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A norm for signed measures, with application to non local transport equation with source term

Benedetto Piccoli ^{*} Francesco Rossi [†] Magali Tournus [‡]

Abstract

We provide a norm on the space of signed Radon measures with finite mass, based on the generalized Wasserstein distance for measures with different masses. We describe converging sequences for this norm, that might be neither uniformly bounded nor uniformly tight.

This norm enables us to obtain existence and uniqueness for solutions to non-local and non-linear transport equations with source terms, when the initial condition is a signed measure. It can be found as the limit of a convergent scheme.

Key-words. Wasserstein distance, Transport equation, Signed measures.

AMS subject classifications. 28A33, 35A01.

1 Introduction

The goal of the paper is to provide a framework to guarantee the well-posedness of the non local transport equation with source term,

$$\partial_t \mu_t(x) + \operatorname{div}(v[\mu_t](x)\mu_t(x)) = h[\mu_t](x), \quad \mu_{|t=0}(x) = \mu_0(x), \quad (1)$$

for $x \in \mathbb{R}^d$ and $\mu_0 \in \mathcal{M}^s(\mathbb{R}^d)$, where $\mathcal{M}^s(\mathbb{R}^d)$ is the space of signed Radon measures with finite mass on \mathbb{R}^d . Equation (1) has already been studied in the framework of positive measures where it has natural interpretations such that the evolution of a crowd density. The motivation for studying equation (1) in the framework of signed measure is the interpretation of μ_t as the spatial derivative of the entropy solution $\rho(x, t)$ to a scalar conservation law. A link between scalar conservation laws and non local transport equation has been initiated [2], but until now, studies are restricted to convex fluxes and a monotonous initial conditions, so that the spatial derivative μ_t is a positive measure for all $t > 0$ with a preserved mass.

1.1 A family of generalized Wasserstein norms

The first main content of this paper is the definition of the norm $\|\cdot\|^{a,b}$ on $\mathcal{M}^s(\mathbb{R}^d)$ where a and b are any positive real numbers, (Definition 5) and the statement of the topological properties of $(\mathcal{M}^s(\mathbb{R}^d), \|\cdot\|^{a,b})$ in Theorems 1 and 2. The usual Wasserstein distance $W_p(\mu, \nu)$ was defined between two measures μ and ν of same mass $|\mu| = |\nu|$, see more details in [11]. A transference plan between

^{*}Department of Mathematical Sciences, Rutgers University Camden, Camden, NJ, USA. Email: piccoli@camden.rutgers.edu.

[†]Dipartimento di Matematica “Tullio Levi-Civita” Università degli Studi di Padova, Via Trieste 63, 35121 Padova, Italy. Email: francesco.rossi@math.unipd.it

[‡]Aix Marseille Univ, CNRS, Centrale Marseille, I2M, Marseille, France. Email: mtournus@math.cnrs.fr.

two measures of same mass μ and ν is a probability measure $\pi \in \mathcal{P}(\mathbb{R}^d, \mathbb{R}^d)$ which satisfies for all $A, B \in \mathcal{B}(\mathbb{R}^d)$

$$\pi(A \times \mathbb{R}^d) = \frac{\mu(A)}{|\mu|}, \quad \pi(\mathbb{R}^d \times B) = \frac{\nu(B)}{|\nu|}.$$

We denote by $\Pi(\mu, \nu)$ the set of transference plans between μ and ν . The p-Wasserstein distance for positive Radon measures of same mass is defined as

$$W_p(\mu, \nu) = |\mu|^{\frac{1}{p}} \left(\min_{\pi \in \Pi(\mu, \nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^p d\pi(x, y) \right)^{\frac{1}{p}}.$$

It was extended to positive measures having possibly different mass in [10], where the authors introduce the distance $W_p^{a,b}$ on the space $\mathcal{M}(\mathbb{R}^d)$ of Radon measures with finite mass.

Definition 1 (Generalized Wasserstein distance [9]). *Let μ, ν be two positive measures in $\mathcal{M}(\mathbb{R}^d)$. The generalized Wasserstein distance between μ and ν is given by*

$$W_p^{a,b} = \left(\inf_{\substack{\tilde{\mu}, \tilde{\nu} \in \mathcal{M}(\mathbb{R}^d) \\ |\tilde{\mu}| = |\tilde{\nu}|}} a^p (|\mu - \tilde{\mu}| + |\nu - \tilde{\nu}|)^p + b^p W_p^p(\tilde{\mu}, \tilde{\nu}) \right)^{1/p}. \quad (2)$$

A part of the mass $\tilde{\mu} \leq \mu$ is transported onto a mass $\tilde{\nu} \leq \nu$ with a weight b , the rest being canceled with a weight a . Notice that for $a/b = a'/b'$ the distance $W_p^{a,b} = b/b' W_p^{a',b'}$. Notice that the infimum in (2) is always attained, moreover at least one minimizer satisfies the additional constraint $\tilde{\mu} \leq \mu, \tilde{\nu} \leq \nu$. Alternative distances on the space of positive measures were introduced at the same period (see [4] [6], [7]).

The authors of [1] suggested to extend the usual Wasserstein distance W_1 to the couples of signed measures $\mu = \mu^+ - \mu^-$ and $\nu = \nu^+ - \nu^-$ such that $|\mu^+| + |\nu^-| = |\mu^-| + |\nu^+|$ by the formula $\mathbb{W}_1(\mu, \nu) = \mathbb{W}_1(\mu^+ + \nu^-, \mu^- + \nu^+)$. This procedure fails for $p \neq 1$, since triangular inequality is lost (see counter-example in [1]). We use the same trick to turn the generalized Wasserstein distance into a distance for signed measure and define $\mathbb{W}_1^{a,b}(\mu, \nu)$ as $W_1^{a,b}(\mu^+ + \nu^-, \mu^- + \nu^+)$. The space of signed measures being a vectorial space, we also define a norm $\|\mu\|^{a,b} = \mathbb{W}_1^{a,b}(\mu, 0)$. Notice that to define the norm $\|\cdot\|^{a,b}$, we need to restrict ourselves to Radon measures with finite mass, since the generalized Wasserstein distance [9] is not defined for Radon measures with infinite mass.

1.2 A framework for the transport equation for signed measures

The second main content of the present paper is the well-posedness of the transport equation (1). The regularity assumptions made in this paper on the velocity field and on the source term are the following

H-1 There exists K such that for all $\mu, \nu \in \mathcal{M}^s(\mathbb{R}^d)$ it holds

$$\|v[\mu] - v[\nu]\|_{C^0(\mathbb{R}^d)} \leq K \|\mu - \nu\|^{a,b}. \quad (3)$$

H-2 There exist L, M such that for all $x, y \in \mathbb{R}^d$, for all $\mu \in \mathcal{M}^s(\mathbb{R}^d)$ it holds

$$|v[\mu](x) - v[\mu](y)| \leq L|x - y|, \quad |v[\mu](x)| \leq M. \quad (4)$$

H-3 There exist Q, P, R such that for all $\mu \in \mathcal{M}^s(\mathbb{R}^d)$ it holds

$$\|h[\mu] - h[\nu]\|^{a,b} \leq Q \|\mu - \nu\|^{a,b}, \quad |h[\mu]| \leq P, \quad \text{supp}(h[\mu]) \subset B_0(R). \quad (5)$$

We emphasize that the assumptions (H-2) and (H-3) are incompatible with a direct interpretation of the solution of (1) as the spatial derivative of a conservation law and need to be relaxed in a future work. Indeed, to draw a parallel between conservation laws and non-local equations, discontinuous velocity fields need to be considered.

Throughout the paper, $\mathcal{B}(\mathbb{R}^d)$ is the space of Borelian sets on \mathbb{R}^d , $\mathcal{M}(\mathbb{R}^d)$ is the space of Radon measures (i.e. Borel regular, positive, and finite on every compact set), $\mathcal{M}^s(\mathbb{R}^d)$ is the space of signed Radon measures, i.e. measures μ that can be written as $\mu = \mu_+ - \mu_-$ with $\mu_+, \mu_- \in \mathcal{M}(\mathbb{R}^d)$. For $\mu \in \mathcal{M}^s(\mathbb{R}^d)$, we define $|\mu| = |\mu_+^J| + |\mu_-^J|$ where (μ_+^J, μ_-^J) is the unique Jordan decomposition of μ .

In Section 2, we state and prove preliminary results which are needed for the rest of the paper. In Section 3, we define the generalized Wasserstein distance for signed measures, we show that it can be used to define a norm, and prove some topological properties. Section 4 is devoted to the use of the norm defined here to guarantee existence, uniqueness, and stability to initial condition for the transport equation (1).

2 Preliminary results

In this section, we introduce the notations and state preliminary results.

2.1 Recall on measure theory

In this section, μ and ν are in $\mathcal{M}(\mathbb{R}^d)$.

Definition 2. *We say that*

- $\mu \ll \nu$ if $\forall A \in \mathcal{B}(\mathbb{R}^d)$, $(\nu(A) = 0) \Rightarrow (\mu(A) = 0)$
- $\mu \leq \nu$ if $\forall A \in \mathcal{B}(\mathbb{R}^d)$, $\mu(A) \leq \nu(A)$
- $\mu \perp \nu$ if there exists $E \in \mathcal{B}(\mathbb{R}^d)$ such that $\forall A \in \mathcal{B}(\mathbb{R}^d)$, $\mu(A) = \mu(A \cap E)$ and $\nu(A) = \nu(A \cap E^c)$

The concept of largest common mass between measures is now recalled.

Lemma 1. *We consider μ and ν two measures in $\mathcal{M}(\mathbb{R}^d)$. Then, there exists a unique measure $\mu \wedge \nu$ which satisfies*

$$\mu \wedge \nu \leq \mu, \quad \mu \wedge \nu \leq \nu, \quad (\eta \leq \mu \text{ and } \eta \leq \nu) \Rightarrow \eta \leq \mu \wedge \nu. \quad (6)$$

We refer to $\mu \wedge \nu$ as the largest common mass to μ and ν . Moreover, denoting by f the Radon-Nikodym derivative of μ with respect to ν , i.e. the unique measurable function f such that $\mu = f\nu + \nu_\perp$, with $\nu_\perp \perp \nu$, we have

$$\mu \wedge \nu = \min\{f, 1\}\nu. \quad (7)$$

Proof. The uniqueness is clear using (6). Existence is given by formula (7) as follows. First, it is obvious that $\min\{f, 1\}\nu \leq \nu$ and using $\mu = f\nu + \nu_\perp$, it is also clear that $\min\{f, 1\}\nu \leq \mu$. Let us now assume by contradiction the existence of a measure η and of $A \in \mathcal{B}(\mathbb{R}^d)$ such that

$$\eta \leq \mu, \quad \eta \leq \nu, \quad \eta(A) > \int_A \min\{f(x), 1\}d\nu(x). \quad (8)$$

Since $\nu_\perp \perp \nu$, there exists $E \in \mathcal{B}(\mathbb{R}^d)$ such that $\forall A \in \mathcal{B}(\mathbb{R}^d)$, $\nu(A) = \nu(A \cap E)$ and $\nu_\perp(A) = \nu_\perp(A \cap E^c)$. Since $\eta \leq \nu$, we have

$$\eta(A \cap E) = \eta(A) > \int_{A \cap E} \min\{f(x), 1\} d\nu(x).$$

We define

$$B = \{x \in A \cap E, f(x) > 1\}.$$

Then

$$\begin{aligned} \eta(B) + \eta((A \cap E) \setminus B) &= \eta(A \cap E) > \int_B \min\{f(x), 1\} d\nu(x) + \int_{(A \cap E) \setminus B} \min\{f(x), 1\} d\nu(x) \\ &= \nu(B) + \mu((A \cap E) \setminus B) - \nu_\perp((A \cap E) \setminus B) \\ &= \nu(B) + \mu((A \cap E) \setminus B) \end{aligned}$$

which contradicts the fact that $\eta \leq \nu$ and $\eta \leq \mu$. This implies that η satisfying (8) does not exist, and then (7) holds. \square

Lemma 2. *We consider μ and ν two measures in $\mathcal{M}(\mathbb{R}^d)$. The following properties hold.*

1. $\mu \perp \nu \Leftrightarrow \mu \wedge \nu = 0$
2. $(\mu - \mu \wedge \nu) \perp (\nu - \mu \wedge \nu)$
3. $\eta \leq \mu + \nu$ implies $\eta - (\mu \wedge \eta) \leq \nu$
4. $\mu \leq \eta$ and $\nu \leq \eta$ implies $\mu + \nu - (\mu \wedge \nu) \leq \eta$.

Proof. 1. If $\mu \perp \nu$, then the unique Radon Nikodym decomposition gives $f = 0$, ν a.e. and $\mu = \nu_\perp$, and thus using (7), $\mu \wedge \nu = 0$. Conversely, if $\mu \wedge \nu = 0$, then, using again (7), $\min\{f, 1\}\nu = 0$, thus the Radon decomposition gives $\mu = f\nu + \nu_\perp = \nu_\perp$ with $\nu_\perp \perp \nu$, which implies $\mu \perp \nu$.

2. We write $\mu = f\nu + \nu_\perp$ with $\nu_\perp \perp \nu$. Then, $\mu - (\mu \wedge \nu) = \nu_\perp + \max\{0, f - 1\}\nu$ and $\mu - (\mu \wedge \nu) = \max\{0, 1 - f\}\nu$. We pick E such that $\nu_\perp(A) = \nu_\perp(A \cap E^c)$ and $\nu(A) = \nu(A \cap E)$. Considering $F = \{x \in E | f(x) < 1\}$, it is easy to check that $(\mu - (\mu \wedge \nu))(A) = (\mu - (\mu \wedge \nu))(A \cap F^c)$ and $(\nu - (\mu \wedge \nu))(A) = (\nu - (\mu \wedge \nu))(A \cap F)$, and the result holds.

3. We write $\mu = f\eta + \eta_\perp$, with $\eta_\perp \perp \eta$. Then $\eta \wedge \mu = \min\{f, 1\}\eta$, and we can write $\eta(A) - (\eta \wedge \mu)(A) = \int_A (1 - \min\{f(x), 1\}) d\eta(x)$. Defining $B = \{x \in A | f(x) < 1\}$, and E such that $\eta(A \cap E) = \eta(A)$ and $\eta_\perp(A \cap E^c) = \eta_\perp(A)$, we have

$$\eta(A) - (\eta \wedge \mu)(A) = \int_{B \cap E} (1 - f(x)) d\eta(x) = \eta(B \cap E) + \eta_\perp(B \cap E) - \mu(B \cap E) \leq \nu(B \cap E) \leq \nu(A).$$

4. We write $\mu \wedge \nu = \min\{f, 1\}\nu$, with $\mu = f\nu + \nu_\perp$. Then $w := \mu + \nu - (\mu \wedge \nu) = f\nu + \nu_\perp - \min\{f, 1\}\nu + \nu$, which is $w(A) = \int_A (1 + f(x) - \min\{f(x), 1\}) d\nu(x) + \nu_\perp(A)$. Defining $B = \{x \in A | f(x) < 1\}$, and E such that $\nu(A \cap E) = \nu(A)$ and $\nu_\perp(A \cap E^c) = \nu_\perp(A)$, we have

$$\begin{aligned} w(A) &= \nu(B) + \int_{A \cap B^c} f(x) d\nu(x) + \nu_\perp(A) = \nu(B) + \mu(A \cap B^c) - \nu_\perp(A \cap B^c) + \nu_\perp(A) \\ &= \nu(B) + \mu(A \cap B^c) + \nu_\perp(B) \end{aligned}$$

and then, since $\nu \leq \eta$ and $\nu_\perp \leq \mu \leq \eta$,

$$w(A) = \nu(B \cap E) + \mu(A \cap B^c) + \nu_\perp(B \cap E^c) \leq \eta(B \cap E) + \eta(A \cap B^c) + \eta(B \cap E^c) = \eta(A).$$

\square

To end this section, we recall the definition of tightness for a sequence of $\mathcal{M}^s(\mathbb{R}^d)$.

Definition 3. A sequence $(\mu_n)_{n \in \mathbb{N}}$ of measures of $\mathcal{M}(\mathbb{R}^d)$ is tight if for all $\varepsilon > 0$, there is a compact set $K \subset \mathbb{R}^d$ such that for all $n \geq 0$, $\mu_n(\mathbb{R}^d \setminus K) < \varepsilon$. A sequence $(\mu_n)_{n \in \mathbb{N}}$ of signed measures of $\mathcal{M}^s(\mathbb{R}^d)$ is tight if the sequences $(\mu_n^+)_{n \in \mathbb{N}}$ and $(\mu_n^-)_{n \in \mathbb{N}}$ given by the Jordan decomposition are both tight.

2.2 Properties of the generalized Wasserstein distance

In this section, we recall key properties of the generalized Wasserstein distance.

Lemma 3. Let μ, ν be two measures of $\mathcal{M}(\mathbb{R}^d)$ of same mass. Given $\eta \in \mathcal{M}(\mathbb{R}^d)$ a measure of finite mass, there exists a transference plan π realizing the 1-Wasserstein distance $W_1(\mu + \eta, \nu + \eta)$ such that η is the image of η under π , i.e. π can be decomposed as $\pi = \pi_1 + \pi_2$ with $\pi_1 \in \Pi(\mu, \nu)$ and $\pi_2 \in \Pi(\eta, \eta)$ and the restriction of π to $\pi_2 \in \Pi(\eta, \eta)$ is supported on the diagonal $\{x = y\}$.

Proof. First, writing $\mu = |\mu|\bar{\mu}$ and $\nu = |\nu|\bar{\nu}$, the Kantorovich-Rubinstein duality gives us the equality

$$\begin{aligned} W_1(\mu, \nu) &= |\mu|W_1(\bar{\mu}, \bar{\nu}) = |\mu| \sup_{\|\varphi\|_{Lip} \leq 1} \int_{\mathbb{R}^d} \varphi(x) d(\bar{\mu} - \bar{\nu})(x) = \sup_{\|\varphi\|_{Lip} \leq 1} \int_{\mathbb{R}^d} \varphi(x) d(\mu - \nu)(x) \\ &= \sup_{\|\varphi\|_{Lip} \leq 1} \int_{\mathbb{R}^d} \varphi(x) d(\mu + \eta - \nu - \eta)(x) = W_1(\mu + \eta, \nu + \eta). \end{aligned}$$

Let us now denote by $\bar{\pi} \in \Pi(\mu, \nu)$ one optimal plan realizing $W_1(\mu, \nu)$. We claim here that there is a plan $\pi \in \Pi(\mu + \eta, \nu + \eta)$ such that η is the image of η under π and such that the restriction of π to $\pi' \in \Pi(\eta, \eta)$ is supported on the diagonal $\{x = y\}$, π being given by the formula

$$\pi = \frac{1}{|\mu + \eta|} (|\mu|\bar{\pi} + (Id, Id)\#\eta), \quad \text{where } ((Id, Id)\#\eta)(A \times B) = \frac{\eta(A)\eta(B)}{|\eta|}.$$

Then we have

$$|\mu + \eta| \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y| d\pi(x, y) = \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y| d\bar{\pi}(x, y) = W_1(\mu, \nu) = W_1(\mu + \eta, \nu + \eta),$$

and thus the plan π realizes the distance $W_1(\mu + \eta, \nu + \eta)$. \square

We notice that Lemma 3 is only valid for $p = 1$ and is trivially false for $p \neq 1$ since $W_p(\delta_0 + \delta_1, \delta_1 + \delta_2) = 2^{\frac{1}{p}} (\frac{1}{2}|1 - 0|^p + \frac{1}{2}|2 - 1|^p) = 2^{\frac{1}{p}} < 2 = W_p(\delta_0, \delta_2)$.

Lemma 4 (Properties of the generalized Wasserstein distance). *Let $\mu, \nu, \eta, \mu_1, \mu_2, \nu_1, \nu_2$ be some positive measures with finite mass on \mathbb{R}^d . The following properties hold*

1. $W_p^{a,b}(\mu_1 + \mu_2, \nu_1 + \nu_2) \leq W_p^{a,b}(\mu_1, \nu_1) + W_p^{a,b}(\mu_2, \nu_2)$.
2. $W_1^{a,b}(\mu + \eta, \nu + \eta) = W_1^{a,b}(\mu, \nu)$,

Proof. The first property is taken from [9] Proposition 11. We only prove the second property. The first inequality is clear,

$$\begin{aligned}
W_1^{a,b}(\mu + \eta, \nu + \eta) &= \min_{\substack{m_\mu \leq \mu + \eta \\ m_\nu \leq \nu + \eta}} \{a(|\mu + \eta - m_\mu| + |\nu + \eta - m_\nu|) + bW_1(m_\mu, m_\nu)\} \\
&\leq \min_{\substack{\tilde{\mu} \leq \mu \\ \tilde{\nu} \leq \nu}} \{a(|\mu + \eta - (\tilde{\mu} + \eta)| + |\nu + \eta - (\tilde{\nu} + \eta)|) + bW_1(\tilde{\mu} + \eta, \tilde{\nu} + \eta)\} \\
&= \min_{\substack{\tilde{\mu} \leq \mu \\ \tilde{\nu} \leq \nu}} \{a(|\mu - \tilde{\mu}| + |\nu - \tilde{\nu}|) + bW_1(\tilde{\mu}, \tilde{\nu})\} = W_1^{a,b}(\mu, \nu).
\end{aligned} \tag{9}$$

To prove the reverse inequality, we need to check that the minimum realizing $W_1^{a,b}(\mu + \eta, \nu + \eta)$ can be attained for some $\tilde{\mu} \geq \eta$, $\tilde{\nu} \geq \eta$.

Step 1. As a first step, we notice that we can assume that $\mu \perp \nu$. Indeed, if the result holds for μ, ν satisfying $\mu \perp \nu$, then, for any μ, ν positive measures,

$$\begin{aligned}
W_1^{a,b}(\mu, \nu) &\leq W_1^{a,b}(\mu - \mu \wedge \nu, \nu - \mu \wedge \nu) \quad \text{using (9)} \\
&= W_1^{a,b}(\mu - \mu \wedge \nu + \mu \wedge \nu + \eta, \nu - \mu \wedge \nu + \mu \wedge \nu + \eta) \quad \text{using Lemma 2, 2,} \\
&= W_1^{a,b}(\mu + \eta, \nu + \eta).
\end{aligned}$$

From now on, we assume $\mu \perp \nu$.

Step 2. We choose two minimizers m_μ and m_ν satisfying $W_1^{a,b}(\mu, \nu) = a(|\mu - m_\mu| + |\nu - m_\nu|) + bW_1(m_\mu, m_\nu)$. We decompose here m_μ and m_ν as

$$\begin{aligned}
m_\mu &= \bar{\mu} + \eta_\mu + \eta_c, \\
m_\nu &= \bar{\nu} + \eta_\nu + \eta_c,
\end{aligned} \tag{10}$$

with

$$\bar{\mu} \leq \mu, \quad \bar{\nu} \leq \nu, \quad \eta_\mu + \eta_\nu + \eta_c \leq \eta,$$

defined above, see Figure 1. This decomposition may not be unique, but there exists at least one that we exhibit. We first define

$$\bar{\mu} = m_\mu \wedge \mu, \quad \bar{\nu} = m_\nu \wedge \nu.$$

By construction, it holds $\bar{\mu} \leq \mu$ and $\bar{\nu} \leq \nu$. Then, calling $\hat{\mu} := m_\mu - \bar{\mu}$ and $\hat{\nu} := m_\nu - \bar{\nu}$, we set

$$\eta_c = \hat{\mu} \wedge \hat{\nu}, \quad \eta_\mu = \hat{\mu} - \eta_c, \quad \eta_\nu = \hat{\nu} - \eta_c.$$

We have $m_\mu \leq \mu + \eta$ and then $\hat{\mu} = m_\mu - m_\mu \wedge \mu \leq \eta$ using Lemma 2,3, and samely, $\hat{\nu} \leq \eta$. As a consequence,

$$\eta_\mu + \eta_\nu + \eta_c = \hat{\mu} + \hat{\nu} - \hat{\mu} \wedge \hat{\nu} \leq \eta$$

using Lemma 2,4. Finally, we check that

$$\bar{\mu} + \eta_c + \eta_\mu = m_\mu \wedge \mu + \hat{\mu} \wedge \hat{\nu} + \hat{\mu} - \hat{\mu} \wedge \hat{\nu} = m_\mu \wedge \mu + m_\mu - m_\mu \wedge \mu = m_\mu, \quad \text{and } \bar{\nu} + \eta_c + \eta_\nu = m_\nu.$$

Step 3. Using decomposition (10), we write

$$W_1^{a,b}(\mu + \eta, \nu + \eta) = a(|\mu - \bar{\mu}| + |\eta - \eta_\mu - \eta_c| + |\nu - \bar{\nu}| + |\eta - \eta_\nu - \eta_c|) + bW_1(\bar{\mu} + \eta_\mu + \eta_c, \bar{\nu} + \eta_\nu + \eta_c)$$

We denote by π the optimal plan realizing $W_1(\bar{\mu} + \eta_\mu + \eta_c, \bar{\nu} + \eta_\nu + \eta_c)$. Using Lemma 3, we can assume that π sends η_c to η_c . The mass $\bar{\mu}$ is then split by π into two pieces. Let us denote by

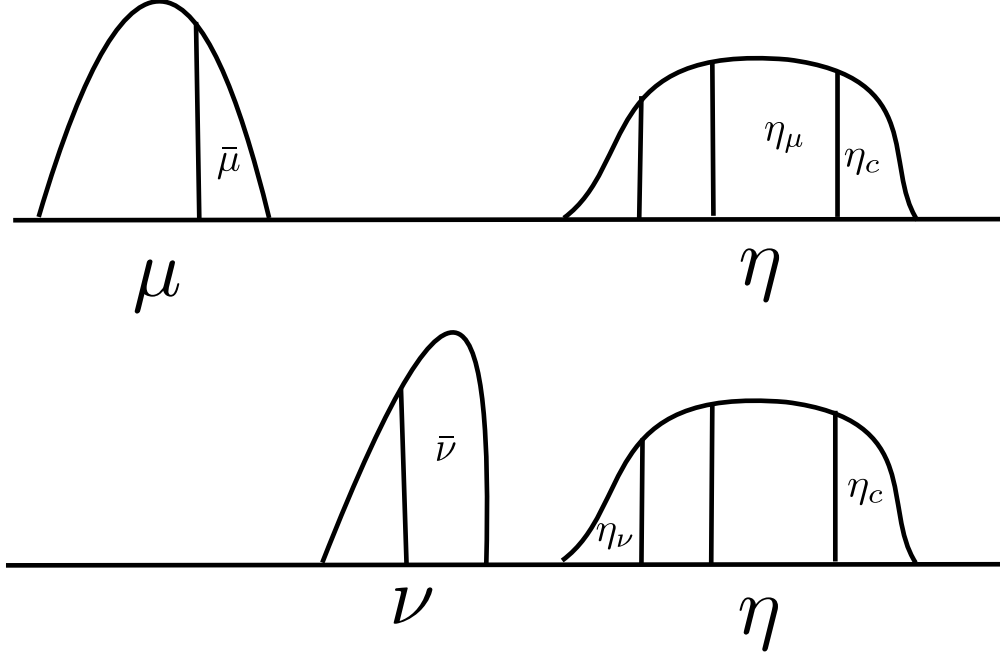


Figure 1: Representation of the measures μ , ν and η in the simple case where their supports do not overlap.

$\mu^* \leq \bar{\mu}$ the mass which is sent to ν , the remainder $(\bar{\mu} - \mu^*)$ being sent on η_ν . Let us also denote by ν^* the image of μ^* under π . We have then

$$|\mu^*| = |\nu^*|, \quad |\bar{\mu} - \mu^*| \leq |\eta_\nu|, \quad |\bar{\mu} - \mu^*| + |\eta_\mu| = |\bar{\nu} - \nu^*| + |\eta_\nu|, \quad \text{and thus} \quad |\bar{\nu} - \nu^*| \leq |\eta_\mu|. \quad (11)$$

We claim now that the minimum realizing $W_1^{a,b}(\mu + \eta, \nu + \eta)$ is also attained for

$$\mu' = \mu^* + \eta, \quad \nu' = \nu^* + \eta.$$

Indeed, we have

$$\begin{aligned} |\mu - \mu^*| + |\nu - \nu^*| &= |\mu - \bar{\mu}| + |\bar{\mu} - \mu^*| + |\nu - \bar{\nu}| + |\bar{\nu} - \nu^*| \\ &\leq |\mu - \bar{\mu}| + |\eta_\nu| + |\nu - \bar{\nu}| + |\eta_\mu|, \quad \text{using (11)} \\ &\leq |\mu - \bar{\mu}| + |\eta - \eta_c - \eta_\mu| + |\nu - \bar{\nu}| + |\eta - \eta_c - \eta_\nu|, \end{aligned}$$

and

$$W_1(\mu^* + \eta, \nu^* + \eta) \leq W_1(\mu^*, \nu^*) \leq W_1(\bar{\mu} + \eta_\mu + \eta_c, \bar{\nu} + \eta_\nu + \eta_c),$$

since the optimal plan $\pi \in \Pi(\bar{\mu} + \eta_\mu + \eta_c, \bar{\nu} + \eta_\nu + \eta_c)$, can be restricted to $\pi \in \Pi(\mu^*, \nu^*)$ (by definition of μ^* and ν^*). Then, we can assume that the minimum realizing $W_1^{a,b}(\mu + \eta, \nu + \eta)$ is attained for $m_\mu \geq \eta$, $m_\nu \geq \eta$, and the inequality in (9) in an equality. \square

Notice that this property is not verified for $p \neq 1$: the results only holds true for $p = 1$ because the cost $c(x, y) = |x - y|$ is a metric (see [11], Corollary 1.16).

3 Generalized Wasserstein norm for signed measures

In this section, we define the generalized Wasserstein distance for signed measures and prove some of its properties. The idea is to follow what was already done in [1] for generalizing the classical Wasserstein distance.

Definition 4 (Generalized Wasserstein distance extended to signed measures). *For μ, ν two signed measures with finite mass over \mathbb{R}^d , we define*

$$\mathbb{W}_1^{a,b}(\mu, \nu) = W_1^{a,b}(\mu_+ + \nu_-, \mu_- + \nu_+),$$

where μ_+, μ_-, ν_+ and ν_- are any measures in $\mathcal{M}(\mathbb{R}^d)$ such that $\mu = \mu_+ - \mu_-$ and $\nu = \nu_+ - \nu_-$.

Proposition 1. *The operator $\mathbb{W}_1^{a,b}$ is a distance on the space $\mathcal{M}^s(\mathbb{R}^d)$ of signed measures with finite mass on \mathbb{R}^d .*

Proof. First, we point out that the definition does not depend on the decomposition. Indeed, if we consider two distinct decompositions, $\mu = \mu_+ - \mu_- = \mu_+^J - \mu_-^J$, and $\nu = \nu_+ - \nu_- = \nu_+^J - \nu_-^J$, with the second one being the Jordan decomposition, then we have $(\mu_+ + \nu_-) - (\mu_+^J + \nu_-^J) = (\mu_- + \nu_+) - (\mu_-^J + \nu_+^J)$, and this is a positive measure since $\mu_+ \geq \mu_+^J$ and $\nu_+ \geq \nu_+^J$. The first property of Lemma 4 then gives us $W_1^{a,b}(\mu_+^J + \nu_-^J, \mu_-^J + \nu_+^J) = W_1^{a,b}(\mu_+^J + \nu_-^J + (\mu_- + \nu_+) - (\mu_-^J + \nu_+^J), \mu_-^J + \nu_+^J + (\mu_- + \nu_+) - (\mu_-^J + \nu_+^J)) = W_1^{a,b}(\mu_+ + \nu_-, \mu_- + \nu_+)$.

We now prove that $\mathbb{W}_1^{a,b}(\mu, \nu) = 0$ implies $\mu = \nu$. We saw that we can assume that we use the Jordan decomposition. Since $W_1^{a,b}$ is a distance, we obtain $\mu_+ + \nu_- = \mu_- + \nu_+$. The orthogonality of μ_+ and μ_- and of ν_+ and ν_- implies that $\mu_+ = \nu_+$ and $\mu_- = \nu_-$, and thus $\mu = \nu$.

We now prove the triangle inequality. We have $\mathbb{W}_1^{a,b}(\mu, \eta) = W_1^{a,b}(\mu_+ + \eta_-, \mu_- + \eta_+)$. Using Lemma 4, we have $\mathbb{W}_1^{a,b}(\mu, \eta) = W_1^{a,b}(\mu_+ + \eta_- + \nu_+ + \nu_-, \mu_- + \eta_+ + \nu_+ + \nu_-) \leq W_1^{a,b}(\mu_+ + \nu_-, \mu_- + \nu_+) + W_1^{a,b}(\eta_- + \nu_+, \eta_+ + \nu_-)$ which is $\mathbb{W}_1^{a,b}(\mu, \nu) + \mathbb{W}_1^{a,b}(\nu, \eta)$. \square

As a particular case, for any $x, y \in \mathbb{R}^d$ it holds $\mathbb{W}_1^{a,b}(\delta_x, -\delta_y) = 2a$.

We also state the following lemma about adding and removing masses.

Lemma 5. *Let $\mu, \nu, \eta, \mu_1, \mu_2, \nu_1, \nu_2$ in $\mathcal{M}^s(\mathbb{R}^d)$ with finite mass on \mathbb{R}^d . The following properties hold*

- $\mathbb{W}_1^{a,b}(\mu + \eta, \nu + \eta) = \mathbb{W}_1^{a,b}(\mu, \nu)$,
- $\mathbb{W}_1^{a,b}(\mu_1 + \mu_2, \nu_1 + \nu_2) \leq \mathbb{W}_1^{a,b}(\mu_1, \nu_1) + \mathbb{W}_1^{a,b}(\mu_2, \nu_2)$.

Proof. The proof is direct. For the first item, it holds $\mathbb{W}_1^{a,b}(\mu + \eta, \nu + \eta) = W_1^{a,b}(\mu_+ + \nu_+ + \eta_+ + \eta_-, \mu_- + \nu_- + \eta_+ + \eta_-)$ which is $W_1^{a,b}(\mu_+ + \nu_+, \mu_- + \nu_-) = \mathbb{W}_1^{a,b}(\mu, \nu)$.

For the second item, it holds $\mathbb{W}_1^{a,b}(\mu_1 + \mu_2, \nu_1 + \nu_2) = W_1^{a,b}(\mu_{1,+} + \mu_{2,+} + \nu_{1,-} + \nu_{2,-}, \nu_{1,+} + \nu_{2,+} + \mu_{1,-} + \mu_{2,-}) \leq W_1^{a,b}(\mu_{1,+} + \nu_{1,-}, \nu_{1,+} + \mu_{1,-}) + W_1^{a,b}(\mu_{2,+} + \nu_{2,-}, \nu_{2,+} + \mu_{2,-})$ which is $\mathbb{W}_1^{a,b}(\mu_1, \nu_1) + \mathbb{W}_1^{a,b}(\mu_2, \nu_2)$. \square

Definition 5. *For $\mu \in \mathcal{M}^s(\mathbb{R}^d)$ and $a > 0, b > 0$, we define*

$$\|\mu\|^{a,b} = \mathbb{W}_1^{a,b}(\mu, 0) = W_1^{a,b}(\mu_+, \mu_-),$$

where μ_+ and μ_- are any measures of $\mathcal{M}(\mathbb{R}^d)$ such that $\mu = \mu_+ - \mu_-$.

Proposition 2. *The space of signed measures $(\mathcal{M}^s(\mathbb{R}^d), \|\cdot\|^{a,b})$ is a normed vector space.*

Proof. First, we notice that $\|\mu\|^{a,b} = 0$ implies that $W_1^{a,b}(\mu, \mu_-) = 0$, which is $\mu_+ = \mu_-$ so that $\mu = \mu_+ - \mu_- = 0$. For triangular inequality, we write that for $\mu, \nu, \eta \in \mathcal{M}^s(\mathbb{R}^d)$, $\|\mu + \eta\|^{a,b} = \mathbb{W}_1^{a,b}(\mu + \eta, 0) \leq \mathbb{W}_1^{a,b}(\mu, 0) + \mathbb{W}_1^{a,b}(\eta, 0)$ using the second property of Lemma 5, and this is equal to $\|\mu\|^{a,b} + \|\eta\|^{a,b}$. Homogeneity is obtained by writing for $\lambda > 0$, $\|\lambda\mu\|^{a,b} = \mathbb{W}_1^{a,b}(\lambda\mu, 0) = W_1^{a,b}(\lambda\mu_+, \lambda\mu_-)$ where $\mu = \mu_+ - \mu_-$. We denote by $\bar{\mu}_+, \bar{\mu}_-$ the minimum realizing $W_1^{a,b}(\mu_+, \mu_-)$, which is $W_1^{a,b}(\mu_+, \mu_-) = C(\bar{\mu}_+, \bar{\mu}_-) = a(|\mu_+ - \bar{\mu}_+| + |\mu_- - \bar{\mu}_-|) + bW_1(\bar{\mu}_+, \bar{\mu}_-)$, then, one minimum realizing $W_1^{a,b}(\lambda\mu_+, \lambda\mu_-)$ for $\lambda > 0$ is $\lambda\bar{\mu}_+, \lambda\bar{\mu}_-$ so that $W_1^{a,b}(\lambda\mu_+, \lambda\mu_-) = C_\lambda(\lambda\bar{\mu}_+, \lambda\bar{\mu}_-) = a(|\lambda\mu_+ - \lambda\bar{\mu}_+| + |\lambda\mu_- - \lambda\bar{\mu}_-|) + bW_1(\lambda\bar{\mu}_+, \lambda\bar{\mu}_-)$. This comes from the fact that for all positive Radon measures μ, ν , we have $\lambda C(\mu, \nu) = C_\lambda(\lambda\mu, \lambda\nu)$. Then $W_1^{a,b}(\lambda\mu_+, \lambda\mu_-) = \lambda W_1^{a,b}(\mu_+, \mu_-) = \lambda \mathbb{W}_1^{a,b}(\mu, 0) = \lambda \|\mu\|^{a,b}$. \square

3.1 Topological properties

Proposition 3. *For $a > 0, b > 0$, the norm $\|\cdot\|^{a,b}$ is equivalent to $\|\cdot\|^{1,1}$.*

Proof. For $\mu \in \mathcal{M}^s(\mathbb{R}^d)$ denote by $(m_+^{a,b}, m_-^{a,b})$ the positive measures such that

$$\|\mu\|^{a,b} = a|\mu_+ - m_+^{a,b}| + a|\mu_- - m_-^{a,b}| + bW_1(m_+^{a,b}, m_-^{a,b}).$$

By definition of the minimizers, we have

$$\begin{aligned} \|\mu\|^{a,b} &= a|\mu_+ - m_+^{a,b}| + a|\mu_- - m_-^{a,b}| + bW_1(m_+^{a,b}, m_-^{a,b}) \\ &\leq a|\mu_+ - m_+^{1,1}| + a|\mu_- - m_-^{1,1}| + bW_1(m_+^{1,1}, m_-^{1,1}) \leq \max\{a, b\}\|\mu\|^{1,1}, \end{aligned}$$

and in the same way, we obtain

$$\min\{a, b\}\|\mu\|^{1,1} \leq \|\mu\|^{a,b} \leq \max\{a, b\}\|\mu\|^{1,1}.$$

\square

We give now, as it was done in [10], an equivalent of the Rubinstein-Kantorovich duality for the new distance. We denote by $\mathcal{C}_c^0(\mathbb{R}^d; \mathbb{R})$ the set of continuous functions on \mathbb{R}^d with compact support. For $f \in \mathcal{C}_c^0(\mathbb{R}^d; \mathbb{R})$, we define

$$\|f\|_{Lip} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|}.$$

We introduce

$$\mathcal{C}_c^{0,Lip} = \{f \in \mathcal{C}_c^0(\mathbb{R}^d; \mathbb{R}) \mid \|f\|_{Lip} < \infty\}.$$

In the next proposition, we prove the identity between $\mathbb{W}_1^{a,b}$ and the flat metric extended to signed measures.

Proposition 4 (The flat metric). *The signed generalized Wasserstein distance $\mathbb{W}_1^{a,b}$ coincides with the flat metric in the sense that for μ, ν in $\mathcal{M}^s(\mathbb{R}^d)$,*

$$\mathbb{W}_1^{a,b}(\mu, \nu) = \sup \left\{ \int_{\mathbb{R}^d} \varphi d(\mu - \nu); \varphi \in \mathcal{C}_c^{0,Lip}, \|\varphi\|_\infty \leq 1, \|\varphi\|_{Lip} \leq 1 \right\}$$

Proof. The proof comes directly from the equivalent property for $W_1^{1,1}$ given in [10]. We have

$$\begin{aligned} W_1^{a,b}(\mu, \nu) &= W_1^{a,b}(\mu_+ + \nu_-, \nu_+ + \mu_-) \\ &= \sup \left\{ \int_{\mathbb{R}^d} \varphi d(\mu_+ - \mu_- - (\nu_+ - \nu_-)); \varphi \in \mathcal{C}_c^{0,Lip}, \|\varphi\|_\infty \leq 1, \|\varphi\|_{Lip} \leq 1 \right\} \\ &= \sup \left\{ \int_{\mathbb{R}^d} \varphi d(\mu - \nu); \varphi \in \mathcal{C}_c^{0,Lip}, \|\varphi\|_\infty \leq 1, \|\varphi\|_{Lip} \leq 1 \right\}. \end{aligned}$$

□

Proposition 5. For $\mu_n, \mu \in \mathcal{M}^s(\mathbb{R}^d)$, we have

$$\|\mu_n - \mu\|^{a,b} \xrightarrow{n \rightarrow \infty} 0 \quad \Rightarrow \quad \forall \varphi \in \mathcal{C}_c^{0,Lip}(\mathbb{R}^d), \int_{\mathbb{R}^d} \varphi d\nu_n \xrightarrow{n \rightarrow \infty} \int_{\mathbb{R}^d} \varphi d\nu$$

Proof. The proof comes directly from Proposition 4. □

The converse statement is not true, take for instance $\mu_n = \delta_n$.

We emphasize that a sequence μ_n of $\mathcal{M}^s(\mathbb{R})$ which satisfies $\|\mu_n\|^{a,b} \xrightarrow{n \rightarrow \infty} 0$ is not necessarily tight, and its mass is not necessarily bounded. For instance, we have

$$\mu_n = n \delta_{\frac{1}{n^2}} - n \delta_{-\frac{1}{n^2}}$$

satisfies $\|\mu_n\|^{a,b} = \frac{2bn}{n^2}$ for n sufficiently large (depending on a and b , it may be less expensive to cancel the mass than to transport it), so that $\|\mu_n\|^{a,b} \xrightarrow{n \rightarrow \infty} 0$ whereas $|\mu_n| = 2n$ is not bounded. The sequence

$$\nu_n = \delta_n - \delta_{n+\frac{1}{n}}$$

is not tight, whereas it satisfies for n sufficiently large $\|\nu_n\|^{a,b} = \frac{b}{n} \xrightarrow{n \rightarrow \infty} 0$.

Proposition 6. Assume that $\|\mu_n\|^{a,b} \xrightarrow{n \rightarrow \infty} 0$, then $\Delta m_n = |\mu_n^+| - |\mu_n^-| \xrightarrow{n \rightarrow \infty} 0$.

Proof. We have by definition $\|\mu_n\|^{a,b} = W_1^{a,b}(\mu_n^+, \mu_n^-)$. We denote by $\bar{\mu}_n^+, \bar{\mu}_n^-$ the minimizers realizing the distance $W_1^{a,b}(\mu_n^+, \mu_n^-)$. We have

$$\|\mu_n\|^{a,b} = a (|\mu_n^+ - \bar{\mu}_n^+| + |\mu_n^- - \bar{\mu}_n^-|) + b W_1(\bar{\mu}_n^+, \bar{\mu}_n^-), \quad |\bar{\mu}_n^+| = |\bar{\mu}_n^-|.$$

Since $\|\mu_n\|^{a,b} \xrightarrow{n \rightarrow \infty} 0$, each of the three terms is going to zero as well. Thus,

$$\begin{aligned} \left| |\mu_n^+| - |\mu_n^-| \right| &= \left| |\mu_n^+ - \bar{\mu}_n^+ + \bar{\mu}_n^+| - |\mu_n^- - \bar{\mu}_n^- + \bar{\mu}_n^-| \right| \\ &= \left| |\mu_n^+ - \bar{\mu}_n^+| + |\bar{\mu}_n^+| - |\mu_n^- - \bar{\mu}_n^-| - |\bar{\mu}_n^-| \right| \\ &= \left| |\mu_n^+ - \bar{\mu}_n^+| - |\mu_n^- - \bar{\mu}_n^-| \right| \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

□

Theorem 1. The two following statements are equivalent:

$$(i) \quad \|\mu_n - \mu\|^{a,b} \xrightarrow{n \rightarrow \infty} 0.$$

(ii) There exists $z_n^+, z_n^-, m_n^+, m_n^- \in \mathcal{M}(\mathbb{R}^d)$

$$\begin{aligned} \mu_n^+ &= z_n^+ + m_n^+, \\ \mu_n^- &= z_n^- + m_n^-, \end{aligned} \quad \text{with} \quad \begin{aligned} W_1^{a,b}(z_n^+, z_n^-) &\xrightarrow{n \rightarrow \infty} 0, \\ W_1^{a,b}(m_n^+, \mu^+) &\xrightarrow{n \rightarrow \infty} 0, \\ W_1^{a,b}(m_n^-, \mu^-) &\xrightarrow{n \rightarrow \infty} 0, \end{aligned} \\ &\{m_n^+\}_n \text{ and } \{m_n^-\}_n \text{ are tight and bounded in mass,} \end{aligned}$$

where $\mu = \mu^+ - \mu^-$ is the Jordan decomposition, and $\mu_n = \mu_n^+ - \mu_n^-$ is any decomposition.

Proof. We start by proving (i) \Rightarrow (ii). We have $\|\mu_n - \mu\|^{a,b} = \mathbb{W}_1^{a,b}(\mu_n, \mu) = W_1^{a,b}(\mu_n^+ + \mu^-, \mu_n^- + \mu^+) \xrightarrow{n \rightarrow \infty} 0$. Let us denote by $a_n \leq (\mu_n^+ + \mu^+)$ and $b_n \leq (\mu_n^- + \mu^+)$ the minimizers realizing $W_1^{a,b}(\mu_n^+ + \mu^-, \mu_n^- + \mu^+)$. We call π_n the transference plan from a_n to b_n .

Step 1. The removed mass. We define by a_n^+ and b_n^- the largest transported mass which is respectively below μ_n^+ and μ_n^-

$$\begin{aligned} a_n^+ &= \mu_n^+ \wedge a_n, & b_n^- &= \mu_n^- \wedge b_n, \\ a_n^- &= a_n - a_n^+, & b_n^+ &= b_n - b_n^-. \end{aligned}$$

The mass which is removed is then $r_n = r_n^+ + r_n^- := (\mu_n^+ - a_n^+) + (\mu_n^- - a_n^-)$ and $r_n^* = r_n^{*, -} + r_n^{*, +} := (\mu_n^- - b_n^-) + (\mu_n^+ - b_n^+)$. The removed mass r_n and r_n^* are expressed here as the sum of two positive measures. Indeed, it is clear by definition that $a_n^+ \leq \mu_n^+$, and since $a_n \leq \mu_n^+ + \mu^+$, Lemma 2, 3 gives that $a_n^- = a_n - a_n \wedge \mu_n^+ \leq \mu^-$. We reason the same way for r_n^* . Then, we have $W_1^{a,b}(\mu_n^+ + \mu^-, \mu_n^- + \mu^+) = a(|\mu_n^+ - a_n^+| + |\mu^- - a_n^-| + |\mu_n^- - b_n^-| + |\mu^+ - b_n^+|) + bW_1(a_n, b_n)$. Since $W_1^{a,b}(\mu_n^+ + \mu^-, \mu_n^- + \mu^+)$ goes to zero, each of the five terms of the above decomposition goes to zero, and in particular, $|\mu_n^+ - a_n^+| \xrightarrow{n \rightarrow \infty} 0$ and $|\mu_n^- - b_n^-| \xrightarrow{n \rightarrow \infty} 0$ which implies that that

$$W_1^{a,b}(\mu_n^+ - a_n^+, 0) \xrightarrow{n \rightarrow \infty} 0, \quad W_1^{a,b}(\mu_n^- - b_n^-, 0) \xrightarrow{n \rightarrow \infty} 0. \quad (12)$$

Step 2. The transported mass. The mass a_n^+ is split into two pieces: ν_n is sent to μ_n^- , and ξ_n is sent to μ^+ . Denote by \bar{a}_n^+ the image of a_n^+ under π_n , then we define $\nu_n^* = \bar{a}_n^+ \wedge \mu_n^-$. We denote by ν_n the image of ν_n^* under π_n . Then, we define ξ_n such that $a_n^+ = \nu_n + \xi_n$, and we denote by ξ_n^* the image of ξ_n under π_n . By definition, we have

$$W_1(a_n, b_n) = W_1(\nu_n, \nu_n^*) + W_1(\xi_n, \xi_n^*) + W_1(w_n, w_n^*) + W_1(\alpha_n, \alpha_n^*), \quad (13)$$

with $a_n^+ = \nu_n + \xi_n$, w_n^* is defined so that $b_n^- = \nu_n^* + w_n^*$, w_n is the image of w_n^* under π_n , α_n is defined so that $\mu^- = w_n + \alpha_n$, α_n^* is the image of α_n under π_n , and it can be checked that $\mu^+ = \xi_n^* + \alpha_n^*$. Since $W_1(a_n, b_n) \xrightarrow{n \rightarrow \infty} 0$, each of the four term of the sum (13) is going to zero.

Step 3. Conclusion.

Let us write

$$z_n^+ = \nu_n + (\mu_n^+ - a_n^+), \quad z_n^- = \nu_n^* + (\mu_n^- - b_n^-), \quad m_n^+ = \xi_n, \quad m_n^- = w_n^*.$$

We show here that the sequences defined hereinabove satisfy the conditions stated in (ii). First, we have $z_n^+ + m_n^+ = \nu_n + (\mu_n^+ - a_n^+) + \xi_n = \mu_n^+$ and similarly, $z_n^- + m_n^- = \nu_n^* + (\mu_n^- - b_n^-) + w_n^* = \mu_n^-$.

Then, we have

$$\begin{aligned}
W_1^{a,b}(z_n^+, z_n^-) &= W_1^{a,b}(\nu_n + (\mu_n^+ - a_n^+), \nu_n^* + (\mu_n^- - b_n^-)) \\
&\leq W_1^{a,b}(\nu_n, \nu_n^*) + W_1^{a,b}(\mu_n^+ - a_n^+, \mu_n^- - b_n^-) \quad \text{using Lemma 4} \\
&\leq W_1(\nu_n, \nu_n^*) + W_1^{a,b}(\mu_n^+ - a_n^+, 0) + W_1^{a,b}(0, \mu_n^- - b_n^-) \\
&\xrightarrow{n \rightarrow \infty} 0, \quad \text{using (12) and (13)}.
\end{aligned}$$

Here, we also used that for $|\mu| = |\nu|$, $W_1^{a,b}(\mu, \nu) \leq W_1(\mu, \nu)$. This is trivial with the definition of $W_1^{a,b}$. Now, we also have

$$\begin{aligned}
W_1^{a,b}(m_n^+, \mu^+) &= W_1^{a,b}(\xi_n, \mu^+) \leq W_1^{a,b}(\xi_n, \xi_n^*) + W_1^{a,b}(\xi_n^*, b_n^+) + W_1^{a,b}(b_n^+, \mu^+) \quad (\text{triangular inequality}) \\
&= W_1^{a,b}(\xi_n, \xi_n^*) + W_1^{a,b}(\alpha_n^*, 0) + W_1^{a,b}(\mu^+ - b_n^+, 0)
\end{aligned}$$

since $\alpha_n^* + \xi_n^* = b_n^+$. We know that $W_1^{a,b}(\xi_n, \xi_n^*) \leq W_1(\xi_n, \xi_n^*) \xrightarrow{n \rightarrow \infty} 0$ using (13), and that $W_1^{a,b}(\mu^+ - b_n^+, 0) \xrightarrow{n \rightarrow \infty} 0$ using (12). Let us explain now why $W_1^{a,b}(\alpha_n^*, 0) \xrightarrow{n \rightarrow \infty} 0$. We recall that $W_1(\alpha_n, \alpha_n^*) \xrightarrow{n \rightarrow \infty} 0$, $\alpha_n \leq a_n^- \leq \mu^-$, $\alpha_n^* \leq b_n^+ \leq \mu^+$. Since $(\alpha_n)_n$ is uniformly bounded in mass, then there exists $\alpha \in \mathcal{M}(\mathbb{R}^d)$ such that $\alpha_{\varphi(n)} \xrightarrow{n \rightarrow \infty} \alpha$ (weak compactness of uniformly bounded in mass Radon measures, see [5]). We have also that $(\alpha_{\varphi(n)})_n$ is tight, since $\alpha_{\varphi(n)} \leq \mu^-$ which has a finite mass. Using Theorem 13 of [8], we deduce that $W_1^{a,b}(\alpha_{\varphi(n)}, \alpha) \xrightarrow{n \rightarrow \infty} 0$. Then, $W_1^{a,b}(\alpha_{\varphi(n)}^*, \alpha) \leq W_1^{a,b}(\alpha_{\varphi(n)}^*, \alpha_{\varphi(n)}) + W_1^{a,b}(\alpha_{\varphi(n)}, \alpha) \leq W_1(\alpha_{\varphi(n)}^*, \alpha_{\varphi(n)}) + W_1^{a,b}(\alpha_{\varphi(n)}, \alpha) \xrightarrow{n \rightarrow \infty} 0$. Then, using again Theorem 13 of [8], we deduce that $\alpha_{\varphi(n)}^* \xrightarrow{n \rightarrow \infty} \alpha$. Since $\alpha_n \leq \mu^-$, we have $\alpha \leq \mu^-$. Likewise, $\alpha_n^* \leq \mu^+$ implies $\alpha \leq \mu^+$. Since $\mu^- \perp \mu^+$, we have $\alpha = 0$. We have $W_1^{a,b}(\alpha_{\varphi(n)}, 0) \xrightarrow{n \rightarrow \infty} 0$ and $W_1^{a,b}(\alpha_{\varphi(n)}^*, 0) \xrightarrow{n \rightarrow \infty} 0$. The sequence $(\alpha_n)_n$ is such that every of its subsequence has a converging subsequence going to zero. Thus, we have that the whole sequence is converging to zero, i.e. $W_1^{a,b}(\alpha_n, 0) \xrightarrow{n \rightarrow \infty} 0$ and $W_1^{a,b}(\alpha_n^*, 0) \xrightarrow{n \rightarrow \infty} 0$. Lastly, the tightness of $(m_n^+)_n$ and $(m_n^-)_n$ is given again by Theorem 13 of [8], since $W_1^{a,b}(m_n^\pm, \mu^\pm) \xrightarrow{n \rightarrow \infty} 0$.

We prove now that (ii) \Rightarrow (i). Let us assume (ii). We have

$$\begin{aligned}
\|\mu_n - \mu\|^{a,b} &= W_1^{a,b}(\mu_n^+ + \mu^-, \mu_n^- + \mu^+) = W_1^{a,b}(z_n^+ + m_n^+ + \mu^-, z_n^- + m_n^- + \mu^+) \\
&\leq W_1^{a,b}(z_n^+, z_n^-) + W_1^{a,b}(m_n^+, \mu^+) + W_1^{a,b}(\mu^-, m_n^-) \\
&\xrightarrow{n \rightarrow \infty} 0,
\end{aligned}$$

which is (i). □

The space $(\mathcal{M}(\mathbb{R}^d), W_p^{a,b})$ is a Banach space. The proof is based on the fact that a Cauchy sequence of positive measures is both uniformly bounded in mass and tight. This is not true anymore for a Cauchy sequence of signed measures.

Remark 1. Observe that $(\mathcal{M}^s(\mathbb{R}^d), \|\cdot\|^{a,b})$ is not a Banach space. Indeed, take the sequence

$$\mu_n = \sum_{i=1}^n \left(\delta_{i+\frac{1}{2^i}} - \delta_{i-\frac{1}{2^i}} \right).$$

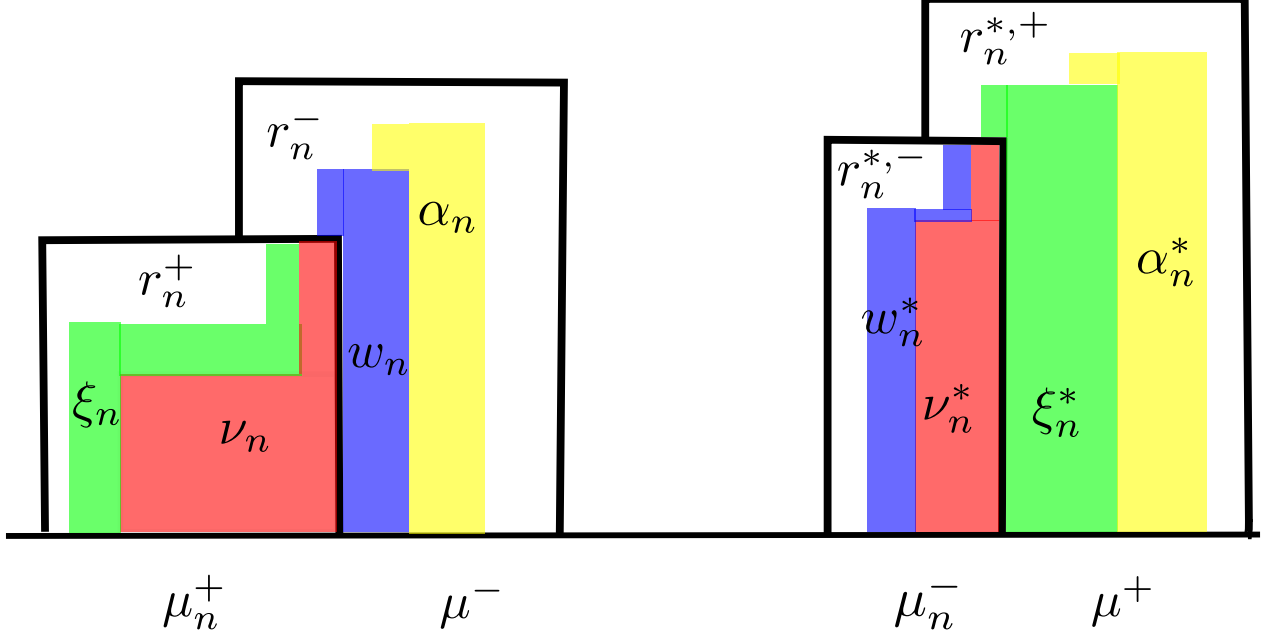


Figure 2: Illustration of the decomposition introduced in the proof of Theorem 1. This picture is only a help for visualization, the shape of the measures represented here may be more complex.

It is a Cauchy sequence in $(\mathcal{M}^s(\mathbb{R}^d), \|\cdot\|^{a,b})$, since it holds

$$W_1^{a,b}(\mu_n, \mu_{n+k}) \leq 2b \sum_{i=n+1}^{n+k} \frac{1}{2^i} \leq 2b \sum_{i=n+1}^{+\infty} \frac{1}{2^i} \xrightarrow{n \rightarrow \infty} 0.$$

However, such sequence does not converge in $(\mathcal{M}^s(\mathbb{R}^d), \|\cdot\|^{a,b})$. As seen in Proposition 5, the convergence for the norm $\|\cdot\|^{a,b}$ implies the convergence in the sense of distributions. In the sense of distributions we have

$$\mu_n \rightharpoonup \mu^* := \sum_{i=1}^{+\infty} \left(\delta_{i+\frac{1}{2^i}} - \delta_{i-\frac{1}{2^i}} \right) \notin \mathcal{M}^s(\mathbb{R}).$$

Indeed, for all $\varphi \in C_c^\infty(\mathbb{R})$, since φ is compactly supported, it holds

$$\langle \mu_n - \mu, \varphi \rangle = \sum_{i=n+1}^{+\infty} \left(\varphi\left(i + \frac{1}{2^i}\right) - \varphi\left(i - \frac{1}{2^i}\right) \right) \xrightarrow{n \rightarrow \infty} 0.$$

Theorem 2. A Cauchy sequence in $(\mathcal{M}^s(\mathbb{R}^d), \|\cdot\|^{a,b})$ uniformly bounded in mass and tight converges in $(\mathcal{M}^s(\mathbb{R}^d), \|\cdot\|^{a,b})$.

Take a tight Cauchy sequence $(\mu_n)_n \in \mathcal{M}^s(\mathbb{R}^d)$ such that the sequences given by the Jordan decomposition $|\mu_n^+|$ and $|\mu_n^-|$ are uniformly bounded. Then, by weak compactness of bounded sets in $\mathcal{M}(\mathbb{R}^d)$, there exists μ^+ and μ^- in $\mathcal{M}(\mathbb{R}^d)$ and φ non decreasing such that, $\mu_{\varphi(n)}^+ \xrightarrow{n \rightarrow \infty} \mu^+$ and $\mu_{\varphi(n)}^- \xrightarrow{n \rightarrow \infty} \mu^-$. Since μ_n^+ and μ_n^- are assumed to be tight, $W_1^{a,b}(\mu_{\varphi(n)}^+, \mu^+) \xrightarrow{n \rightarrow \infty} 0$ and

$W_1^{a,b}(\mu_{\varphi(n)}^-, \mu^-) \xrightarrow{n \rightarrow \infty} 0$ (see Theorem 13 [9]). Then we have

$$\begin{aligned} \|\mu_n - (\mu^+ - \mu^-)\|^{a,b} &= \|\mu_n - \mu_{\varphi(n)}\|^{a,b} + \|\mu_{\varphi(n)} - (\mu^+ - \mu^-)\|^{a,b} \\ &\leq \|\mu_n - \mu_{\varphi(n)}\|^{a,b} + W_1^{a,b}(\mu_{\varphi(n)}^+ + \mu^-, \mu_{\varphi(n)}^- + \mu^+) \\ &\leq \|\mu_n - \mu_{\varphi(n)}\|^{a,b} + W_1^{a,b}(\mu_{\varphi(n)}^+, \mu^+) + W_1^{a,b}(\mu_{\varphi(n)}^-, \mu^-) \xrightarrow{n \rightarrow \infty} 0 \end{aligned}$$

since $(\mu_n)_n$ is a Cauchy sequence.

4 Application to the transport equation with source term

This section is devoted to the use of the norm defined in Definition 5 to guarantee existence, uniqueness, and stability to initial condition for the transport equation (1).

4.1 Preliminary definitions and estimates

In this section, we extend the action of flows on probability measures to signed measures, and state some estimates about the variation of $\|\mu - \nu\|^{a,b}$ after action of a flow on μ and ν . Notice that for $\mu \in \mathcal{M}^s(\mathbb{R}^d)$ and T a map, we have $T\#\mu = T\#\mu^+ - T\#\mu^-$, where $\mu = \mu^+ - \mu^-$ is the Jordan decomposition of μ . Then for $\mu \in \mathcal{M}^s(\mathbb{R}^d)$, we only have $|T\#\mu| \leq |\mu|$, even though it holds $|T\#\mu| = |\mu|$ if T is injective.

Lemma 6. *For $v(t, x)$ continuous in time and Lipschitz in space, we denote by Φ_t^v the flow it generates, i.e. the unique solution to*

$$\frac{d}{dt}\Phi_t^v = v(\Phi_t^v), \quad \Phi_0^v = I_d.$$

Given by $\mu_0 \in \mathcal{M}^s(\mathbb{R}^d)$, then, $\mu_t = \Phi_t^v\#\mu_0$ is the unique solution of the linear transport equation

$$\begin{cases} \frac{\partial}{\partial t}\mu_t + \nabla \cdot (v(t, x)\mu_t) = 0, \\ \mu|_{t=0} = \mu_0 \end{cases}$$

in $\mathcal{C}((0, T), \mathcal{M}^s(\mathbb{R}^d))$.

Proof. The proof is a direct consequence of Theorem 5.34 [11] combined with Theorem 2.1.1 [3]. \square

Lemma 7. *Let v and w be two vector fields, both satisfying for $t \in \mathbb{R}$ and $x, y \in \mathbb{R}^d$, $|v(t, x) - v(t, y)| \leq L|x - y|$ and $|v(t, x)| \leq M$. Let μ and ν be two measures of $\mathcal{M}^s(\mathbb{R}^d)$. Then*

- $\|\phi_t^v\#\mu - \phi_t^v\#\nu\|^{a,b} \leq e^{Lt}\|\mu - \nu\|^{a,b}$
- $\|\mu - \phi_t^v\#\mu\|^{a,b} \leq b t M \mu(\mathbb{R}^d)$, $\mu(\mathbb{R}^d) = \mu^+(\mathbb{R}^d) + \mu^-(\mathbb{R}^d)$
- $\|\phi_t^v\#\mu - \phi_t^w\#\mu\|^{a,b} \leq \mu(\mathbb{R}^d) \frac{(e^{Lt}-1)}{L} \|v - w\|_{\mathcal{C}^0}$
- $\|\phi_t^v\#\mu - \phi_t^w\#\nu\|^{a,b} \leq e^{Lt}\|\mu - \nu\|^{a,b} + \mu(\mathbb{R}^d) \frac{(e^{Lt}-1)}{L} \|v - w\|_{\mathcal{C}^0}$

Proof. The first three inequalities follow from Proposition 10 [10]. For the first inequality, we write

$$\begin{aligned}
\|\phi_t^v \# \mu - \phi_t^v \# \nu\|^{a,b} &= \mathbb{W}_1^{a,b}(\phi_t^v \# \mu, \phi_t^v \# \nu) = \mathbb{W}_1^{a,b}(\phi_t^v \# \mu^+ - \phi_t^v \# \mu^-, \phi_t^v \# \nu^+ - \phi_t^v \# \nu^-) \\
&= W_1^{a,b}(\phi_t^v \# (\mu^+ + \nu^-), \phi_t^v \# (\mu^- + \nu^+)) \\
&= W_1^{a,b}(\phi_t^v \# (\mu^+ + \nu^-), \phi_t^v \# (\mu^- + \nu^+)) \quad (\text{Prop. 10 [10]}) \\
&\leq e^{Lt} W_1^{a,b}(\mu^+ + \nu^-, \mu^- + \nu^+) \\
&= e^{Lt} \|\mu - \nu\|^{a,b}.
\end{aligned}$$

For the second inequality,

$$\begin{aligned}
\mathbb{W}_1^{a,b}(\mu, \phi_t^v \# \mu) &= W_1^{a,b}(\mu^+ + \phi_t^v \# \mu^-, \mu^- + \phi_t^v \# \mu^+) \\
&\leq W_1^{a,b}(\mu^+, \phi_t^v \# \mu^+) + W_1^{a,b}(\mu^-, \phi_t^v \# \mu^-) \quad (\text{Lemma 4}) \\
&\leq b t \|v\|_{\mathcal{C}^0}(\mu^+(\mathbb{R}^d) + \mu^-(\mathbb{R}^d)) \quad (\text{Prop. 10 [10]}) \\
&= b t \|v\|_{\mathcal{C}^0}(\mathbb{R}^d) \quad \text{since } \mu = \mu^+ - \mu^- \text{ is the Jordan decomposition.}
\end{aligned}$$

The third inequality is given by

$$\begin{aligned}
\|\phi_t^v \# \mu - \phi_t^w \# \mu\|^{a,b} &= \mathbb{W}_1^{a,b}(\phi_t^v \# \mu^+ + \phi_t^w \# \mu^-, \phi_t^w \# \mu^+ + \phi_t^v \# \mu^-) \\
&\leq W_1^{a,b}(\phi_t^v \# \mu^+, \phi_t^w \# \mu^+) + W_1^{a,b}(\phi_t^w \# \mu^-, \phi_t^v \# \mu^-) \\
&= W_1(\phi_t^v \# \mu^+, \phi_t^w \# \mu^+) + W_1(\phi_t^w \# \mu^-, \phi_t^v \# \mu^-) \\
&\leq (|\mu^+| + |\mu^-|) \frac{(e^{Lt} - 1)}{L} \|v - w\|_{\mathcal{C}^0(\mathbb{R}^d)} \quad \text{using Prop. 10 [10] with } \mu = \nu.
\end{aligned}$$

The last inequality is deduced from the first and the third one using triangular inequality. \square

4.2 Existence and uniqueness of the solution to the transport equation

In this section, we build a solution to (1) as the limit of a sequence of approximated solutions defined in the following scheme. We then prove that (1) admits a unique solution.

Consider $\mu_0 \in \mathcal{M}^s(\mathbb{R}^d)$ such that $\text{supp}(\mu_0) \in \mathcal{K}$, \mathcal{K} compact, and consider $v \in \mathcal{C}^0(\mathcal{M}^s(\mathbb{R}^d), \mathcal{C}^0(\mathbb{R}^d))$ and $h \in \mathcal{C}^0(\mathcal{M}^s(\mathbb{R}^d), \mathcal{M}^s(\mathbb{R}^d))$ satisfying (H-1), (H-2), (H-3). We define a sequence $(\mu_t^k)_k$ through the following iteration scheme.

SCHEME

Initialization. Fix $k \in \mathbb{N}$. Define $\Delta t = \frac{1}{2^k}$. Set $\mu_0^k = \mu_0$.

Induction. Given $\mu_{i\Delta t}^k$ for $i \in [0, 2^k]$, define $v_{i\Delta t}^k := v[\mu_{i\Delta t}^k]$ and

$$\mu_t^k = \Phi_t^{v_{i\Delta t}^k} \# \mu_{i\Delta t}^k + (t - i\Delta t)h[\mu_{i\Delta t}^k], \quad t \in [i\Delta t, (i+1)\Delta t]. \quad (14)$$

Proposition 7. *The sequence $(\mu_t^k)_k$ defined via the scheme is a Cauchy sequence in the space $\mathcal{C}^0([0, 1], \mathcal{M}^s(\mathbb{R}^d), \|\cdot\|)$ with*

$$\|\mu_t\| = \sup_{t \in [0, 1]} \|\mu_t\|^{a,b}.$$

Moreover, it is uniformly bounded in mass, i.e.

$$\sup_{t \in [0, 1]} |\mu_t^k| < \infty. \quad (15)$$

Proof. We first notice that the sequence built by the scheme satisfies

$$|\mu_t^k| \leq P + |\mu_0|, \quad t \in [0, 1]. \quad (16)$$

Indeed, for $t \in [i\Delta t, (i+1)\Delta t]$

$$|\mu_t^k| \leq |\Phi_t^{v_i \Delta t} \# \mu_{i\Delta t}^k| + \Delta t |h[\mu_{i\Delta t}^k]| \leq |\mu_{i\Delta t}^k| + \Delta t P,$$

and the result follows by induction. This proves (15). The sequence $(\mu_t^k)_{k \in \mathbb{N}}$ is also uniformly tight with respect to time, since using (14) and (H-2), (H-3),

$$\text{supp}\{\mu_t^k\} \subset B_0(R) \cup \mathcal{K}_{t,M},$$

with

$$\text{supp}\{\mu\} = \text{supp}\{\mu^+\} \cup \text{supp}\{\mu^-\}, \quad \text{where } (\mu^+, \mu^-) \text{ is the Jordan decomposition of } \mu,$$

and

$$\mathcal{K}_{t,M} := \{x \in \mathbb{R}^d, x = x_{\mathcal{K}} + x', x_{\mathcal{K}} \in \mathcal{K}, \|x'\| \leq tM\}.$$

We now set the same notations than in [8] and define $m_j^k := \mu_{\frac{j}{2^k}}^k$, $v_j^k := v[m_j^k]$ and the corresponding flow $f_t^{j,k} := \phi_t^{v_j^k}$. Fix $k \in \mathbb{N}$ and $t \in [0, 1]$. Define $j \in [0, 2^k]$ such that $t \in \left] \frac{j}{2^k}, \frac{j+1}{2^k} \right]$

First case. If $t \in \left] \frac{j}{2^k}, \frac{2j+1}{2^{k+1}} \right]$, we call $t' = t - \frac{j}{2^k} \leq \frac{1}{2^{k+1}}$ and we obtain

$$\begin{aligned} \mathbb{W}_1^{a,b}(\mu_t^k, \mu_t^{k+1}) &= \mathbb{W}_1^{a,b}(f_{t'}^{j,k} \# m_j^k + t'h[m_j^k], f_{t'}^{2j,k+1} \# m_{2j}^{k+1} + t'h[m_{2j}^{k+1}]) \\ &\leq \mathbb{W}_1^{a,b}(f_{t'}^{j,k} \# m_j^k, f_{t'}^{2j,k+1} \# m_{2j}^{k+1}) + \mathbb{W}_1^{a,b}(t'h[m_j^k], t'h[m_{2j}^{k+1}]) \\ &\leq e^{Lt'} \mathbb{W}_1^{a,b}(m_j^k, m_{2j}^{k+1}) + |m_j^k| \frac{(e^{Lt'} - 1)}{L} \|v_j^k - v_{2j}^{k+1}\|_{C^0(\mathbb{R}^d)} + t'Q \mathbb{W}_1^{a,b}(m_j^k, m_{2j}^{k+1}) \\ &\leq \mathbb{W}_1^{a,b}(m_j^k, m_{2j}^{k+1}) \left(e^{Lt'} + (P + |\mu^0|) \frac{1}{L} (e^{Lt'} - 1) + t'Q \right) \end{aligned}$$

Since

$$e^{Lt'} \leq 1 + 2Lt' \leq 2L2^{-(k+1)}, \quad \frac{(e^{Lt'} - 1)}{L} \leq 22^{-(k+1)},$$

we have

$$\|\mu_t^k - \mu_t^{k+1}\|^{a,b} \leq \|\mu_{\frac{j}{2^k}}^k - \mu_{\frac{2j}{2^{k+1}}}^{k+1}\|^{a,b} \left(1 + 2^{-(k+1)} (2L + 2(P + |\mu^0|) + Q) \right), \quad t \in \left[\frac{j}{2^k}, \frac{2j+1}{2^{k+1}} \right]. \quad (17)$$

Second case. If $t \in \left] \frac{2j+1}{2^{k+1}}, \frac{j+1}{2^k} \right]$, we call $t' = t - \frac{2j+1}{2^{k+1}} \leq \frac{1}{2^{k+1}}$ and we have,

$$\begin{aligned} \mu_t^k &= f_{t' + \frac{1}{2^{k+1}}}^{j,k} \# m_j^k + \left(t' + \frac{1}{2^{k+1}} \right) h[m_j^k] = f_{t'}^{j,k} \# f_{\frac{1}{2^{k+1}}}^{j,k} \# m_j^k + t'h[m_j^k] + \frac{1}{2^{k+1}} h[m_j^k], \\ \mu_t^{k+1} &= f_{t'}^{2j+1,k+1} \# \left(f_{\frac{1}{2^{k+1}}}^{2j,k+1} \# m_{2j}^{k+1} + \frac{1}{2^{k+1}} h[m_{2j}^{k+1}] \right) + t'h \left[f_{\frac{1}{2^{k+1}}}^{2j,k+1} \# m_{2j}^{k+1} + \frac{1}{2^{k+1}} h[m_{2j}^{k+1}] \right] \\ &= f_{t'}^{2j+1,k+1} \# f_{\frac{1}{2^{k+1}}}^{2j,k+1} \# m_{2j}^{k+1} + \frac{1}{2^{k+1}} f_{t'}^{2j+1,k+1} \# h[m_{2j}^{k+1}] + t'h \left[f_{\frac{1}{2^{k+1}}}^{2j,k+1} \# m_{2j}^{k+1} + \frac{1}{2^{k+1}} h[m_{2j}^{k+1}] \right] \end{aligned}$$

and thus

$$\begin{aligned}
\|\mu_t^k - \mu_t^{k+1}\|^{a,b} &\leq \mathbb{W}_1^{a,b} \left(f_{t'}^{j,k} \# f_{\frac{1}{2^{k+1}}}^{j,k} \# m_j^k, f_{t'}^{2j+1,k+1} \# f_{\frac{1}{2^{k+1}}}^{2j,k+1} \# m_{2j}^{k+1} \right) \\
&\quad + \frac{1}{2^{k+1}} \mathbb{W}_1^{a,b} \left(h[m_j^k], f_{t'}^{2j+1,k+1} \# h[m_{2j}^{k+1}] \right) \\
&\quad + t' \mathbb{W}_1^{a,b} \left(h[m_j^k], h \left[f_{\frac{1}{2^{k+1}}}^{2j,k+1} \# m_{2j}^{k+1} + \frac{1}{2^{k+1}} h[m_{2j}^{k+1}] \right] \right).
\end{aligned} \tag{18}$$

We have first, using Lemma 7,

$$\begin{aligned}
&\mathbb{W}_1^{a,b} \left(f_{t'}^{j,k} \# f_{\frac{1}{2^{k+1}}}^{j,k} \# m_j^k, f_{t'}^{2j+1,k+1} \# f_{\frac{1}{2^{k+1}}}^{2j,k+1} \# m_{2j}^{k+1} \right) \\
&\leq (1 + 2L2^{-(k+1)}) \mathbb{W}_1^{a,b} \left(f_{\frac{1}{2^{k+1}}}^{j,k} \# m_j^k, f_{\frac{1}{2^{k+1}}}^{2j,k+1} \# m_{2j}^{k+1} \right) + 2^{-(k+1)} 2P \|v_j^k - v_{2j+1}^{k+1}\|_{C^0(\mathbb{R}^d)}
\end{aligned}$$

Since, according to the first case,

$$\mathbb{W}_1^{a,b} \left(f_{\frac{1}{2^{k+1}}}^{j,k} \# m_j^k, f_{\frac{1}{2^{k+1}}}^{2j,k+1} \# m_{2j}^{k+1} \right) \leq \|m_j^k - m_{2j}^{k+1}\|^{a,b} \left(1 + 2^{-(k+1)} (2L + 2(P + |\mu^0|)) \right)$$

and

$$\begin{aligned}
\|v_j^k - v_{2j+1}^{k+1}\|_{C^0(\mathbb{R}^d)} &\leq K \mathbb{W}_1^{a,b}(m_j^k, m_{2j+1}^{k+1}) \leq K \mathbb{W}_1^{a,b}(m_j^k, m_{2j}^{k+1}) + K \mathbb{W}_1^{a,b}(m_{2j}^{k+1}, m_{2j+1}^{k+1}) \\
&\leq K \mathbb{W}_1^{a,b}(m_j^k, m_{2j}^{k+1}) + K \mathbb{W}_1^{a,b}(m_{2j}^{k+1}, m_{2j+1}^{k+1}) \\
&= K \mathbb{W}_1^{a,b}(m_j^k, m_{2j}^{k+1}) + K \mathbb{W}_1^{a,b}(m_{2j}^{k+1}, f_{\frac{1}{2^{k+1}}}^{2j,k+1} \# m_{2j}^{k+1}) \\
&= K \mathbb{W}_1^{a,b}(m_j^k, m_{2j}^{k+1}) + KM 2^{-(k+1)},
\end{aligned}$$

we have

$$\begin{aligned}
&\mathbb{W}_1^{a,b} \left(f_{t'}^{j,k} \# f_{\frac{1}{2^{k+1}}}^{j,k} \# m_j^k, f_{t'}^{2j+1,k+1} \# f_{\frac{1}{2^{k+1}}}^{2j,k+1} \# m_{2j}^{k+1} \right) \\
&\leq \|m_j^k - m_{2j}^{k+1}\|^{a,b} \left(1 + 2^{-(k+1)} (4L + 2(P + |\mu^0|)(1 + L) + 2KP) \right) + 2^{-2(k+1)} 2PKM
\end{aligned} \tag{19}$$

Secondly,

$$\begin{aligned}
&\mathbb{W}_1^{a,b} \left(h[m_j^k], f_{t'}^{2j+1,k+1} \# h[m_{2j}^{k+1}] \right) \\
&\leq \mathbb{W}_1^{a,b} \left(h[m_j^k], f_{t'}^{2j+1,k+1} \# h[m_j^k] \right) + \mathbb{W}_1^{a,b} \left(f_{t'}^{2j+1,k+1} \# h[m_j^k], f_{t'}^{2j+1,k+1} \# h[m_{2j}^{k+1}] \right) \\
&\leq t' MP + e^{Lt'} Q \|m_j^k - m_{2j}^{k+1}\|^{a,b} \leq +MP 2^{-(k+1)} + (1 + 2L2^{-(k+1)}) \|m_j^k - m_{2j}^{k+1}\|^{a,b},
\end{aligned} \tag{20}$$

and third

$$\begin{aligned}
&\mathbb{W}_1^{a,b} \left(m_j^k, f_{\frac{1}{2^{k+1}}}^{2j,k+1} \# m_{2j}^{k+1} + \frac{1}{2^{k+1}} h[m_{2j}^{k+1}] \right) \\
&\leq \mathbb{W}_1^{a,b} \left(m_j^k, f_{\frac{1}{2^{k+1}}}^{2j,k+1} \# m_{2j}^{k+1} \right) + 2^{-(k+1)} \mathbb{W}_1^{a,b} \left(0, h[m_{2j}^{k+1}] \right) \\
&\leq \mathbb{W}_1^{a,b} \left(m_j^k, m_{2j}^{k+1} \right) + \mathbb{W}_1^{a,b} \left(m_{2j}^{k+1}, f_{\frac{1}{2^{k+1}}}^{2j,k+1} \# m_{2j}^{k+1} \right) + 2^{-(k+1)} aP \\
&\leq \|m_j^k - m_{2j}^{k+1}\|^{a,b} + 2^{-(k+1)} (|\mu^0| + P(1 + a)).
\end{aligned} \tag{21}$$

Plugging (19), (20) and (21) into (18), and combining with (17) gives us

$$\|\mu_t^k - \mu_t^{k+1}\|^{a,b} \leq (1 + 2^{-k}C_1)\|m_j^k - m_{2j}^{k+1}\|^{a,b} + C_22^{-2k}, \quad t \in \left] \frac{j}{2^k}, \frac{j+1}{2^k} \right],$$

with

$$C_1 = (1 + 3L + (P + |\mu^0|)(1 + L) + KP + Q), \quad C_2 = \frac{1}{4}(MP(1 + 2K) + |\mu^0| + P(1 + a)).$$

By induction on j , we obtain

$$\|\mu_t^k - \mu_t^{k+1}\| \leq \|m_{2^k}^k - m_{2^{k+1}}^{k+1}\|^{a,b} \leq \sum_{j=0}^{2^k-1} (1 + 2^{-k}C_1)^j 2^{-2k} C_2 \leq \frac{C_2}{C_1} (e^{C_1} - 1) 2^{-k}.$$

Since the right hand side is the term of a convergent series, then $(\mu_t^k)_k$ is a Cauchy sequence. \square

Theorem 3 (Existence and uniqueness). *Let v and h satisfy (H-1), (H-2), (H-3) and $\mu_0 \in \mathcal{M}^s(\mathbb{R}^d)$ compactly supported given. Then, there exists a unique distributional solution to (1) in $\mathcal{C}^0((0, 1), \mathcal{M}^s(\mathbb{R}^d))$.*

Proof. The proof is based on the proof of the same result for positive measures written in [10]. We first focus on existence.

Step 1. Existence. Since the sequence given by the scheme $(\mu_t^k)_k$ is a Cauchy sequence (Proposition 7) which is uniformly bounded in mass, in the space $(\mathcal{C}^0(0, 1), \mathcal{M}^s(\mathbb{R}^d))$, we can define using Theorem 2

$$\mu_t := \lim_{k \rightarrow \infty} \mu_t^k, \quad \mathcal{C}^0\left((0, 1), \mathcal{M}^s(\mathbb{R}^d)\right).$$

The goal is to prove that for all $\varphi \in \mathcal{D}((0, 1) \times \mathbb{R}^d)$

$$\int_0^1 dt (\langle \mu_t, \partial_t \varphi(t, x) + v[\mu_t] \cdot \nabla \varphi(t, x) \rangle + \langle h[\mu_t], \varphi(t, x) \rangle) = 0, \quad \langle \mu, \varphi \rangle := \int_{\mathbb{R}^d} \varphi(t, x) d\mu_t(x) \quad (22)$$

We first notice that

$$\sum_{j=0}^{2^k-1} \int_{j\Delta t}^{(j+1)\Delta t} dt \left(\langle \mu_t^k, \partial_t \varphi(t, x) + v[\mu_{j\Delta t}^k] \cdot \nabla \varphi(t, x) \rangle + \langle h[\mu_t^k], \varphi(t, x) \rangle \right) \xrightarrow[k \rightarrow \infty]{} 0$$

Indeed, $\mu_t := \phi_t^v \# \mu_0$ is a weak solution of $\frac{\partial}{\partial t} \mu_t + \nabla \cdot (v(x)\mu_t)$ and $\mu_t = \mu_0 + th(x)$ is a weak solution of $\frac{\partial}{\partial t} \mu_t = h(x)$, so that

$$\begin{aligned} & \left| \sum_{j=0}^{2^k-1} \int_{j\Delta t}^{(j+1)\Delta t} dt \left(\langle \mu_t^k, \partial_t \varphi(t, x) + v[\mu_{j\Delta t}^k] \cdot \nabla \varphi(t, x) \rangle + \langle h[\mu_t^k], \varphi(t, x) \rangle \right) \right| \\ &= \left| \sum_{j=0}^{2^k-1} \int_{j\Delta t}^{(j+1)\Delta t} dt \langle (t - j\Delta t)h[\mu_{j\Delta t}^k], v[\mu_{j\Delta t}^k] \cdot \nabla \varphi(t, x) \rangle \right| \\ &\leq MP \|\nabla \varphi\|_\infty 2^{-(k+1)} \xrightarrow[k \rightarrow \infty]{} 0. \end{aligned}$$

Now, to guarantee (22), it is enough to prove that

$$\begin{aligned} & \lim_{k \rightarrow \infty} \left| \int_0^1 dt (\langle \mu_t, \partial_t \varphi(t, x) + v[\mu_t] \cdot \nabla \varphi(t, x) \rangle + \langle h[\mu_t], \varphi(t, x) \rangle) \right. \\ & \quad \left. - \sum_{j=0}^{2^k-1} \int_{j\Delta t}^{(j+1)\Delta t} dt (\langle \mu_t^k, \partial_t \varphi(t, x) + v[\mu_{j\Delta t}^k] \cdot \nabla \varphi(t, x) \rangle + \langle h[\mu_t^k], \varphi(t, x) \rangle) \right| = 0 \end{aligned}$$

We have

$$\left| \int_0^1 dt (\langle \mu_t, \partial_t \varphi(t, x) \rangle) - \sum_{j=0}^{2^k-1} \int_{j\Delta t}^{(j+1)\Delta t} dt (\langle \mu_t^k, \partial_t \varphi(t, x) \rangle) \right| \leq \|\partial_t \varphi\|_\infty \|\mu_t - \mu_t^k\| \xrightarrow[k \rightarrow \infty]{} 0,$$

$$\left| \int_0^1 dt \langle h[\mu_t], \varphi(t, x) \rangle - \sum_{j=0}^{2^k-1} \int_{j\Delta t}^{(j+1)\Delta t} dt \langle h[\mu_t^k], \varphi(t, x) \rangle \right| \leq Q \|\varphi\|_\infty \|\mu_t - \mu_t^k\| \xrightarrow[k \rightarrow \infty]{} 0,$$

and

$$\begin{aligned} & \left| \int_0^1 dt \langle \mu_t, v[\mu_t] \cdot \nabla \varphi(t, x) \rangle - \sum_{j=0}^{2^k-1} \int_{j\Delta t}^{(j+1)\Delta t} dt \langle \mu_t^k, v[\mu_{j\Delta t}^k] \cdot \nabla \varphi(t, x) \rangle \right| \\ & \leq \left| \sum_{j=0}^{2^k-1} \int_{j\Delta t}^{(j+1)\Delta t} dt \langle \mu_t^k - \mu_t, v[\mu_{j\Delta t}^k] \cdot \nabla \varphi(t, x) \rangle \right| + \left| \sum_{j=0}^{2^k-1} \int_{j\Delta t}^{(j+1)\Delta t} dt \langle \mu_t^k, (v[\mu_{j\Delta t}^k] - v[\mu_t^k]) \cdot \nabla \varphi(t, x) \rangle \right| \\ & \quad + \left| \sum_{j=0}^{2^k-1} \int_{j\Delta t}^{(j+1)\Delta t} dt \langle \mu_t^k, (v[\mu_t] - v[\mu_t^k]) \cdot \nabla \varphi(t, x) \rangle \right| \\ & \leq \|\nabla \varphi\|_\infty \left(M \|\mu_t - \mu_t^k\| + LM(P + |\mu_0|) 2^{-(k+1)} + (P + |\mu_0|) L \|\mu_t - \mu_t^k\| \right) \xrightarrow[k \rightarrow \infty]{} 0. \end{aligned}$$

Step 2. Any weak solution to (1) is Lipschitz in time. In this step, we prove that any weak solution to the transport equation (1) satisfies for some generic constant $C \geq 0$

$$\|\mu_{t+\tau} - \mu_t\|^{a,b} \leq L_1 \tau, \quad t \geq 0, \tau \geq 0, \quad (23)$$

with $C = P + bM(P + |\mu_0|)$. To do so, we consider a solution μ_t to (1). We define the velocity field $w(t, x) := v[\mu_t](x)$ and the signed measure $b_t = h[\mu_t]$. The velocity field w is continuous in time, since by definition, μ_t is continuous in time and v is Lipschitz with respect to μ . Then, μ_t is the unique solution of

$$\partial_t \mu_t(x) + \operatorname{div} \cdot (w(t, x) \mu_t(x)) = b_t(x), \quad \mu_{|t=0}(x) = \mu_0(x). \quad (24)$$

Uniqueness of the linear equation (24) is a direct consequence of Lemma 6. Thus, the unique solution μ_t to (24) can be obtained as the limit of scheme defined above. We have for $k \geq 0$

$$\|\mu_{t+\tau} - \mu_t\|^{a,b} = \|\mu_t - \mu_t^k\|^{a,b} + \|\mu_t^k - \mu_{t+\tau}^k\|^{a,b} + \|\mu_{t+\tau}^k - \mu_{t+\tau}\|^{a,b},$$

where μ_t^k is given by the scheme. The two outside terms can be rendered as small as desired for $k \geq k_0$ large enough, and for $\ell := \min\{i \in \{1, \dots, 2^k\}, t \leq \frac{i}{2^k}\}$, $j := \min\{i \in \{1, \dots, 2^k\}, t + \tau \leq \frac{i}{2^k}\}$

$\frac{i}{2^k}$ with the notations of the scheme

$$\begin{aligned} \|\mu_{t+\tau}^k - \mu_t^k\|^{a,b} &= \|m_j^k - m_\ell^k\|^{a,b} = \left\| \sum_{i=\ell}^{j-1} (m_{i+1}^k - m_i^k) \right\|^{a,b} = \left\| \sum_{i=\ell}^{j-1} (\phi_{\Delta t}^{v[m_i^k]} \# m_i^k + \Delta t h[m_i^k] - m_i^k) \right\|^{a,b} \\ &\leq \sum_{i=\ell}^{j-1} \|\phi_{\Delta t}^{v[m_i^k]} \# m_i^k - m_i^k\|^{a,b} + \Delta t \left\| \sum_{i=\ell}^{j-1} h[m_i^k] \right\|^{a,b}. \end{aligned}$$

Using (H-3), we have

$$\Delta t \left\| \sum_{i=\ell}^{j-1} h[m_i^k] \right\|^{a,b} \leq \frac{j-\ell}{2^k} P \leq P\tau + \frac{P}{2^k},$$

and using Lemma 7 and (16)

$$\sum_{i=\ell}^{j-1} \|\phi_{\Delta t}^{v[m_i^k]} \# m_i^k - m_i^k\|^{a,b} \leq \frac{j-\ell}{2^k} bM(P + |\mu_0|) \leq bM(P + |\mu_0|)\tau + \frac{bM(P + |\mu_0|)}{2^k},$$

which proves (23).

Step 3. Any weak solution to (1) satisfies

$$\|\mu_{t+\tau} - (\phi_\tau^{v[\mu_t]} \# \mu_t + \tau h[\mu_t])\|^{a,b} \leq K_1 \tau^2, \quad (25)$$

for some $K_1 > 0$. Indeed, let us consider a solution μ_t to (1). As in the previous step, μ_t is the unique solution to (24), and thus it can be obtained as the limit of the sequence provided by the scheme. With the notations used in Step 2 and using Lemma 7

$$\begin{aligned} \|\mu_{t+\tau} - (\phi_\tau^{v[\mu_t]} \# \mu_t + \tau h[\mu_t])\|^{a,b} &\leq \|\mu_{t+\tau} - \mu_{t+\tau}^k\|^{a,b} + \|\mu_{t+\tau}^k - (\phi_\tau^{v[\mu_t^k]} \# \mu_t^k + \tau h[\mu_t^k])\|^{a,b} \\ &\quad + \tau \|h[\mu_t^k] - h[\mu_t]\|^{a,b} + \|\phi_\tau^{v[\mu_t]} \# \mu_t - \phi_\tau^{v[\mu_t^k]} \# \mu_t^k\|^{a,b}. \end{aligned}$$

The first, third and fourth terms can be rendered as small as needed for k sufficiently large, we focus then on the second term. Assume for simplicity that $t = \ell\Delta t$ and $t + \tau = (\ell + n)\Delta t$, we have

$$\|\mu_{t+\tau}^k - (\phi_\tau^{v[\mu_t^k]} \# \mu_t^k + \tau h[\mu_t^k])\|^{a,b} = \|m_{\ell+n}^k - (\phi_{n\Delta t}^{v[m_\ell^k]} \# m_\ell^k + n\Delta t h[m_\ell^k])\|^{a,b}.$$

For $n = 2$, we have

$$\begin{aligned} \|m_{\ell+2}^k - (\phi_{2\Delta t}^{v[m_\ell^k]} \# m_\ell^k + 2\Delta t h[m_\ell^k])\|^{a,b} &= \|\phi_{\Delta t}^{v[m_{\ell+1}^k]} \# m_{\ell+1}^k + \Delta t h[m_{\ell+1}^k] - \phi_{\Delta t}^{v[m_\ell^k]} \# \phi_{\Delta t}^{v[m_\ell^k]} \# m_\ell^k - 2\Delta t h[m_\ell^k]\|^{a,b} \\ &= \|\phi_{\Delta t}^{v[m_{\ell+1}^k]} \# (\phi_{\Delta t}^{v[m_\ell^k]} \# m_\ell^k + \Delta t h[m_\ell^k]) + \Delta t h[m_{\ell+1}^k] - \phi_{\Delta t}^{v[m_\ell^k]} \# \phi_{\Delta t}^{v[m_\ell^k]} \# m_\ell^k - 2\Delta t h[m_\ell^k]\|^{a,b} \\ &= \|\phi_{\Delta t}^{v[m_{\ell+1}^k]} \# \phi_{\Delta t}^{v[m_\ell^k]} \# m_\ell^k + \Delta t \phi_{\Delta t}^{v[m_{\ell+1}^k]} \# h[m_\ell^k] + \Delta t h[m_{\ell+1}^k] - \phi_{\Delta t}^{v[m_\ell^k]} \# \phi_{\Delta t}^{v[m_\ell^k]} \# m_\ell^k - 2\Delta t h[m_\ell^k]\|^{a,b} \\ &\leq \|\phi_{\Delta t}^{v[m_{\ell+1}^k]} \# \phi_{\Delta t}^{v[m_\ell^k]} \# m_\ell^k - \phi_{\Delta t}^{v[m_\ell^k]} \# \phi_{\Delta t}^{v[m_\ell^k]} \# m_\ell^k\|^{a,b} + \Delta t \|\phi_{\Delta t}^{v[m_{\ell+1}^k]} \# h[m_\ell^k] + h[m_{\ell+1}^k] - 2h[m_\ell^k]\|^{a,b} \end{aligned}$$

Using Step 3, we have $\|m_{\ell+n}^k - m_\ell^k\| \leq L_1 n \Delta t$, then, using Lemma 7

$$\|m_{\ell+2}^k - (\phi_{2\Delta t}^{v[m_\ell^k]} \# m_\ell^k + 2\Delta t h[m_\ell^k])\|^{a,b} \leq C \Delta^2$$

By induction on $i = 1 \dots n$, we prove that there is a constant C such that

$$\|m_{\ell+n}^k - (\phi_{n\Delta t}^{v[m_\ell^k]} \# m_\ell^k + n\Delta t h[m_\ell^k])\|^{a,b} \leq C(n\Delta t)^2,$$

and (25) follows.

Step 4. Uniqueness of the solution to (1). Assume that μ_t and ν_t are two solutions to (1) and define $\varepsilon(t) = \|\mu_t - \nu_t\|^{a,b}$. We denote

$$R_\mu(t, \tau) = \mu_{t+\tau} - (\phi_\tau^{v[\mu_t]} \# \mu_t + \tau h[\mu_t]), \quad R_\nu(t, \tau) = \nu_{t+\tau} - (\phi_\tau^{v[\nu_t]} \# \nu_t + \tau h[\nu_t]).$$

Using Lemma 7 and Step 3, and $e^a \leq 1 + 2a$ for $0 \leq a \leq \ln(2)$, we have

$$\begin{aligned} \varepsilon(t + \tau) &= \|\mu_{t+\tau} - \nu_{t+\tau}\|^{a,b} = \|\phi_\tau^{v[\mu_t]} \# \mu_t + \tau h[\mu_t] + R_\mu(t, \tau) - \phi_\tau^{v[\nu_t]} \# \nu_t - \tau h[\nu_t] - R_\nu(t, \tau)\|^{a,b} \\ &\leq \|\phi_\tau^{v[\mu_t]} \# \mu_t - \phi_\tau^{v[\nu_t]} \# \nu_t\|^{a,b} + \tau \|h[\mu_t] - h[\nu_t]\|^{a,b} + \|R_\mu(t, \tau)\|^{a,b} + \|R_\nu(t, \tau)\|^{a,b} \\ &\leq \left(e^{L\tau} + (P + |\mu_0|) \frac{e^{L\tau} - 1}{L} K + Q \right) \|\mu_t - \nu_t\|^{a,b} + 2K_1\tau^2 \\ &\leq (1 + \tau(2L + 2K(P + |\mu_0|) + Q)) \|\mu_t - \nu_t\|^{a,b} + 2K_1\tau^2, \end{aligned}$$

which is

$$\frac{\varepsilon(t + \tau) - \varepsilon(t)}{\tau} \leq M\varepsilon(t) + 2K_1\tau, \quad t > 0, \tau \leq \frac{\ln(2)}{L}, \quad M = 2L + 2K(P + |\mu_0|) + Q. \quad (26)$$

Letting τ go to zero, we deduce $\varepsilon'(t) \leq M\varepsilon(t)$ and Gronwall Lemma implies that $\varepsilon(t) \leq M\varepsilon(0)$. This proves uniqueness of the solution. □

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