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Frequency decay for Navier-Stokes stationary solutions

Diego Chamorro¹, Oscar Jarrín¹, and Pierre-Gilles Lemarié-Rieusset^{*1}

¹Laboratoire de Mathématiques et Modélisation d'Evry (LaMME), UMR 8071, Université d'Evry Val d'Essonne, 23 Boulevard de France 91037, Evry

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Abstract

We consider stationary Navier-Stokes equations in \mathbb{R}^3 with a regular external force and we prove exponential frequency decay of the solutions. Moreover, if the external force is small enough, we give a pointwise exponential frequency decay for such solutions according to the K41 theory. If a damping term is added to the equation, a pointwise decay is obtained without the smallness condition over the force.

1 Introduction

Gevrey regularity for solutions of the Navier-Stokes equations has been studied in many different frameworks: for a periodic setting with external force see [1], [6]; for the stationary problem in \mathbb{T}^3 with frequency localized forces see [2]. For the evolution problem in \mathbb{R}^3 (with a null force) a pointwise analysis is obtained in [4].

In this article we generalize some of these previous results in the framework of stationary Navier-Stokes equations in \mathbb{R}^3

$$-\nu\Delta\vec{U} + \mathbb{P}(\operatorname{div}(\vec{U} \otimes \vec{U})) = \vec{F}, \quad \operatorname{div}(\vec{U}) = 0, \quad \operatorname{div}(\vec{F}), \quad (1)$$

where $\nu > 0$ is the fluid's viscosity parameter, $\vec{U} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is the velocity, \mathbb{P} is the Leray's projector and $\vec{F} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is a time-independent external force.

*Corresponding author: plemarie@univ-evry.fr

If the external force is regular enough we prove in Theorem 1.1 an exponential frequency decay. Moreover, if the external force is small enough, we give in Theorem 1.2 a pointwise exponential frequency decay for such solutions. Finally, if a damping term is added to the equation, a pointwise decay is obtained in Theorem 1.3 without the smallness condition over the force.

Theorem 1.1 *Let $\vec{F} \in \dot{H}^{-1}(\mathbb{R}^3)$ be such that for $\varepsilon_0 > 0$ we have*

$$\int_{\mathbb{R}^3} e^{2\varepsilon_0|\xi|} |\widehat{\vec{F}}(\xi)|^2 |\xi|^{-2} d\xi < +\infty.$$

Then there exists $\vec{U} \in \dot{H}^1(\mathbb{R}^3)$ a solution to the stationary Navier-Stokes equations (1), such that \vec{U} verifies the following exponential frequency decay:

$$\int_{\mathbb{R}^3} e^{2\varepsilon_1|\xi|} |\widehat{\vec{U}}(\xi)|^2 |\xi|^2 d\xi < +\infty, \quad \text{where } \varepsilon_1 = \varepsilon_1(\varepsilon_0, \vec{F}, \nu) > 0. \quad (2)$$

In the laminar setting we obtain a sharper *pointwise* exponential frequency decay. For $0 \leq a < 3$, we define the pseudo-measures space by

$$\mathcal{PM}^a = \left\{ \vec{g} \in \mathcal{S}'(\mathbb{R}^3) : \widehat{\vec{g}} \in L^1_{loc}(\mathbb{R}^3) \quad \text{and} \quad |\xi|^a \widehat{\vec{g}} \in L^\infty(\mathbb{R}^3) \right\},$$

which is a Banach space endowed with the norm $\|\vec{g}\|_{\mathcal{PM}^a} = \| |\xi|^a \widehat{\vec{g}} \|_{L^\infty}$, for $a = 0$ we will simply denote the space \mathcal{PM}^0 by \mathcal{PM} .

Theorem 1.2 *Let $\vec{F} \in \mathcal{PM}$. There exists a (small) constant $\eta > 0$ such that if*

$$\sup_{\xi \in \mathbb{R}^3} e^{|\xi|} |\widehat{\vec{F}}(\xi)| < \eta,$$

then there exists $\vec{U} \in \mathcal{PM}^2$ a solution to the stationary Navier-Stokes equations (1) such that \vec{U} verifies the following pointwise exponential frequency decay:

$$|\widehat{\vec{U}}(\xi)| \leq ce^{-|\xi|} |\xi|^{-2}, \quad \text{for all } \xi \neq 0. \quad (3)$$

If a damping term is added to the stationary Navier-Stokes system, we have the following result

Theorem 1.3 *Let $\vec{F} \in H^{-1}(\mathbb{R}^3)$ and for $\alpha > 0$ consider the damped stationary Navier-Stokes equations*

$$-\nu \Delta \vec{U} + \mathbb{P}(\text{div}(\vec{U} \otimes \vec{U})) = \vec{F} - \alpha \vec{U}, \quad \text{div}(\vec{U}) = 0. \quad (4)$$

If the external force \vec{F} is such that $|\widehat{\vec{F}}(\xi)| \leq e^{-\varepsilon_0|\xi|}$ for a fixed $\varepsilon_0 > 0$, then the stationary solution $\vec{U} \in H^1(\mathbb{R}^3)$ satisfies the following pointwise exponential frequency decay

$$|\widehat{\vec{U}}(\xi)| \leq ce^{-\varepsilon_1|\xi|} |\xi|^{-\frac{5}{2}}, \quad \text{for all } \xi \neq 0, \quad \text{where } \varepsilon_1 = \varepsilon_1(\varepsilon_0, \vec{F}, \nu) > 0. \quad (5)$$

2 Proof of Theorem 1.1

Lemma 2.1 *If $\vec{F} \in \dot{H}^{-1}(\mathbb{R}^3)$, then there exists at least one solution $\vec{U} \in \dot{H}^1(\mathbb{R}^3)$ to the stationary Navier-Stokes equation (1).*

Lemma 2.2 *Let $T_0 > 0$. For $\vec{u}_0 \in \dot{H}^1(\mathbb{R}^3)$ a divergence-free initial data and a divergence-free external force $\vec{f} \in \mathcal{C}([0, T_0[, \dot{H}^1(\mathbb{R}^3))$ there exists a time $0 < T_1 < T_0$ and a function $\vec{u} \in \mathcal{C}([0, T_1[, \dot{H}^1(\mathbb{R}^3))$ which is a unique solution to the Navier-Stokes equations*

$$\partial_t \vec{u} - \nu \Delta \vec{u} + \mathbb{P}(\operatorname{div}(\vec{u} \otimes \vec{u})) = \vec{f}, \quad \operatorname{div}(\vec{u}) = 0, \quad \vec{u}(0, \cdot) = \vec{u}_0. \quad (6)$$

Existence and uniqueness issues are classical, see [5] for details.

In the following proposition we prove the frequency decay for the solution \vec{u} obtained in Lemma 2.2.

Proposition 2.1 *Let $\alpha > 0$ and consider the Poisson kernel $e^{\alpha\sqrt{t}\sqrt{-\Delta}}$. Within the framework of Lemma 2.2, if the external force \vec{f} is such that*

$$e^{\alpha\sqrt{t}\sqrt{-\Delta}} \vec{f} \in \mathcal{C}([0, T_0[, \dot{H}^1(\mathbb{R}^3)),$$

then the unique solution of equations (6) satisfies $e^{\alpha\sqrt{t}\sqrt{-\Delta}} \vec{u} \in \mathcal{C}([0, T_1[, \dot{H}^1(\mathbb{R}^3))$ for all time $t \in [0, T_1[$ where $0 < T_1 < T_0$ is small enough.

Proof. Consider the space

$$E = \left\{ \vec{u} \in \mathcal{C}([0, T_1[, \dot{H}^1(\mathbb{R}^3)) : e^{\alpha\sqrt{t}\sqrt{-\Delta}} \vec{u} \in \mathcal{C}([0, T_1[, \dot{H}^1(\mathbb{R}^3)) \right\},$$

endowed with the norm $\|\cdot\|_E = \|e^{\alpha\sqrt{t}\sqrt{-\Delta}}(\cdot)\|_{L_t^\infty \dot{H}_x^1}$. We study the quantity

$$\|\vec{u}_1\|_E = \left\| h_{\nu t} * \vec{u}_0 + \int_0^t h_{\nu(t-s)} * \vec{f}(s, \cdot) ds - \int_0^t h_{\nu(t-s)} * \mathbb{P}(\operatorname{div}(\vec{u}_1 \otimes \vec{u}_1))(s, \cdot) ds \right\|_E \quad (7)$$

where $h_{\nu t}$ is the heat kernel. The two first terms of this expression are easy to estimate and we have

$$\left\| h_{\nu t} * \vec{u}_0 + \int_0^t h_{\nu(t-s)} * \vec{f}(s, \cdot) ds \right\|_E \leq c(\nu, \alpha, T_0) \left(\|\vec{u}_0\|_{\dot{H}_x^1} + \|e^{\alpha\sqrt{t}\sqrt{-\Delta}} \vec{f}\|_{L_t^\infty \dot{H}_x^1} \right). \quad (8)$$

For the last term of (7), by definition of the norm $\|\cdot\|_E$, by the Plancherel formula and by the boundedness of the Leray projector we have

$$\begin{aligned} (I) &= \left\| \int_0^t h_{\nu(t-s)} * \mathbb{P}(\operatorname{div}(\vec{u}_1 \otimes \vec{u}_1)) ds \right\|_E \\ &= \sup_{0 < t < T_1} \left\| e^{\alpha\sqrt{t}\sqrt{-\Delta}} \left(\int_0^t h_{\nu(t-s)} * \mathbb{P}(\operatorname{div}(\vec{u}_1 \otimes \vec{u}_1)) ds \right) \right\|_{\dot{H}_x^1} \\ &\leq \sup_{0 < t < T_1} c \left\| |\xi|^2 \int_0^t e^{-\nu(t-s)|\xi|^2} e^{\alpha\sqrt{t}|\xi|} |(\mathcal{F}[\vec{u}_1] * \mathcal{F}[\vec{u}_1])(s, \cdot)| ds \right\|_{L_x^2}. \end{aligned}$$

Since we have the pointwise inequality

$$e^{\alpha\sqrt{t}|\xi|} |(\mathcal{F}[\vec{u}_1] * \mathcal{F}[\vec{u}_1])(s, \xi)| \leq \left[\left(e^{\alpha\sqrt{t}|\xi|} |\mathcal{F}[\vec{u}_1]| \right) * \left(e^{\alpha\sqrt{t}|\xi|} |\mathcal{F}[\vec{u}_1]| \right) \right] (s, \xi), \quad (9)$$

due to the fact that $e^{\alpha\sqrt{t}|\xi|} \leq e^{\alpha\sqrt{t}|\xi-\eta|} e^{\alpha\sqrt{t}|\eta|}$ for all $\xi, \eta \in \mathbb{R}^3$, then we obtain

$$(I) \leq \sup_{0 < t < T_1} c \int_0^t \left\| |\xi|^{\frac{3}{2}} e^{-\nu(t-s)|\xi|^2} |\xi|^{\frac{1}{2}} \left[\left(e^{\alpha\sqrt{t}|\xi|} |\mathcal{F}[\vec{u}_1]| \right) * \left(e^{\alpha\sqrt{t}|\xi|} |\mathcal{F}[\vec{u}_1]| \right) \right] \right\|_{L_x^2} ds.$$

Getting back to the spatial variable we can write

$$\begin{aligned} (I) &\leq \sup_{0 < t < T_1} c \int_0^t \left\| (-\Delta)^{\frac{3}{4}} h_{\nu(t-s)} * (-\Delta)^{\frac{1}{4}} \left\{ \left(\mathcal{F}^{-1} \left[e^{\alpha\sqrt{t}|\xi|} |\mathcal{F}[\vec{u}_1]| \right] \right) \otimes \right. \right. \\ &\quad \left. \left. \left(\mathcal{F}^{-1} \left[e^{\alpha\sqrt{t}|\xi|} |\mathcal{F}[\vec{u}_1]| \right] \right) \right\} \right\|_{L_x^2} ds \\ &\leq \left(c \int_0^{T_1} \left\| (-\Delta)^{\frac{3}{4}} h_{\nu(t-s)} \right\|_{L^1} ds \right) \left\| \left(\mathcal{F}^{-1} \left[e^{\alpha\sqrt{t}|\xi|} |\mathcal{F}[\vec{u}_1]| \right] \right) \otimes \right. \\ &\quad \left. \left(\mathcal{F}^{-1} \left[e^{\alpha\sqrt{t}|\xi|} |\mathcal{F}[\vec{u}_1]| \right] \right) \right\|_{L_t^\infty \dot{H}_x^{\frac{1}{2}}} \\ &\leq c \frac{T_1^{\frac{1}{4}}}{\nu^{\frac{3}{4}}} \left\| \mathcal{F}^{-1} \left[e^{\alpha\sqrt{t}|\xi|} |\mathcal{F}[\vec{u}_1]| \right] \right\|_{L_t^\infty \dot{H}_x^1} \left\| \mathcal{F}^{-1} \left[e^{\alpha\sqrt{t}|\xi|} |\mathcal{F}[\vec{u}_1]| \right] \right\|_{L_t^\infty \dot{H}_x^1} \\ &\leq c \frac{T_1^{\frac{1}{4}}}{\nu^{\frac{3}{4}}} \|\vec{u}_1\|_E \|\vec{u}_1\|_E. \end{aligned} \quad (10)$$

With estimates (8) and (10) at hand, we fix T_1 small enough in order to apply Picard's contraction principle and we obtain a solution $\vec{u}_1 \in E$ of (6). Since $E \subset \mathcal{C}([0, T_1[, \dot{H}^1(\mathbb{R}^3))$ we have $\vec{u}_1 \in \mathcal{C}([0, T_1[, \dot{H}^1(\mathbb{R}^3))$ and by uniqueness of the solution \vec{u} we have $\vec{u}_1 = \vec{u}$, and thus $\vec{u} \in E$. \blacksquare

Now, we come back to the stationary Navier-Stokes equations (1) and we will prove that the solution $\vec{U} \in \dot{H}^1(\mathbb{R}^3)$ (given by Lemma 2.1) satisfies the exponential frequency decay given in (2). In the space $\mathcal{C}([0, 1[, \dot{H}^1(\mathbb{R}^3))$ we consider the evolution problem (6) with the initial data $\vec{u}_0 = \vec{U}$ where the external force \vec{f} is now given by with the expression

$$\vec{f} = e^{-\alpha\sqrt{t}\sqrt{-\Delta}} (e^{\alpha\sqrt{t}\sqrt{-\Delta}} \vec{F}),$$

for the particular value $\alpha = \frac{2}{3}\varepsilon_0 > 0$ where $\varepsilon_0 > 0$ is given in the hypothesis of the force \vec{F} . To obtain a unique solution $\vec{u} \in \mathcal{C}([0, 1[, \dot{H}^1(\mathbb{R}^3))$ to the equations (6) such that

$$e^{\alpha\sqrt{t}\sqrt{-\Delta}} \vec{u} \in \mathcal{C}([0, 1[, \dot{H}^1(\mathbb{R}^3)),$$

we prove that the external force \vec{f} verifies the hypotheses of Lemma 2.2 and Proposition 2.1 above:

$$\begin{aligned}
\left\| e^{\alpha\sqrt{t}\sqrt{-\Delta}} \vec{F} \right\|_{L_t^\infty \dot{H}_x^1}^2 &= \sup_{0 < t < 1} \int_{\mathbb{R}^3} |\xi|^2 e^{2\alpha\sqrt{t}|\xi|} |\widehat{\vec{F}}(\xi)|^2 d\xi \\
&\leq \frac{1}{\alpha^4} \int_{\mathbb{R}^3} (\alpha|\xi|)^4 e^{2\alpha|\xi|} |\widehat{\vec{F}}(\xi)|^2 |\xi|^{-2} d\xi \\
&\leq \frac{1}{\alpha^4} \int_{\mathbb{R}^3} e^{3\alpha|\xi|} |\widehat{\vec{F}}(\xi)|^2 |\xi|^{-2} d\xi \\
&\leq \frac{1}{\alpha^4} \int_{\mathbb{R}^3} e^{2\varepsilon_0|\xi|} |\widehat{\vec{F}}(\xi)|^2 |\xi|^{-2} d\xi < +\infty.
\end{aligned}$$

Thus, once we have $e^{\alpha\sqrt{t}\sqrt{-\Delta}} \vec{F} \in \mathcal{C}([0, 1[, \dot{H}^1(\mathbb{R}^3))$, since the operator $e^{-\alpha\sqrt{t}\sqrt{-\Delta}}$ is bounded in the space $\mathcal{C}([0, 1[, \dot{H}^1(\mathbb{R}^3))$ we have

$$\vec{f} = e^{-\alpha\sqrt{t}\sqrt{-\Delta}} (e^{\alpha\sqrt{t}\sqrt{-\Delta}} \vec{F}) \in \mathcal{C}([0, 1[, \dot{H}^1(\mathbb{R}^3)).$$

Moreover, we have

$$e^{\alpha\sqrt{t}\sqrt{-\Delta}} \vec{f} = e^{\alpha\sqrt{t}\sqrt{-\Delta}} \vec{F} \in \mathcal{C}([0, 1[, \dot{H}^1(\mathbb{R}^3)).$$

By Lemma 2.2 there exists a time $0 < T_1 < 1$ and a unique solution $\vec{u} \in \mathcal{C}([0, T_1[, \dot{H}^1(\mathbb{R}^3))$ to the equation (6). Moreover, since $e^{\alpha\sqrt{t}\sqrt{-\Delta}} \vec{f} \in \mathcal{C}([0, 1[, \dot{H}^1(\mathbb{R}^3))$ by Proposition 2.1 we have $e^{\alpha\sqrt{t}\sqrt{-\Delta}} \vec{u} \in \mathcal{C}([0, T_1[, \dot{H}^1(\mathbb{R}^3))$. Since the solution $\vec{U} \in \dot{H}^1(\mathbb{R}^3)$ of the stationary Navier-Stokes equations (1) is a constant in time, we have $\vec{U} \in \mathcal{C}([0, T_1[, \dot{H}^1(\mathbb{R}^3))$ and since $\partial_t \vec{U} \equiv 0$ and

$$\vec{f} = e^{-\alpha\sqrt{t}\sqrt{-\Delta}} (e^{\alpha\sqrt{t}\sqrt{-\Delta}} \vec{F}) = \vec{F},$$

we find that $\vec{U} \in \mathcal{C}([0, T_1[, \dot{H}^1(\mathbb{R}^3))$ is also a solution to the equation (6) and thus, by uniqueness we get $\vec{U} = \vec{u}$. Then, since $e^{\alpha\sqrt{t}\sqrt{-\Delta}} \vec{u} \in \mathcal{C}([0, T_1[, \dot{H}^1(\mathbb{R}^3))$ we have

$$e^{\alpha\sqrt{t}\sqrt{-\Delta}} \vec{U} \in \mathcal{C}([0, T_1[, \dot{H}^1(\mathbb{R}^3)),$$

for all time $t \in [0, T_1[$. Thus, if $\varepsilon_1 = \alpha\sqrt{\frac{T_1}{2}} > 0$, we have

$$\int_{\mathbb{R}^3} e^{2\varepsilon_1|\xi|} |\vec{U}(\xi)|^2 |\xi|^2 d\xi = \left\| e^{\alpha\sqrt{\frac{T_1}{2}}\sqrt{-\Delta}} \vec{U} \right\|_{\dot{H}_x^1}^2 \leq \sup_{0 < t < T_1} \left\| e^{\alpha\sqrt{t}\sqrt{-\Delta}} \vec{U} \right\|_{\dot{H}_x^1}^2 < +\infty,$$

and we obtain the frequency decay given in (2). ■

3 Proof of Theorem 1.2

We consider now the space $A = \{\vec{U} \in \mathcal{PM}^2 : e^{\sqrt{-\Delta}}\vec{U} \in \mathcal{PM}^2\}$, endowed with the norm

$$\|\cdot\|_A = \|e^{\sqrt{-\Delta}}(\cdot)\|_{\mathcal{PM}^2}, \quad (11)$$

and in this space we study the existence of a solution of equations (1) under the hypotheses of Theorem 1.2. For this we study the quantity

$$\|\vec{U}\|_A = \left\| \frac{1}{\nu} \mathbb{P} \left(\frac{1}{\Delta} \operatorname{div}(\vec{U} \otimes \vec{U}) \right) - \frac{1}{\nu} \frac{1}{\Delta} \vec{F} \right\|_A \leq \frac{1}{\nu} \left\| \mathbb{P} \left(\frac{1}{\Delta} \operatorname{div}(\vec{U} \otimes \vec{U}) \right) \right\|_A + \frac{1}{\nu} \left\| \frac{1}{\Delta} \vec{F} \right\|_A, \quad (12)$$

where, for the first term of the inequality above we have the following estimate:

$$\frac{1}{\nu} \left\| \mathbb{P} \left(\frac{1}{\Delta} \operatorname{div}(\vec{U} \otimes \vec{U}) \right) \right\|_A \leq \frac{c}{\nu} \|\vec{U}\|_A \|\vec{U}\|_A. \quad (13)$$

Indeed, by the expression (11) and by the continuity of the Leray projector we have

$$\begin{aligned} \frac{1}{\nu} \left\| \mathbb{P} \left(\frac{1}{\Delta} \operatorname{div}(\vec{U} \otimes \vec{U}) \right) \right\|_A &= \frac{1}{\nu} \left\| |\xi|^2 e^{|\xi|} \mathcal{F} \left[\mathbb{P} \left(\frac{1}{\Delta} \operatorname{div}(\vec{U} \otimes \vec{U}) \right) \right] \right\|_{L^\infty} \\ &\leq \frac{c}{\nu} \left\| |\xi|^2 e^{|\xi|} \frac{1}{|\xi|} |\mathcal{F}[\vec{U}] * \mathcal{F}[\vec{U}]| \right\|_{L^\infty} \\ &\leq \frac{c}{\nu} \left\| |\xi| \left[\left(e^{|\xi|} \mathcal{F}[|\vec{U}|] \right) * \left(e^{|\xi|} \mathcal{F}[|\vec{U}|] \right) \right] \right\|_{L^\infty}, \end{aligned} \quad (14)$$

where the last inequality can be deduced from (9). Now we remark that

$$\begin{aligned} \left[\left(e^{|\xi|} \mathcal{F}[|\vec{U}|] \right) * \left(e^{|\xi|} \mathcal{F}[|\vec{U}|] \right) \right] (\xi) &= \int_{\mathbb{R}^3} e^{|\xi-\eta|} \mathcal{F}[|\vec{U}|] (\xi-\eta) e^{|\eta|} \mathcal{F}[|\vec{U}|] (\eta) d\eta \\ &\leq \|\vec{U}\|_A \|\vec{U}\|_A \int_{\mathbb{R}^3} \frac{d\eta}{|\xi-\eta|^2 |\eta|^2} \leq \frac{c}{|\xi|} \|\vec{U}\|_A \|\vec{U}\|_A, \end{aligned}$$

and thus, using this inequality in (14) we easily obtain the estimate (13). For the second term in the RHS of (12) we have

$$\frac{1}{\nu} \left\| \frac{1}{\Delta} \vec{F} \right\|_A = \frac{1}{\nu} \left\| e^{\sqrt{-\Delta}} \left(\frac{1}{\Delta} \vec{F} \right) \right\|_{\mathcal{PM}^2} = \frac{c_1}{\nu} \sup_{\xi \in \mathbb{R}^3} |\xi|^2 e^{|\xi|} \frac{1}{|\xi|^2} |\vec{F}(\xi)| = \frac{c_1}{\nu} \sup_{\xi \in \mathbb{R}^3} e^{|\xi|} |\vec{F}(\xi)|.$$

Thus, if the external force \vec{F} satisfies $\sup_{\xi \in \mathbb{R}^3} e^{|\xi|} |\vec{F}(\xi)| < \eta$, for η small enough, we obtain $\vec{U} \in A$ a solution to the stationary Navier-Stokes equations (1) for which we have the pointwise estimate (3). \blacksquare

4 Proof of Theorem 1.3

For $\alpha > 0$ and under the hypotheses of Theorem 1.3, the existence of solutions of equation (4) is given by applying the Scheafer fixed point theorem. Now, for $\vec{u}_0 \in \mathcal{PM}^{\frac{5}{2}}$ we consider the non-stationary damped Navier-Stokes equations

$$\partial_t \vec{u} + \mathbb{P}(\operatorname{div}(\vec{u} \otimes \vec{u})) - \nu \Delta \vec{u} = \vec{f} - \alpha \vec{u}, \quad \operatorname{div}(\vec{u}) = 0, \quad \vec{u}(0, \cdot) = \vec{u}_0, \quad (15)$$

where the divergence-free external force \vec{f} belongs to the space $\mathcal{C}([0, T_0[, \mathcal{PM}^{\frac{5}{2}})$. For this problem there exists a unique solution $\vec{u} \in \mathcal{C}([0, T_1[, \mathcal{PM}^{\frac{5}{2}})$ with $0 < T_1 < T_0$. For existence issues for equations (4) and (15) see the details in [5].

Following essentially the same lines of Proposition 2.1 above, we prove that if the external force is such that $e^{\beta\sqrt{t}\sqrt{-\Delta}}\vec{f} \in \mathcal{C}([0, T_0[, \mathcal{PM}^{\frac{5}{2}})$ then the unique solution of (15) is such that $e^{\beta\sqrt{t}\sqrt{-\Delta}}\vec{u} \in \mathcal{C}([0, T_1[, \mathcal{PM}^{\frac{5}{2}})$. As in the proof of Theorem 1.2, we consider

$$\vec{f} = e^{-\beta\sqrt{t}\sqrt{-\Delta}} \left(e^{\beta\sqrt{t}\sqrt{-\Delta}} \vec{F} \right) = \vec{F},$$

and for a suitable value of the parameter $\beta > 0$ we can prove that $\vec{f} \in \mathcal{C}([0, 1[, \mathcal{PM}^{\frac{5}{2}})$ and $e^{\beta\sqrt{t}\sqrt{-\Delta}}\vec{f} \in \mathcal{C}([0, 1[, \mathcal{PM}^{\frac{5}{2}})$.

In order to link the stationary solution to the non-stationary problem, we must prove that the solution $\vec{U} \in H^1(\mathbb{R}^3)$ of (4) is such that $\vec{U} \in \mathcal{PM}^{\frac{5}{2}}$, and in this step we use the extra damping term. Indeed, rewriting (4) we consider the equation

$$\vec{U} = \frac{-\nu\Delta}{\alpha Id - \nu\Delta} \left(\mathbb{P} \left(\frac{1}{\nu\Delta} \operatorname{div}(\vec{U} \otimes \vec{U}) \right) \right) + \frac{1}{\alpha Id - \nu\Delta} \left(\vec{F} \right), \quad (16)$$

and we obtain

$$\|\vec{U}\|_{\dot{H}^{\frac{3}{2}}} \leq \left\| \frac{-\nu\Delta}{\alpha Id - \nu\Delta} \left(\mathbb{P} \left(\frac{1}{\nu\Delta} \operatorname{div}(\vec{U} \otimes \vec{U}) \right) \right) \right\|_{\dot{H}^{\frac{3}{2}}} + \left\| \frac{1}{\alpha Id - \nu\Delta} \left(\vec{F} \right) \right\|_{\dot{H}^{\frac{3}{2}}}.$$

Since the operator $\frac{-\nu\Delta}{\alpha Id - \nu\Delta}$ is bounded in $\dot{H}^{\frac{3}{2}}(\mathbb{R}^3)$ and by the properties of \vec{F} we can write

$$\begin{aligned} \|\vec{U}\|_{\dot{H}^{\frac{3}{2}}} &\leq \left\| \frac{1}{\nu\Delta} \operatorname{div}(\vec{U} \otimes \vec{U}) \right\|_{\dot{H}^{\frac{3}{2}}} + \left\| \frac{1}{\alpha Id - \nu\Delta} \left(\vec{F} \right) \right\|_{H^2} \\ &\leq c \|\vec{U} \otimes \vec{U}\|_{\dot{H}^{\frac{1}{2}}} + c(\alpha) \|\vec{F}\|_{L^2} \\ &\leq c \|\vec{U}\|_{H^1} \|\vec{U}\|_{H^1} + c(\alpha) \|\vec{F}\|_{L^2}. \end{aligned}$$

We thus have $\vec{U} \in \dot{H}^{\frac{3}{2}}(\mathbb{R}^3)$ and we prove now $\vec{U} \in \mathcal{PM}^{\frac{5}{2}}$: from equation (16) we obtain

$$\left| \mathcal{F} [\vec{U}] (\xi) \right| \leq c \frac{1}{\nu |\xi|} \left| \left(\mathcal{F} [\vec{U}] * \mathcal{F} [\vec{U}] \right) (\xi) \right| + \frac{1}{\nu |\xi|^2} \left| \mathcal{F} [\vec{F}] (\xi) \right|,$$

and then, multiplying by $|\xi|^{\frac{5}{2}}$ and by hypothesis on \vec{F} we get the estimate

$$\begin{aligned} |\xi|^{\frac{5}{2}} \left| \mathcal{F} [\vec{U}] (\xi) \right| &\leq \int_{\mathbb{R}^3} |\xi|^{\frac{3}{2}} \left| \mathcal{F} [\vec{U}] (\xi - \eta) \right| \left| \mathcal{F} [\vec{U}] (\eta) \right| d\eta + \frac{1}{\nu} |\xi|^{\frac{1}{2}} \left| \mathcal{F} [\vec{F}] (\xi) \right| \\ &\leq 2 \|\vec{U}\|_{\dot{H}^{\frac{3}{2}}} \|\vec{U}\|_{L^2} + \frac{1}{\nu} |\xi|^{\frac{1}{2}} e^{-\varepsilon_0 |\xi|}, \end{aligned}$$

from which we deduce that $\vec{U} \in \mathcal{PM}^{\frac{5}{2}}$. Then, we study (15) with $\vec{u}_0 = \vec{U}$ and we have $\vec{U} \in \mathcal{C}([0, T_1[, \mathcal{PM}^{\frac{5}{2}})$, but since \vec{U} verifies the equations (4), $\partial_t \vec{U} \equiv 0$ and $\vec{f} = \vec{F}$, we obtain that \vec{U} is also a solution of (15) and by uniqueness we have $\vec{U} = \vec{u}$. Finally, we have $e^{\beta\sqrt{t}\sqrt{-\Delta}} \vec{U} \in \mathcal{C}([0, T_1[, \mathcal{PM}^{\frac{5}{2}})$ for $0 < t < T_1$ and if $\varepsilon_1 = \beta\sqrt{\frac{T_1}{2}}$ we can write

$$\|e^{\varepsilon_1\sqrt{-\Delta}} \vec{U}\|_{\mathcal{PM}^{\frac{5}{2}}} \leq \|e^{\beta\sqrt{t}\sqrt{-\Delta}} \vec{U}\|_{L^\infty([0, T_1[, \mathcal{PM}^{\frac{5}{2}})} < +\infty,$$

and we obtain the frequency decay stated in the formula (5). ■

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