty}(\mathbb{R}^d)$. We also have that for $f \in H^s$, $g \in H^s$, and $s > d/2$

$$\|\Lambda_s(fg)\|_{L^2} \lesssim \|f\|_{H^s} \|g\|_{H^s}.$$

So, by the decomposition of the symmetrizer matrix, we may write that

$$\begin{aligned} \left| \int_{\mathbb{R}^d} [\Lambda^s, S(\mathcal{U}, X_S)] D_\mu \partial_t \mathcal{U} \cdot \Lambda^s \mathcal{U} \, dx \right| &= \left| \int_{\mathbb{R}^d} [\Lambda^s, \varepsilon \bar{S}(\mathcal{U}, X_S)] D_\mu \partial_t \mathcal{U} \cdot \Lambda^s \mathcal{U} \, dx \right| \\ &\lesssim \varepsilon (\|\mathcal{U}\|_{H^s} + \|\mathbf{b}\|_{H^s}) \left\| \sum_{j=1}^d S A_j(\mathcal{U}, X_S) \partial_j \mathcal{U} + S B(\mathcal{U}, X_S) \right\|_{H^{s-1}} \cdot \|\mathcal{U}\|_{H^s} \\ &\lesssim \varepsilon c(\|\mathcal{U}\|_{H^s}, \|\mathbf{b}\|_{H^s}) \|\mathcal{U}\|_{H^s}^2 + \varepsilon c(\|\mathcal{U}\|_{H^s}, \|\mathbf{b}\|_{H^s}) \|S B(\mathcal{U}, X_S)\|_{H^s} \|\mathcal{U}\|_{H^s}. \end{aligned}$$

Here we made use of equation (3.9) to handle the time derivative. The additional term will be absorbed by the source term in equation (3.27).

Additionally we have that

$$\begin{aligned} \left| \int_{\mathbb{R}^d} \sum_{j=1}^d [\Lambda^s, S A_j(\mathcal{U}, X_S)] \partial_j \mathcal{U} \cdot \Lambda^s \mathcal{U} \, dx \right| &= \left| \int_{\mathbb{R}^d} \sum_{j=1}^d [\Lambda^s, \varepsilon S \bar{A}_j(\mathcal{U}, X_S)] \partial_j \mathcal{U} \cdot \Lambda^s \mathcal{U} \, dx \right| \\ &\lesssim \varepsilon \sum_{j=1}^d \left\| S \bar{A}_j(\mathcal{U}, X_S) \right\|_{H^s} \|\partial_j \mathcal{U}\|_{H^{s-1}} \|\mathcal{U}\|_{H^s} \lesssim \varepsilon c(\|\mathcal{U}\|_{H^s}, \|\mathbf{b}\|_{H^s}) \|\mathcal{U}\|_{H^s}^2. \end{aligned}$$

In both these commutator estimates, we made use of the fact that the constant diagonal matrix component cancels out in the commutator, by definition.

Finally, attention has to be paid to the source term too, since for instance now we can't apply the ODE (3.5b) to treat the original right hand side of the first equation due to the presence of the operator Λ^s . Thus we are left with

$$\begin{aligned} \int_{\mathbb{R}^d} \Lambda^s \zeta \Lambda^s \nabla_x \mathbf{b}(x - X_S) \, dx \cdot \dot{X}_S &\leq \sqrt{\varepsilon} (\|\mathcal{U}_{in}\|_{\mathcal{X}^0} + |V_{S_0}|) \|\zeta\|_{H^s} \|\nabla_x \mathbf{b}\|_{H^s} \\ &\leq \sqrt{\varepsilon} c(\|\mathcal{U}_{in}\|_{\mathcal{X}^0}, |v_{S_0}|) \|\zeta\|_{H^s}^2 + \sqrt{\varepsilon} \|\nabla_x \mathbf{b}\|_{H^s}^2. \end{aligned} \tag{3.28}$$

Here we made use of Remark 3.5. and the smallness assumption on the initial velocity of the solid. Notice that it is at this point that we can no longer use the cancellation, thus loosing a smallness factor.

Moreover, we are required to estimate terms which involve the operator Λ^s applied to a product, this is handled by the commutator estimates, giving us

$$\varepsilon \int_{\mathbb{R}^d} \Lambda^s (\nabla_x \mathbf{b}(x - X_S) \cdot \bar{V}) \Lambda^s \zeta \, dx \leq \varepsilon c(\|\mathbf{b}\|_{H^{s+1}}) \|\mathcal{U}\|_{H^s}^2,$$

and with a simple upper bound, we have

$$\frac{\mu}{2} \int_{\mathbb{R}^d} h \Lambda^s \nabla_x \left(\nabla_x^2 \mathbf{b}(x - X_S) \dot{X}_S \cdot \dot{X}_S \right) \cdot \Lambda^s \bar{V} \, dx \leq \mu c(\|\mathcal{U}\|_{H^s}, \|\mathbf{b}\|_{H^{s+3}}) \left| \dot{X}_S \right|^2.$$

Finally, just as with Lemma 3.2. we can deduce the following estimate

$$\begin{aligned}
\mu c(\|\mathcal{U}\|_{T,H^s}) \int_{\mathbb{R}^d} \int_0^t \Lambda^s \nabla_x^2 \mathbf{b}(x - X_S(\tau)) \ddot{X}_S(\tau) \cdot \Lambda^s \bar{V} \, d\tau \, dx \leq \\
+ \mu c(\|\mathcal{U}\|_{T,H^s}, \|\mathbf{b}\|_{H^{s+2}}) (E_B^s(t) + E_B^s(0)) \\
+ \sqrt{\varepsilon^2} c(\|\mathcal{U}\|_{T,H^s}, \|\mathbf{b}\|_{H^{s+3}}) \int_0^t E_B^s(\tau) \, d\tau.
\end{aligned} \tag{3.29}$$

To sum it up, after an integration with respect to the time variable, with the definition of the energy functional $E_B^s(t)$ and the velocity estimate obtained from the L^2 estimate (Corollary 3.1.), we may write that

$$E_B^s(t) \leq 3E_B^s(0) + \sqrt{\varepsilon} t \|\nabla_x b\|_{H^s}^2 + \sqrt{\varepsilon} \tilde{c}_s \int_0^t E_B^s(\tau) \, d\tau, \tag{3.30}$$

with the constant

$$\tilde{c}_s = c(\|\mathcal{U}\|_{T,H^s}, \|\mathbf{b}\|_{H^{s+3}}, \|\mathcal{U}_{in}\|_{\mathcal{X}^0}, |V_{S_0}|).$$

So we have the right terms in order to complete the estimation, again with Grönwall's lemma. \square

3.5 Local in time existence theorem

The energy estimate allows us to establish the main existence theorem for the coupled Boussinesq system, which states as follows

Theorem 3.1. *Let us consider the coupled system defined by equations (3.5). Let us suppose that for the initial value ζ_{in} and \mathbf{b} the lower bound condition (1.1)*

$$\exists h_{min} > 0, \forall X \in \mathbb{R}^d, 1 + \varepsilon \zeta(X) - \varepsilon \mathbf{b}(X) \geq h_{min} \tag{3.31}$$

is satisfied. If the initial values ζ_{in} and \bar{V}_{in} are in $\mathcal{X}^s(\mathbb{R}^d)$ with $s \in \mathbb{R}$, $s > d/2 + 1$, and $V_{S_0} \in \mathbb{R}^d$ then there exists a maximal $T_0 > 0$ independent of ε such that there is a unique solution

$$\begin{aligned}
(\zeta, \bar{V}) &\in C\left(\left[0, \frac{T_0}{\sqrt{\varepsilon}}\right]; \mathcal{X}^s(\mathbb{R}^d)\right) \cap C^1\left(\left[0, \frac{T_0}{\sqrt{\varepsilon}}\right]; \mathcal{X}^{s-1}(\mathbb{R}^d)\right), \\
X_S &\in C^1\left(\left[0, \frac{T_0}{\sqrt{\varepsilon}}\right]\right)
\end{aligned}$$

with uniformly bounded norms for the system (3.5) with initial conditions $(\zeta_{in}, \bar{V}_{in})$ and $(0, \sqrt{\varepsilon} V_{S_0})$.

Proof: For this demonstration we shall follow the footsteps of a classical Friedrichs type reasoning for (in general) symmetric hyperbolic systems, found for example in chapter 16 of [Tay97]. The reason for this is has already been evoked in the previous section, an iterative scheme is not adapted to the nonlinear coupled Boussinesq system because it destroys the cancellation of the coupling terms in the energy estimates. With a carefully chosen Friedrichs smoothing of the equations, these cancellations can be preserved.

1. A regularized system: We shall first of all regularize the system with the help of the Friedrichs mollifier J_δ .

Definition 3.2. *For every $u \in L^2(\mathbb{R}^d)$ we have that*

$$\widehat{J_\delta u}(\xi) = \varphi(\delta\xi) \hat{u}(\xi) \quad \forall \xi \in \mathbb{R}^d,$$

with φ a regular real valued even function defined on \mathbb{R}^d with compact support, such that $\varphi(0) = 1$.

A slightly modified classical property of the mollifier entails the followings

Lemma 3.3. (1) For every $s, t \in \mathbb{R}$, the operator J_δ acts from \mathcal{X}^s onto \mathcal{X}^t , moreover there exists a constant $C(s, t, \delta)$ such that

$$\|J_\delta u\|_{\mathcal{X}^t} \leq C(s, t, \delta) \|u\|_{\mathcal{X}^s} \quad \forall u \in \mathcal{X}^s.$$

(2) J_δ as a linear operator is continuous every $L^p(\mathbb{R}^d)$, $1 \leq p \leq \infty$, furthermore for all $u \in L^p(\mathbb{R}^d)$

$$\|J_\delta u\|_{L^p} \leq C \|u\|_{L^p}$$

with a constant C independent of δ .

Using the mollifier, we propose the following symmetric regularized system

$$\begin{cases} S(J_\delta \mathcal{U}^\delta, X_S^\delta) D_\mu \partial_t \mathcal{U}^\delta + \sum_{j=1}^d J_\delta S(J_\delta \mathcal{U}^\delta, X_S^\delta) A_j(J_\delta \mathcal{U}^\delta, X_S^\delta) J_\delta \partial_j \mathcal{U}^\delta = & (3.32a) \\ \quad \quad \quad = S(J_\delta \mathcal{U}^\delta, X_S^\delta) J_\delta B(J_\delta \mathcal{U}^\delta, X_S^\delta), \\ \ddot{X}_S^\delta(t) = \mathcal{F}[J_\delta \mathcal{U}_0^\delta] \left(t, X_S^\delta, \dot{X}_S^\delta \right), & (3.32b) \\ \mathcal{U}^\delta(0, \cdot) = \mathcal{U}_{in}, \left(X_S^\delta, \dot{X}_S^\delta \right) (0) = (0, v_{S_0}). \end{cases}$$

Based on Lemma 3.3, we may deduce that the regularized system (3.32) is in fact an ODE (in the Fourier space) on any \mathcal{X}^s Banach-space, the regularization assures that the nonlinear operator on the right hand side in its canonical form is regular thus uniformly Lipschitz and continuous in time. So by the Picard–Lindelöf theorem we may deduce that there exists a solution $\mathcal{U}^\delta \in C([0, T_\delta]; \mathcal{X}^s)$ and $X_S^\delta \in C^1([0, T_\delta])$.

2. A priori estimate for the regularization: Following the steps of the a priori estimates proved in the previous section, the estimation in Proposition 3.2 holds for our regularized system as well, since by the careful choice of regularization in system (3.32) the cancellations are preserved. So we have that

$$\|\mathcal{U}^\delta\|_{\mathcal{X}^s}^2 + \frac{1}{\varepsilon} \left| \dot{X}_S^\delta(t) \right|^2 \leq e^{\lambda t} \left(\|\mathcal{U}_{in}\|_{\mathcal{X}^s}^2 + |v_{S_0}|^2 + \sqrt{\varepsilon} t \|\nabla_x b\|_{H^s}^2 \right), \quad (3.33)$$

with $\lambda = \sqrt{\varepsilon} c_s$, $t \in [0, T_\delta]$.

3. Uniformization of the time interval: Here the hypothesis $s > d/2 + 1$ is important since we want to make use of the Sobolev embedding $H^s \hookrightarrow W^{1, \infty}$.

Lemma 3.4. The regularized problem (3.32) has a solution on $\left[0, \frac{T_0}{\sqrt{\varepsilon}}\right]$ with T_0 independent of δ and ε .

We have an estimate of the form

$$\frac{d}{dt} E_B^s(\mathcal{U}^\delta, X_S^\delta)(t) \leq \sqrt{\varepsilon} c (\|\mathcal{U}^\delta\|_{H^s}, \|\mathbf{b}\|_{W^{3, \infty}}, \|\mathcal{U}_{in}\|_{\mathcal{X}^0}, |V_{S_0}|) E_B^s(\mathcal{U}^\delta, X_S^\delta)(t) + \sqrt{\varepsilon} \|\nabla_x b\|_{H^s},$$

just before using Grönwall's lemma in the higher order energy estimates. By a change of variable of the time parameter of the form $t = \frac{t'}{\sqrt{\varepsilon}}$, it is clear to see that we have an uniform upper bound for a long time regime (with the variable t'). This implies that solutions to the regularized system (3.32) exist for a time $\frac{T_{0, \delta}}{\sqrt{\varepsilon}}$ with $T_{0, \delta}$ independent of ε .

Furthermore, we have that for every $\tilde{T} > 0$, $\forall t \in \left[0, \min\left\{\tilde{T}, \frac{T_{0,\delta}}{\sqrt{\varepsilon}}\right\}\right]$

$$\frac{d}{dt} \left(\|\mathcal{U}^\delta(t, \cdot)\|_{\mathcal{X}^s}^2 + \frac{1}{\varepsilon} \left| \dot{X}_S^\delta(t) \right|^2 \right) \leq F \left(\|\mathcal{U}^\delta(t, \cdot)\|_{\mathcal{X}^s}^2 + \frac{1}{\varepsilon} \left| \dot{X}_S^\delta(t) \right|^2 \right) \quad (3.34)$$

with F being a regular function independent of δ . By the Picard–Lindelöf theorem, there exists $T_0 > 0$ ($\frac{T_0}{\sqrt{\varepsilon}} \leq \tilde{T}$) such that the ordinary differential equation

$$\begin{cases} y'(t) = F(y(t)) \\ y(0) = \|\mathcal{U}_{in}\|_{\mathcal{X}^s}^2 + |V_{S_0}|^2 \end{cases}$$

has a unique solution on $\left[0, \frac{T_0}{\sqrt{\varepsilon}}\right]$. By Grönwall's lemma and a standard comparison theorem for ODEs we can deduce that

$$\|\mathcal{U}^\delta(t, \cdot)\|_{\mathcal{X}^s}^2 + \frac{1}{\varepsilon} \left| \dot{X}_S^\delta(t) \right|^2 \leq y(t).$$

□

4. Convergence: Let us define the following supplementary function space

$$E_{T_0}^s = L^\infty \left(\left[0, \frac{T_0}{\sqrt{\varepsilon}}\right]; \mathcal{X}^s \right) \cap W^{1,\infty} \left(\left[0, \frac{T_0}{\sqrt{\varepsilon}}\right]; \mathcal{X}^{s-1} \right).$$

Solutions \mathcal{U}^δ of the regularized problem clearly belong to $E_{T_0}^s$ with $X_S^\delta \in W^{1,\infty}$. Thus, the family $\{\mathcal{U}^\delta\}_\delta$ is bounded in $E_{T_0}^s$ so it has a weakly convergent subsequence in $E_{T_0}^s$ towards a function $\mathcal{U} \in E_{T_0}^s$.

Since the inclusion $H_{loc}^s(\mathbb{R}^d) \hookrightarrow H_{loc}^{s-1}(\mathbb{R}^d)$ is compact, by the Arzela–Ascoli theorem we may extract a strongly convergent subsequence from it in $C \left(\left[0, \frac{T_0}{\sqrt{\varepsilon}}\right]; \mathcal{X}_{loc}^{s-1} \right)$ locally.

By interpolation inequalities we have that $\mathcal{U} \in C^\sigma \left(\left[0, \frac{T_0}{\sqrt{\varepsilon}}\right]; \mathcal{X}_{loc}^{s-\sigma} \right)$ for each $\sigma \in (0, 1)$. Moreover, since the inclusion $H^{s-\sigma}(\mathbb{R}^d) \hookrightarrow C^1(\mathbb{R}^d)$ is also compact for sufficiently small $\sigma > 0$, we may deduce that the subsequence is converging in $C \left(\left[0, \frac{T_0}{\sqrt{\varepsilon}}\right]; C_{loc}^1(\mathbb{R}^d) \right)$, with X_S^δ converging in $C^1 \left[0, \frac{T_0}{\sqrt{\varepsilon}}\right]$. With this subsequence we shall no problem in passing to the limit in δ for the regularized system (3.32) in the sense of distributions, leading to a solution of the problem in $E_{T_0}^s$.

5. Additional regularity: In fact, by being more careful with the estimates, we may deduce that the solution \mathcal{U} is in

$$C \left(\left[0, \frac{T_0}{\sqrt{\varepsilon}}\right]; \mathcal{X}^s \right) \cap C^1 \left(\left[0, \frac{T_0}{\sqrt{\varepsilon}}\right]; \mathcal{X}^{s-1} \right).$$

Essentially, the main idea is to prove that the norm $\|\mathcal{U}(t, \cdot)\|_{\mathcal{X}^s}$ is in fact a (Lipschitz-)continuous function of t , since it is the limit of $\|J_\delta \mathcal{U}(t, \cdot)\|_{\mathcal{X}^s}$. For more details, we refer to Chapter 16 of [Tay97].

6. Uniqueness: By taking the difference of two solutions for the system (3.5), they consequently satisfy a similar system, thus by the previous a priori estimate, with 0 right hand side, we may conclude that this difference has to be 0 as well. □

This concludes the proof of the well-posedness theorem concerning the coupled Boussinesq system (3.5). As one can clearly see from the demonstration, Remark 3.6. on the nature of the time of existence stays valid, so solutions are assured over a time of order $\mathcal{O} \left(\frac{1}{\sqrt{\varepsilon}} \right)$.

4 Conclusion

In the present paper, we established a coupled physical model of the water waves problem with a freely moving object on the bottom of the fluid domain. We deduced the exact coupled system and analysed two different shallow water asymptotic regimes (with respect to the shallowness parameter μ): the nonlinear Saint-Venant system and the Boussinesq system. We established local in time existence results as well as a uniqueness theorem for both cases and we improved the existence time for the weakly nonlinear Boussinesq regime.

Another possible approach would be to consider the full Green–Naghdi system for the $\mathcal{O}(\mu^2)$ asymptotic regime and establish the coupled system and possibly well-posedness results for it. This would yield to a nonhydrostatic pressure formula, and consequently a more complex equation for the solid motion, incorporating the added mass effect which is not present in our asymptotic regimes.

Even a more general scenario can be envisioned, that is to handle the full problem formulated in the first section, treating the coupled problem (1.24) with (1.28).

To complement the theoretical results, a numerical study is to follow this article in order to verify the applicability of the system as well as to compare it with other existing methods to treat wave-structure interaction problems ([DNZ15], [ACDn17]).

Contents

1	Introduction	1
1.1	The case of a fixed bottom	3
1.1.1	The free surface Euler equations	4
1.1.2	The free surface Bernoulli equations	4
1.1.3	The Zakharov / Craig–Sulem formulation	5
1.2	The case of a moving bottom with prescribed motion	5
1.2.1	The Euler / Bernoulli frameworks	6
1.2.2	The Zakharov / Craig–Sulem framework	6
1.3	The coupled problem with a freely moving object	7
1.4	Dimensionless form of the equations	10
1.4.1	The different scales of the problem	10
1.4.2	Nondimensionalised equations	12
2	The $\mathcal{O}(\mu)$ asymptotic regime: The nonlinear Saint-Venant equations	13
2.1	Asymptotics for the fluid model	13
2.2	Formal derivation of a first order asymptotic equation for the solid motion	14
2.3	The wave-structure interaction problem at first order	15
2.4	Local in time existence of the solution	15
2.4.1	Reformulation of the coupled system	16
2.4.2	The iterative scheme	17
2.4.3	Existence and a priori estimates	17
2.4.4	Convergence	22
3	The $\mathcal{O}(\mu^2)$ asymptotic regime: The Boussinesq system	24
3.1	Formal derivation of the corresponding solid motion equation	25
3.2	The coupled Boussinesq system	25
3.3	A reformulation of the coupled system	26
3.4	A priori estimate for the coupled Boussinesq system	27
3.5	Local in time existence theorem	35
4	Conclusion	38

List of Figures

1	The water waves setting for a fixed bottom	3
2	The water waves setting for a moving bottom	6
3	The coupled water waves setting in the presence of a solid	8
4	The characteristic scales of the coupled water waves problem	11

References

- [ABZ11] Thomas Alazard, Nicolas Burq, and Claude Zuily. On the Cauchy problem for the water waves with surface tension. *Duke Mathematical Journal*, 158:413–499, 2011.
- [ABZ14] Thomas Alazard, Nicolas Burq, and Claude Zuily. On the Cauchy problem for gravity water waves. *Inventiones Mathematicae*, 198:71–163, 2014.
- [ACDn17] Stéphane Abadie, Marcela Cruchaga, Benoit Ducassou, and Jonathan Nuñez. A fictitious domain approach based on a viscosity penalty method to simulate wave/structure interaction. *Journal of Hydraulic Research*, pages 1–16, 2017.
- [AELS14] Reza Alam, Ryan Elandt, Marcus Lehmann, and Mostafa Shakeri. The Wave Carpet: Development of a Submerged Pressure Differential Wave Energy Converter. 2014. 30th Symposium on Naval Hydrodynamics, Hobart, Australia.
- [AG07] Serge Alinhac and Patrick Gérard. *Pseudo-differential operators and the Nash–Moser theorem*, volume 82 of *Graduate Studies in Mathematics*. American Mathematical Society, 2007.
- [ASL08a] Borys Alvarez-Samaniego and David Lannes. Large time existence for 3d water-waves and asymptotics. *Inventiones Mathematicae*, 171:485–541, 2008.
- [ASL08b] Borys Alvarez-Samaniego and David Lannes. A Nash–Moser theorem for singular evolution equations. Applications to the Serre and Green–Naghdi equations. *Indiana University Mathematics Journal*, 57:97–131, 2008.
- [BCL05] Jerry Lloyd Bona, Thierry Colin, and David Lannes. Long wave approximation for water waves. *Archive for Rational Mechanics and Analysis*, 178:373–410, 2005.
- [BCS04] Jerry Lloyd Bona, Min Chen, and Jean-Claud Saut. Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media. II: The nonlinear theory. *Nonlinearity*, 17:925–952, 2004.
- [Ber06] Jean-Marie Berthelot. *Mécanique des solides rigides*. Tec and Doc Lavoisier, 2006.
- [Bur16] Cosmin Burtea. New long time existence results for a class of Boussinesq-type systems. *Journal de Mathématiques Pures et Appliquées*, 109(2):203–236, 2016.
- [Cha07] Florent Chazel. Influence of bottom topography on long water waves. *ESAIM: Mathematical Modeling and Numerical Analysis*, 41(4):771–799, 2007.
- [Che03] Min Chen. Equations for bi-directional waves over an uneven bottom. *Mathematics and Computers in Simulation*, 62:3–9, 2003.
- [CLS12] Walter Craig, David Lannes, and Catherine Sulem. Water waves over a rough bottom in the shallow water regime. *Annales de l’Institut Henri Poincaré (C) Non Linear Analysis*, 29:233–259, 2012.
- [CM06] Georges-Henri Cottet and Emmanuel Maitre. A level set method for fluid-structure interactions with immersed surfaces. *Mathematical Models and Methods in Applied Sciences*, 16: 415–438, 2006.

- [Cra85] Walter Craig. An existence theory for water waves and the Boussinesq and Korteweg–de Vries scaling limits. *Communications in Partial Differential Equations*, 10(8):787–1003, 1985.
- [CS93] Walter Craig and Catherine Sulem. Numerical simulation of gravity waves. *Journal of Computational Physics*, 108:73–83, 1993.
- [CSS92] Walter Craig, Catherine Sulem, and Pierre-Louis Sulem. Nonlinear modulation of gravity waves: a rigorous approach. *Nonlinearity*, 5:497–522, 1992.
- [DNZ15] Denys Dutykh, Hayk Nersisyan, and Enrique Zuazua. Generation of two-dimensional water waves by moving bottom disturbances. *IMA Journal of Applied Mathematics*, 80(4):1235–1253, 2015.
- [dP16] Thibault de Poyferré. A priori estimates for water waves with emerging bottom. 2016. arXiv:1612.04103.
- [GIL⁺14] Mateus das Neves Gomes, Liércio André Isoldi, Max Letzow, Luiz Alberto Oliveira Rocha, Elizaldo Domingues dos Santos, Flávio Medeiros Seibt, and Jeferson Avila Souza. Computational modeling applied to the study of wave energy converters (WEC). *Marine Systems and Ocean Technology*, 9:77–84, 2014.
- [GKSW95] Stephan T. Grilli, James T. Kirby, Ravishankar Subramanya, and Ge Wei. A fully nonlinear Boussinesq model for surface waves. Part 1. Highly nonlinear unsteady waves. *Journal of Fluid Mechanics*, 294:71–92, 1995.
- [HI15] Fujiwara Hiroyasu and Tatsuo Iguchi. A shallow water approximation for water waves over a moving bottom. *Advanced studies in pure mathematics*, 64:77–88, 2015.
- [Igu09] Tatsuo Iguchi. A shallow water approximation for water waves. *Journal of Mathematics of Kyoto University*, 49(1):13–55, 2009.
- [Igu11] Tatsuo Iguchi. A mathematical analysis of tsunami generation in shallow water due to seabed deformation. *Proceedings of the Royal Society of Edinburgh Section A: Mathematics*, 141:551–608, 2011.
- [KN86] Tadayoshi Kano and Takaaki Nishida. A mathematical justification for Korteweg–de Vries equation and Boussinesq equation of water surface waves. *Osaka Journal of Mathematics*, 23(2):389–413, 1986.
- [Lan05] David Lannes. Well-posedness of the water-waves equations. *Journal of the American Mathematical Society*, 18(3):605–654, 2005.
- [Lan13] David Lannes. *The water waves problem: mathematical analysis and asymptotics*, volume 188 of *Mathematical Surveys and Monographs*. American Mathematical Society, 2013.
- [Lan17] David Lannes. On the dynamics of floating structures. *Annals of Partial Differential Equations*, 3(1), 2017.
- [Li06] Yi A. Li. A shallow-water approximation to the full water wave problem. *Communications on Pure and Applied Mathematics*, 59(9):1225–1285, 2006.
- [LM17] David Lannes and Guy Métivier. The shoreline problem for the one-dimensional shallow water and Green–Naghdi equations. 2017. HAL Archives Ouvertes, <hal-01614321>.

- [Mel15] Benjamin Melinand. A mathematical study of meteo and landside tsunamis: the Proudman resonance. *Nonlinearity*, 28:4037–4080, 2015.
- [Mit09] Dimitrios E. Mitsotakis. Boussinesq systems in two space dimensions over a variable bottom for the generation and propagation of tsunami waves. *Mathematics and Computers in Simulation*, 80(4):860–873, 2009.
- [Ovs74] Lev V. Ovsjannikov. To the shallow water theory foundation. *Archives of Mechanics*, 26:407–422, 1974.
- [Per67] D. Howell Peregrine. Long waves on a beach. *Journal of Fluid Mechanics*, 27(4):815–827, 1967.
- [PR83] George C. Papanicolaou and Rodolfo R. Rosales. Gravity waves in a channel with a rough bottom. *Studies in Applied Mathematics*, 68:89–102, 1983.
- [SW00] Guido Schneider and C. Eugene Wayne. The long-wave limit for the water wave problem I. The case of zero surface tension. *Communications on Pure and Applied Mathematics*, 53(12):1475–1535, 2000.
- [SX12] Jean-Claude Saut and Li Xu. The Cauchy problem on large time for surface waves Boussinesq systems. *Journal de Mathématiques Pures et Appliquées*, 97(6):635–662, 2012.
- [Tay97] Michael E. Taylor. *Partial Differential Equations: Nonlinear Equations*, volume 117 of *Applied Mathematical Sciences*. Springer, 1997.
- [TW92] Michelle H. Teng and Theodore Y. Wu. Nonlinear water waves in channels of arbitrary shape. *Journal of Fluid Mechanics*, 242:211–233, 1992.
- [Wu87] Theodore Y. Wu. Generation of upstream advancing solitons by moving disturbances. *Journal of Fluid Mechanics*, 184:75–99, 1987.
- [Wu97] Sijue Wu. Well-posedness in Sobolev spaces of the full water wave problem in 2-D. *Inventiones Mathematicae*, 130(1):39–72, 1997.
- [Wu99] Sijue Wu. Well-posedness in Sobolev spaces of the full water wave problem in 3-D. *Journal of the American Mathematical Society*, 12(2):445–495, 1999.
- [Zak68] Vladimir E. Zakharov. Stability of periodic water waves of finite amplitude on the surface of a deep fluid. *Journal of Applied Mechanics and Technical Physics*, 9:190–194, 1968.