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# Newtonian fluid flow in a thin porous medium with a non-homogeneous Fourier boundary condition

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## Abstract

We consider the Stokes system in a thin porous medium  $\Omega_\varepsilon$  of thickness  $\varepsilon$  which is perforated by periodically distributed solid cylinders of size  $\varepsilon$ . On the boundary of the cylinders we prescribe non-homogeneous Fourier boundary condition depending on a parameter  $\gamma$ . The aim is to give the asymptotic behavior of the velocity and the pressure of the fluid as  $\varepsilon$  goes to zero. Using an adaptation of the unfolding method, we give, following the values of  $\gamma$ , different limit systems.

**AMS classification numbers:** 76A20, 76M50, 35B27.

**Keywords:** Homogenization; Stokes system; Darcy's law; thin porous medium; Non-homogeneous Fourier boundary condition.

## 1 Introduction

We consider a viscous fluid obeying the Stokes system in a thin porous medium  $\Omega_\varepsilon$  of thickness  $\varepsilon$  which is perforated by periodically distributed solid cylinders of size  $\varepsilon$ . On the boundary of the solid cylinders, we prescribe a non-homogeneous Fourier boundary condition depending on a parameter  $\gamma \in \mathbb{R}$ . The aim of this work is to prove the convergence of the homogenization process depending on the different values of  $\gamma$ .

**The domain:** the periodic porous medium is defined by a domain  $\omega$  and an associated microstructure, or periodic cell  $Y' = [-1/2, 1/2]^2$ , which is made of two complementary parts: the fluid part  $Y'_f$ , and the solid part  $T'$  ( $Y'_f \cup T' = Y'$  and  $Y'_f \cap T' = \emptyset$ ). More precisely, we assume that  $\omega$  is a smooth, bounded, connected set in  $\mathbb{R}^2$ , and that  $T'$  is an open connected subset of  $Y'$  with a smooth boundary  $\partial T'$ , such that  $\bar{T}'$  is strictly included in  $Y'$ .

The microscale of a porous medium is a small positive number  $\varepsilon$ . The domain  $\omega$  is covered by a regular mesh of square of size  $\varepsilon$ : for  $k' \in \mathbb{Z}^2$ , each cell  $Y'_{k',\varepsilon} = \varepsilon k' + \varepsilon Y'$  is divided in a fluid part  $Y'_{f_{k',\varepsilon}}$  and a solid part  $T'_{k',\varepsilon}$ , i.e. is similar to the unit cell  $Y'$  rescaled to size  $\varepsilon$ . We define  $Y = Y' \times (0, 1) \subset \mathbb{R}^3$ , which is divided in a fluid part  $Y_f = Y'_f \times (0, 1)$  and a solid part  $T = T' \times (0, 1)$ , and consequently  $Y_{k',\varepsilon} = Y'_{k',\varepsilon} \times (0, 1) \subset \mathbb{R}^3$ , which is also divided in a fluid part  $Y_{f_{k',\varepsilon}}$  and a solid part  $T_{k',\varepsilon}$ .

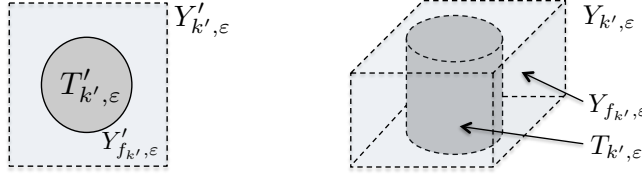


Figure 1: Views of a periodic cell in 2D (left) and 3D (right)

The fluid part  $\omega_\varepsilon \subset \mathbb{R}^2$  of a porous medium is defined by  $\omega_\varepsilon = \omega \setminus \bigcup_{k' \in \mathcal{K}_\varepsilon} \bar{T}'_{k',\varepsilon}$ , where  $\mathcal{K}_\varepsilon = \{k' \in \mathbb{Z}^2 : Y'_{k',\varepsilon} \cap \omega \neq \emptyset\}$ .

The whole fluid part  $\Omega_\varepsilon \subset \mathbb{R}^3$  in the thin porous medium is defined by (see Figures 3 and 2)

$$\Omega_\varepsilon = \{(x_1, x_2, x_3) \in \omega_\varepsilon \times \mathbb{R} : 0 < x_3 < \varepsilon\}. \quad (1.1)$$

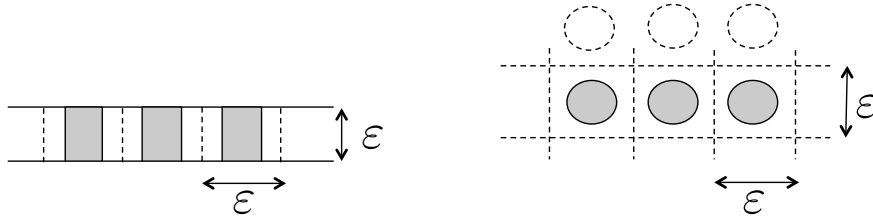


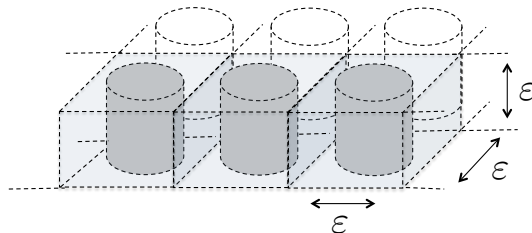
Figure 2: Views from lateral (left) and from above (right)

We denote by  $S_\varepsilon$  the set of the solid cylinders contained in  $\Omega_\varepsilon$ , i.e.  $S_\varepsilon = \bigcup_{k' \in \mathcal{K}_\varepsilon} T'_{k',\varepsilon} \times (0, \varepsilon)$ .

We define

$$\tilde{\Omega}_\varepsilon = \omega_\varepsilon \times (0, 1), \quad \Omega = \omega \times (0, 1), \quad \Lambda_\varepsilon = \omega \times (0, \varepsilon). \quad (1.2)$$

We observe that  $\tilde{\Omega}_\varepsilon = \Omega \setminus \bigcup_{k' \in \mathcal{K}_\varepsilon} \bar{T}'_{k',\varepsilon}$ , and we define  $T_\varepsilon = \bigcup_{k' \in \mathcal{K}_\varepsilon} T_{k',\varepsilon}$  as the set of the solid cylinders contained in  $\tilde{\Omega}_\varepsilon$ .


 Figure 3: View of the domain  $\Omega_\varepsilon$ 

**The problem:** let us consider the following Stokes system in  $\Omega_\varepsilon$ , with a Dirichlet boundary condition on the exterior boundary  $\partial\Lambda_\varepsilon$  and a non-homogeneous Fourier boundary condition on the cylinders  $\partial S_\varepsilon$ :

$$\left\{ \begin{array}{ll} -\mu\Delta u_\varepsilon + \nabla p_\varepsilon = f_\varepsilon & \text{in } \Omega_\varepsilon, \\ \operatorname{div} u_\varepsilon = 0 & \text{in } \Omega_\varepsilon, \\ u_\varepsilon = 0 & \text{on } \partial\Lambda_\varepsilon, \\ -p_\varepsilon \cdot n + \mu \frac{\partial u_\varepsilon}{\partial n} + \alpha \varepsilon^\gamma u_\varepsilon = g_\varepsilon & \text{on } \partial S_\varepsilon, \end{array} \right. \quad (1.3)$$

where  $u_\varepsilon = (u_{\varepsilon,1}, u_{\varepsilon,2}, u_{\varepsilon,3})$  denotes the velocity field,  $p_\varepsilon$  is the (scalar) pressure,  $f_\varepsilon = (f_{\varepsilon,1}(x_1, x_2), f_{\varepsilon,2}(x_1, x_2), 0)$  is the field of exterior body force and  $g_\varepsilon = (g_{\varepsilon,1}(x_1, x_2), g_{\varepsilon,2}(x_1, x_2), 0)$  is the field of exterior surface forces. The constants  $\alpha$  and  $\gamma$  are given, with  $\alpha \geq 0$ ,  $\mu$  is the viscosity and  $n$  is the outward normal to  $S_\varepsilon$ .

This choice of  $f$  and  $g$  is usual when we deal with thin domains. Since the thickness of the domain,  $\varepsilon$ , is small then vertical component of the forces can be neglected and, moreover the force can be considered independent of vertical variable.

The boundary condition on  $\partial S_\varepsilon$  means that the stress vector gives rise to a breaking phenomenon due to the presence of the term  $\alpha \varepsilon^\gamma u_\varepsilon$  and to a proportionality with the exterior surface forces due to the presence of  $g_\varepsilon$  (see [1] for more details).

Problem (1.3) models in particular the flow of an incompressible viscous fluid through a thin porous medium under the action of an exterior electric field. This system is derived from a physical model well detailed in the literature (see Cioranescu *et al.* [1], Sanchez-Palencia [2] and references therein for more details).

The Stokes problem in a periodically perforated domain with holes of the same size as the periodic has been treated in the literature. More precisely, with Dirichlet conditions on the boundary of the holes was studied by Ene and Sanchez-Palencia [3], where the limit law describing the homogenized medium is a Darcy's law. And with non-homogeneous Fourier boundary condition was studied by Cioranescu *et al.* [1], where using the variational method introduced by Tartar [4], a Darcy's law, a Brinkmann-type equation or a Stokes-type equation are obtained depending of the valued of  $\gamma$ .

The Stokes and Navier-Stokes equations in a perforated domain with holes of size  $r_\varepsilon$ , with  $r_\varepsilon \ll \varepsilon$  is considered by Allaire [5]. On the boundary of the holes, the normal component of the velocity is equal to zero and the tangential velocity is proportional to the tangential component of the normal stress. The type of the limit law is determined by the size  $r_\varepsilon$ , i.e. by the geometry of the domain.

The earlier results relate to a fixed height domain. For a thin domain, in [6] Anguiano considers a non-stationary incompressible non-Newtonian Stokes system, in a thin porous medium of thickness  $\varepsilon$  that is perforated by periodically distributed solid cylinders of size  $a_\varepsilon$ , with Dirichlet conditions on the boundary of the cylinders. Applying an adaptation of the unfolding method introduced by Cioranescu *et al.* [7], time-dependent Darcy's laws are obtained rigorously. The behavior observed when  $a_\varepsilon \approx \varepsilon$  has motivated the fact of considering non-homogeneous Fourier conditions on the boundary of the cylinders. In this sense, our aim in the present paper is to consider a Newtonian fluid flow, in the thin porous medium (1.1), with a non-homogeneous Fourier boundary condition. We

prove the convergence of the homogenization process depending on the different values of  $\gamma$ . One of the major difficulty in the present paper is to treat the surface integrals. The papers mentioned above about problems with non-homogeneous boundary conditions use a generalization (see Cioranescu and Donato [8]) of the technique introduced by Vanninathan [9] for the Steklov problem, which transforms the surface integrals into volume integrals. An excellent alternative, in our opinion, of this technique was made possible with the development of the unfolding method (see Cioranescu *et al.* [7]), which allows to treat quite elementary in the surface integrals. In the present paper, we extend some abstract results for thin domains, using an adaptation of the unfolding method, in order to treat all the surface integrals and we obtain in this way directly the corresponding homogenized terms. A similar approach is made by Cioranescu *et al.* [10], Zaki [11] and Capatina and Ene [12] for a fixed height domain.

We show that the asymptotic behavior of the system (1.3) depends on the values of  $\gamma$ :

- For  $\gamma < -1$ , we obtained a 2D Darcy's law as an homogenized model. The flow is only driven by the pressure.
- For  $-1 \leq \gamma < 1$ , we also obtain a 2D Darcy's law but in this case the flow depends on the pressure, the external body forces and the mean value of the external surface forces.
- For  $\gamma \geq 1$ , we obtain a 2D Darcy's law where the flow is only driven by the pressure. The flow behaves as if there were not any solid cylinders.

The paper is organized as follows. In Section 2, we formulate the problem and state our main result, which is proved in Section 3.

## 2 Main result

Along this paper, the points  $x \in \mathbb{R}^3$  will be decomposed as  $x = (x', x_3)$  with  $x' \in \mathbb{R}^2$ ,  $x_3 \in \mathbb{R}$ . We also use the notation  $x'$  to denote a generic vector of  $\mathbb{R}^2$ .

In this section we describe the asymptotic behavior of a viscous fluid obeying (1.3) in the geometry  $\Omega_\varepsilon$  described in (1.1). The proof of the corresponding results will be given in the next section.

**The variational formulation:** let us introduce the spaces

$$H_\varepsilon = \{ \varphi \in H^1(\Omega_\varepsilon)^3 : \varphi = 0 \text{ on } \partial\Lambda_\varepsilon \},$$

and

$$\tilde{H}_\varepsilon = \{ \tilde{\varphi} \in H^1(\tilde{\Omega}_\varepsilon)^3 : \tilde{\varphi} = 0 \text{ on } \partial\Omega \}.$$

Then, the variational formulation of system (1.3) is the following one:

$$\left\{ \begin{array}{l} \mu \int_{\Omega_\varepsilon} Du_\varepsilon : D\varphi \, dx - \int_{\Omega_\varepsilon} p_\varepsilon \operatorname{div} \varphi \, dx + \alpha \varepsilon^\gamma \int_{\partial S_\varepsilon} u_\varepsilon \cdot \varphi \, d\sigma(x) = \int_{\Omega_\varepsilon} f'_\varepsilon \cdot \varphi' \, dx + \int_{\partial S_\varepsilon} g'_\varepsilon \cdot \varphi' \, d\sigma(x), \quad \forall \varphi \in H_\varepsilon, \\ \int_{\Omega_\varepsilon} u_\varepsilon \cdot \nabla \psi \, dx = 0, \quad \forall \psi \in \mathcal{D}(\Omega_\varepsilon). \end{array} \right. \quad (2.4)$$

Assume that  $f_\varepsilon \in L^2(\Omega)^2$  and  $g_\varepsilon(x) = g(x'/\varepsilon)$ , where  $g$  is a  $Y'$ -periodic function in  $L^2(\partial T)^2$ . Under these assumptions, it is well known that (2.4) has a unique solution  $(u_\varepsilon, p_\varepsilon) \in H_\varepsilon \times L^2(\Omega_\varepsilon)$ . This solution is unique up to an additive constant for  $p_\varepsilon$ , i.e. it is unique if we consider the corresponding equivalence class:  $p_\varepsilon \in L^2(\Omega_\varepsilon)/\mathbb{R}$  (see [13] for more details).

Our aim is to study the asymptotic behavior of  $u_\varepsilon$  and  $p_\varepsilon$  when  $\varepsilon$  tends to zero. For this purpose, we use the dilatation in the variable  $x_3$ , i.e.

$$y_3 = \frac{x_3}{\varepsilon}, \quad (2.5)$$

in order to have the functions defined in the open set with fixed height  $\tilde{\Omega}_\varepsilon$  defined by (1.2).

Namely, we define  $\tilde{u}_\varepsilon \in \tilde{H}_\varepsilon$ ,  $\tilde{p}_\varepsilon \in L^2(\tilde{\Omega}_\varepsilon)/\mathbb{R}$  by

$$\tilde{u}_\varepsilon(x', y_3) = u_\varepsilon(x', \varepsilon y_3), \quad \tilde{p}_\varepsilon(x', y_3) = p_\varepsilon(x', \varepsilon y_3), \quad a.e. (x', y_3) \in \tilde{\Omega}_\varepsilon.$$

Using the transformation (2.5), the system (1.3) can be rewritten as

$$\begin{cases} -\mu \Delta_{x'} \tilde{u}_\varepsilon - \varepsilon^{-2} \mu \partial_{y_3}^2 \tilde{u}_\varepsilon + \nabla_{x'} \tilde{p}_\varepsilon + \varepsilon^{-1} \partial_{y_3} \tilde{p}_\varepsilon e_3 = f_\varepsilon & \text{in } \tilde{\Omega}_\varepsilon, \\ \operatorname{div}_{x'} \tilde{u}'_\varepsilon + \varepsilon^{-1} \partial_{y_3} \tilde{u}_{\varepsilon,3} = 0 & \text{in } \tilde{\Omega}_\varepsilon, \\ \tilde{u}_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \quad (2.6)$$

with non-homogeneous Fourier boundary condition,

$$-\tilde{p}_\varepsilon \cdot n + \mu \frac{\partial \tilde{u}_\varepsilon}{\partial n} + \alpha \varepsilon^\gamma \tilde{u}_\varepsilon = g_\varepsilon \quad \text{on } \partial T_\varepsilon, \quad (2.7)$$

where  $e_3 = (0, 0, 1)^t$ .

Taking in (2.4) as test function  $\tilde{\varphi}(x', x_3/\varepsilon)$  with  $\tilde{\varphi} \in \tilde{H}_\varepsilon$  and  $\tilde{\psi}(x', x_3/\varepsilon)$  with  $\tilde{\psi} \in \mathcal{D}(\tilde{\Omega}_\varepsilon)$ , applying the change of variables (2.5) and taking into account that  $d\sigma(x) = \varepsilon d\sigma(x') dy_3$ , the variational formulation of system (2.6)-(2.7) is then the following one:

$$\begin{cases} \mu \int_{\tilde{\Omega}_\varepsilon} D_\varepsilon \tilde{u}_\varepsilon : D_\varepsilon \tilde{\varphi} \, dx' dy_3 - \int_{\tilde{\Omega}_\varepsilon} \tilde{p}_\varepsilon \operatorname{div}_\varepsilon \tilde{\varphi} \, dx' dy_3 + \alpha \varepsilon^\gamma \int_{\partial T_\varepsilon} \tilde{u}_\varepsilon \cdot \tilde{\varphi} \, d\sigma(x') dy_3 \\ = \int_{\tilde{\Omega}_\varepsilon} f'_\varepsilon \cdot \tilde{\varphi}' \, dx' dy_3 + \int_{\partial T_\varepsilon} g'_\varepsilon \cdot \tilde{\varphi}' \, d\sigma(x') dy_3, \quad \forall \tilde{\varphi} \in \tilde{H}_\varepsilon, \\ \int_{\tilde{\Omega}_\varepsilon} (\tilde{u}'_\varepsilon \cdot \nabla_{x'} \tilde{\psi} + \varepsilon^{-1} \tilde{u}_{\varepsilon,3} \partial_{y_3} \tilde{\psi}) \, dx = 0, \quad \forall \tilde{\psi} \in \mathcal{D}(\tilde{\Omega}_\varepsilon). \end{cases} \quad (2.8)$$

In the sequel, we assume that the data  $f'_\varepsilon$  satisfies that there exists  $f' \in L^2(\Omega)^2$  such that

$$\varepsilon f'_\varepsilon \rightharpoonup f' \quad \text{weakly in } L^2(\Omega)^2. \quad (2.9)$$

**Main result:** our goal then is to describe the asymptotic behavior of this new sequence  $(\tilde{u}_\varepsilon, \tilde{p}_\varepsilon)$ . The sequence of solutions  $(\tilde{u}_\varepsilon, \tilde{p}_\varepsilon) \in \tilde{H}_\varepsilon \times L^2(\tilde{\Omega}_\varepsilon)/\mathbb{R}$  is not defined in a fixed domain independent of  $\varepsilon$  but rather in a varying set  $\tilde{\Omega}_\varepsilon$ . In order to pass the limit if  $\varepsilon$  tends to zero, convergences in fixed Sobolev spaces (defined in  $\Omega$ ) are used which requires first that  $(\tilde{u}_\varepsilon, \tilde{p}_\varepsilon)$  be extended to the whole domain  $\Omega$ . Then, an extension  $(\tilde{U}_\varepsilon, \tilde{P}_\varepsilon) \in H_0^1(\Omega)^3 \times L^2(\Omega)/\mathbb{R}$  of  $(\tilde{u}_\varepsilon, \tilde{p}_\varepsilon)$  is defined on  $\Omega$  and coincides with  $(\tilde{u}_\varepsilon, \tilde{p}_\varepsilon)$  on  $\tilde{\Omega}_\varepsilon$ .

Our main result referred to the asymptotic behavior of the solution of (2.8) is given by the following theorem.

**Theorem 2.1.** *We distinguish three cases:*

- i) For  $\gamma < -1$ , then  $(\varepsilon^{-1} \tilde{U}_\varepsilon, \varepsilon^{-\gamma} \tilde{P}_\varepsilon)$  converges weakly to  $(\tilde{u}(x', y_3), \tilde{p}(x'))$ , in  $H^1(0, 1; L^2(\omega)^3) \times L^2(\omega)/\mathbb{R}$ , with  $\tilde{u}_3 = 0$ . Moreover, we have that  $(\tilde{v}, \tilde{p})$  satisfies the following Darcy law:

$$\begin{cases} \tilde{v}'(x') = -\frac{|Y'_f|}{\alpha |\partial T'|} \nabla_{x'} \tilde{p}(x') & \text{in } \omega, \\ \tilde{v}_3(x') = 0, \end{cases} \quad (2.10)$$

where  $\tilde{v}(x') = \int_0^1 \tilde{u}(x', y_3) dy_3$ .

ii) For  $-1 \leq \gamma < 1$ , then  $(\varepsilon^\gamma \tilde{U}_\varepsilon, \varepsilon \tilde{P}_\varepsilon)$  converges weakly to  $(\tilde{u}(x', y_3), \tilde{p}(x'))$  in  $H^1(0, 1; L^2(\omega)^3) \times L^2(\omega)/\mathbb{R}$ , with  $\tilde{u}_3 = 0$  and  $\tilde{u}' = 0$  on  $y_3 = \{0, 1\}$ . Moreover, we have that  $(\tilde{v}', \tilde{p})$  satisfies the following Darcy law:

$$\begin{cases} \tilde{v}'(x') = \frac{|Y'_f|}{\alpha |\partial T'|} \left( f' - \nabla_{x'} \tilde{p}(x') + \frac{|\partial T'|}{|Y'|} \mathcal{M}_{\partial T'}[g'] \right) \\ \tilde{v}_3(x') = 0. \end{cases} \quad \text{in } \omega, \quad (2.11)$$

where  $\tilde{v}(x') = \int_0^1 \tilde{u}(x', y_3) dy_3$ .

iii) For  $\gamma \geq 1$ , then  $(\tilde{U}_\varepsilon, \varepsilon^2 \tilde{P}_\varepsilon)$  converges weakly to  $(\tilde{u}(x', y_3), \tilde{p}(x'))$  in  $H^1(0, 1; L^2(\omega)^3) \times L^2(\omega)/\mathbb{R}$ , with  $\tilde{u} = 0$  on  $y_3 = \{0, 1\}$ . Moreover, we have that  $(\tilde{v}', \tilde{p})$  satisfies the following Darcy law:

$$\tilde{v}'(x') = -\frac{1}{12\mu} \nabla_{x'} \tilde{p}(x') \quad \text{in } \omega, \quad (2.12)$$

where  $\tilde{v}'(x') = \int_0^1 \tilde{u}'(x', y_3) dy_3$ .

**Remark 2.2.** We have to point out that in the homogenized problems related to system (2.6)-(2.7), the limit function does not satisfy the incompressibility condition. This is a consequence of the fact that the normal component of  $\tilde{u}_\varepsilon$  does not vanish on the boundary of the solid cylinders.

### 3 Proof of the main result

In this section we prove our main result. In particular, Theorem 2.1 is proved in Subsection 3.3 by means of an adaptation of the unfolding method (see Arbogast *et al.* [14] and Cioranescu *et al.* [7, 10]), which is strongly related to the two-scale convergence method (see Allaire [15] and Nguetseng [16]). To apply this method, *a priori* estimates are established in Subsection 3.1 and some compactness results are proved in Subsection 3.2.

**Some notations:** in order to apply the unfolding method, we need the following notation: for  $k' \in \mathbb{Z}^2$ , we define  $\kappa : \mathbb{R}^2 \rightarrow \mathbb{Z}^2$  by

$$\kappa(x') = k' \iff x' \in Y'_{k', 1}. \quad (3.13)$$

Remark that  $\kappa$  is well defined up to a set of zero measure in  $\mathbb{R}^2$ , which is given by  $\cup_{k' \in \mathbb{Z}^2} \partial Y'_{k', 1}$ . Moreover, for every  $\varepsilon > 0$ , we have

$$\kappa\left(\frac{x'}{\varepsilon}\right) = k' \iff x' \in Y'_{k', \varepsilon}.$$

Now, let us introduce some notation which will be useful in the following: for a vectorial function  $v = (v', v_3)$  and a scalar function  $w$ , we introduce the operators:  $D_\varepsilon$ ,  $\nabla_\varepsilon$  and  $\text{div}_\varepsilon$ , by

$$(D_\varepsilon v)_{i,j} = \partial_{x_j} v_i \text{ for } i = 1, 2, 3, j = 1, 2, \quad (D_\varepsilon v)_{i,3} = \frac{1}{\varepsilon} \partial_{y_3} v_i \text{ for } i = 1, 2, 3,$$

$$\nabla_\varepsilon w = (\nabla_{x'} w, \frac{1}{\varepsilon} \partial_{y_3} w)^t, \quad \text{div}_\varepsilon v = \text{div}_{x'} v' + \frac{1}{\varepsilon} \partial_{y_3} v_3.$$

We denote by  $|\mathcal{O}|$  the Lebesgue measure of  $|\mathcal{O}|$  (3-dimensional if  $\mathcal{O}$  is a 3-dimensional open set, 2-dimensional if  $\mathcal{O}$  is a curve).

For every bounded set  $\mathcal{O}$  and if  $\varphi \in L^1(\mathcal{O})$ , we define the average of  $\varphi$  on  $\mathcal{O}$  by

$$\mathcal{M}_\mathcal{O}[\varphi] = \frac{1}{|\mathcal{O}|} \int_\mathcal{O} \varphi dx. \quad (3.14)$$

Similarly, for every compact set  $K$  of  $Y$ , if  $\varphi \in L^1(\partial K)$  then

$$\mathcal{M}_{\partial K}[\varphi] = \frac{1}{|\partial K|} \int_{\partial K} \varphi d\sigma,$$

is the average of  $\varphi$  over  $\partial K$ .

We denote by  $L_{\#}^2(Y)$ ,  $H_{\#}^1(Y)$ , the functional spaces

$$L_{\#}^2(Y) = \left\{ v \in L_{loc}^2(Y) : \int_Y |v|^2 dy < +\infty, v(y' + k', y_3) = v(y) \quad \forall k' \in \mathbb{Z}^2, \text{ a.e. } y \in Y \right\},$$

and

$$H_{\#}^1(Y) = \left\{ v \in H_{loc}^1(Y) \cap L_{\#}^2(Y) : \int_Y |\nabla_y v|^2 dy < +\infty \right\}.$$

We denote by  $:$  the full contraction of two matrices, i.e. for  $A = (a_{i,j})_{1 \leq i,j \leq 2}$  and  $B = (b_{i,j})_{1 \leq i,j \leq 2}$ , we have  $A : B = \sum_{i,j=1}^2 a_{ij} b_{ij}$ .

Finally, we denote by  $O_\varepsilon$  a generic real sequence, which tends to zero with  $\varepsilon$  and can change from line to line, and by  $C$  a generic positive constant which also can change from line to line.

### 3.1 Some abstract results for thin domains and *a priori* estimates

The *a priori* estimates independent of  $\varepsilon$  for  $\tilde{u}_\varepsilon$  and  $\tilde{p}_\varepsilon$  will be obtained by using an adaptation of the unfolding method.

**Some abstract results for thin domains:** let us introduce the adaption of the unfolding method in which we divide the domain  $\tilde{\Omega}_\varepsilon$  in cubes of lateral lengths  $\varepsilon$  and vertical length 1. For this purpose, given  $\tilde{\varphi} \in L^p(\tilde{\Omega}_\varepsilon)^3$ ,  $1 \leq p < +\infty$ , (assuming  $\tilde{\varphi}$  extended by zero outside of  $\omega_\varepsilon$ ), we define  $\hat{\varphi}_\varepsilon \in L^p(\mathbb{R}^2 \times Y_f)^3$  by

$$\hat{\varphi}_\varepsilon(x', y) = \tilde{\varphi} \left( \varepsilon \kappa \left( \frac{x'}{\varepsilon} \right) + \varepsilon y', y_3 \right), \quad \text{a.e. } (x', y) \in \mathbb{R}^2 \times Y_f, \quad (3.15)$$

where the function  $\kappa$  is defined in (3.13).

**Remark 3.1.** The restriction of  $\hat{\varphi}_\varepsilon$ , to  $Y_{f_k, \varepsilon} \times Y_f$  does not depend on  $x'$ , whereas as a function of  $y$  it is obtained from  $\tilde{v}_\varepsilon$ , by using the change of variables

$$y' = \frac{x' - \varepsilon k'}{\varepsilon}, \quad (3.16)$$

which transforms  $Y_{f_k, \varepsilon}$  into  $Y_f$ .

**Proposition 3.2.** We have the following estimates:

i) for every  $\tilde{\varphi} \in L^p(\tilde{\Omega}_\varepsilon)$ ,  $1 \leq p < +\infty$ , we have

$$\|\hat{\varphi}_\varepsilon\|_{L^p(\mathbb{R}^2 \times Y_f)^3} = |Y'|^{\frac{1}{p}} \|\tilde{\varphi}\|_{L^p(\tilde{\Omega}_\varepsilon)^3}, \quad (3.17)$$

where  $\hat{\varphi}_\varepsilon$  is given by (3.15),

ii) for every  $\tilde{\varphi} \in W^{1,p}(\tilde{\Omega}_\varepsilon)^3$ ,  $1 \leq p < +\infty$ , we have that  $\hat{\varphi}_\varepsilon$ , given by (3.15), belongs to  $L^p(\mathbb{R}^2; W^{1,p}(Y_f)^3)$ , and

$$\|D_y \hat{\varphi}_\varepsilon\|_{L^p(\mathbb{R}^2 \times Y_f)^{3 \times 3}} = \varepsilon |Y'|^{\frac{1}{p}} \|D_\varepsilon \tilde{\varphi}\|_{L^p(\tilde{\Omega}_\varepsilon)^{3 \times 3}}. \quad (3.18)$$



*Proof.* Let us prove *i*). Using Remark 3.1 and definition (3.15), we have

$$\begin{aligned} \int_{\mathbb{R}^2 \times Y_f} |\hat{\varphi}_\varepsilon(x', y)|^p dx' dy &= \sum_{k' \in \mathbb{Z}^2} \int_{Y'_{k', \varepsilon}} \int_{Y_f} |\hat{\varphi}_\varepsilon(x', y)|^p dx' dy \\ &= \sum_{k' \in \mathbb{Z}^2} \int_{Y'_{k', \varepsilon}} \int_{Y_f} |\tilde{\varphi}(\varepsilon k' + \varepsilon y', y_3)|^p dx' dy. \end{aligned}$$

We observe that  $\tilde{\varphi}$  does not depend on  $x'$ , then we can deduce

$$\int_{\mathbb{R}^2 \times Y_f} |\hat{\varphi}_\varepsilon(x', y)|^p dx' dy = \varepsilon^2 |Y'| \sum_{k' \in \mathbb{Z}^2} \int_{Y_f} |\tilde{\varphi}(\varepsilon k' + \varepsilon y', y_3)|^p dy.$$

For every  $k' \in \mathbb{Z}^2$ , by the change of variable (3.16), we have

$$k' + y' = \frac{x'}{\varepsilon}, \quad dy' = \frac{dx'}{\varepsilon^2} \quad \partial_{y'} = \varepsilon \partial_{x'}, \quad (3.19)$$

and we obtain

$$\int_{\mathbb{R}^2 \times Y_f} |\hat{\varphi}_\varepsilon(x', y)|^p dx' dy = |Y'| \int_{\omega_\varepsilon \times (0,1)} |\tilde{\varphi}(x', y_3)|^p dx' dy_3$$

which gives (3.17).

Let us prove *ii*). Taking into account the definition (3.15) of  $\hat{\varphi}_\varepsilon$  and observing that  $\tilde{\varphi}$  does not depend on  $x'$ , then we can deduce

$$\int_{\mathbb{R}^2 \times Y_f} |D_{y'} \hat{\varphi}_\varepsilon(x', y)|^p dx' dy = \varepsilon^2 |Y'| \sum_{k' \in \mathbb{Z}^2} \int_{Y_f} |D_{y'} \tilde{\varphi}(\varepsilon k' + \varepsilon y', y_3)|^p dy.$$

By (3.19), we obtain

$$\begin{aligned} \int_{\mathbb{R}^2 \times Y_f} |D_{y'} \hat{\varphi}_\varepsilon(x', y)|^p dx' dy &= \varepsilon^p |Y'| \sum_{k' \in \mathbb{Z}^2} \int_{Y'_{k', \varepsilon}} \int_0^1 |D_{x'} \tilde{\varphi}(x', y_3)|^p dx' dy_3 \\ &= \varepsilon^p |Y'| \int_{\omega_\varepsilon \times (0,1)} |D_{x'} \tilde{\varphi}(x', y_3)|^p dx' dy_3. \end{aligned} \quad (3.20)$$

For the partial of the vertical variable, proceeding similarly to (3.17), we obtain

$$\begin{aligned} \int_{\mathbb{R}^2 \times Y_f} |\partial_{y_3} \hat{\varphi}_\varepsilon(x', y)|^p dx' dy &= |Y'| \int_{\omega_\varepsilon \times (0,1)} |\partial_{y_3} \tilde{\varphi}(x', y_3)|^p dx' dy_3 \\ &= \varepsilon^p |Y'| \int_{\omega_\varepsilon \times (0,1)} \left| \frac{1}{\varepsilon} \partial_{y_3} \tilde{\varphi}(x', y_3) \right|^p dx' dy_3, \end{aligned}$$

which together with (3.20) gives (3.18).  $\square$

In a similar way, let us introduce the adaption of the unfolding method on the boundary of the solid cylinders  $\partial T_\varepsilon$  (see Cioranescu *et al.* [10] for more details). For this purpose, given  $\tilde{\varphi} \in L^p(\partial T_\varepsilon)^3$ ,  $1 \leq p < +\infty$ , we define  $\hat{\varphi}_\varepsilon^b \in L^p(\mathbb{R}^2 \times \partial T)^3$  by

$$\hat{\varphi}_\varepsilon^b(x', y) = \tilde{\varphi} \left( \varepsilon \kappa \left( \frac{x'}{\varepsilon} \right) + \varepsilon y', y_3 \right), \quad \text{a.e. } (x', y) \in \mathbb{R}^2 \times \partial T, \quad (3.21)$$

where the function  $\kappa$  is defined in (3.13).

**Remark 3.3.** Observe that from this definition, if we consider  $\tilde{\varphi} \in L^p(\partial T)$ ,  $1 \leq p < +\infty$ , a  $Y'$ -periodic function, and we define  $\tilde{\varphi}_\varepsilon(x', y_3) = \tilde{\varphi}(x'/\varepsilon, y_3)$ , it follows that  $\hat{\varphi}_\varepsilon^b(x', y) = \tilde{\varphi}(y)$ .

Observe that for  $\tilde{\varphi} \in W^{1,p}(\tilde{\Omega}_\varepsilon)^3$ ,  $\hat{\varphi}_\varepsilon^b$  is the trace on  $\partial T$  of  $\hat{\varphi}_\varepsilon$ . Therefore  $\hat{\varphi}_\varepsilon^b$  has a similar properties as  $\hat{\varphi}_\varepsilon$ .

We have the following property.

**Proposition 3.4.** *If  $\tilde{\varphi} \in L^p(\partial T_\varepsilon)^3$ ,  $1 \leq p < +\infty$ , then*

$$\|\hat{\varphi}_\varepsilon^b\|_{L^p(\mathbb{R}^2 \times \partial T)^3} = \varepsilon^{\frac{1}{p}} |Y'|^{\frac{1}{p}} \|\tilde{\varphi}\|_{L^p(\partial T_\varepsilon)^3}, \quad (3.22)$$

where  $\hat{\varphi}_\varepsilon^b$  is given by (3.21).

*Proof.* We take  $(x', y_3) \in \partial T_{k', \varepsilon}$ . Taking into account (3.21), we obtain

$$\int_{\mathbb{R}^2 \times \partial T} |\hat{\varphi}_\varepsilon^b(x', y)|^p dx' d\sigma(y) = \varepsilon^2 |Y'| \sum_{k' \in \mathbb{Z}^2} \int_{\partial T} |\tilde{\varphi}(\varepsilon k' + \varepsilon y', y_3)|^p d\sigma(y).$$

For every  $k' \in \mathbb{Z}^2$ , by taking  $x' = \varepsilon(k' + y')$ , we have  $d\sigma(x') = \varepsilon d\sigma(y')$ . Since the thickness of the solid cylinders is one, we have that  $d\sigma(x') dy_3 = \varepsilon d\sigma(y)$ . Hence

$$\int_{\mathbb{R}^2 \times \partial T} |\hat{\varphi}_\varepsilon^b(x', y)|^p dx' d\sigma(y) = \varepsilon |Y'| \int_{\partial T_\varepsilon} |\tilde{\varphi}(x', y_3)|^p d\sigma(x') dy_3,$$

which gives (3.22).  $\square$

Now, let us give two results which will be useful for obtaining *a priori* estimates of the solution  $(\tilde{u}_\varepsilon, \tilde{p}_\varepsilon)$  of problem (2.6)-(2.7). These results are an extension of Cioranescu *et al.* (Proposition 5.3 and Corollary 5.4 in [17]) to the thin domain case.

**Proposition 3.5.** *Let  $g \in L^2(\partial T')^3$  and  $\tilde{\varphi} \in H^1(\tilde{\Omega}_\varepsilon)^3$ , extended by zero in outside of  $w_\varepsilon$ . Let  $\hat{\varphi}_\varepsilon$  be given by (3.15). Then, there exists a positive constant  $C$ , independent of  $\varepsilon$ , such that*

$$\left| \int_{\mathbb{R}^2 \times \partial T} g(y') \cdot \hat{\varphi}_\varepsilon(x', y) dx' d\sigma(y) \right| \leq C \mathcal{M}_{\partial T'}[g] \left( \|\tilde{\varphi}\|_{L^2(\tilde{\Omega}_\varepsilon)^3} + \varepsilon \|D_\varepsilon \tilde{\varphi}\|_{L^2(\tilde{\Omega}_\varepsilon)^{3 \times 3}} \right). \quad (3.23)$$

*In particular, if  $g = 1$ , there exists a positive constant  $C$ , independent of  $\varepsilon$ , such that*

$$\left| \int_{\mathbb{R}^2 \times \partial T} \hat{\varphi}_\varepsilon(x', y) dx' d\sigma(y) \right| \leq C \left( \|\tilde{\varphi}\|_{L^1(\tilde{\Omega}_\varepsilon)^3} + \varepsilon \|D_\varepsilon \tilde{\varphi}\|_{L^1(\tilde{\Omega}_\varepsilon)^{3 \times 3}} \right). \quad (3.24)$$

*Proof.* Due to density properties, it is enough to prove this estimate for functions in  $\mathcal{D}(\mathbb{R}^3)^3$ . Let  $\tilde{\varphi} \in \mathcal{D}(\mathbb{R}^3)^3$ , one has

$$\begin{aligned} & \left| \int_{\mathbb{R}^2 \times \partial T} g(y') \cdot \hat{\varphi}_\varepsilon(x', y) dx' d\sigma(y) \right| \\ &= \left| \int_{\mathbb{R}^2 \times \partial T} g(y') \cdot \tilde{\varphi} \left( \varepsilon \kappa \left( \frac{x'}{\varepsilon} \right) + \varepsilon y', y_3 \right) dx' d\sigma(y) \right| \\ &\leq \left| \int_{\mathbb{R}^2 \times \partial T} g(y') \cdot \tilde{\varphi} \left( \varepsilon \kappa \left( \frac{x'}{\varepsilon} \right), y_3 \right) dx' d\sigma(y) \right| \\ &\quad + \left| \int_{\mathbb{R}^2 \times \partial T} g(y') \cdot \left( \tilde{\varphi} \left( \varepsilon \kappa \left( \frac{x'}{\varepsilon} \right) + \varepsilon y', y_3 \right) - \tilde{\varphi} \left( \varepsilon \kappa \left( \frac{x'}{\varepsilon} \right), y_3 \right) \right) dx' d\sigma(y) \right| \\ &\leq C \mathcal{M}_{\partial T'}[g] \left( \|\tilde{\varphi}\|_{L^2(\tilde{\Omega}_\varepsilon)^3} + \varepsilon \|D_{x'} \tilde{\varphi}\|_{L^2(\tilde{\Omega}_\varepsilon)^{3 \times 3}} \right) \\ &\leq C \mathcal{M}_{\partial T'}[g] \left( \|\tilde{\varphi}\|_{L^2(\tilde{\Omega}_\varepsilon)^3} + \varepsilon \|D_\varepsilon \tilde{\varphi}\|_{L^2(\tilde{\Omega}_\varepsilon)^{3 \times 3}} \right), \end{aligned}$$

which implies (3.23). In particular, if  $g = 1$ , proceeding as above, we have

$$\begin{aligned}
 & \left| \int_{\mathbb{R}^2 \times \partial T} \hat{\varphi}_\varepsilon(x', y) dx' d\sigma(y) \right| \\
 &= \left| \int_{\mathbb{R}^2 \times \partial T} \tilde{\varphi} \left( \varepsilon \kappa \left( \frac{x'}{\varepsilon} \right) + \varepsilon y', y_3 \right) dx' d\sigma(y) \right| \\
 &\leq \left| \int_{\mathbb{R}^2 \times \partial T} \tilde{\varphi} \left( \varepsilon \kappa \left( \frac{x'}{\varepsilon} \right), y_3 \right) dx' d\sigma(y) \right| \\
 &\quad + \left| \int_{\mathbb{R}^2 \times \partial T} \left( \tilde{\varphi} \left( \varepsilon \kappa \left( \frac{x'}{\varepsilon} \right) + \varepsilon y', y_3 \right) - \tilde{\varphi} \left( \varepsilon \kappa \left( \frac{x'}{\varepsilon} \right), y_3 \right) \right) dx' d\sigma(y) \right| \\
 &\leq C \left( \|\tilde{\varphi}\|_{L^1(\tilde{\Omega}_\varepsilon)^3} + \varepsilon \|D_{x'} \tilde{\varphi}\|_{L^1(\tilde{\Omega}_\varepsilon)^{3 \times 3}} \right) \\
 &\leq C \left( \|\tilde{\varphi}\|_{L^1(\tilde{\Omega}_\varepsilon)^3} + \varepsilon \|D_\varepsilon \tilde{\varphi}\|_{L^1(\tilde{\Omega}_\varepsilon)^{3 \times 3}} \right),
 \end{aligned}$$

which implies (3.24). □

**Corollary 3.6.** *Let  $g \in L^2(\partial T)^3$  be a  $Y'$ -periodic function. Then, for every  $\tilde{\varphi} \in H^1(\tilde{\Omega}_\varepsilon)^3$ , we have that there exists a positive constant  $C$ , independent of  $\varepsilon$ , such that*

$$\left| \int_{\partial T_\varepsilon} g(x'/\varepsilon) \cdot \tilde{\varphi}(x', y_3) d\sigma(x') dy_3 \right| \leq \frac{C}{\varepsilon} \left( \|\tilde{\varphi}\|_{L^2(\tilde{\Omega}_\varepsilon)^3} + \varepsilon \|D_\varepsilon \tilde{\varphi}\|_{L^2(\tilde{\Omega}_\varepsilon)^{3 \times 3}} \right). \quad (3.25)$$

In particular, if  $g = 1$ , there exists a positive constant  $C$ , independent of  $\varepsilon$ , such that

$$\left| \int_{\partial T_\varepsilon} \tilde{\varphi}(x', y_3) d\sigma(x') dy_3 \right| \leq \frac{C}{\varepsilon} \left( \|\tilde{\varphi}\|_{L^1(\tilde{\Omega}_\varepsilon)^3} + \varepsilon \|D_\varepsilon \tilde{\varphi}\|_{L^1(\tilde{\Omega}_\varepsilon)^{3 \times 3}} \right). \quad (3.26)$$

*Proof.* Since  $\tilde{\varphi} \in H^1(\tilde{\Omega}_\varepsilon)^3$ , then  $\hat{\varphi}_\varepsilon^b$  has similar properties as  $\hat{\varphi}_\varepsilon$ . By using Proposition 3.4 with  $p = 1$  and Remark 3.3, we have

$$\left| \int_{\partial T_\varepsilon} g(x'/\varepsilon) \cdot \tilde{\varphi}(x', y_3) d\sigma(x') dy_3 \right| = \frac{1}{\varepsilon |Y'|} \left| \int_{\mathbb{R}^2 \times \partial T} g(y') \cdot \hat{\varphi}_\varepsilon(x', y) dx' d\sigma(y) \right|,$$

and by Proposition 3.5, we can deduce estimates (3.25) and (3.26). □

Moreover, for the proof of the *a priori* estimates for the velocity, we need the following lemma due to Conca [18] generalized to a thin domain  $\Omega_\varepsilon$ .

**Lemma 3.7.** *There exists a constant  $C$  independent of  $\varepsilon$ , such that, for any function  $\varphi \in H_\varepsilon$ , one has*

$$\|\varphi\|_{L^2(\Omega_\varepsilon)^3} \leq C \left( \varepsilon \|D\varphi\|_{L^2(\Omega_\varepsilon)^{3 \times 3}} + \varepsilon^{\frac{1}{2}} \|\varphi\|_{L^2(\partial S_\varepsilon)^3} \right). \quad (3.27)$$

*Proof.* We observe that the microscale of the porous medium  $\varepsilon$  is similar than the thickness of the domain  $\varepsilon$ , which lead us to divide the domain  $\Omega_\varepsilon$  in small cubes of lateral length  $\varepsilon$  and vertical length  $\varepsilon$ . We consider the periodic cell  $Y$ . The function  $\varphi \rightarrow \left( \|D\varphi\|_{L^2(Y_f)^{3 \times 3}}^2 + \|\varphi\|_{L^2(\partial T)^3}^2 \right)^{1/2}$  is a norm in  $H^1(Y_f)^3$ , equivalent to the  $H^1(Y_f)^3$ -norm (see Nečas [19]). Therefore, for any function  $\varphi(z) \in H^1(Y_f)^3$ , we have

$$\int_{Y_f} |\varphi|^2 dz \leq C \left( \int_{Y_f} |D_z \varphi|^2 dz + \int_{\partial T} |\varphi|^2 d\sigma(z) \right), \quad (3.28)$$

where the constant  $C$  depends only on  $Y_f$ .

Then, for every  $k' \in \mathbb{Z}^2$ , by the change of variable

$$k' + z' = \frac{x'}{\varepsilon}, \quad z_3 = \frac{x_3}{\varepsilon}, \quad dz = \frac{dx}{\varepsilon^3}, \quad \partial_z = \varepsilon \partial_x, \quad d\sigma(x) = \varepsilon^2 d\sigma(z), \quad (3.29)$$

we rescale (3.28) from  $Y_f$  to  $Q_{f_{k'},\varepsilon} = Y'_{f_{k'},\varepsilon} \times (0, \varepsilon)$ . This yields that, for any function  $\varphi(x) \in H^1(Q_{f_{k'},\varepsilon})^3$ , one has

$$\int_{Q_{f_{k'},\varepsilon}} |\varphi|^2 dx \leq C \left( \varepsilon^2 \int_{Q_{f_{k'},\varepsilon}} |D_x \varphi|^2 dx + \varepsilon \int_{T'_{k'},\varepsilon \times (0,\varepsilon)} |\varphi|^2 d\sigma(x) \right), \quad (3.30)$$

with the same constant  $C$  as in (3.28). Summing the inequality (3.30) for all the periods  $Q_{f_{k'},\varepsilon}$  and  $T'_{k'},\varepsilon \times (0, \varepsilon)$ , gives

$$\int_{\Omega_\varepsilon} |\varphi|^2 dx \leq C \left( \varepsilon^2 \int_{\Omega_\varepsilon} |D_x \varphi|^2 dx + \varepsilon \int_{\partial S_\varepsilon} |\varphi|^2 d\sigma(x) \right).$$

In fact, we must consider separately the periods containing a portion of  $\partial\omega$ , but they yield at a distance  $O(\varepsilon)$  of  $\partial\omega$ , where  $\varphi$  is zero. Therefore, using Poincaré's inequality one can easily verify that in this part (3.27) holds without considering the boundary term occurring in (3.27). □

Considering the change of variables given in (2.5) and taking into account that  $d\sigma(x) = \varepsilon d\sigma(x') dy_3$ , we obtain the following result for the domain  $\tilde{\Omega}_\varepsilon$ .

**Corollary 3.8.** *There exists a constant  $C$  independent of  $\varepsilon$ , such that, for any function  $\tilde{\varphi} \in \tilde{H}_\varepsilon$ , one has*

$$\|\tilde{\varphi}\|_{L^2(\tilde{\Omega}_\varepsilon)^3} \leq C \left( \varepsilon \|D_\varepsilon \tilde{\varphi}\|_{L^2(\tilde{\Omega}_\varepsilon)^{3 \times 3}} + \varepsilon^{\frac{1}{2}} \|\tilde{\varphi}\|_{L^2(\partial T_\varepsilon)^3} \right). \quad (3.31)$$

The presence in (1.3) of the stress tensor in the boundary condition implies that the extension of the velocity is no longer obvious. If we consider the Stokes system with Dirichlet boundary condition on the solid cylinders, the velocity would be extended by zero in the solid cylinders. However, in this case, we need another kind of extension (see Conca [18] for more details).

**Lemma 3.9.** *There exists an extension operator  $\Pi_\varepsilon \in \mathcal{L}(H_\varepsilon; H_0^1(\Lambda_\varepsilon)^3)$  and a positive constant  $C$ , independent of  $\varepsilon$ , such that*

$$\begin{aligned} \Pi_\varepsilon \varphi(x) &= \varphi(x), \quad \text{if } x \in \Omega_\varepsilon, \\ \|D\Pi_\varepsilon \varphi\|_{L^2(\Lambda_\varepsilon)^{3 \times 3}} &\leq C \|D\varphi\|_{L^2(\Omega_\varepsilon)^{3 \times 3}}, \quad \forall \varphi \in H_\varepsilon. \end{aligned}$$

*Proof.* From Cioranescu and Saint-Jean Paulin [20], we have that there exist an extension operator  $\Pi \in \mathcal{L}(H^1(Y_f)^3; H^1(Y)^3)$  such that

$$\Pi \varphi(z) = \varphi(z), \quad \text{if } z \in Y_f,$$

and a positive constant  $C$ , independent of  $\varepsilon$ , such that

$$\|D\Pi \varphi\|_{L^2(Y)^{3 \times 3}} \leq C \|D\varphi\|_{L^2(Y_f)^{3 \times 3}}, \quad \forall \varphi \in H^1(Y_f)^3. \quad (3.32)$$

We define  $\Pi_\varepsilon$  by applying  $\Pi$  to each  $Q_{k',\varepsilon} = Y'_{k',\varepsilon} \times (0, \varepsilon)$ . Using the change of variable (3.29), we rescale (3.32) from  $Y, Y_f$  to  $Q_{k',\varepsilon}, Q_{f_{k'},\varepsilon}$ , respectively. And proceeding as the proof of Lemma 3.7, we deduce the desired result. □

Considering the change of variables given in (2.5), we obtain the following result for the domain  $\tilde{\Omega}_\varepsilon$ .

**Corollary 3.10.** *There exists an extension operator  $\tilde{\Pi}_\varepsilon \in \mathcal{L}(\tilde{H}_\varepsilon; H_0^1(\Omega)^3)$  and a positive constant  $C$ , independent of  $\varepsilon$ , such that*

$$\begin{aligned} \tilde{\Pi}_\varepsilon \tilde{\varphi}(x', y_3) &= \tilde{\varphi}(x', y_3), \quad \text{if } (x', y_3) \in \tilde{\Omega}_\varepsilon, \\ \|D_\varepsilon \tilde{\Pi}_\varepsilon \tilde{\varphi}\|_{L^2(\Omega)^{3 \times 3}} &\leq C \|D_\varepsilon \tilde{\varphi}\|_{L^2(\tilde{\Omega}_\varepsilon)^{3 \times 3}}, \quad \forall \tilde{\varphi} \in \tilde{H}_\varepsilon. \end{aligned}$$

Using Corollary 3.10, we obtain a Poincaré inequality in  $\tilde{H}_\varepsilon$ .

**Corollary 3.11.** *There exists a constant  $C$  independent of  $\varepsilon$ , such that, for any function  $\tilde{\varphi} \in \tilde{H}_\varepsilon$ , one has*

$$\|\tilde{\varphi}\|_{L^2(\tilde{\Omega}_\varepsilon)^3} \leq C \|D_\varepsilon \tilde{\varphi}\|_{L^2(\tilde{\Omega}_\varepsilon)^{3 \times 3}}. \quad (3.33)$$

*Proof.* We observe that

$$\|\tilde{\varphi}\|_{L^2(\tilde{\Omega}_\varepsilon)^3} \leq \|\tilde{\Pi}_\varepsilon \tilde{\varphi}\|_{L^2(\Omega)^3}, \quad \forall \tilde{\varphi} \in \tilde{H}_\varepsilon. \quad (3.34)$$

Since  $\tilde{\Pi}_\varepsilon \tilde{\varphi} \in H_0^1(\Omega)^3$ , we can apply the Poincaré inequality in  $H_0^1(\Omega)$  and then taking into account Corollary 3.10, we get

$$\|\tilde{\Pi}_\varepsilon \tilde{\varphi}\|_{L^2(\Omega)^3} \leq C \|D \tilde{\Pi}_\varepsilon \tilde{\varphi}\|_{L^2(\Omega)^{3 \times 3}} \leq C \|D_\varepsilon \tilde{\Pi}_\varepsilon \tilde{\varphi}\|_{L^2(\Omega)^{3 \times 3}} \leq C \|D_\varepsilon \tilde{\varphi}\|_{L^2(\tilde{\Omega}_\varepsilon)^{3 \times 3}}.$$

This together with (3.34) gives (3.33).  $\square$

Now, for the proof of the *a priori* estimates for the pressure, we also need the following lemma due to Conca [18] generalized to a thin domain  $\Omega_\varepsilon$ .

**Lemma 3.12.** *There exists a constant  $C$  independent of  $\varepsilon$ , such that, for each  $q \in L^2(\Omega_\varepsilon)$ , there exists  $\varphi = \varphi(q) \in H_\varepsilon$ , such that*

$$\operatorname{div} \varphi = q \text{ in } \Omega_\varepsilon, \quad (3.35)$$

$$\|\varphi\|_{L^2(\Omega_\varepsilon)^3} \leq C \|q\|_{L^2(\Omega_\varepsilon)}, \quad \|D\varphi\|_{L^2(\Omega_\varepsilon)^{3 \times 3}} \leq \frac{C}{\varepsilon} \|q\|_{L^2(\Omega_\varepsilon)}. \quad (3.36)$$

*Proof.* Let  $q \in L^2(\Omega_\varepsilon)$  be given. We extend  $q$  inside the cylinders by means of the function:

$$Q(x) = \begin{cases} q(x) & \text{if } x \in \Omega_\varepsilon \\ \frac{-1}{|\Lambda_\varepsilon - \Omega_\varepsilon|} \int_{\Omega_\varepsilon} q(x) dx & \text{if } x \in \Lambda_\varepsilon - \Omega_\varepsilon. \end{cases}$$

It follows that  $Q \in L^2(\Lambda_\varepsilon)/\mathbb{R}$  and

$$\|Q\|_{L^2(\Lambda_\varepsilon)}^2 = \|q\|_{L^2(\Omega_\varepsilon)}^2 + \frac{1}{|\Lambda_\varepsilon - \Omega_\varepsilon|} \left( \int_{\Omega_\varepsilon} q(x) dx \right)^2. \quad (3.37)$$

Since  $|\Lambda_\varepsilon - \Omega_\varepsilon|$  is bounded from below by a positive number, it follows from (3.37) and Cauchy-Schwartz inequality that

$$\|Q\|_{L^2(\Lambda_\varepsilon)} \leq C \|q\|_{L^2(\Omega_\varepsilon)}. \quad (3.38)$$

Besides that, since  $Q \in L^2(\Lambda_\varepsilon)/\mathbb{R}$ , it follows from Marušić and Marušić-Paloka (Lemma 20 in [21]) that we can find  $\varphi \in H_0^1(\Lambda_\varepsilon)^3$  such that

$$\operatorname{div} \varphi = Q \text{ in } \Lambda_\varepsilon, \quad (3.39)$$

$$\|\varphi\|_{L^2(\Lambda_\varepsilon)^3} \leq C \|Q\|_{L^2(\Lambda_\varepsilon)}, \quad \|D\varphi\|_{L^2(\Lambda_\varepsilon)^{3 \times 3}} \leq \frac{C}{\varepsilon} \|Q\|_{L^2(\Lambda_\varepsilon)}. \quad (3.40)$$

Let us consider  $\varphi|_{\Omega_\varepsilon}$ : it belongs to  $H_\varepsilon$ . Moreover, (3.35) follows from (3.39) and the estimates (3.36) follows from (3.40) and (3.38).  $\square$

Considering the change of variables given in (2.5), we obtain the following result for the domain  $\tilde{\Omega}_\varepsilon$ .

**Corollary 3.13.** *There exists a constant  $C$  independent of  $\varepsilon$ , such that, for each  $\tilde{q} \in L^2(\tilde{\Omega}_\varepsilon)$ , there exists  $\tilde{\varphi} = \tilde{\varphi}(\tilde{q}) \in \tilde{H}_\varepsilon$ , such that*

$$\begin{aligned} \operatorname{div}_\varepsilon \tilde{\varphi} &= \tilde{q} \text{ in } \tilde{\Omega}_\varepsilon, \\ \|\tilde{\varphi}\|_{L^2(\tilde{\Omega}_\varepsilon)^3} &\leq C \|\tilde{q}\|_{L^2(\tilde{\Omega}_\varepsilon)}, \quad \|D_\varepsilon \tilde{\varphi}\|_{L^2(\tilde{\Omega}_\varepsilon)^{3 \times 3}} \leq \frac{C}{\varepsilon} \|\tilde{q}\|_{L^2(\tilde{\Omega}_\varepsilon)}. \end{aligned}$$

**A priori estimates for  $(\tilde{u}_\varepsilon, \tilde{p}_\varepsilon)$  in  $\tilde{\Omega}_\varepsilon$ :** first, let us obtain some *a priori* estimates for  $\tilde{u}_\varepsilon$  for different values of  $\gamma$ .

**Lemma 3.14.** *We distinguish three cases:*

i) For  $\gamma < -1$ , then there exists a constant  $C$  independent of  $\varepsilon$ , such that

$$\|\tilde{u}_\varepsilon\|_{L^2(\tilde{\Omega}_\varepsilon)^3} \leq C\varepsilon, \quad \|D_\varepsilon \tilde{u}_\varepsilon\|_{L^2(\tilde{\Omega}_\varepsilon)^{3 \times 3}} \leq C. \quad (3.41)$$

ii) For  $-1 \leq \gamma < 1$ , then there exists a constant  $C$  independent of  $\varepsilon$ , such that

$$\|\tilde{u}_\varepsilon\|_{L^2(\tilde{\Omega}_\varepsilon)^3} \leq C\varepsilon^{-\gamma}, \quad \|D_\varepsilon \tilde{u}_\varepsilon\|_{L^2(\tilde{\Omega}_\varepsilon)^{3 \times 3}} \leq C\varepsilon^{-\frac{1+\gamma}{2}}. \quad (3.42)$$

iii) For  $\gamma \geq 1$ , then there exists a constant  $C$  independent of  $\varepsilon$ , such that

$$\|\tilde{u}_\varepsilon\|_{L^2(\tilde{\Omega}_\varepsilon)^3} \leq C\varepsilon^{-1}, \quad \|D_\varepsilon \tilde{u}_\varepsilon\|_{L^2(\tilde{\Omega}_\varepsilon)^{3 \times 3}} \leq C\varepsilon^{-1}. \quad (3.43)$$

*Proof.* Taking  $\tilde{u}_\varepsilon \in \tilde{H}_\varepsilon$  as function test in (2.8), we have

$$\mu \|D_\varepsilon \tilde{u}_\varepsilon\|_{L^2(\tilde{\Omega}_\varepsilon)^{3 \times 3}}^2 + \alpha \varepsilon^\gamma \|\tilde{u}_\varepsilon\|_{L^2(\partial T_\varepsilon)^3}^2 = \int_{\tilde{\Omega}_\varepsilon} f'_\varepsilon \cdot \tilde{u}'_\varepsilon dx' dy_3 + \int_{\partial T_\varepsilon} g'_\varepsilon \cdot \tilde{u}'_\varepsilon d\sigma(x') dy_3. \quad (3.44)$$

Using Cauchy-Schwarz's inequality and  $f'_\varepsilon \in L^2(\Omega)^2$ , we obtain that

$$\int_{\tilde{\Omega}_\varepsilon} f'_\varepsilon \cdot \tilde{u}'_\varepsilon dx' dy_3 \leq C \|\tilde{u}_\varepsilon\|_{L^2(\tilde{\Omega}_\varepsilon)^3},$$

and by using that  $g' \in L^2(\partial T)^2$  is a  $Y'$ -periodic function and estimate (3.25), we have

$$\left| \int_{\partial T_\varepsilon} g'_\varepsilon \cdot \tilde{u}'_\varepsilon d\sigma(x') dy_3 \right| \leq \frac{C}{\varepsilon} \left( \|\tilde{u}_\varepsilon\|_{L^2(\tilde{\Omega}_\varepsilon)^3} + \varepsilon \|D_\varepsilon \tilde{u}_\varepsilon\|_{L^2(\tilde{\Omega}_\varepsilon)^{3 \times 3}} \right).$$

Putting these estimates in (3.44), we get

$$\mu \|D_\varepsilon \tilde{u}_\varepsilon\|_{L^2(\tilde{\Omega}_\varepsilon)^{3 \times 3}}^2 + \alpha \varepsilon^\gamma \|\tilde{u}_\varepsilon\|_{L^2(\partial T_\varepsilon)^3}^2 \leq C \left( \|D_\varepsilon \tilde{u}_\varepsilon\|_{L^2(\tilde{\Omega}_\varepsilon)^{3 \times 3}} + \varepsilon^{-1} \|\tilde{u}_\varepsilon\|_{L^2(\tilde{\Omega}_\varepsilon)^3} \right). \quad (3.45)$$

In particular, if we use the Poincaré inequality (3.33) in (3.45), we have

$$\|D_\varepsilon \tilde{u}_\varepsilon\|_{L^2(\tilde{\Omega}_\varepsilon)^{3 \times 3}} \leq \frac{C}{\varepsilon}, \quad (3.46)$$

therefore (independently of  $\gamma \in \mathbb{R}$ ), using again (3.33), we get

$$\|\tilde{u}_\varepsilon\|_{L^2(\tilde{\Omega}_\varepsilon)^3} \leq \frac{C}{\varepsilon}. \quad (3.47)$$

These estimates can be refined following the different values of  $\gamma$ . To do so, observe that from estimate (3.31) we have

$$\varepsilon^{-1} \|\tilde{u}_\varepsilon\|_{L^2(\tilde{\Omega}_\varepsilon)^3} \leq C \left( \|D_\varepsilon \tilde{u}_\varepsilon\|_{L^2(\tilde{\Omega}_\varepsilon)^{3 \times 3}} + \varepsilon^{-\frac{1}{2}} \|\tilde{u}_\varepsilon\|_{L^2(\partial T_\varepsilon)^3} \right).$$

Using Young's inequality, we get

$$\varepsilon^{-\frac{1}{2}} \|\tilde{u}_\varepsilon\|_{L^2(\partial T_\varepsilon)^3} \leq \varepsilon^{-\frac{1+\gamma}{2}} \varepsilon^{\frac{\gamma}{2}} \|\tilde{u}_\varepsilon\|_{L^2(\partial T_\varepsilon)^3} \leq \frac{2}{\alpha} \varepsilon^{-1-\gamma} + \frac{\alpha}{2} \varepsilon^\gamma \|\tilde{u}_\varepsilon\|_{L^2(\partial T_\varepsilon)^3}^2.$$

Consequently, from (3.45), we get

$$\mu \|D_\varepsilon \tilde{u}_\varepsilon\|_{L^2(\tilde{\Omega}_\varepsilon)^{3 \times 3}}^2 + \frac{\alpha}{2} \varepsilon^\gamma \|\tilde{u}_\varepsilon\|_{L^2(\partial T_\varepsilon)^3}^2 \leq C \left( \|D_\varepsilon \tilde{u}_\varepsilon\|_{L^2(\tilde{\Omega}_\varepsilon)^{3 \times 3}} + \varepsilon^{-1-\gamma} \right),$$

which applying in a suitable way the Young inequality gives

$$\mu \|D_\varepsilon \tilde{u}_\varepsilon\|_{L^2(\tilde{\Omega}_\varepsilon)^{3 \times 3}}^2 + \alpha \varepsilon^\gamma \|\tilde{u}_\varepsilon\|_{L^2(\partial T_\varepsilon)^3}^2 \leq C (1 + \varepsilon^{-1-\gamma}). \quad (3.48)$$

For the case when  $\gamma < -1$ , estimate (3.48) reads

$$\|D_\varepsilon \tilde{u}_\varepsilon\|_{L^2(\tilde{\Omega}_\varepsilon)^{3 \times 3}} \leq C, \quad \|\tilde{u}_\varepsilon\|_{L^2(\partial T_\varepsilon)^3} \leq C \varepsilon^{-\frac{\gamma}{2}}.$$

Then, estimate (3.31) gives

$$\|\tilde{u}_\varepsilon\|_{L^2(\tilde{\Omega}_\varepsilon)^3} \leq C(\varepsilon + \varepsilon^{\frac{1-\gamma}{2}}) \leq C\varepsilon,$$

since  $1 \leq (1 - \gamma)/2$ , and so, we have proved (3.41).

For  $\gamma \geq -1$ , estimate (3.48) reads

$$\|D_\varepsilon \tilde{u}_\varepsilon\|_{L^2(\tilde{\Omega}_\varepsilon)^{3 \times 3}} \leq C \varepsilon^{-\frac{1+\gamma}{2}}, \quad \|\tilde{u}_\varepsilon\|_{L^2(\partial T_\varepsilon)^3} \leq C \varepsilon^{-\frac{1}{2}-\gamma}.$$

Applying estimate (3.31), we get

$$\|\tilde{u}_\varepsilon\|_{L^2(\tilde{\Omega}_\varepsilon)^3} \leq C(\varepsilon^{\frac{1-\gamma}{2}} + \varepsilon^{-\gamma}) \leq C\varepsilon^{-\gamma}$$

since  $-\gamma \leq (1 - \gamma)/2$ . Then, we have proved (3.42) for  $-1 \leq \gamma < 1$ . Observe that for  $\gamma \geq 1$ , the estimates (3.46)-(3.47) are the optimal ones, so we have (3.43). □

We will prove now *a priori* estimates for the pressure  $\tilde{p}_\varepsilon$  for different values of  $\gamma$ .

**Lemma 3.15.** *We distinguish three cases:*

*i) For  $\gamma < -1$ , then there exists a constant  $C$  independent of  $\varepsilon$ , such that*

$$\|\tilde{p}_\varepsilon\|_{L^2(\tilde{\Omega}_\varepsilon)/\mathbb{R}} \leq C \varepsilon^\gamma. \quad (3.49)$$

*ii) For  $-1 \leq \gamma < 1$ , then there exists a constant  $C$  independent of  $\varepsilon$ , such that*

$$\|\tilde{p}_\varepsilon\|_{L^2(\tilde{\Omega}_\varepsilon)/\mathbb{R}} \leq C \varepsilon^{-1}. \quad (3.50)$$

*iii) For  $\gamma \geq 1$ , then there exists a constant  $C$  independent of  $\varepsilon$ , such that*

$$\|\tilde{p}_\varepsilon\|_{L^2(\tilde{\Omega}_\varepsilon)/\mathbb{R}} \leq C \varepsilon^{-2}. \quad (3.51)$$

*Proof.* Let  $\tilde{\Phi} \in L^2(\tilde{\Omega}_\varepsilon)$ . From Corollary 3.13, there exists  $\tilde{\varphi} \in \tilde{H}_\varepsilon$  such that

$$\operatorname{div}_\varepsilon \tilde{\varphi} = \tilde{\Phi} \text{ in } \tilde{\Omega}_\varepsilon, \quad \|\tilde{\varphi}\|_{L^2(\tilde{\Omega}_\varepsilon)^3} \leq C \|\tilde{\Phi}\|_{L^2(\tilde{\Omega}_\varepsilon)}, \quad \|D_\varepsilon \tilde{\varphi}\|_{L^2(\tilde{\Omega}_\varepsilon)^{3 \times 3}} \leq \frac{C}{\varepsilon} \|\tilde{\Phi}\|_{L^2(\tilde{\Omega}_\varepsilon)}. \quad (3.52)$$

Taking  $\tilde{\varphi} \in \tilde{H}_\varepsilon$  as function test in (2.8), we have

$$\begin{aligned} \left| \int_{\tilde{\Omega}_\varepsilon} \tilde{p}_\varepsilon \tilde{\Phi} dx' dy_3 \right| &\leq \mu \|D_\varepsilon \tilde{u}_\varepsilon\|_{L^2(\tilde{\Omega}_\varepsilon)^{3 \times 3}} \|D_\varepsilon \tilde{\varphi}\|_{L^2(\tilde{\Omega}_\varepsilon)^{3 \times 3}} + \alpha \varepsilon^\gamma \left| \int_{\partial T_\varepsilon} \tilde{u}_\varepsilon \cdot \tilde{\varphi} d\sigma(x') dy_3 \right| \\ &\quad + C \|\tilde{\varphi}\|_{L^2(\tilde{\Omega}_\varepsilon)^3} + \left| \int_{\partial T_\varepsilon} g'_\varepsilon \cdot \tilde{\varphi}' d\sigma(x') dy_3 \right|. \end{aligned} \quad (3.53)$$

By using that  $g \in L^2(\partial T)^3$  is a  $Y'$ -periodic function and estimate (3.25), we have

$$\left| \int_{\partial T_\varepsilon} g'_\varepsilon \cdot \tilde{\varphi}' d\sigma(x') dy_3 \right| \leq C \left( \varepsilon^{-1} \|\tilde{\varphi}\|_{L^2(\tilde{\Omega}_\varepsilon)^3} + \|D_\varepsilon \tilde{\varphi}\|_{L^2(\tilde{\Omega}_\varepsilon)^{3 \times 3}} \right).$$

Analogously, using estimate (3.26) and the Cauchy-Schwarz inequality, a simple computation gives

$$\begin{aligned} \alpha \varepsilon^\gamma \left| \int_{\partial T_\varepsilon} \tilde{u}_\varepsilon \cdot \tilde{\varphi} d\sigma(x') dy_3 \right| &\leq \varepsilon^{\gamma-1} C \|\tilde{u}_\varepsilon\|_{L^2(\tilde{\Omega}_\varepsilon)} \|\tilde{\varphi}\|_{L^2(\tilde{\Omega}_\varepsilon)} + \varepsilon^\gamma C \|\tilde{u}_\varepsilon\|_{L^2(\tilde{\Omega}_\varepsilon)} \|D_\varepsilon \tilde{\varphi}\|_{L^2(\tilde{\Omega}_\varepsilon)} \\ &\quad + \varepsilon^\gamma C \|D_\varepsilon \tilde{u}_\varepsilon\|_{L^2(\tilde{\Omega}_\varepsilon)} \|\tilde{\varphi}\|_{L^2(\tilde{\Omega}_\varepsilon)}. \end{aligned}$$

Then, turning back to (3.53) and using (3.52), one has

$$\begin{aligned} \left| \int_{\tilde{\Omega}_\varepsilon} \tilde{p}_\varepsilon \tilde{\Phi} dx' dy_3 \right| &\leq C (\varepsilon^{-1} + \varepsilon^\gamma) \|D_\varepsilon \tilde{u}_\varepsilon\|_{L^2(\tilde{\Omega}_\varepsilon)^{3 \times 3}} \|\tilde{\Phi}\|_{L^2(\tilde{\Omega}_\varepsilon)} \\ &\quad + C \left( \varepsilon^{\gamma-1} \|\tilde{u}_\varepsilon\|_{L^2(\tilde{\Omega}_\varepsilon)^3} + \varepsilon^{-1} \right) \|\tilde{\Phi}\|_{L^2(\tilde{\Omega}_\varepsilon)}. \end{aligned} \quad (3.54)$$

The *a priori* estimates for the pressure follow now from (3.54) and estimates (3.41)-(3.42) and (3.43), corresponding to the different values of  $\gamma$ . □

**A priori estimates for the extension of  $(\tilde{u}_\varepsilon, \tilde{p}_\varepsilon)$  to the whole domain  $\Omega$ :** for the velocity, we will denote  $\tilde{U}_\varepsilon = \tilde{I}_\varepsilon \tilde{u}_\varepsilon$  the continuation of  $\tilde{u}_\varepsilon$  in  $\Omega$ , where  $\tilde{I}_\varepsilon$  is given in Corollary 3.10. For the pressure, we proceed as in the proof of Lemma 3.12, and we will denote by  $\tilde{P}_\varepsilon$  the continuation of  $\tilde{p}_\varepsilon$  in  $\Omega$ .

**Lemma 3.16.** *We distinguish three cases:*

i) For  $\gamma < -1$ , then there exists a constant  $C$  independent of  $\varepsilon$ , such that

$$\|\tilde{U}_\varepsilon\|_{L^2(\Omega)^3} \leq C\varepsilon, \quad \|D_\varepsilon \tilde{U}_\varepsilon\|_{L^2(\Omega)^{3 \times 3}} \leq C, \quad (3.55)$$

$$\|\tilde{P}_\varepsilon\|_{L^2(\Omega)/\mathbb{R}} \leq C\varepsilon^\gamma. \quad (3.56)$$

ii) For  $-1 \leq \gamma < 1$ , then there exists a constant  $C$  independent of  $\varepsilon$ , such that

$$\|\tilde{U}_\varepsilon\|_{L^2(\Omega)^3} \leq C\varepsilon^{-\gamma}, \quad \|D_\varepsilon \tilde{U}_\varepsilon\|_{L^2(\Omega)^{3 \times 3}} \leq C\varepsilon^{-\frac{1+\gamma}{2}}, \quad (3.57)$$

$$\|\tilde{P}_\varepsilon\|_{L^2(\Omega)/\mathbb{R}} \leq C\varepsilon^{-1}. \quad (3.58)$$

iii) For  $\gamma \geq 1$ , then there exists a constant  $C$  independent of  $\varepsilon$ , such that

$$\|\tilde{U}_\varepsilon\|_{L^2(\Omega)^3} \leq C\varepsilon^{-1}, \quad \|D_\varepsilon \tilde{U}_\varepsilon\|_{L^2(\Omega)^{3 \times 3}} \leq C\varepsilon^{-1}, \quad (3.59)$$

$$\|\tilde{P}_\varepsilon\|_{L^2(\Omega)/\mathbb{R}} \leq C\varepsilon^{-2}. \quad (3.60)$$

*Proof.* For the velocity, using Corollary 3.10 and the *a priori estimates* in  $\tilde{\Omega}_\varepsilon$  given in Lemma 3.14, we deduce the *a priori estimates* in  $\Omega$ . For the pressure, taking into account the change of variable (2.5) in (3.38) and the *a priori estimates* in  $\tilde{\Omega}_\varepsilon$  given in Lemma 3.15, we deduce the *a priori estimates* in  $\Omega$ . □



**A priori estimates of the unfolding functions**  $(\hat{u}_\varepsilon, \hat{p}_\varepsilon)$ : let us obtain some *a priori* estimates for the sequences  $(\hat{u}_\varepsilon, \hat{p}_\varepsilon)$  where  $\hat{u}_\varepsilon$  and  $\hat{p}_\varepsilon$  are obtained by applying the change of variable (3.15) to  $(\tilde{u}_\varepsilon, \tilde{p}_\varepsilon)$ .

**Lemma 3.17.** *We distinguish three cases:*

i) For  $\gamma < -1$ , then there exists a constant  $C$  independent of  $\varepsilon$ , such that

$$\|\hat{u}_\varepsilon\|_{L^2(\mathbb{R}^2 \times Y_f)^3} \leq C\varepsilon, \quad \|D_y \hat{u}_\varepsilon\|_{L^2(\mathbb{R}^2 \times Y_f)^{3 \times 3}} \leq C\varepsilon, \quad (3.61)$$

$$\|\hat{p}_\varepsilon\|_{L^2(\mathbb{R}^2 \times Y_f)} \leq C\varepsilon^\gamma. \quad (3.62)$$

ii) For  $-1 \leq \gamma < 1$ , then there exists a constant  $C$  independent of  $\varepsilon$ , such that

$$\|\hat{u}_\varepsilon\|_{L^2(\mathbb{R}^2 \times Y_f)^3} \leq C\varepsilon^{-\gamma}, \quad \|D_y \hat{u}_\varepsilon\|_{L^2(\mathbb{R}^2 \times Y_f)^{3 \times 3}} \leq C\varepsilon^{\frac{1-\gamma}{2}}, \quad (3.63)$$

$$\|\hat{p}_\varepsilon\|_{L^2(\mathbb{R}^2 \times Y_f)} \leq C\varepsilon^{-1}. \quad (3.64)$$

iii) For  $\gamma \geq 1$ , then there exists a constant  $C$  independent of  $\varepsilon$ , such that

$$\|\hat{u}_\varepsilon\|_{L^2(\mathbb{R}^2 \times Y_f)^3} \leq C\varepsilon^{-1}, \quad \|D_y \hat{u}_\varepsilon\|_{L^2(\mathbb{R}^2 \times Y_f)^{3 \times 3}} \leq C, \quad (3.65)$$

$$\|\hat{p}_\varepsilon\|_{L^2(\mathbb{R}^2 \times Y_f)} \leq C\varepsilon^{-2}. \quad (3.66)$$

*Proof.* Using properties (3.17) and (3.18) with  $p = 2$  and the *a priori* estimates given in Lemma 3.14 and Lemma 3.15, we have the desired result. □

## 3.2 Some compactness results

In this subsection we obtain some compactness results about the behavior of the sequences  $(\tilde{U}_\varepsilon, \tilde{P}_\varepsilon)$  and  $(\hat{u}_\varepsilon, \hat{p}_\varepsilon)$  satisfying *a priori* estimates given in Lemma 3.16 and Lemma 3.17, respectively.

Let us start giving a convergence result for the extended pressure  $\tilde{P}_\varepsilon$ .

**Lemma 3.18.** *We distinguish three cases:*

i) For  $\gamma < -1$ , then for a subsequence of  $\varepsilon$  still denote by  $\varepsilon$ , there exists  $\tilde{p} \in L^2(\Omega)/\mathbb{R}$  such that

$$\varepsilon^{-\gamma} \tilde{P}_\varepsilon \rightharpoonup \tilde{p} \text{ in } L^2(\Omega)/\mathbb{R}. \quad (3.67)$$

ii) For  $-1 \leq \gamma < 1$ , then for a subsequence of  $\varepsilon$  still denote by  $\varepsilon$ , there exists  $\tilde{p} \in L^2(\Omega)/\mathbb{R}$  such that

$$\varepsilon \tilde{P}_\varepsilon \rightharpoonup \tilde{p} \text{ in } L^2(\Omega)/\mathbb{R}. \quad (3.68)$$

iii) For  $\gamma \geq 1$ , then for a subsequence of  $\varepsilon$  still denote by  $\varepsilon$ , there exists  $\tilde{p} \in L^2(\Omega)/\mathbb{R}$  such that

$$\varepsilon^2 \tilde{P}_\varepsilon \rightharpoonup \tilde{p} \text{ in } L^2(\Omega)/\mathbb{R}. \quad (3.69)$$

*Proof.* These convergences are a consequence of the *a priori* estimates of the extended pressure (3.56), (3.58) and (3.60). □

Now, we have a convergence result for the extended velocity  $\tilde{U}_\varepsilon$ .

**Lemma 3.19.** *We distinguish three cases:*

i) For  $\gamma < -1$ , then for a subsequence of  $\varepsilon$  still denote by  $\varepsilon$ , there exists  $\tilde{u} \in H^1(0, 1; L^2(\omega)^3)$  where  $\tilde{u} = 0$  on  $y_3 = \{0, 1\}$ , such that

$$\varepsilon^{-1}\tilde{U}_\varepsilon \rightharpoonup \tilde{u} \text{ in } H^1(0, 1; L^2(\omega)^3). \quad (3.70)$$

ii) For  $-1 \leq \gamma < 1$ , then for a subsequence of  $\varepsilon$  still denote by  $\varepsilon$ , there exists  $\tilde{u} \in L^2(\Omega)^3$  where  $\tilde{u} = 0$  on  $y_3 = \{0, 1\}$ , such that

$$\varepsilon^\gamma \tilde{U}_\varepsilon \rightharpoonup \tilde{u} \text{ in } L^2(\Omega)^3. \quad (3.71)$$

iii) For  $\gamma \geq 1$ , then for a subsequence of  $\varepsilon$  still denote by  $\varepsilon$ , there exists  $\tilde{u} \in H^1(0, 1; L^2(\omega)^3)$  where  $\tilde{u} = 0$  on  $y_3 = \{0, 1\}$ , such that

$$\tilde{U}_\varepsilon \rightharpoonup \tilde{u} \text{ in } H^1(0, 1; L^2(\omega)^3). \quad (3.72)$$

*Proof.* From the *a priori* estimates (3.55) and (3.57), we deduce the desired results for the two first cases. For the last one, using the *a priori* estimate (3.59) and Dirichlet boundary condition, we deduce  $\|\tilde{U}_\varepsilon\|_{L^2(\Omega)^3} \leq \|\partial_{y_3}\tilde{U}_\varepsilon\|_{L^2(\Omega)^3} \leq C$ , and then we obtain the desired result.  $\square$

Finally, we give a convergence result for  $\hat{u}_\varepsilon$ .

**Lemma 3.20.** *We distinguish three cases:*

i) For  $\gamma < -1$ , then for a subsequence of  $\varepsilon$  still denote by  $\varepsilon$ , there exists  $\hat{u} \in L^2(\mathbb{R}^2; H_{\sharp}^1(Y_f)^3)$ , such that

$$\varepsilon^{-1}\hat{u}_\varepsilon \rightharpoonup \hat{u} \text{ in } L^2(\mathbb{R}^2; H^1(Y_f)^3), \quad (3.73)$$

$$\varepsilon^{-1}\hat{u}_\varepsilon \rightharpoonup \hat{u} \text{ in } L^2(\mathbb{R}^2; H^{\frac{1}{2}}(\partial T)^3), \quad (3.74)$$

$$\frac{|Y'_f|}{|Y_f|} \mathcal{M}_{Y'_f}[\hat{u}] = \tilde{u} \text{ a.e. in } \Omega. \quad (3.75)$$

ii) For  $-1 \leq \gamma < 1$ , then for a subsequence of  $\varepsilon$  still denote by  $\varepsilon$ , there exists there exists  $\hat{u} \in L^2(\mathbb{R}^2; H_{\sharp}^1(Y_f)^3)$ , independent of  $y$ , such that

$$\varepsilon^\gamma \hat{u}_\varepsilon \rightharpoonup \hat{u} \text{ in } L^2(\mathbb{R}^2; H^1(Y_f)^3), \quad (3.76)$$

$$\varepsilon^\gamma \hat{u}_\varepsilon \rightharpoonup \hat{u} \text{ in } L^2(\mathbb{R}^2; H^{\frac{1}{2}}(\partial T)^3), \quad (3.77)$$

$$\hat{u} = \frac{|Y'_f|}{|Y_f|} \tilde{u} \text{ a.e. in } \Omega. \quad (3.78)$$

iii) For  $\gamma \geq 1$ , then for a subsequence of  $\varepsilon$  still denote by  $\varepsilon$ , there exists there exists  $\hat{u} \in L^2(\mathbb{R}^2; H_{\sharp}^1(Y_f)^3)$ , independent of  $y$ , such that

$$\varepsilon \hat{u}_\varepsilon \rightharpoonup \hat{u} \text{ in } L^2(\mathbb{R}^2; H^1(Y_f)^3), \quad (3.79)$$

$$\varepsilon \hat{u}_\varepsilon \rightharpoonup \hat{u} \text{ in } L^2(\mathbb{R}^2; H^{\frac{1}{2}}(\partial T)^3). \quad (3.80)$$

*Proof.* We proceed in three steps:

*Step 1.* For  $\gamma < -1$ , using (3.61), there exists  $\hat{u} : \mathbb{R}^2 \times Y_f \rightarrow \mathbb{R}^3$ , such that convergence (3.73) holds. Passing to the limit by semicontinuity and using estimates (3.61), we get

$$\int_{\mathbb{R}^2 \times Y_f} |\hat{u}|^2 dx' dy \leq C, \quad \int_{\mathbb{R}^2 \times Y_f} |D_y \hat{u}|^2 dx' dy \leq C,$$

which, once we prove the  $Y'$ -periodicity of  $\hat{u}$  in  $y'$ , shows that  $\hat{u} \in L^2(\mathbb{R}^2; H_{\#}^1(Y_f)^3)$ .

It remains to prove the  $Y'$ -periodicity of  $\hat{u}$  in  $y'$ . To do this, we observe that by definition of  $\hat{u}_\varepsilon$  given by (3.15) applied to  $\tilde{u}_\varepsilon$ , we have

$$\hat{u}_\varepsilon(x_1 + \varepsilon, x_2, -1/2, y_2, y_3) = \hat{u}_\varepsilon(x', 1/2, y_2, y_3) \text{ a.e. } (x', y_2, y_3) \in \mathbb{R}^2 \times (-1/2, 1/2) \times (0, 1).$$

Multiplying by  $\varepsilon^{-1}$  and passing to the limit by (3.73), we get

$$\hat{u}(x', -1/2, y_2, y_3) = \hat{u}(x', 1/2, y_2, y_3) \text{ a.e. } (x', y_2, y_3) \in \mathbb{R}^2 \times (-1/2, 1/2) \times (0, 1).$$

Analogously, we can prove

$$\hat{u}(x', y_1, -1/2, y_3) = \hat{u}(x', y_1, 1/2, y_3) \text{ a.e. } (x', y_1, y_3) \in \omega \times (-1/2, 1/2) \times (0, 1).$$

These equalities prove that  $\hat{u}$  is periodic with respect to  $Y'$ . Convergence (3.74) is straightforward from the definition (3.21) and the Sobolev injections.

Finally, using Proposition 3.2, we can deduce

$$\frac{1}{|Y'|} \int_{\mathbb{R}^2 \times Y_f} \hat{u}_\varepsilon(x', y) dx' dy = \int_{\tilde{\Omega}_\varepsilon} \tilde{u}_\varepsilon(x', y_3) dx' dy_3,$$

and multiplying by  $\varepsilon^{-1}$  and taking into account the first item in Corollary 3.10, we have

$$\frac{1}{\varepsilon|Y'|} \int_{\mathbb{R}^2 \times Y_f} \hat{u}_\varepsilon(x', y) dx' dy = \frac{1}{\varepsilon} \int_{\Omega} \tilde{u}_\varepsilon(x', y_3) dx' dy_3.$$

Using convergences (3.70) and (3.73), we have (3.75).

*Step 2.* For  $-1 \leq \gamma < 1$ , using (3.63) and taking into account that  $\varepsilon^{\frac{1-\gamma}{2}} \leq \varepsilon^{-\gamma}$ , then there exists  $\hat{u} : \mathbb{R}^2 \times Y_f \rightarrow \mathbb{R}^3$ , such that convergence (3.76) holds. Convergence (3.77) is straightforward from the definition (3.21) and the Sobolev injections.

On the other hand, since  $\varepsilon^\gamma D_y \hat{u}_\varepsilon = \varepsilon^{\frac{\gamma+1}{2}} \varepsilon^{\frac{\gamma-1}{2}} D_y \hat{u}_\varepsilon$  and  $\varepsilon^{\frac{\gamma-1}{2}} D_y \hat{u}_\varepsilon$  is bounded in  $L^2(\mathbb{R}^2 \times Y_f)^{3 \times 3}$ , using (3.76) we can deduce that  $D_y \hat{u} = 0$ . This implies that  $\hat{u}$  is independent of  $y$ . Finally, (3.78) is obtained similarly to the Step 1.

*Step 3.* For  $\gamma \geq 1$ , using the first *a priori* estimate in (3.65), we can deduce that there exists  $\hat{u} \in L^2(\mathbb{R}^2 \times Y_f)^3$  such that

$$\varepsilon \hat{u}_\varepsilon \rightharpoonup \hat{u} \quad \text{in } L^2(\mathbb{R}^2 \times Y_f)^3,$$

and by the second *a priori* estimate in (3.65) we deduce that

$$\varepsilon D_y \hat{u}_\varepsilon \rightharpoonup 0 \quad \text{in } L^2(\mathbb{R}^2 \times Y_f)^{3 \times 3},$$

which implies that  $\hat{u}$  does not depend on  $y$ . Convergence (3.80) is straightforward from the definition (3.21) and the Sobolev injections. □

### 3.3 Proof of Theorem 2.1

In this section, we will multiply system (2.8) by a test function having the form of the limit  $\hat{u}$  (as explicated in Lemma 3.20), and we will use the convergences given in the previous section in order to identify the homogenized model in every cases.

**Proof of Theorem 2.1:** We proceed in three steps.

*Step 1.* For  $\gamma < -1$ . Let  $\tilde{\varphi} \in \mathcal{D}(\Omega)^3$  be a test function in (2.8). By Proposition 3.4, one has

$$\begin{aligned} & \mu \int_{\tilde{\Omega}_\varepsilon} D_{x'} \tilde{u}_\varepsilon : D_\varepsilon \tilde{\varphi} \, dx' dy_3 - \int_{\tilde{\Omega}_\varepsilon} \tilde{p}_\varepsilon \operatorname{div}_\varepsilon \tilde{\varphi} \, dx' dy_3 + \alpha \frac{\varepsilon^{\gamma-1}}{|Y'|} \int_{\omega \times \partial T} \hat{u}_\varepsilon \cdot \hat{\varphi}_\varepsilon \, dx' d\sigma(y) \\ &= \int_{\tilde{\Omega}_\varepsilon} f'_\varepsilon \cdot \tilde{\varphi}' \, dx' dy_3 + \frac{\varepsilon^{-1}}{|Y'|} \int_{\omega \times \partial T} \tilde{g}' \cdot \hat{\varphi}'_\varepsilon \, dx' d\sigma(y), \end{aligned}$$

i.e.,

$$\begin{aligned} & \mu \int_{\tilde{\Omega}_\varepsilon} D_{x'} \tilde{u}_\varepsilon : D_{x'} \tilde{\varphi} \, dx' dy_3 + \frac{\mu}{\varepsilon^2} \int_{\tilde{\Omega}_\varepsilon} \partial_{y_3} \tilde{u}_\varepsilon \cdot \partial_{y_3} \tilde{\varphi} \, dx' dy_3 - \int_{\tilde{\Omega}_\varepsilon} \tilde{p}_\varepsilon \operatorname{div}_{x'} \tilde{\varphi}' \, dx' dy_3 - \frac{1}{\varepsilon} \int_{\tilde{\Omega}_\varepsilon} \tilde{p}_\varepsilon \partial_{y_3} \tilde{\varphi}_3 \, dx' dy_3 \\ &+ \alpha \frac{\varepsilon^{\gamma-1}}{|Y'|} \int_{\omega \times \partial T} \hat{u}_\varepsilon \cdot \hat{\varphi}_\varepsilon \, dx' d\sigma(y) = \int_{\tilde{\Omega}_\varepsilon} f'_\varepsilon \cdot \tilde{\varphi}' \, dx' dy_3 + \frac{\varepsilon^{-1}}{|Y'|} \int_{\omega \times \partial T} \tilde{g}' \cdot \hat{\varphi}'_\varepsilon \, dx' d\sigma(y). \end{aligned} \quad (3.81)$$

First, we prove that  $\tilde{p}$  does not depend on  $y_3$ . Let  $\tilde{\varphi} = (0, \varepsilon^{-\gamma+1} \tilde{\varphi}_3) \in \mathcal{D}(\Omega)^3$  be a test function in (3.81), we have

$$\begin{aligned} & \mu \varepsilon^{-\gamma+1} \int_{\tilde{\Omega}_\varepsilon} \nabla_{x'} \tilde{u}_{\varepsilon,3} \cdot \nabla_{x'} \tilde{\varphi}_3 \, dx' dy_3 + \mu \varepsilon^{-\gamma-1} \int_{\tilde{\Omega}_\varepsilon} \partial_{y_3} \tilde{u}_{\varepsilon,3} \partial_{y_3} \tilde{\varphi}_3 \, dx' dy_3 - \varepsilon^{-\gamma} \int_{\tilde{\Omega}_\varepsilon} \tilde{p}_\varepsilon \partial_{y_3} \tilde{\varphi}_3 \, dx' dy_3 \\ &+ \frac{\alpha}{|Y'|} \int_{\omega \times \partial T} \hat{u}_{\varepsilon,3} \cdot \hat{\varphi}_{\varepsilon,3} \, dx' d\sigma(y) = 0. \end{aligned}$$

Taking into account that  $\tilde{P}_\varepsilon$  is the extension of  $\tilde{p}_\varepsilon$  in  $\Omega$ , we have

$$\int_{\tilde{\Omega}_\varepsilon} \tilde{p}_\varepsilon \partial_{y_3} \tilde{\varphi}_3 \, dx' dy_3 = \int_{\Omega} \tilde{P}_\varepsilon \partial_{y_3} \tilde{\varphi}_3 \, dx' dy_3,$$

and by the second *a priori* estimate (3.41), the convergences (3.67) and (3.74), passing to the limit we have

$$\int_{\Omega} \tilde{p} \partial_{y_3} \tilde{\varphi}_3 \, dx' dy_3 = 0,$$

so  $\tilde{p}$  does not depend on  $y_3$ .

Let  $\tilde{\varphi} = (\varepsilon^{-\gamma} \varphi'(x', y_3), \varepsilon^{-\gamma} \tilde{\varphi}_3(x')) \in \mathcal{D}(\Omega)^3$  be a test function in (3.81), we have

$$\begin{aligned} & \mu \varepsilon^{-\gamma} \int_{\tilde{\Omega}_\varepsilon} D_{x'} \tilde{u}'_\varepsilon : D_{x'} \tilde{\varphi}' \, dx' dy_3 + \mu \varepsilon^{-\gamma-2} \int_{\tilde{\Omega}_\varepsilon} \partial_{y_3} \tilde{u}'_\varepsilon \cdot \partial_{y_3} \tilde{\varphi}' \, dx' dy_3 - \varepsilon^{-\gamma} \int_{\tilde{\Omega}_\varepsilon} \tilde{p}_\varepsilon \operatorname{div}_{x'} \tilde{\varphi}' \, dx' dy_3 \\ &+ \alpha \frac{\varepsilon^{-1}}{|Y'|} \int_{\omega \times \partial T} \hat{u}'_\varepsilon \cdot \hat{\varphi}'_\varepsilon \, dx' d\sigma(y) = \varepsilon^{-\gamma} \int_{\tilde{\Omega}_\varepsilon} f'_\varepsilon \cdot \tilde{\varphi}' \, dx' dy_3 + \frac{\varepsilon^{-\gamma-1}}{|Y'|} \int_{\omega \times \partial T} \tilde{g}' \cdot \hat{\varphi}'_\varepsilon \, dx' d\sigma(y), \end{aligned}$$

and

$$\mu \varepsilon^{-\gamma} \int_{\tilde{\Omega}_\varepsilon} \nabla_{x'} \tilde{u}_{\varepsilon,3} \cdot \nabla_{x'} \tilde{\varphi}_3 \, dx' dy_3 + \alpha \frac{\varepsilon^{-1}}{|Y'|} \int_{\omega \times \partial T} \hat{u}_{\varepsilon,3} \cdot \hat{\varphi}_{\varepsilon,3} \, dx' d\sigma(y) = 0.$$

Taking into account that  $\tilde{P}_\varepsilon$  is the extension of  $\tilde{p}_\varepsilon$  in  $\Omega$ , we have

$$\int_{\tilde{\Omega}_\varepsilon} \tilde{p}_\varepsilon \operatorname{div}_{x'} \tilde{\varphi}' \, dx' dy_3 = \int_{\Omega} \tilde{P}_\varepsilon \operatorname{div}_{x'} \tilde{\varphi}' \, dx' dy_3.$$

Using that  $\hat{\varphi}_\varepsilon$  converges strongly to  $\tilde{\varphi}$  in  $L^2(\omega \times \partial T)^3$  (see Proposition 2.6 in [17] for more details) and by the second *a priori* estimate (3.41), the convergences (3.67) and (3.74), passing to the limit we have

$$- \int_{\Omega} \tilde{p}(x') \operatorname{div}_{x'} \tilde{\varphi}'(x', y_3) \, dx' dy_3 + \frac{\alpha}{|Y'|} \int_{\omega \times \partial T'} \int_0^1 \hat{u}'(x', y) \cdot \tilde{\varphi}'(x', y_3) \, dx' d\sigma(y') dy_3 = 0,$$

and

$$\frac{\alpha}{|Y'|} \int_{\omega \times \partial T'} \hat{u}_3(x', y) \tilde{\varphi}_3(x') dx' d\sigma(y) = 0,$$

which implies that  $\mathcal{M}_{\partial T'}[\hat{u}_3] = 0$ .

Taking into account that

$$\int_{\omega \times \partial T'} \int_0^1 \hat{u}'(x', y) \cdot \tilde{\varphi}'(x', y_3) dx' d\sigma(y') dy_3 = |\partial T'| \int_{\Omega} \mathcal{M}_{\partial T'}[\hat{u}'](x', y_3) \cdot \tilde{\varphi}'(x', y_3) dx' dy_3,$$

implies that

$$\int_{\Omega} \nabla_{x'} \tilde{p}(x') \cdot \tilde{\varphi}'(x', y_3) dx' dy_3 = -\frac{\alpha |\partial T'|}{|Y'|} \int_{\Omega} \mathcal{M}_{\partial T'}[\hat{u}'](x', y_3) \cdot \tilde{\varphi}'(x', y_3) dx' dy_3. \quad (3.82)$$

In order to obtain the homogenized system (2.10), we introduce the auxiliary problem

$$\begin{cases} -\Delta_{y'} \chi(y) = -\frac{|\partial T'|}{|Y'_f|} \mathcal{M}_{Y'_f}[\hat{u}](x', y_3), & \text{in } Y'_f, \\ \frac{\partial \chi}{\partial n} = \hat{u}, & \text{on } \partial T', \\ \mathcal{M}_{Y'_f}[\chi] = 0, \\ \chi(y) \text{ } Y' \text{-periodic,} \end{cases}$$

for a.e.  $(x', y_3) \in \Omega$ , which has a unique solution in  $H^1(Y'_f)$  (see Chapter 2, Section 7.3 in Lions and Magenes [22]). Using this auxiliary problem, we conclude that

$$\int_{\Omega} \mathcal{M}_{\partial T'}[\hat{u}] \cdot \tilde{\varphi} dx' dy_3 = \int_{\Omega} \mathcal{M}_{Y'_f}[\hat{u}] \cdot \tilde{\varphi} dx' dy_3, \quad (3.83)$$

which together with (3.82) and  $\mathcal{M}_{\partial T'}[\hat{u}_3] = 0$  gives

$$\mathcal{M}_{Y'_f}[\hat{u}'](x', y_3) = -\frac{|Y'|}{\alpha |\partial T'|} \nabla_{x'} \tilde{p}(x'),$$

and

$$\mathcal{M}_{Y'_f}[\hat{u}_3] = 0,$$

which together with (3.75) gives

$$\tilde{u}'(x', y_3) = -\frac{|Y'_f|}{\alpha |\partial T'|} \nabla_{x'} \tilde{p}(x'), \quad \tilde{u}_3(x', y_3) = 0.$$

This implies (2.10).

*Step 2.* For  $-1 \leq \gamma < 1$ . First, we prove that  $\tilde{p}$  does not depend on  $y_3$ . Let  $\tilde{\varphi} = (0, \varepsilon^2 \tilde{\varphi}_3) \in \mathcal{D}(\Omega)^3$  be a test function in (3.81). Reasoning as Step 1 and by the second *a priori* estimate (3.42), the convergence (3.68) and (3.77), passing to the limit we deduce that  $\tilde{p}$  does not depend on  $y_3$ .

Let  $\tilde{\varphi} = (\varepsilon \varphi'(x', y_3), \varepsilon \tilde{\varphi}_3(x')) \in \mathcal{D}(\Omega)^3$  be a test function in (3.81). Reasoning as Step 1 and by the second *a priori* estimate (3.42), the convergences (2.9), (3.68) and (3.77), passing to the limit we have

$$\begin{aligned} & - \int_{\Omega} \tilde{p}(x') \operatorname{div}_{x'} \tilde{\varphi}'(x', y_3) dx' dy_3 + \frac{\alpha}{|Y'|} \int_{\omega \times \partial T'} \int_0^1 \hat{u}'(x') \cdot \tilde{\varphi}'(x', y_3) dx' d\sigma(y') dy_3 \\ & = \int_{\Omega} f'(x') \cdot \tilde{\varphi}'(x', y_3) dx' dy_3 + \frac{1}{|Y'|} \int_{\omega \times \partial T'} \int_0^1 g'(y') \cdot \tilde{\varphi}'(x', y_3) dx' d\sigma(y') dy_3, \end{aligned}$$

and

$$\frac{\alpha}{|Y'|} \int_{\omega \times \partial T} \hat{u}_3(x') \tilde{\varphi}_3(x') dx' d\sigma(y) = 0,$$

which implies that  $\hat{u}_3 = 0$ .

Taking into account that

$$\int_{\omega \times \partial T'} \int_0^1 \hat{u}'(x') \cdot \tilde{\varphi}'(x', y_3) dx' d\sigma(y') dy_3 = |\partial T'| \int_{\Omega} \hat{u}'(x') \cdot \tilde{\varphi}'(x', y_3) dx' dy_3,$$

implies that

$$\begin{aligned} & \int_{\Omega} \nabla_{x'} \tilde{p}(x') \cdot \varphi'(x', y_3) dx' dy_3 + \frac{\alpha |\partial T'|}{|Y'|} \int_{\Omega} \hat{u}'(x') \cdot \tilde{\varphi}'(x', y_3) dx' dy_3 \\ &= \int_{\Omega} f'(x') \cdot \tilde{\varphi}'(x', y_3) dx' dy_3 + \frac{|\partial T'|}{|Y'|} \int_{\Omega} \mathcal{M}_{\partial T'}[g'] \cdot \tilde{\varphi}'(x', y_3) dx' dy_3, \end{aligned}$$

which together with (3.78) gives (2.11) after integrating the vertical variable  $y_3$  between 0 and 1. gives

*Step 3.* For  $\gamma \geq 1$ . First, we prove that  $\tilde{p}$  does not depend on  $y_3$ . Let  $\tilde{\varphi} = (0, \varepsilon^3 \tilde{\varphi}_3) \in \mathcal{D}(\Omega)^3$  be a test function in (3.81). Reasoning as Step 1 and by the second *a priori* estimate (3.59), the convergences (3.69) and (3.80), passing to the limit we deduce that  $\tilde{p}$  does not depend on  $y_3$ .

Let  $\tilde{\varphi} = (\varepsilon^2 \varphi'(x', y_3), 0) \in \mathcal{D}(\Omega)^3$  be a test function in (3.81). Reasoning as Step 1 and by the second *a priori* estimate (3.59), the convergences (2.9), (3.69), (3.72) and (3.80), passing to the limit we have

$$\mu \int_{\Omega} \partial_{y_3} \tilde{u}' \cdot \partial_{y_3} \tilde{\varphi}' dx' dy_3 - \int_{\Omega} \tilde{p} \operatorname{div}_{x'} \tilde{\varphi}' dx' dy_3 = 0,$$

which is equivalent

$$\tilde{u}'(x', y_3) = \frac{y_3(y_3 - 1)}{2\mu} \nabla_{x'} \tilde{p}(x') \quad \text{a.e. in } \Omega.$$

Integrating the vertical variable  $y_3$  between 0 and 1 gives (2.12).

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