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# CONDITIONAL CORES AND CONDITIONAL CONVEX HULLS OF RANDOM SETS

EMMANUEL LEPINETTE AND ILYA MOLCHANOV

ABSTRACT. We define two non-linear operations with random (not necessarily closed) sets in Banach space: the conditional core and the conditional convex hull. While the first is sublinear, the second one is superlinear (in the reverse set inclusion ordering). Furthermore, we introduce the generalised conditional expectation of random closed sets and show that it is sandwiched between the conditional core and the conditional convex hull. The results rely on measurability properties of not necessarily closed random sets considered from the point of view of the families of their selections. Furthermore, we develop analytical tools suitable to handle random convex (not necessarily compact) sets in Banach spaces; these tools are based on considering support functions as functions of random arguments. The paper is motivated by applications to assessing multivariate risks in mathematical finance.

## 1. INTRODUCTION

Each almost surely non-empty random closed set  $X$  (see definition in Section 2.1) in a Banach space admits a measurable *selection*, that is, a random element  $\xi$  that almost surely belongs to  $X$ . Moreover,  $X$  equals the closure of a countable family of its selections, called a *Castaing representation* of  $X$ , see [15]. If at least one selection is Bochner integrable, then  $X$  becomes the closure of a countable family of *integrable* selections, and the (selection) *expectation*  $\mathbf{E}X$  of  $X$  is defined as the closure of the set of expectations for its all integrable selections, see [11, 15].

Almost surely deterministic selections (if they exist) constitute the set of *fixed points* of  $X$ ; this set is always a subset of  $\mathbf{E}X$ . The union of supports of all selections is a superset of  $\mathbf{E}X$ ; it can be regarded as the *support* of  $X$ .

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In this paper we work out conditional variants of these concepts which also apply to not necessarily closed graph measurable random sets. Given a probability space  $(\Omega, \mathcal{F}, \mathbf{P})$  and a sub- $\sigma$ -algebra  $\mathcal{H}$  of  $\mathcal{F}$ , we introduce the concept of the *conditional core*  $\mathbf{m}(X|\mathcal{H})$ , which relies on considering selections measurable with respect to  $\mathcal{H}$ . If  $\mathcal{H}$  is trivial, then the conditional core becomes the set of fixed points of  $X$ ; it is also related to the essential intersection considered in [10]. The conditional core corresponds to the concept of the conditional essential supremum (infimum) for a family of random variables, see [2].

If  $X$  is a.s. convex, its conditional core can be obtained by taking the conditional essential infimum of its support function. Taking the conditional essential maximum leads to the dual concept of the *conditional convex hull*  $\mathbf{M}(X|\mathcal{H})$ . While the conditional core of the sum of sets is a superset of the sum of their conditional cores, the opposite inclusion holds for the conditional convex hull. In other words, the conditional core and the conditional convex hull are *non-linear* set-valued expectations.

The conventional conditional (selection) expectation  $\mathbf{E}(X|\mathcal{H})$  is a well-known concept for integrable random closed sets, see [9, 11]. We introduce a *generalised* conditional expectation  $\mathbf{E}^g(X|\mathcal{H})$  based on working with the set of generalised conditional expectation of all selections and show how it relates to the conventional one. In particular, it is shown that  $\mathbf{E}^g(X|\mathcal{H}) = X$  if  $X$  is  $\mathcal{H}$ -measurable, no matter if  $X$  is integrable or not.

The presented results are motivated by applications in mathematical finance, where multiasset portfolios are represented as sets and their *risks* are also set-valued, see [7, 8, 16]. Working in the dynamic setting requires a better understanding of conditioning operation with random sets, and its iterative properties. In this relation, the conditional core provides a simple *conditional risk measure* which generalises the concept of the essential infimum for multiasset portfolios. In order to make a parallel with classical financial concept (where real-valued risk measures are sublinear), the sets are ordered by *reverse* inclusion, so that the conditional core is sublinear and the conditional convex hull is superlinear.

Section 2 introduces random sets, their selections and treats various measurability issues, in particular, it is shown that each closed set-valued map admits a measurable version. A special attention is devoted to the decomposability and infinite decomposability properties, which are the key concepts suitable to relate families of random vectors and selections of random sets.

Section 3 develops various analytical tools suitable to handle random *convex* sets. Random *compact* convex sets in Euclidean space can be efficiently explored by passing to their support functions, and the same tool works well for weakly compact sets in separable Banach spaces. Otherwise (e.g., for unbounded closed sets in Euclidean space), the support function is only lower semicontinuous and may become a non-separable random function on the dual space. For instance, the support function of a random half-space in Euclidean space with an isotropic normal almost surely vanishes on all deterministic arguments. We show that this complication can be circumvented by viewing the support function as a function applied to *random* elements in the dual space. In particular, we prove a random variant of the well-known result saying that a closed convex set is given by intersection of a countable number of half-spaces. It is also shown that random convex closed sets can be alternatively described as measurable epigraphs of their support functions, thereby extending the fact known in the Euclidean setting for compact random sets to all random convex closed sets in separable Banach spaces.

Section 4 introduces and elaborates the properties of conditional cores of random sets. Given a sub- $\sigma$ -algebra  $\mathcal{H}$ , the conditional core  $\mathbf{m}(X|\mathcal{H})$  is the largest  $\mathcal{H}$ -measurable random closed set contained in  $X$ . While we work with random sets in Banach spaces, the conditional core may be defined for random sets in general Polish spaces. In linear spaces, the conditional core is subadditive for the reverse inclusion, that is the core of the sum of two random sets is a superset of the sum of their conditional cores.

While the conditional core is the largest random set contained in the given one and is measurable with respect to a sub- $\sigma$ -algebra  $\mathcal{H}$ , the conditional convex hull  $\mathbf{M}(X|\mathcal{H})$  is a smallest  $\mathcal{H}$ -measurable random convex closed set containing  $X$ . Section 5 establishes the existence of the conditional convex hull. It is shown that the support function of the conditional convex hull is given by the essential supremum of the support function of  $X$ . Duality relationships between the core and the convex hull are also obtained.

Section 6 introduces the concept of a generalised conditional expectation for random sets and shows that it is sandwiched between the conditional core and the conditional convex hull. By taking the intersection (or closed convex hull) of generalised conditional expectations with respect to varying probability measures, it is possible to obtain a rich collection of conditional set-valued non-linear expectations.

Random convex cones have a particular property that their support functions either vanish or are infinite. Conditional cores and convex hulls of random convex cones are considered in Section 7.

Some useful facts about conditional essential supremum and infimum of random variables and the generalised conditional expectation are collected in the appendices.

## 2. DECOMPOSABILITY AND MEASURABLE VERSIONS

**2.1. Graph measurable random sets and their selections.** Let  $\mathfrak{X}$  be a separable (real) Banach space with norm  $\|\cdot\|$  and the Borel  $\sigma$ -algebra  $\mathcal{B}(\mathfrak{X})$  generated by its strong topology. The norm-closure of a set  $A \subset \mathfrak{X}$  is denoted by  $\text{cl } A$  and the interior by  $\text{int } A$ .

Fix a complete probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . Let  $\mathcal{H}$  be a sub- $\sigma$ -algebra of  $\mathcal{F}$ , which may coincide with  $\mathcal{F}$ . Denote by  $\mathcal{L}^p(\mathfrak{X}, \mathcal{H})$  the family of  $\mathcal{H}$ -measurable random elements in  $\mathfrak{X}$  with  $p$ -integrable norm for  $p \in [1, \infty)$ , essentially bounded if  $p = \infty$ , and all random elements if  $p = 0$ . The closure in the strong topology in  $\mathcal{L}^p$  for  $p \in [1, \infty)$  is denoted by  $\text{cl}_p$  and  $\text{cl}_0$  is the closure in probability for  $p = 0$ . If  $p = \infty$ , the closure is considered in the  $\sigma(\mathcal{L}^\infty, \mathcal{L}^1)$ -topology.

An  $\mathcal{H}$ -measurable random set (shortly, random set) is a set-valued function  $\omega \mapsto X(\omega) \subset \mathfrak{X}$  from  $\Omega$  to the family of all subsets of  $\mathfrak{X}$ , such that its graph

$$(2.1) \quad \text{Gr } X = \{(\omega, x) \in \Omega \times \mathfrak{X} : x \in X(\omega)\}$$

belongs to the product  $\sigma$ -algebra  $\mathcal{H} \otimes \mathcal{B}(\mathfrak{X})$ ; in this case  $X$  is often called *graph measurable*, see [15, Sec. 1.2.5]. Unless otherwise stated, by the measurability we always understand the measurability with respect to  $\mathcal{F}$ . The random set  $X$  is said to be closed (convex, open) if  $X(\omega)$  is a closed (convex, open) set for almost all  $\omega$ . If  $X$  is closed, then (2.1) holds if and only if  $X$  is *Effros measurable*, that is  $\{\omega : X(\omega) \cap G \neq \emptyset\} \in \mathcal{F}$  for each open set  $G$ , see [15, Def. 1.2.1].

**Definition 2.1.** An  $\mathcal{H}$ -measurable random element  $\xi$  such that  $\xi(\omega) \in X(\omega)$  for almost all  $\omega \in \Omega$  is said to be an  $\mathcal{H}$ -measurable *selection* (selection in short) of  $X$ ,  $\mathcal{L}^0(X, \mathcal{H})$  denotes the family of all  $\mathcal{H}$ -measurable selections of  $X$ , and  $\mathcal{L}^p(X, \mathcal{H})$  is the family of  $p$ -integrable ones for  $p \in [1, \infty]$ .

It is known that each almost surely non-empty random set has at least one selection, see [9, Th. 4.4].

**Lemma 2.2.** Let  $\{\xi_n, n \geq 1\}$  be a sequence from  $\mathcal{L}^0(\mathfrak{X}, \mathcal{F})$ , so that  $X(\omega) = \text{cl}\{\xi_n(\omega), n \geq 1\}$  is a random closed set. Let  $\xi \in \mathcal{L}^0(X, \mathcal{F})$ .

Then, for each  $\varepsilon > 0$ , there exists a measurable partition  $A_1, \dots, A_n$  of  $\Omega$  such that

$$\mathbb{E} \left[ \left\| \xi - \sum_{i=1}^n \mathbf{1}_{A_i} \xi_i \right\| \wedge 1 \right] \leq \varepsilon.$$

*Proof.* Consider a measurable countable partition  $\{B_i, i \geq 1\}$  of  $\Omega$ , such that  $\|\xi - \xi_i\| < \varepsilon/2$  on  $B_i$  and choose a large enough  $n$ , so that

$$\mathbb{E} \left[ \mathbf{1}_{\cup_{i \geq n+1} B_i} \|\xi - \xi_1\| \wedge 1 \right] \leq \varepsilon/2.$$

Define  $A_1 = B_1 \cup (\cup_{i \geq n+1} B_i)$  and  $A_i = B_i$  for  $i = 2 \dots, n$ . Since the mapping  $x \mapsto x \wedge 1$  is increasing,

$$\mathbb{E} \left[ \left\| \xi - \sum_{i=1}^n \mathbf{1}_{A_i} \xi_i \right\| \wedge 1 \right] \leq \mathbb{E} \left[ \left( \sum_{i=1}^n \|\xi - \xi_i\| \mathbf{1}_{A_i} \right) \wedge 1 \right].$$

Then

$$\begin{aligned} \mathbb{E} \left[ \left\| \xi - \sum_{i=1}^n \mathbf{1}_{A_i} \xi_i \right\| \wedge 1 \right] &\leq \sum_{i=1}^n \mathbb{E} [\|\xi - \xi_i\| \mathbf{1}_{A_i} \wedge 1] \\ &\leq \sum_{i=1}^n \mathbb{E} [\|\xi - \xi_i\| \mathbf{1}_{B_i} \wedge 1] + \mathbb{E} [\mathbf{1}_{\cup_{i \geq n+1} B_i} \|\xi - \xi_1\| \wedge 1] \leq \varepsilon. \quad \square \end{aligned}$$

**Definition 2.3.** A family  $\Xi \subset \mathcal{L}^0(\mathfrak{X}, \mathcal{F})$  is said to be  $\mathcal{H}$ -decomposable if, for each  $\xi, \eta \in \Xi$  and each  $A \in \mathcal{H}$ , the random element  $\mathbf{1}_A \xi + \mathbf{1}_{A^c} \eta$  belongs to  $\Xi$ .

Decomposable subsets of  $\mathcal{L}^0(\mathbb{R}^d, \mathcal{F})$  are studied under the name stable sets in [3]. The following result is well known for  $p = 1$  [11], for  $p \in [1, \infty]$  [15, Th.2.1.6], and is mentioned in [13, Prop. 5.4.3] without proof for  $p = 0$ . We give below the proof in the latter case and provide its variant for random *convex* sets.

**Theorem 2.4.** Let  $\Xi$  be a non-empty subset of  $\mathcal{L}^p(\mathfrak{X}, \mathcal{F})$  for  $p = 0$  or  $p \in [1, \infty]$ . Then  $\Xi = \mathcal{L}^p(X, \mathcal{F})$  for a random closed set  $X$  if and only if  $\Xi$  is  $\mathcal{F}$ -decomposable and closed. The family  $\Xi$  is convex (is a cone in  $\mathcal{L}^p(\mathfrak{X}, \mathcal{F})$ ) if and only if  $X$  is convex (is a cone in  $\mathfrak{X}$ ).

*Proof.* The necessity is trivial. Let  $p = 0$  and assume that  $\Xi$  is  $\mathcal{F}$ -decomposable and closed. Consider a countable dense set  $\{x_i, i \geq 1\} \subset \mathfrak{X}$  and define

$$a_i = \inf_{\eta \in \Xi} \mathbb{E} [\|\eta - x_i\| \wedge 1], \quad i \geq 1.$$

For all  $i, j \geq 1$ , there exists an  $\eta_{ij} \in \Xi$  such that

$$\mathbb{E} [\|\eta_{ij} - x_i\| \wedge 1] \leq a_i + j^{-1}.$$

Define  $X(\omega) = \text{cl}\{\eta_{ij}(\omega), i, j \geq 1\}$  for all  $\omega \in \Omega$ . Since  $\Xi$  is decomposable and closed,  $\mathcal{L}^0(X, \mathcal{F}) \subset \Xi$  by Lemma 2.2. If  $\Xi$  is convex or is a cone, the same inclusion holds for  $X$  being the closed convex hull of  $\{\eta_{ij}(\omega), i, j \geq 1\}$  or the closed cone generated by these random elements.

Assume that there exists a  $\xi \in \Xi$ , which does not belong to  $\mathcal{L}^0(X, \mathcal{F})$ . Then there exists a  $\delta \in (0, 1)$  such that

$$A = \bigcap_{i, j \geq 1} \{\|\xi - \eta_{ij}\| \wedge 1 > \delta\}$$

has positive probability. Since  $\Omega = \cup_i \{\|\xi - x_i\| \wedge 1 < \delta/3\}$ , the event  $B_i = A \cap \{\|\xi - x_i\| \wedge 1 < \delta/3\}$  has a positive probability for some  $i \geq 1$ . Recall that

$$\|a - b\| \wedge 1 \leq \|a - c\| \wedge 1 + \|c - b\| \wedge 1.$$

Then, on the set  $B_i$ ,

$$\|x_i - \eta_{ij}\| \wedge 1 \geq \|\xi - \eta_{ij}\| \wedge 1 - \|\xi - x_i\| \wedge 1 \geq \frac{2\delta}{3}.$$

Furthermore,  $\eta'_{ij} = \xi \mathbf{1}_{B_i} + \eta_{ij} \mathbf{1}_{B_i^c} \in \Xi$  by decomposability, and

$$\begin{aligned} j^{-1} &\geq \mathbb{E}[\|\eta_{ij} - x_i\| \wedge 1] - a_i \geq \mathbb{E}[\|\eta_{ij} - x_i\| \wedge 1] - \mathbb{E}[\|\eta'_{ij} - x_i\| \wedge 1] \\ &\geq \mathbb{E}\left[(\|\eta_{ij} - x_i\| \wedge 1 - \|\xi - x_i\| \wedge 1) \mathbf{1}_{B_i}\right] \geq \frac{\delta}{3} \mathbb{P}(B_i). \end{aligned}$$

Since  $B_i$  and  $\delta$  do not depend on  $j$ , letting  $j \rightarrow \infty$  yields a contradiction.  $\square$

**Corollary 2.5.** *If  $\Xi \subset \mathcal{L}^p(\mathfrak{X}, \mathcal{F})$  with  $p = 0$  or  $p \in [1, \infty]$  is closed and  $\mathcal{H}$ -decomposable, then there exists an  $\mathcal{H}$ -measurable random closed set  $X$  such that*

$$\Xi \cap \mathcal{L}^p(\mathfrak{X}, \mathcal{H}) = \mathcal{L}^p(X, \mathcal{H}).$$

**Proposition 2.6.** *If  $X$  is a random set, then its pointwise closure  $\text{cl } X(\omega)$ ,  $\omega \in \Omega$ , is a random closed set, and  $\mathcal{L}^0(\text{cl } X, \mathcal{F}) = \text{cl}_0 \mathcal{L}^0(X, \mathcal{F})$ .*

*Proof.* Since the probability space is complete and the graph of  $X$  is measurable in the product space, the projection theorem yields that  $\{X \cap G \neq \emptyset\} \in \mathcal{F}$  for any open set  $G$ . Finally, note that  $X$  hits any open set  $G$  if and only if  $\text{cl } X$  hits  $G$ . Thus,  $\text{cl } X$  is Effros measurable and so is a random closed set.

The inclusion  $\text{cl}_0 \mathcal{L}^0(X, \mathcal{F}) \subset \mathcal{L}^0(\text{cl } X, \mathcal{F})$  obviously holds. Since  $\text{cl}_0 \mathcal{L}^0(X, \mathcal{F})$  is decomposable, there exists a random closed set  $Y$  such that  $\text{cl}_0 \mathcal{L}^0(X, \mathcal{F}) = \mathcal{L}^0(Y, \mathcal{F})$ . Since  $\mathcal{L}^0(X, \mathcal{F}) \subset \mathcal{L}^0(Y, \mathcal{F})$ , we have  $X \subset Y$  a.s. Therefore,  $\text{cl } X \subset Y$  a.s. and the conclusion follows.  $\square$

The following result is well known for random closed sets as a *Castaing representation*, see e.g. [15, Th. 1.2.3].

**Proposition 2.7.** *If  $X$  is a non-empty random set, then there exists a countable family  $\{\xi_i, i \geq 1\}$  of measurable selections of  $X$  such that*

$$\text{cl } X = \text{cl}\{\xi_i, i \geq 1\} \quad a.s.$$

*Proof.* It suffices to repeat the part of the proof of Theorem 2.4 with  $\Xi = \mathcal{L}^0(\text{cl } X, \mathcal{F})$  and observe that

$$a_i = \inf_{\eta \in \mathcal{L}^0(X, \mathcal{F})} \mathbf{E}\|\eta - x_i\| \wedge 1 = \inf_{\eta \in \Xi} \mathbf{E}\|\eta - x_i\| \wedge 1, \quad i \geq 1.$$

Indeed, by Proposition 2.6,  $\Xi = \text{cl}_0 \mathcal{L}^0(X, \mathcal{F})$ . □

Proposition 2.7 yields that the norm

$$\|X\| = \sup\{\|x\| : x \in X\} = \sup\{\|\xi\| : \xi \in \mathcal{L}^0(X, \mathcal{F})\}$$

and the Hausdorff distance between any two random sets are random variables with values in  $[0, \infty]$ .

The following result establishes the existence of a measurable version for any closed-valued mapping.

**Proposition 2.8.** *For any closed set-valued mapping  $X(\omega)$ ,  $\omega \in \Omega$ , there exists a random closed set  $Y$  (called the measurable version of  $X$ ) such that  $\mathcal{L}^0(X, \mathcal{F}) = \mathcal{L}^0(Y, \mathcal{F})$ . If  $X$  is convex (respectively, a cone), then  $Y$  is also convex (respectively, a cone).*

*Proof.* Assume that  $\mathcal{L}^0(X, \mathcal{F})$  is non-empty, otherwise, the statement is evident with empty  $Y$ . Since  $\mathcal{L}^0(X, \mathcal{F})$  is closed and decomposable, Theorem 2.4 ensures the existence of  $Y$  that satisfies the required conditions. □

If  $X_i$ ,  $i \in I$ , is an uncountable family of random sets, then Proposition 2.8 makes it possible to define measurable versions of the closure of their union or intersection.

For  $A, B \subset \mathfrak{X}$ , define their pointwise sum as

$$A + B = \{x + y : x \in A, y \in B\}.$$

The same definition applies to the sum of subsets of  $\mathcal{L}^p(\mathfrak{X}, \mathcal{F})$ . Note that the sum of two closed sets is not necessarily closed, unless at least one summand is compact. If  $X$  and  $Y$  are two random closed sets, then the closure of  $X + Y$  is a random closed set too, see [15].



## 2.2. Infinite decomposability.

**Definition 2.9.** A family  $\Xi \subset \mathcal{L}^0(\mathfrak{X}, \mathcal{F})$  is said to be *infinitely  $\mathcal{H}$ -decomposable* if

$$\sum_n \xi_n \mathbf{1}_{A_n} \in \Xi$$

for all sequences  $\{\xi_n, n \geq 1\}$  from  $\Xi$  and all  $\mathcal{H}$ -measurable partitions  $\{A_n, n \geq 1\}$  of  $\Omega$ .

Infinitely  $\mathcal{F}$ -decomposable subsets of  $\mathcal{L}^0(\mathbb{R}^d, \mathcal{F})$  are called  $\sigma$ -stable in [3]. Taking a partition that consists of two sets and letting all other sets be empty, it is immediate that the infinite decomposability implies the decomposability. Observe that an infinitely decomposable family  $\Xi$  is not necessarily closed in  $\mathcal{L}^0$ , e.g.,  $\Xi = \mathcal{L}^0(X, \mathcal{F})$  with a non-closed  $X$ . It is easy to see that if  $\Xi$  is infinitely decomposable, then its closure in  $\mathcal{L}^0$  is also infinitely decomposable.

In Euclidean space, the sum of an open set  $G$  and another set  $M$  equals the sum of  $G$  and the closure of  $M$ . The following result shows that an analogue of this for subsets of  $\mathcal{L}^0(\mathfrak{X}, \mathcal{F})$  holds under the infinite decomposability assumption.

**Proposition 2.10.** *Let  $\Xi$  be an infinitely  $\mathcal{F}$ -decomposable subset of  $\mathcal{L}^0(\mathfrak{X}, \mathcal{F})$ , and let  $X$  be an  $\mathcal{F}$ -measurable a.s. non-empty random open set. Then*

$$\mathcal{L}^0(X, \mathcal{F}) + \Xi = \mathcal{L}^0(X, \mathcal{F}) + \text{cl}_0 \Xi.$$

*Proof.* Consider  $\gamma \in \mathcal{L}^0(X, \mathcal{F})$  and  $\xi \in \text{cl}_0 \Xi$ , so that  $\xi_n \rightarrow \xi$  a.s. for  $\xi_n \in \Xi$ ,  $n \geq 1$ . By a measurable selection argument, there exists an  $\alpha \in \mathcal{L}^0((0, \infty), \mathcal{F})$  such that the ball of radius  $\alpha$  centred at  $\gamma$  is a subset of  $X$  a.s. Let us define, up to a null set,

$$k(\omega) = \inf\{n : \|\xi(\omega) - \xi_n(\omega)\| \leq \alpha(\omega)\}, \quad \omega \in \Omega.$$

Since the mapping  $\omega \mapsto k(\omega)$  is  $\mathcal{F}$ -measurable,

$$\hat{\xi}(\omega) = \xi_{k(\omega)}(\omega) = \sum_{j=1}^{\infty} \xi_j \mathbf{1}_{k(\omega)=j},$$

is also  $\mathcal{F}$ -measurable and belongs to  $\Xi$  by the infinite decomposability assumption. Since  $\|\hat{\xi} - \xi\| \leq \alpha$  a.s.,

$$\xi + \gamma = (\xi + \gamma - \hat{\xi}) + \hat{\xi} \in \mathcal{L}^0(X, \mathcal{F}) + \Xi. \quad \square$$

**Corollary 2.11.** *Let  $\Xi$  be an infinitely  $\mathcal{F}$ -decomposable subset of  $\mathcal{L}^0(\mathfrak{X}, \mathcal{F})$ . For every  $\gamma \in \text{cl}_0 \Xi$  and  $\alpha \in \mathcal{L}^0((0, \infty), \mathcal{F})$ , there exists a  $\xi \in \Xi$  such that  $\|\gamma - \xi\| \leq \alpha$  a.s.*

*Proof.* Apply Proposition 2.10 with  $X$  being the open ball of radius  $\alpha/2$  centred at zero.  $\square$

### 3. RANDOM CONVEX SETS

**3.1. Support function.** Let  $\mathfrak{X}^*$  be the dual space to  $\mathfrak{X}$  with the pairing  $\langle u, x \rangle$  for  $x \in \mathfrak{X}$  and  $u \in \mathfrak{X}^*$ . The space  $\mathfrak{X}^*$  is equipped with the  $\sigma(\mathfrak{X}^*, \mathfrak{X})$ -topology (see [1, Sec. 5.14]) and the corresponding Borel  $\sigma$ -algebra, so that  $\zeta$  is a random element in  $\mathfrak{X}^*$  if  $\langle \zeta, x \rangle$  is a random variable for all  $x \in \mathfrak{X}$ . Let  $\mathfrak{X}_0^*$  be a countable total subset of  $\mathfrak{X}^*$  (which always exists). The separability of  $\mathfrak{X}$  ensures that the  $\sigma(\mathfrak{X}^*, \mathfrak{X})$ -topology is metrisable, and the corresponding metric space is complete separable, see [4] and [14, Th. 7.8.3].

The *support function* of a random set  $X$  in  $\mathfrak{X}$  is defined by

$$h_X(u) = \sup\{\langle u, x \rangle : x \in X\}, \quad u \in \mathfrak{X}^*.$$

The support function of the empty set is set to be  $-\infty$ . It is easy to see that the support function does not discern between  $X$  and its closed convex hull. The support function is a lower semicontinuous sublinear function of  $u$ ; if  $X$  is weakly compact, then the support function is  $\sigma(\mathfrak{X}^*, \mathfrak{X})$ -continuous, see [1, Th. 7.52]. Recall that all topologies consistent with the pairing have the same lower semicontinuous sublinear functions.

If  $X$  is  $p$ -integrably bounded, that is  $\|X\| \in \mathcal{L}^p(\mathbb{R}, \mathcal{F})$  for  $p \in [1, \infty]$ , then  $h_X(\zeta) \in \mathcal{L}^1(\mathbb{R}, \mathcal{F})$  for  $\zeta \in \mathcal{L}^q(\mathfrak{X}^*, \mathcal{F})$  with  $p^{-1} + q^{-1} = 1$ . If  $X$  is not bounded, then the support function may take infinite values, even with probability one for each given  $u \in \mathfrak{X}^*$ , e.g., if  $X$  is a line in  $\mathfrak{X} = \mathbb{R}^2$  with a uniformly distributed direction. This fact calls for letting the argument of  $h_X$  be random. The following result is well known for deterministic arguments of the support function; it refers to the definition of the essential supremum from Appendix A.

**Lemma 3.1.** *For every  $\zeta \in \mathcal{L}^0(\mathfrak{X}^*, \mathcal{F})$  and a random closed convex set  $X$ ,  $h_X(\zeta)$  is a random variable in  $[-\infty, \infty]$ , and*

$$h_X(\zeta) = \text{ess sup}_{\mathcal{F}}\{\langle \zeta, \xi \rangle : \xi \in \mathcal{L}^0(X, \mathcal{F})\} \quad \text{a.s.}$$

*if  $X$  is a.s. non-empty.*

*Proof.* Since  $\{X = \emptyset\} \in \mathcal{F}$ , it is possible to assume that  $X$  is a.s. non-empty. Taking a Castaing representation  $X = \text{cl}\{\xi_i, i \geq 1\}$ , we confirm that  $h_X(\zeta) = \sup_i \langle \zeta, \xi_i \rangle$  is  $\mathcal{F}$ -measurable. It is immediate that  $h_X(\zeta) \geq \langle \zeta, \xi \rangle$  for all  $\xi \in \mathcal{L}^0(X, \mathcal{F})$ , so that

$$h_X(\zeta) \geq \text{ess sup}_{\mathcal{F}}\{\langle \zeta, \xi \rangle : \xi \in \mathcal{L}^0(X, \mathcal{F})\}.$$

Assume that  $X$  is a.s. bounded, so that  $|h_X(\zeta)| < \infty$  a.s. For any  $\varepsilon > 0$ , the random closed set  $X \cap \{x : \langle \zeta, x \rangle \geq h_X(\zeta) - \varepsilon\}$  is a.s. non-empty and so possesses a selection  $\eta$ . Then

$$\text{ess sup}_{\mathcal{F}}\{\langle \zeta, \xi \rangle : \xi \in \mathcal{L}^0(X, \mathcal{F})\} \geq \langle \zeta, \eta \rangle \geq h_X(\zeta) - \varepsilon.$$

Letting  $\varepsilon \downarrow 0$  yields that

$$(3.1) \quad h_X(\zeta) = \text{ess sup}_{\mathcal{F}}\{\langle \zeta, \xi \rangle : \xi \in \mathcal{L}^0(X, \mathcal{F})\} \quad \text{a.s.}$$

For a general closed set  $X$ ,  $h_X(\zeta)$  is the limit of  $h_{X^n}(\zeta)$  as  $n \rightarrow \infty$ , where  $X^n$  is the intersection of  $X$  with the centred ball of radius  $n$ . Since (3.1) holds for  $X = X^n$  and  $X^n \subset X$  a.s.,

$$\langle \zeta, \xi_n \rangle \leq \text{ess sup}_{\mathcal{F}}\{\langle \zeta, \xi \rangle : \xi \in \mathcal{L}^0(X, \mathcal{F})\} \quad \text{a.s.}$$

for all  $\xi_n \in \mathcal{L}^0(X^n, \mathcal{F})$ . Hence,

$$\begin{aligned} \text{ess sup}_{\mathcal{F}}\{\langle \zeta, \xi_n \rangle : \xi_n \in \mathcal{L}^0(X^n, \mathcal{F})\} \\ \leq \text{ess sup}_{\mathcal{F}}\{\langle \zeta, \xi \rangle : \xi \in \mathcal{L}^0(X, \mathcal{F})\} \quad \text{a.s.} \end{aligned}$$

Therefore,

$$h_X(\zeta) = \lim_{n \rightarrow \infty} h_{X^n}(\zeta) \leq \text{ess sup}_{\mathcal{F}}\{\langle \zeta, \xi \rangle : \xi \in \mathcal{L}^0(X, \mathcal{F})\},$$

and the conclusion follows.  $\square$

*Remark 3.2.* For  $\xi_1, \xi_2 \in \mathcal{L}^0(X, \mathcal{F})$  and  $\zeta \in \mathcal{L}^0(\mathfrak{X}^*, \mathcal{F})$ , define

$$\xi = \xi_1 1_{\langle \zeta, \xi_1 \rangle > \langle \zeta, \xi_2 \rangle} + \xi_2 1_{\langle \zeta, \xi_1 \rangle \leq \langle \zeta, \xi_2 \rangle}.$$

Then  $\xi \in \mathcal{L}^0(X, \mathcal{F})$  and  $\langle \zeta, \xi \rangle = \langle \zeta, \xi_1 \rangle \vee \langle \zeta, \xi_2 \rangle$ . Therefore, the family  $\{\langle \zeta, \xi \rangle : \xi \in \mathcal{L}^0(X, \mathcal{F})\}$  is directed upward, so that  $\langle \zeta, \xi_n \rangle \uparrow h_X(\zeta)$ , where  $\{\xi_n, n \geq 1\} \subset \mathcal{L}^0(X, \mathcal{F})$ .

**3.2. Polar sets.** The *polar* set to a random set  $X$  is defined by

$$X^o = \{u \in \mathfrak{X}^* : h_X(u) \leq 1\}.$$

The polar to  $X$  is a convex  $\sigma(\mathfrak{X}^*, \mathfrak{X})$ -closed set, which coincides with the polar to the closed convex hull of  $X$ .

**Lemma 3.3.** *If  $X$  is a random set in  $\mathfrak{X}$ , then its polar  $X^o$  is a random  $\sigma(\mathfrak{X}^*, \mathfrak{X})$ -closed convex set.*

*Proof.* By Proposition 2.7,  $\text{cl} X$  admits a Castaing representation  $\{\xi_i, i \geq 1\}$ . Then

$$\text{Gr } X^o = \bigcap_{i \geq 1} \{(\omega, u) \in \Omega \times \mathfrak{X}^* : \langle u, \xi_i(\omega) \rangle \leq 1\} \in \mathcal{F} \otimes \mathcal{B}(\mathfrak{X}^*),$$

and it remains to note that  $X^o$  is closed for all  $\omega$  and to note that  $\mathfrak{X}^*$  is Polish in the  $\sigma(\mathfrak{X}^*, \mathfrak{X})$ -topology.  $\square$

The following result is a variant of the well-known fact saying that each convex closed set equals the intersection of at most a countable number of half-spaces. In the setting of random convex sets, these half-spaces become random and are determined by selections of  $X^\circ$ .

**Theorem 3.4.** *Each almost surely non-empty random closed convex set  $X$  in a separable Banach space is obtained as the intersection of an at most countable number of random half-spaces, that is, there exists a countable set  $\{\zeta_n, n \geq 1\} \subset \mathcal{L}^0(\mathfrak{X}^*, \mathcal{F})$  such that*

$$(3.2) \quad X = \bigcap_{n \geq 1} \{x \in \mathfrak{X} : \langle \zeta_n, x \rangle \leq h_X(\zeta_n)\}.$$

If  $X$  is weakly compact, then it is possible to let  $\zeta_n$  be deterministic from a countable total set.

*Proof.* Assume first that  $X$  almost surely contains the origin, and let

$$X^\circ = \text{cl}\{\zeta_n, n \geq 1\}$$

be a Castaing representation of  $X^\circ$ , which is graph measurable by Lemma 3.3. Since  $X$  is convex, it is also weakly closed, and the bipolar theorem yields that  $X = (X^\circ)^\circ$ . The second polar does not make a difference between the set  $\{\zeta_n, n \geq 1\}$  and its closure, whence  $X$  is the polar set to  $\{\zeta_n, n \geq 1\}$ , i.e.

$$(3.3) \quad X = \bigcap_{n \geq 1} \{x \in \mathfrak{X} : \langle \zeta_n, x \rangle \leq 1\}.$$

Denote by  $\tilde{X}$  the right-hand side of (3.2). Since  $h_X(\zeta_n) \leq 1$ , the right-hand side of (3.3) is a superset of  $\tilde{X}$ . It remains to note that  $X$  is a subset of  $\{x \in \mathfrak{X} : \langle \zeta_n, x \rangle \leq h_X(\zeta_n)\}$  for all  $n$  and so is a subset of  $\tilde{X}$ .

If  $X$  does not necessarily contain the origin, consider  $Y = X - \xi$  for any  $\xi \in \mathcal{L}^0(X, \mathcal{F})$ , so that

$$\begin{aligned} Y &= \bigcap_{n \geq 1} \{y \in \mathfrak{X} : \langle \zeta_n, y \rangle \leq h_Y(\zeta_n)\} \\ &= \bigcap_{n \geq 1} \{y \in \mathfrak{X} : \langle \zeta_n, y + \xi \rangle \leq h_X(\zeta_n)\}. \end{aligned}$$

It remains to note that  $x \in X$  if and only if  $x - \xi \in Y$ .

If  $X$  is weakly compact, then the support function is  $\sigma(\mathfrak{X}^*, \mathfrak{X})$ -continuous on  $\mathfrak{X}^*$ , and  $h_X(u)$  is a finite random variable for each  $u \in \mathfrak{X}^*$ . Consequently,  $X$  equals the intersection of half-spaces  $\{x : \langle u, x \rangle \leq h_X(u)\}$  for all  $u \in \mathfrak{X}_0^*$ .  $\square$

**Corollary 3.5.** *Each almost surely non-empty random closed convex set  $X$  in a separable Banach space satisfies*

$$(3.4) \quad X = \bigcap_{\zeta \in \mathcal{L}^0(\mathfrak{X}^*, \mathcal{F})} \{x \in \mathfrak{X} : \langle \zeta, x \rangle \leq h_X(\zeta)\}.$$

*Proof.* It suffices to note that  $X$  is a subset of the right-hand side of (3.4), and the uncountable intersection is a subset of the right-hand side of (3.3).  $\square$

**Corollary 3.6.** *If  $X$  and  $Y$  are two random convex closed sets and  $h_Y(\zeta) \leq h_X(\zeta)$  a.s. for each  $\zeta \in \mathcal{L}^0(\mathfrak{X}^*, \mathcal{F})$ , then  $Y \subset X$  a.s.*

*Proof.* Consider the representation of  $X$  given by (3.3). Then

$$X \supset \bigcap_{n \geq 1} \{x \in \mathfrak{X} : \langle \zeta_n, x \rangle \leq h_Y(\zeta_n)\} \supset Y,$$

where the latter inclusion follows from (3.4).  $\square$

Unless the random sets are weakly compact, it does not suffice to consider deterministic  $\zeta$  in Corollary 3.6.

**3.3. Epigraphs.** The *epigraph* of a function  $f : \mathfrak{X}^* \mapsto [-\infty, \infty]$  is defined as

$$\text{epi } f = \{(u, t) \in \mathfrak{X}^* \times \mathbb{R} : f(u) \leq t\}.$$

The epigraphs of support functions can be characterised as subsets of  $\mathfrak{X}^* \times \mathbb{R}$  closed in the product of the  $\sigma(\mathfrak{X}^*, \mathfrak{X})$ -topology and the Euclidean topology on  $\mathbb{R}$ , and which are convex cones that, with each element  $(u, t)$ , contain  $(u, s)$  for all  $s \geq t$ . We denote the family of such subsets by  $\mathcal{E}$ . The following result characterises epigraphs of random closed sets in  $\mathfrak{X}$  and is interesting on its own. Its version for Euclidean spaces is known, see [15, Prop. 5.3.6].

**Theorem 3.7.** *A closed convex set-valued mapping  $X$  in  $\mathfrak{X}$  is  $\mathcal{F}$ -measurable if and only if  $\text{epi } h_X$  is an  $\mathcal{F}$ -measurable random closed set with values in  $\mathcal{E}$ .*

*Proof. Necessity.* Consider a Castaing representation  $X = \text{cl}\{\xi_i, i \geq 1\}$  of an  $\mathcal{F}$ -measurable random closed convex set  $X$ . Then

$$h_X(u) = \sup\{\langle x, u \rangle : x \in X\} = \sup_{i \geq 1} \langle \xi_i, u \rangle, \quad u \in \mathfrak{X}^*.$$

Therefore, the graph of  $\text{epi } h_X$  is given by

$$(3.5) \quad \text{Gr}(\text{epi } h_X) = \bigcap_{i \geq 1} \{(\omega, u, t) \in \Omega \times \mathfrak{X}^* \times \mathbb{R} : t \geq \langle \xi_i(\omega), u \rangle\}$$

Finally, note that the mapping  $u \mapsto \langle \xi_i(\omega), u \rangle$  is measurable with respect to the product of  $\mathcal{F}$  and  $\mathcal{B}(\mathfrak{X}^*)$ .

*Sufficiency.* Let  $Y$  be a random closed set with values in  $\mathcal{E}$ . Then  $Y$  is the epigraph of a lower semicontinuous sublinear function

$$h(u) = \inf\{t : (u, t) \in Y\}.$$

Thus,  $h$  is the support function of a set-valued map  $X$  with closed convex values. It remains to show that  $X$  is  $\mathcal{F}$ -measurable.

Let  $\{(\zeta_i, t_i), i \geq 1\}$  be a Castaing representation of  $Y$ . Define an  $\mathcal{F}$ -measurable random closed set by letting

$$Z = \bigcap_{n \geq 1} \{x \in \mathfrak{X} : \langle \zeta_n, x \rangle \leq t_n\}.$$

Let  $(\zeta, t)$  be a selection of  $Y$ , that is  $h(\zeta) \leq t$  a.s., and assume that  $t = h(\zeta)$ . By Lemma 2.2,  $(\zeta, t)$  is the a.s. limit of a sequence  $(\zeta'_m, t'_m)$ , where  $(\zeta'_m, t'_m)$  are obtained as combinations of the members of a Castaing representation of  $Y$ . It is easy to see that

$$Z \subset \{x \in \mathfrak{X} : \langle \zeta'_m, x \rangle \leq t'_m\}, \quad m \geq 1,$$

whence  $h_Z(\zeta'_m) \leq t'_m$  for all  $m \geq 1$ . Note that  $\zeta'_m \rightarrow \zeta$  in the norm topology on  $\mathfrak{X}^*$ , whence also in  $\sigma(\mathfrak{X}^*, \mathfrak{X})$ . Passing to the limits and using the lower semicontinuity of the support function  $h_Z$  yields that  $h_Z(\zeta) \leq h(\zeta)$ . By Corollary 3.6,  $Z \subset X$ . The other inclusion is obvious.  $\square$

#### 4. CONDITIONAL CORE

**4.1. Existence.** The following concept is related to the measurable versions of random closed sets considered in Proposition 2.8.

**Definition 4.1.** Let  $X$  be any set-valued mapping. The conditional core  $\mathbf{m}(X|\mathcal{H})$  of  $X$  (also called  $\mathcal{H}$ -core) is the largest  $\mathcal{H}$ -measurable random set  $X'$  such that  $X' \subset X$  a.s.

The following result relates the conditional core to the family of  $\mathcal{H}$ -measurable selections of  $X$ .

**Lemma 4.2.** *If  $\mathbf{m}(X|\mathcal{H})$  exists and is almost surely non-empty, then*

$$(4.1) \quad \mathcal{L}^0(X, \mathcal{H}) = \mathcal{L}^0(\mathbf{m}(X|\mathcal{H}), \mathcal{H}),$$

*in particular  $\mathbf{m}(X, \mathcal{H})$  is a.s. non-empty if and only if  $\mathcal{L}^0(X, \mathcal{H}) \neq \emptyset$ .*

*Proof.* In order to show the non-trivial inclusion, consider  $\gamma \in \mathcal{L}^0(X, \mathcal{H})$ . The random set  $X'(\omega) = \{\gamma(\omega)\}$  is  $\mathcal{H}$ -measurable and satisfies  $X' \subset X$  a.s. It follows that  $X' \subset \mathbf{m}(X|\mathcal{H})$  a.s., so that  $\gamma \in \mathbf{m}(X|\mathcal{H})$  a.s.  $\square$

The existence of the  $\mathcal{H}$ -core is the issue of the existence of the *largest*  $\mathcal{H}$ -measurable set  $X' \subset X$ . It does not prevent  $\mathbf{m}(X|\mathcal{H})$  from being empty. If  $\mathcal{H}$  is the trivial  $\sigma$ -algebra, then  $\mathbf{m}(X|\mathcal{H})$  is the set of all points  $x \in \mathfrak{X}$  such that  $x \in X(\omega)$  almost surely. Such points are called *fixed points* of a random set and it is obvious that the set of fixed points may be empty.

**Lemma 4.3.** *If  $X$  is a random closed set, then  $\mathbf{m}(X|\mathcal{H})$  exists and is a random closed set, which is a.s. convex (respectively, is a cone) if  $X$  is a.s. convex (respectively, is a cone).*

*Proof.* We first consider the case where  $\mathcal{L}^0(X, \mathcal{H}) \neq \emptyset$ . By Theorem 2.4,  $\mathcal{L}^0(X, \mathcal{H}) = \mathcal{L}^0(Y, \mathcal{H})$  for an  $\mathcal{H}$ -measurable random closed set  $Y$ . Moreover,  $Y = \text{cl}\{\xi_n, n \geq 1\}$  a.s. for  $\xi_n \in \mathcal{L}^0(X, \mathcal{H})$ ,  $n \geq 1$ . Since  $X$  is closed,  $Y \subset X$  a.s. Since any  $\mathcal{H}$ -measurable random set  $Z \subset X$  satisfies  $\mathcal{L}^0(Z, \mathcal{H}) \subset \mathcal{L}^0(Y, \mathcal{H})$ , we have  $Z \subset X$  a.s., so that  $Y = \mathbf{m}(X|\mathcal{H})$ . If  $X$  is convex, then  $\mathcal{L}^0(X, \mathcal{H}) = \mathcal{L}^0(Y, \mathcal{H})$  is convex, whence  $Y$  is a random convex set by Theorem 2.4.

Let us now consider the case  $\mathcal{L}^0(X, \mathcal{H}) = \emptyset$ . Define

$$\mathcal{I} = \{H \in \mathcal{H} : \mathcal{L}^0(X, \mathcal{H} \cap H) \neq \emptyset\},$$

where  $\mathcal{L}^0(X, \mathcal{H} \cap H)$  designates the  $\mathcal{H} \cap H$ -measurable selections of  $X(\omega)$ ,  $\omega \in H$ , measurable with respect to the trace of  $\mathcal{H}$  on  $H$ . Observe that  $\mathcal{I} \neq \emptyset$  if and only if there exists a closed  $\mathcal{H}$ -measurable subset  $Z_{\mathcal{H}}$  of  $X$  such that  $\mathbf{P}\{Z_{\mathcal{H}} \neq \emptyset\} > 0$ . If  $\mathcal{I} = \emptyset$ , we let  $\mathbf{m}(X|\mathcal{H}) = \emptyset$ . Otherwise, note that  $H_1, H_2 \in \mathcal{I}$  implies that  $H_1 \cup H_2 \in \mathcal{I}$ . Hence

$$\zeta = \text{ess sup}_{\mathcal{H}} \{1_H : H \in \mathcal{I}\} = 1_{H^*},$$

where  $H_n \uparrow H^*$  for  $H_n \in \mathcal{I}$ ,  $n \geq 1$ . In particular,  $H^* \in \mathcal{I}$ . By the result for non-empty cores, we have  $\mathcal{L}^0(X, \mathcal{H} \cap H^*) = \mathcal{L}^0(X_{\mathcal{H}}, \mathcal{H} \cap H^*)$  where  $X_{\mathcal{H}}$  is a closed  $\mathcal{H} \cap H^*$ -measurable subset which is non-empty on  $H^*$ . This is the largest  $\mathcal{H} \cap H^*$ -measurable closed subset of  $X$ .

Define  $\mathbf{m}(X|\mathcal{H})$  by  $\mathbf{m}(X|\mathcal{H})(\omega) = X_{\mathcal{H}}(\omega)$  if  $\omega \in H^*$  and  $\emptyset$  otherwise. Consider a closed  $\mathcal{H}$ -measurable subset  $Z$  of  $X$ . The inclusion  $Z \subset \mathbf{m}(X|\mathcal{H})$  is trivial on  $\{\omega : Z(\omega) = \emptyset\} \in \mathcal{H}$ . The complement  $H$  of this latter set belongs to  $\mathcal{I}$  as soon as  $\mathbf{P}(H) > 0$ , using a measurable selection argument. Therefore, we may modify  $H$  on a non-null set and suppose that  $H \subset H^*$ . We deduce by construction of  $X_{\mathcal{H}}$  that  $Z \subset X_{\mathcal{H}}$  on  $H^*$  while this inclusion is trivial on the complement  $\{\omega : Z(\omega) = \emptyset\}$ . Thus,  $\mathbf{m}(X|\mathcal{H})$  is the largest closed  $\mathcal{H}$ -measurable subset of  $X$ .  $\square$

**Lemma 4.4.** *If  $\mathcal{L}^p(X, \mathcal{H}) \neq \emptyset$  for some  $p \in [1, \infty]$  and a random closed set  $X$ , then  $\mathcal{L}^p(\mathbf{m}(X|\mathcal{H}), \mathcal{H}) = \mathcal{L}^p(X, \mathcal{H})$ .*

*Proof.* By the condition,  $\mathbf{m}(X|\mathcal{H})$  admits a  $p$ -integrable selection, and so has a Castaing representation consisting of  $p$ -integrable selections, see [11] and [15].  $\square$

*Example 4.5.* If  $\mathcal{H}$  is generated by a partition  $\{B_1, \dots, B_m\}$  of a finite probability space, then  $Y = \mathbf{m}(X|\mathcal{H})$  is obtained by letting  $Y(\omega) = \cap_{\omega' \in B_i} X(\omega')$  if  $\omega \in B_i$ ,  $i = 1, \dots, m$ .

#### 4.2. Sublinearity of the conditional core.

**Lemma 4.6.** *Let  $X$  be an  $\mathcal{F}$ -measurable random set.*

- (i)  $\lambda X$  is an  $\mathcal{F}$ -measurable random set for any  $\lambda \in \mathcal{L}^0(\mathbb{R}, \mathcal{F})$ .
- (ii) If  $\lambda \in \mathcal{L}^0(\mathbb{R}, \mathcal{H})$ , then

$$(4.2) \quad \mathbf{m}(\lambda X|\mathcal{H}) = \lambda \mathbf{m}(X|\mathcal{H}).$$

- (iii) If  $X$  is a random closed set and  $\mathcal{H}'$  is a sub- $\sigma$ -algebra of  $\mathcal{H}$ , then

$$\mathbf{m}(\mathbf{m}(X|\mathcal{H})|\mathcal{H}') = \mathbf{m}(X|\mathcal{H}').$$

*Proof.* (i) Since

$$\text{Gr}(\lambda X) = (\{\lambda = 0\} \times \{0\}) \cup (\text{Gr}(\lambda X) \cap (\{\lambda \neq 0\} \times \mathfrak{X})),$$

it suffices to assume that  $\lambda \neq 0$  a.s. The measurability of  $\lambda X$  is immediate, since the map  $\phi : (\omega, x) \mapsto (\omega, \lambda^{-1}x)$  is measurable and  $\text{Gr}(\lambda X) = \phi^{-1}(\text{Gr } X)$ .

(ii) Observe that  $\lambda \mathbf{m}(X|\mathcal{H})$  is  $\mathcal{H}$ -measurable if  $\lambda \in \mathcal{L}^0(\mathbb{R}, \mathcal{H})$  and  $\lambda \mathbf{m}(X|\mathcal{H}) \subset \lambda X$ . Suppose that  $X' \subset \lambda X$  is  $\mathcal{H}$ -measurable. Then

$$X'' = \lambda^{-1}X'\mathbf{1}_{\lambda \neq 0} + X\mathbf{1}_{\lambda=0} \subset X$$

is  $\mathcal{H}$ -measurable. Therefore,  $X'' \subset \mathbf{m}(X|\mathcal{H})$ , so that  $X' \subset \lambda \mathbf{m}(X|\mathcal{H})$ . Thus,  $\mathbf{m}(\lambda X|\mathcal{H})$  exists and (4.2) holds.

(iii) By Lemma 4.2,

$$\begin{aligned} \mathcal{L}^0(\mathbf{m}(\mathbf{m}(X|\mathcal{H})|\mathcal{H}'), \mathcal{H}') &= \mathcal{L}^0(\mathbf{m}(X|\mathcal{H}), \mathcal{H}) \cap \mathcal{L}^0(\mathfrak{X}, \mathcal{H}') \\ &= \mathcal{L}^0(X, \mathcal{H}') = \mathcal{L}^0(\mathbf{m}(X|\mathcal{H}'), \mathcal{H}'). \end{aligned} \quad \square$$

The following result establishes that the conditional core is subadditive for the reverse inclusion ordering; together with Lemma 4.6(ii), they mean that the conditional core is a set-valued conditional *sublinear expectation*.

**Lemma 4.7.** *Let  $X$  and  $Y$  be set-valued mappings. Then*

$$(4.3) \quad \mathbf{m}(X|\mathcal{H}) + \mathbf{m}(Y|\mathcal{H}) \subset \mathbf{m}(X + Y|\mathcal{H}).$$



*Proof.* The inclusion is trivial on the  $\mathcal{H}$ -measurable subset where one of the conditional cores on the left-hand side of (4.3) is empty, since then the sum is also empty. If  $\gamma' \in \mathcal{L}^0(\mathbf{m}(X|\mathcal{H}), \mathcal{H})$  and  $\gamma'' \in \mathcal{L}^0(\mathbf{m}(Y|\mathcal{H}), \mathcal{H})$ , then (4.1) yields that

$$\gamma' + \gamma'' \in \mathcal{L}^0(X + Y, \mathcal{H}) = \mathcal{L}^0(\mathbf{m}(X + Y|\mathcal{H}), \mathcal{H}). \quad \square$$

*Example 4.8.* Let  $\mathcal{H}$  be trivial, and let  $X$  be a line in the plane passing through the origin with a random direction, so that  $\mathbf{m}(X|\mathcal{H}) = \{0\}$ . If  $Y$  is a deterministic centred ball, then the set of fixed points of  $X + Y$  may be strictly larger than  $\{0\} + Y = Y$ .

While Example 4.8 shows that (4.3) does not necessarily turn into the equality if one of the summands is  $\mathcal{H}$ -measurable, the equality holds if one of the summands is an  $\mathcal{H}$ -measurable *singleton*.

**Lemma 4.9.** *If  $X$  is a random closed set and  $\eta \in \mathcal{L}^0(\mathfrak{X}, \mathcal{H})$ , then*

$$\mathbf{m}(X + \{\eta\}|\mathcal{H}) = \mathbf{m}(X|\mathcal{H}) + \eta.$$

*Proof.* Consider  $\xi + \eta \in \mathcal{L}^0(\mathbf{m}(X + \{\eta\}|\mathcal{H}), \mathcal{H})$ . Then  $\xi$  is also  $\mathcal{H}$ -measurable, so that  $\xi \in \mathcal{L}^0(\mathbf{m}(X|\mathcal{H}), \mathcal{H})$ . The opposite inclusion follows from Lemma 4.7.  $\square$

**Lemma 4.10.** *Let  $X$  be a random closed set such that there exists a  $\gamma \in \mathcal{L}^0(X, \mathcal{H})$ . Let  $X^n$  be the intersection of  $X$  with the closed ball of radius  $n \geq 1$  centred at  $\gamma$ . Then*

$$\mathbf{m}(X|\mathcal{H}) = \text{cl} \bigcup_{n \geq 1} \mathbf{m}(X^n|\mathcal{H}).$$

*Proof.* Since  $\mathbf{m}(X^n|\mathcal{H}) \subset X^n \subset X$ , we have  $\text{cl} \bigcup_n \mathbf{m}(X^n|\mathcal{H}) \subset X$ , whence

$$\text{cl} \bigcup_n \mathbf{m}(X^n|\mathcal{H}) \subset \mathbf{m}(X|\mathcal{H}).$$

Reciprocally, consider a selection  $\xi$  of  $\mathbf{m}(X|\mathcal{H}) \subset X$  (otherwise,  $\mathbf{m}(X|\mathcal{H}) = \emptyset$  and the inclusion is trivial). Then

$$\xi^n = \xi \mathbf{1}_{\|\xi - \gamma\| \leq n} + \gamma \mathbf{1}_{\|\xi - \gamma\| > n} \in \mathcal{L}^0(X^n, \mathcal{H}),$$

so that  $\xi^n \in \mathbf{m}(X^n|\mathcal{H})$  a.s. Letting  $n \rightarrow \infty$  finishes the proof.  $\square$

The strong upper limit  $s\text{-lim sup } X_n$  of a sequence  $\{X_n, n \geq 1\}$  of random sets is defined as the set of limits for each almost surely strongly convergent sequence  $\xi_{n_k} \in \mathcal{L}^0(X_{n_k}, \mathcal{F})$ ,  $k \geq 1$ .

**Proposition 4.11.** *If  $s\text{-lim sup } X_n \subset X$  a.s. for a sequence of random closed sets  $\{X_n, n \geq 1\}$  and a random closed set  $X$ , then*

$$(4.4) \quad s\text{-lim sup } \mathbf{m}(X_n|\mathcal{H}) \subset \mathbf{m}(X|\mathcal{H}) \quad \text{a.s.}$$

*Proof.* If  $\gamma_{n_k} \in \mathbf{m}(X_{n_k}|\mathcal{H})$  and  $\gamma_{n_k} \rightarrow \gamma$ , then  $\gamma$  is  $\mathcal{H}$ -measurable, and almost surely belongs to  $X$ , whence  $\gamma$  is a selection of  $\mathbf{m}(X|\mathcal{H})$ .  $\square$

It should be noted that the reverse inclusion in (4.4) does not hold, e.g., if  $X_n = \{\xi_n\}$  with a non- $\mathcal{H}$ -measurable  $\xi_n$  such that  $\xi_n$  a.s. converges to an  $\mathcal{H}$ -measurable  $\xi$ .

**4.3. Essential infimum of the support function.** It is possible to relate the conditional core of random closed convex sets to the conditional essential infimum of their support functions.

**Theorem 4.12.** *Let  $X$  be an a.s. non-empty random closed convex set. Then*

$$(4.5) \quad h_{\mathbf{m}(X|\mathcal{H})}(\zeta) \leq \text{ess inf}_{\mathcal{H}} h_X(\zeta), \quad \zeta \in \mathcal{L}^0(\mathfrak{X}^*, \mathcal{H}),$$

and

$$(4.6) \quad \mathbf{m}(X|\mathcal{H}) = \bigcap_{\zeta \in \mathcal{L}^0(\mathfrak{X}^*, \mathcal{F})} \{x \in \mathfrak{X} : \text{ess inf}_{\mathcal{H}} (h_X(\zeta) - \langle \zeta, x \rangle) \geq 0\}.$$

If  $\{\zeta_n, n \geq 1\}$  is a sequence from (3.2), then

$$(4.7) \quad \mathbf{m}(X|\mathcal{H}) = \bigcap_{n \geq 1} \{x \in \mathfrak{X} : \text{ess inf}_{\mathcal{H}} (h_X(\zeta_n) - \langle \zeta_n, x \rangle) \geq 0\}.$$

If  $X$  is weakly compact, then

$$(4.8) \quad \mathbf{m}(X|\mathcal{H}) = \bigcap_{u \in \mathfrak{X}_0^*} \{x \in \mathfrak{X} : \langle u, x \rangle \leq \text{ess inf}_{\mathcal{H}} h_X(u)\},$$

and the intersection can be taken over all  $u \in \mathfrak{X}^*$ .

*Proof.* Without loss of generality assume that  $\mathbf{m}(X|\mathcal{H}) \neq \emptyset$ . Fix any  $\zeta \in \mathcal{L}^0(\mathfrak{X}^*, \mathcal{H})$ . By Lemma 3.1,

$$\begin{aligned} h_{\mathbf{m}(X|\mathcal{H})}(\zeta) &= \text{ess sup}_{\mathcal{H}} \{\langle \zeta, \xi \rangle : \xi \in \mathcal{L}^0(\mathbf{m}(X|\mathcal{H}), \mathcal{H})\} \\ &= \text{ess sup}_{\mathcal{H}} \{\langle \zeta, \xi \rangle : \xi \in \mathcal{L}^0(X, \mathcal{H})\}. \end{aligned}$$

Moreover,  $\mathbf{m}(X|\mathcal{H}) \subset X \subset \{x : \langle x, \zeta \rangle \leq h_X(\zeta)\}$ . Thus,  $\langle \zeta, \xi \rangle \leq h_X(\zeta)$  for  $\xi \in \mathcal{L}^0(X, \mathcal{H})$ . Since  $\langle \zeta, \xi \rangle$  is  $\mathcal{H}$ -measurable,  $\langle \zeta, \xi \rangle \leq \text{ess inf}_{\mathcal{H}} h_X(\zeta)$ , so that (4.5) holds.

Each selection  $\xi$  of  $\mathbf{m}(X|\mathcal{H})$  is also a selection of the right-hand side of (4.7), since  $h_X(\zeta_n) - \langle \zeta_n, \xi \rangle \geq 0$  a.s. for all  $n$ . The function

$$\eta(x) = \text{ess inf}_{\mathcal{H}} (h_X(\zeta_n) - \langle \zeta_n, x \rangle) = \text{ess inf}_{\mathcal{H}} h_{X-x}(\zeta_n)$$

is Lipschitz in  $x$  and so is continuous. Since  $\eta(x)$  is  $\mathcal{H}$ -measurable for all  $x$ , the function is jointly measurable in  $(\omega, x)$ . Thus, each set on the right-hand side is an  $\mathcal{H}$ -measurable random closed set, whence the intersection is also a random closed set, and the equality follows from

the maximality of the conditional core. The right-hand side of (4.6) is a subset of the right-hand side of (4.7) and contains  $\mathbf{m}(X|\mathcal{H})$ , whence (4.6) holds.

By choosing  $\zeta = u$  in (4.5), we see that the left-hand side of (4.8) is a subset of the right-hand one with the intersection taken over all  $u \in \mathfrak{X}^*$ . The right-hand side of (4.8) is an  $\mathcal{H}$ -measurable random closed set denoted by  $\tilde{X}$ . It suffices to assume that  $\tilde{X}$  is a.s. non-empty and consider its  $\mathcal{H}$ -measurable selection  $\gamma$ . For all  $u \in \mathfrak{X}_0^*$ ,

$$\langle u, \gamma \rangle \leq \text{ess inf}_{\mathcal{H}} h_X(u) \leq h_X(u),$$

whence  $\gamma \in X$  a.s., and  $\tilde{X} \subset \mathbf{m}(X|\mathcal{H})$  a.s.  $\square$

Equation (4.6) may be thought as the dual representation of the conditional core.

*Example 4.13.* The inequality in (4.5) can be strict. Let  $X$  be a line in the plane  $\mathfrak{X} = \mathbb{R}^2$  passing through the origin with the normal vector  $\zeta$  having a non-atomic distribution and such that  $\langle x, \zeta \rangle$  is not  $\mathcal{H}$ -measurable for any  $x \neq 0$ . Furthermore, assume that  $\mathcal{H}$  contains all null-events from  $\mathcal{F}$ . Then the only  $\mathcal{H}$ -measurable selection of  $X$  is the origin, so that the left-hand side of (4.5) vanishes. For each deterministic (and so  $\mathcal{H}$ -measurable) non-vanishing  $u$ , we have  $h_X(u) = \infty$  a.s., so that the right-hand side of (4.5) is infinite. Still (4.7) holds with  $\zeta_1 = \zeta$  and  $\zeta_2 = -\zeta$ . Indeed, then  $\langle \zeta, x \rangle = 0$  a.s., which is only possible for  $x = 0$ .

Note that  $\text{ess inf}_{\mathcal{H}} h_X(u)$  is not necessarily subadditive as function of  $u$  and so may fail to be a support function. Recall that the conjugate of a function  $f : \mathfrak{X} \mapsto (-\infty, \infty]$  is defined by

$$f^\circ(u) = \sup_{x \in \mathfrak{X}} (\langle u, x \rangle - f(x)), \quad u \in \mathfrak{X}^*,$$

and the biconjugate of  $f$  is the conjugate of  $f^\circ : \mathfrak{X}^* \mapsto (-\infty, \infty]$ .

**Proposition 4.14.** *Let  $X$  be a random convex closed set. Then the support function of  $\mathbf{m}(X|\mathcal{H})$  is the largest  $\mathcal{H}$ -measurable lower semi-continuous sublinear function  $h : \mathfrak{X}^* \mapsto (-\infty, \infty]$  such that (4.5) holds. If  $X$  is weakly compact, then the support function of  $\mathbf{m}(X|\mathcal{H})$  is the biconjugate function to  $\text{ess inf}_{\mathcal{H}} h_X(u)$ ,  $u \in \mathfrak{X}^*$ .*

*Proof.* By Theorem 4.12, (4.5) holds. If  $h$  is the largest lower semi-continuous function such that (4.5) holds, then it corresponds to an  $\mathcal{H}$ -measurable random closed set  $Y$ . The conclusion follows from the definition of the conditional core as the largest  $\mathcal{H}$ -measurable subset of  $X$ .

If  $X$  is weakly compact, then it is possible to let the argument of the support function be non-random. The largest sublinear function dominated by that  $\text{ess inf}_{\mathcal{H}} h_X(u)$ ,  $u \in \mathfrak{X}^*$ , is its biconjugate.  $\square$

## 5. CONDITIONAL CONVEX HULL

### 5.1. Existence and construction.

**Definition 5.1.** If  $X$  is a random set, then its conditional convex hull  $\mathbf{M}(X|\mathcal{H})$  is the smallest  $\mathcal{H}$ -measurable random convex closed set which contains  $X$ .

**Theorem 5.2.** *If  $X$  is a random set, then  $\mathbf{M}(X|\mathcal{H})$  exists, and*

$$(5.1) \quad h_{\mathbf{M}(X|\mathcal{H})}(\zeta) = \text{ess sup}_{\mathcal{H}} h_X(\zeta), \quad \zeta \in \mathcal{L}^0(\mathfrak{X}^*, \mathcal{H}).$$

*Proof.* Without loss of generality, assume that  $X$  is closed and convex (with possibly empty values). Then the epigraph  $\text{epi } h_X$  is a closed convex cone in  $\mathfrak{X}^* \times \mathbb{R}$ , so that  $\mathbf{m}(\text{epi } h_X|\mathcal{H})$  exists and is a convex cone by Lemma 4.3. By Theorem 3.7,  $\mathbf{m}(\text{epi } h_X|\mathcal{H})$  is the epigraph of the support function of an  $\mathcal{H}$ -measurable random closed convex set  $Z$ . The support function of  $Z$  is the smallest  $\mathcal{H}$ -measurable support function that dominates  $h_X$  and so  $Z = \mathbf{M}(X|\mathcal{H})$  is the smallest  $\mathcal{H}$ -measurable random closed convex set containing  $X$ .

The left-hand side of (5.1) is greater than or equal to the right-hand one. Since

$$\begin{aligned} \text{ess sup}_{\mathcal{H}} h_X(u+v) &\leq \text{ess sup}_{\mathcal{H}} (h_X(u) + h_X(v)) \\ &\leq \text{ess sup}_{\mathcal{H}} h_X(u) + \text{ess sup}_{\mathcal{H}} h_X(v) \end{aligned}$$

for all  $u, v \in \mathfrak{X}^*$ , the essential supremum retains the subadditivity property and so is a support function. Thus, the right-hand side is a support function of an  $\mathcal{H}$ -measurable random closed convex set, which is a subset of  $\mathbf{M}(X|\mathcal{H})$  by Corollary 3.6. The definition of the conditional convex hull yields the equality.  $\square$

By Corollary 3.5,  $\mathbf{M}(X|\mathcal{H})$  equals the intersection of random half-spaces  $\{x : \langle \zeta, x \rangle \leq \text{ess sup}_{\mathcal{H}} h_X(\zeta)\}$  over  $\zeta \in \mathcal{L}^0(\mathfrak{X}^*, \mathcal{H})$ . If  $\mathbf{M}(X|\mathcal{H})$  is a.s. weakly compact, then it is possible to let  $\zeta$  run over deterministic elements from  $\mathfrak{X}_0^*$ .

**Proposition 5.3.** *If  $X^n$  is the intersection of a random closed set  $X$  with the centred ball of radius  $n$ , then*

$$(5.2) \quad \mathbf{M}(X|\mathcal{H}) = \text{cl} \bigcup_n \mathbf{M}(X^n|\mathcal{H}).$$

*Proof.* If  $X \subset Y$  for an  $\mathcal{H}$ -measurable random closed convex set  $Y$ , then  $X^n \subset Y$ . Thus,  $\mathbf{M}(X^n|\mathcal{H}) \subset Y$  for all  $n$ , whence  $\mathbf{M}(X|\mathcal{H}) \subset Y$ .  $\square$

## 5.2. Superadditivity of the conditional convex hull.

**Proposition 5.4.** *If  $X$  and  $Y$  are random sets, then*

$$(5.3) \quad \mathbf{M}(X + Y|\mathcal{H}) \subset \text{cl} \left( \mathbf{M}(X|\mathcal{H}) + \mathbf{M}(Y|\mathcal{H}) \right).$$

*If  $Y$  is  $\mathcal{H}$ -measurable and convex, then*

$$\mathbf{M}(X + Y|\mathcal{H}) = \text{cl}(\mathbf{M}(X|\mathcal{H}) + Y).$$

*Proof.* For (5.3), it suffices to note that the right-hand side is a convex closed set that is  $\mathcal{H}$ -measurable and contains  $X + Y$ . The second statement follows from (5.1), since

$$\begin{aligned} h_{\mathbf{M}(X+Y|\mathcal{H})}(u) &= \text{ess sup}_{\mathcal{H}} h_{X+Y}(u), \\ &= \text{ess sup}_{\mathcal{H}} (h_X(u)) + h_Y(u) = h_{\mathbf{M}(X|\mathcal{H})}(u) + h_Y(u). \quad \square \end{aligned}$$

Proposition 5.4 together with the rather obvious homogeneity property imply that the conditional convex hull is a set-valued conditional *superlinear expectation*.

**5.3. Duality between the core and the convex hull.** The following result establishes a relationship between the conditional convex hull and the conditional core assuming that the random set  $X$  contains at least one  $\mathcal{H}$ -measurable selection.

**Proposition 5.5.** *If  $\mathcal{L}^0(X, \mathcal{H}) \neq \emptyset$ , then*

$$\mathbf{M}(X - \gamma|\mathcal{H}) = (\mathbf{m}((X - \gamma)^o|\mathcal{H}))^o$$

*for any  $\gamma \in \mathcal{L}^0(X, \mathcal{H})$ .*

*Proof.* It suffices to assume that  $X$  is a random convex closed set and  $0 \in X$  a.s., so let  $\gamma = 0$  a.s. Then  $(\mathbf{m}(X^o|\mathcal{H}))^o$  contains  $X$  a.s. If  $Y$  is an  $\mathcal{H}$ -measurable random convex closed set which contains  $X$ , then  $Y^o \subset X^o$  a.s., whence  $Y^0 \subset \mathbf{m}(X^o|\mathcal{H})$  a.s., and  $Y$  contains the polar to the latter set.  $\square$

A similar duality relationship holds for epigraphs, namely,

$$\begin{aligned} \mathbf{M}(\text{epi } h_X|\mathcal{H}) &= \text{epi } h_{\mathbf{m}(X|\mathcal{H})}, \\ \mathbf{m}(\text{epi } h_X|\mathcal{H}) &= \text{epi } h_{\mathbf{M}(X|\mathcal{H})}. \end{aligned}$$

*Remark 5.6.* Consider a filtration  $(\mathcal{F}_t)_{t=0, \dots, T}$  on  $(\Omega, \mathcal{F}, \mathbf{P})$ . An adapted sequence  $(X_t)_{t=0, \dots, T}$  of random closed set is said to be a *maxingale* if  $X_s = \mathbf{M}(X_t|\mathcal{F}_s)$  for all  $s \leq t$  the same holds for the conditional convex hull. If  $X$  is a random closed set, then  $X_t = \mathbf{M}(X|\mathcal{F}_t)$ ,  $t = 0, \dots, T$ , is a maxingale. Random sets  $X_t = (-\infty, \xi_t]$  form a maxingale if and only if the sequence  $(\xi_t)_{t=0, \dots, T}$  of random variables is a maxingale in

the sense of [2]. A similar concept applies for the conditional core. If the conditional core (or convex hull) is replaced by the expectation, one recovers the concept of a set-valued martingale, see [11] and [15, Sec. 5.1].

## 6. CONDITIONAL EXPECTATION

### 6.1. Integrable random sets.

**Definition 6.1** (see [11]). Let  $X$  be an *integrable* random closed set, that is  $\mathcal{L}^1(X, \mathcal{F}) \neq \emptyset$ . The conditional expectation  $\mathbf{E}(X|\mathcal{H})$  with respect to a  $\sigma$ -algebra  $\mathcal{H} \subset \mathcal{F}$  is the random closed set such that

$$(6.1) \quad \mathcal{L}^1(\mathbf{E}(X|\mathcal{H}), \mathcal{H}) = \text{cl}_1\{\mathbf{E}(\xi|\mathcal{H}) : \xi \in \mathcal{L}^1(X, \mathcal{F})\}.$$

The following result shows that it is possible to take the  $\mathcal{L}^0$ -closure in (6.1).

**Lemma 6.2.**  $\mathcal{L}^0(\mathbf{E}(X|\mathcal{H}), \mathcal{H})$  coincides with the  $\mathcal{L}^0$ -closure of the set  $\{\mathbf{E}(\xi|\mathcal{H}) : \xi \in \mathcal{L}^1(X, \mathcal{F})\}$ .

*Proof.* By definition,  $\tilde{\Xi} = \{\mathbf{E}(\xi|\mathcal{H}) : \xi \in \mathcal{L}^1(X, \mathcal{F})\}$  is a subset of  $\mathcal{L}^0(\mathbf{E}(X|\mathcal{H}), \mathcal{H})$ , which is closed in  $\mathcal{L}^0$  since  $\mathbf{E}(X|\mathcal{H})$  is a.s. closed. Therefore,  $\text{cl}_0 \tilde{\Xi} \subset \mathcal{L}^0(\mathbf{E}(X|\mathcal{H}), \mathcal{H})$ . By Proposition 2.7, the random set  $\mathbf{E}(X|\mathcal{H})$  admits a Castaing representation  $\{\xi_i, i \geq 1\}$ , where  $\xi_i \in \text{cl}_1 \tilde{\Xi}$  for all  $i \geq 1$ . Then  $\{\xi_i, i \geq 1\} \subset \text{cl}_0 \tilde{\Xi}$ , so that  $\mathcal{L}^0(\mathbf{E}(X|\mathcal{H}), \mathcal{H}) \subset \text{cl}_0 \tilde{\Xi}$  by Lemma 2.2.  $\square$

**6.2. Generalised conditional expectation of random sets.** The following definition relies on the concept of the generalised expectation discussed in Appendix B.

**Definition 6.3.** Let  $X$  be a random closed set and let  $\mathcal{H}$  be a sub- $\sigma$ -algebra of  $\mathcal{F}$  such that  $\mathcal{L}_{\mathcal{H}}^1(X, \mathcal{F}) \neq \emptyset$ . The generalised conditional expectation  $\mathbf{E}^g(X|\mathcal{H})$  is the  $\mathcal{H}$ -measurable random closed set such that

$$(6.2) \quad \mathcal{L}^0(\mathbf{E}^g(X|\mathcal{H}), \mathcal{H}) = \text{cl}_0\{\mathbf{E}^g(\xi|\mathcal{H}) : \xi \in \mathcal{L}_{\mathcal{H}}^1(X, \mathcal{F})\}.$$

The existence of the generalised conditional expectation follows from Corollary 2.5, since the family on the right-hand side of (6.2) is  $\mathcal{H}$ -decomposable.

**Lemma 6.4.** *If  $X$  is an integrable random closed set, then  $\mathbf{E}(X|\mathcal{H}) = \mathbf{E}^g(X|\mathcal{H})$ .*

*Proof.* To show the non-trivial inclusion, consider  $\xi \in \mathcal{L}_{\mathcal{H}}^1(X, \mathcal{F})$ , so that  $\mathbf{E}^g(\xi|\mathcal{H}) = \sum_{i=1}^{\infty} \mathbf{E}(\xi \mathbf{1}_{A_i}|\mathcal{H}) \mathbf{1}_{A_i}$  for an  $\mathcal{H}$ -measurable partition

$\{A_i, i \geq 1\}$ , and  $\xi \mathbf{1}_{A_i} \in \mathcal{L}^1(\mathfrak{X}, \mathcal{F})$  for all  $i \geq 1$ . If  $\gamma \in \mathcal{L}^1(X, \mathcal{F})$ , then

$$\mathbf{E}^g(\xi|\mathcal{H}) = \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[ \mathbf{E}(\xi \mathbf{1}_{A_i}|\mathcal{H}) \mathbf{1}_{A_i} + \mathbf{E}(\gamma|\mathcal{H}) \mathbf{1}_{\Omega \setminus \cup_{i \leq n} A_i} \right] \quad \text{a.s.}$$

Since  $\xi \mathbf{1}_{A_i}$  and  $\gamma$  are integrable, the sum under the limit belongs to  $\mathbf{E}(X|\mathcal{H})$ . Therefore,  $\mathbf{E}^g(\xi|\mathcal{H}) \in \mathbf{E}(X|\mathcal{H})$  a.s. Since  $\mathbf{E}(X|\mathcal{H})$  is a random closed set, the family  $\mathcal{L}^0(\mathbf{E}(X|\mathcal{H}), \mathcal{H})$  is closed in  $\mathcal{L}^0$  by Lemma 6.2. Thus,  $\mathbf{E}^g(X|\mathcal{H}) \subset \mathbf{E}(X|\mathcal{H})$  a.s.  $\square$

**Lemma 6.5.** *Let  $X$  be a random closed set such that  $\mathcal{L}_{\mathcal{H}}^1(X, \mathcal{F}) \neq \emptyset$ . Then, for every  $\xi \in \mathcal{L}_{\mathcal{H}}^1(X, \mathcal{F})$ ,  $X - \xi$  is an integrable random closed set, and*

$$\mathbf{E}^g(X|\mathcal{H}) = \mathbf{E}(X - \xi|\mathcal{H}) + \mathbf{E}^g(\xi|\mathcal{H}) \quad \text{a.s.}$$

*Proof.* Since  $X - \xi$  is integrable,  $\mathbf{E}(X - \xi|\mathcal{H}) = \mathbf{E}^g(X - \xi|\mathcal{H})$  by Lemma 6.4. Then

$$\mathbf{E}^g(\eta|\mathcal{H}) = \mathbf{E}^g(\eta - \xi|\mathcal{H}) + \mathbf{E}^g(\xi|\mathcal{H}) \in \mathbf{E}^g(X - \xi|\mathcal{H}) + \mathbf{E}^g(\xi|\mathcal{H}) \quad \text{a.s.}$$

for all  $\eta \in \mathcal{L}_{\mathcal{H}}^1(X, \mathcal{F})$ . Therefore,  $\mathbf{E}^g(X|\mathcal{H}) \subset \mathbf{E}(X - \xi|\mathcal{H}) + \mathbf{E}^g(\xi|\mathcal{H})$  a.s. The reverse inclusion follows from

$$\mathbf{E}^g(X - \xi|\mathcal{H}) \subset \mathbf{E}^g(X|\mathcal{H}) - \mathbf{E}^g(\xi|\mathcal{H}). \quad \square$$

**Corollary 6.6.** *If  $X$  is an  $\mathcal{H}$ -measurable a.s. non-empty random closed convex set, then  $\mathbf{E}^g(X|\mathcal{H}) = X$  a.s.*

It is well known, see [11] and [15], that if  $X$  is a.s. convex and integrable, then  $h_{\mathbf{E}(X|\mathcal{H})}(u) = \mathbf{E}(h_X(u)|\mathcal{H})$  a.s. for all  $u \in \mathfrak{X}^*$ , see [15, Th. 2.1.47]. The following result is a generalisation of this fact for possibly random arguments of the support function.

**Lemma 6.7.** *If  $\mathcal{L}_{\mathcal{H}}^1(X, \mathcal{F}) \neq \emptyset$ , then*

$$\mathbf{E}^g(h_X(\zeta)|\mathcal{H}) = h_{\mathbf{E}^g(X|\mathcal{H})}(\zeta), \quad \zeta \in \mathcal{L}^0(\mathfrak{X}^*, \mathcal{H}).$$

*Proof.* By passing from  $X$  to  $X - \gamma$  for  $\gamma \in \mathcal{L}_{\mathcal{H}}^1(X, \mathcal{F}) \neq \emptyset$ , it is possible to assume that  $X$  contains the origin with probability one and work with the conventional conditional expectation.

Each  $\eta \in \mathcal{L}^0(\mathbf{E}(X|\mathcal{H}), \mathcal{H})$  is the almost sure limit of  $\mathbf{E}(\xi_n|\mathcal{H})$  for  $\xi_n \in \mathcal{L}_{\mathcal{H}}^1(X, \mathcal{F})$ ,  $n \geq 1$ . Then

$$\langle \zeta, \eta \rangle = \lim \mathbf{E}(\langle \zeta, \xi_n \rangle|\mathcal{H}) \leq \mathbf{E}(h_X(\zeta)|\mathcal{H}).$$

Thus,  $h_{\mathbf{E}^g(X|\mathcal{H})}(\zeta) \leq \mathbf{E}^g(h_X(\zeta)|\mathcal{H})$  a.s.

In the other direction, fix  $\varepsilon > 0$  and  $c > 0$  and let

$$Y = \{x \in \mathfrak{X} : \langle \zeta, x \rangle \geq h_X(\zeta) - \varepsilon\} \cup \{x \in \mathfrak{X} : \langle \zeta, x \rangle \geq c\}.$$

Then  $X \cap Y$  is an almost surely non-empty random closed set, which possesses a selection  $\xi$  such that

$$\langle \zeta, \xi \rangle \geq \min(h_X(\zeta) - \varepsilon, c).$$

Passing to conditional expectations yields that

$$\mathbf{E}(\min(h_X(\zeta) - \varepsilon, c) | \mathcal{H}) \leq \mathbf{E}(\langle \zeta, \xi \rangle | \mathcal{H}) \leq h_{\mathbf{E}(X | \mathcal{H})}(\zeta).$$

Since the support function of  $X$  is non-negative, letting  $c \uparrow \infty$  and  $\varepsilon \downarrow 0$  concludes the proof.  $\square$

Note that Lemma 6.7 holds for the conventional conditional expectations if  $X$  is integrable and  $\zeta \in \mathcal{L}^\infty(\mathfrak{X}^*, \mathcal{H})$ , that is, the (strong) norm of  $\zeta$  is essentially bounded.

**Lemma 6.8.** *Let  $X$  be a random closed set such that  $\mathcal{L}_{\mathcal{H}}^1(X, \mathcal{F}) \neq \emptyset$ , and let  $X^n = X \cap B_n$ ,  $n \geq 1$ . Then*

$$\mathbf{E}^g(X | \mathcal{H}) = \text{cl} \bigcup_n \mathbf{E}^g(X^n | \mathcal{H}) \quad \text{a.s.}$$

*Proof.* By passing from  $X$  to  $X - \gamma$  for any  $\gamma \in \mathcal{L}_{\mathcal{H}}^1(X, \mathcal{F})$ , it is possible to assume that  $0 \in X$  a.s., so that  $X$  is integrable.

Denote the right-hand side by  $Y$ . Note that  $Y \subset \mathbf{E}(X | \mathcal{H})$ . To confirm the reverse inclusion, let  $\xi = \mathbf{E}(\eta | \mathcal{H})$  for  $\eta \in \mathcal{L}^1(X, \mathcal{F})$ . Then  $\xi$  is the limit of  $\mathbf{E}(\eta_n | \mathcal{H})$  in  $\mathcal{L}^1$ , where  $\eta_n = \eta \mathbf{1}_{|\eta| \leq n} \in \mathcal{L}^1(X^n, \mathcal{F})$ . Since  $\mathbf{E}(\eta_n | \mathcal{H}) \in \mathbf{E}(X^n | \mathcal{H})$  a.s.,  $\xi \in Y$  a.s. and the conclusion follows.  $\square$

**6.3. Sandwich theorem.** Now we show that the (generalised) conditional expectation is sandwiched between the conditional core and the conditional convex hull.

**Proposition 6.9.** *If  $X$  is an a.s. non-empty random closed set, then*

$$(6.3) \quad \mathbf{m}(X | \mathcal{H}) \subset \mathbf{E}^g(X | \mathcal{H}) \subset \mathbf{M}(X | \mathcal{H}) \quad \text{a.s.}$$

*Proof.* The first inclusion is trivial, unless  $\mathcal{L}^0(X, \mathcal{H}) \neq \emptyset$ . Then each  $\mathcal{H}$ -measurable selection  $\xi$  of  $\mathbf{m}(X | \mathcal{H})$  satisfies  $\xi = \mathbf{E}^g(\xi | \mathcal{H})$ , whence the first inclusion holds.

For the second inclusion, note that  $X \subset \mathbf{M}(X | \mathcal{H})$ , and

$$\mathbf{E}^g(\mathbf{M}(X | \mathcal{H}) | \mathcal{H}) = \mathbf{M}(X | \mathcal{H})$$

by Corollary 6.6.  $\square$

If  $0 \in X$  a.s., then Proposition 5.5 yields that

$$\mathbf{m}(X | \mathcal{H}) \subset \mathbf{E}^g(X | \mathcal{H}) \subset (\mathbf{m}(X^o | \mathcal{H}))^o,$$



whence

$$\mathbf{m}(X|\mathcal{H}) \subset \left( \mathbf{E}^g(X|\mathcal{H}) \cap (\mathbf{E}^g(X^o|\mathcal{H}))^o \right).$$

Consider the family  $\mathcal{Q}$  of all probability measures  $\mathbf{Q}$  absolutely continuous with respect to  $\mathbf{P}$ . The following result can be viewed as an analogue of the representation of superlinear and sublinear functions as suprema and infima of linear functions.

**Theorem 6.10.** *Let  $X$  be a random closed convex set. Then  $\mathbf{M}(X|\mathcal{H})$  (respectively,  $\mathbf{m}(X|\mathcal{H})$ ) is the smallest (respectively, largest)  $\mathcal{H}$ -measurable random closed convex set a.s. containing  $\mathbf{E}_{\mathbf{Q}}^g(X|\mathcal{H})$  for all  $\mathbf{Q} \in \mathcal{Q}$  such that the generalised conditional expectation exists.*

*Proof.* Consider the family  $\{\zeta_n, n \geq 1\}$  which yields  $\mathbf{M}(X|\mathcal{H})$  by an analogue of (3.2). By (5.1) and Theorem A.2,

$$\mathbf{E}_{\mathbf{Q}_{nm}}^g(h_X(\zeta_n)|\mathcal{H}) \uparrow h_{\mathbf{M}(X|\mathcal{H})}(\zeta_n) \quad \text{a.s. as } m \rightarrow \infty$$

for a sequence  $\{\mathbf{Q}_{nm}, m \geq 1\} \subset \mathcal{Q}$ . By Lemma 6.7,

$$h_{\mathbf{M}(X|\mathcal{H})}(\zeta_n) = h_{\bigcup_m \mathbf{E}_{\mathbf{Q}_{nm}}^g(X|\mathcal{H})}(\zeta_n) \leq h_{\bigcup_{n,m} \mathbf{E}_{\mathbf{Q}_{nm}}^g(X|\mathcal{H})}(\zeta_n).$$

In view of (3.2),

$$\mathbf{M}(X|\mathcal{H}) \subset \text{cl co } \bigcup_{n,m \geq 1} \mathbf{E}_{\mathbf{Q}_{nm}}^g(X|\mathcal{H}).$$

Proposition 6.9 yields the reverse inclusion, so that the equality holds. The union can be, equivalently, taken over all  $\mathbf{Q} \in \mathcal{Q}$ , letting the generalised conditional expectation to be empty if  $X$  does not contain any selection which does not admit the generalised conditional expectation under  $\mathbf{Q}$ .

If  $Z$  is an  $\mathcal{H}$ -measurable subset of  $\mathbf{E}_{\mathbf{Q}}^g(X|\mathcal{H})$  for all  $\mathbf{Q} \in \mathcal{Q}$ , then

$$h_Z(\zeta) = \mathbf{E}_{\mathbf{Q}}^g(h_Z(\zeta)|\mathcal{H}) \leq \mathbf{E}_{\mathbf{Q}}^g(h_X(\zeta)|\mathcal{H}).$$

By Theorem A.2,  $h_Z(\zeta) \leq \text{ess inf}_{\mathcal{H}} h_X(\zeta)$  for all  $\zeta \in \mathcal{L}^0(\mathfrak{X}^*, \mathcal{H})$ . By Proposition 4.14,

$$h_Z(\zeta) \leq h_{\mathbf{m}(X|\mathcal{H})}(\zeta).$$

By Corollary 3.6,  $\mathbf{m}(X|\mathcal{H}) \supset Z$ . □

Following the idea of Theorem 6.10, it is possible to come up with a general way of constructing non-linear set-valued expectations. If  $\mathcal{M}$  is a sub-family of  $\mathcal{Q}$ , then a measurable version of

$$\bigcap_{\mathbf{Q} \in \mathcal{M}} \mathbf{E}_{\mathbf{Q}}(X|\mathcal{H})$$

is a set-valued sublinear conditional expectation that satisfies (4.3) and a measurable version of

$$\text{cl} \bigcup_{\mathbb{Q} \in \mathcal{M}} \mathbf{E}_{\mathbb{Q}}(X|\mathcal{H})$$

satisfies (5.3).

*Example 6.11.* Assume that  $\mathcal{M}$  consists of all probability measures  $\mathbb{Q}$  such that  $d\mathbb{Q}/d\mathbb{P} \leq \alpha^{-1}$  for some  $\alpha \in (0, 1)$ . Then the above formula provide set-valued sub- and superlinear analogues of the conditional Average Value-at-Risk, which is a well-known risk measure, see [5, 6].

## 7. POLAR SETS AND RANDOM CONES

If  $X$  is a *convex cone* in  $\mathfrak{X}$ , then  $X^* = -X^\circ$  is called the positive dual cone to  $X$ , so that

$$X^* = \{u \in \mathfrak{X}^* : \langle u, x \rangle \geq 0 \ \forall x \in X\}.$$

**Proposition 7.1.** *Let  $K$  be a random convex closed cone in  $\mathfrak{X}$ . Then both  $\mathbf{m}(K|\mathcal{H})$  and  $\mathbf{M}(K|\mathcal{H})$  are closed convex cones,  $\mathbf{m}(K|\mathcal{H}) = \mathbf{M}(K^*|\mathcal{H})^*$ , and  $\mathbf{E}(K|\mathcal{H}) = \mathbf{M}(K|\mathcal{H})$  a.s.*

*Proof.* The conical properties of the core and the convex hull are obvious. Since  $\mathbf{m}(K|\mathcal{H}) \subset K$ ,

$$K^* \subset \mathbf{M}(K^*|\mathcal{H}) \subset \mathbf{m}(K|\mathcal{H})^*$$

in view of the definition of  $\mathbf{M}(K^*|\mathcal{H})$ . Therefore,  $\mathbf{m}(K|\mathcal{H}) \subset \mathbf{M}(K^*|\mathcal{H})^*$ . The opposite inclusion follows from

$$\mathbf{M}(K^*|\mathcal{H})^* \subset \mathbf{m}(K|\mathcal{H}) \subset K$$

by the definition of the conditional core.

For the last statement, assume that  $\gamma \in \mathcal{L}^1(\mathbf{M}(K|\mathcal{H}), \mathcal{H})$  does not belong to  $\mathcal{L}^1(\mathbf{E}(K|\mathcal{H}), \mathcal{H})$ . By the Hahn-Banach separation theorem, there exist  $\eta \in \mathcal{L}^\infty(\mathfrak{X}^*, \mathcal{H})$  and  $c \in \mathbb{R}$  such that

$$\mathbf{E}\langle \xi, \eta \rangle < c < \mathbf{E}\langle \gamma, \eta \rangle$$

for all  $\xi \in \mathcal{L}^1(\mathbf{E}(K|\mathcal{H}), \mathcal{H})$ . Since  $\mathcal{L}^1(\mathbf{E}(K|\mathcal{H}), \mathcal{H})$  is a cone, we have  $c > 0$  and  $-\eta$  belongs to  $\mathcal{L}^1(\mathbf{E}(K|\mathcal{H})^*, \mathcal{H})$ . Since  $\mathbf{m}(K|\mathcal{H}) \subset \mathbf{E}(K|\mathcal{H})$ ,

$$\mathbf{E}(K|\mathcal{H})^* \subset \mathbf{m}(K|\mathcal{H})^* = \mathbf{M}(K^*|\mathcal{H})$$

by Proposition 7.1. Therefore,  $-\eta \in K^*$  a.s. Since  $\gamma \in K$  a.s.,  $\mathbf{E}\langle \gamma, \eta \rangle \leq 0$  in contradiction with  $c > 0$ .

The opposite inclusion follows from Theorem 6.10(i).  $\square$

Each random convex closed set  $X$  in  $\mathfrak{X}$  gives rise to a random convex cone  $Y = \text{cone}(X)$  in  $\mathbb{R}_+ \times \mathfrak{X}$  given by

$$(7.1) \quad \text{cone}(X) = \{(t, tx) : t \geq 0, x \in X\}.$$

Note that  $Y^*$  is not necessarily a subset of  $\mathbb{R}_+ \times \mathfrak{X}$  and so cannot be represented by (7.1).

**Proposition 7.2.** *If  $Y = \text{cone}(X)$  is given by (7.1), then  $\mathbf{m}(Y|\mathcal{H}) = \text{cone}(\mathbf{m}(X|\mathcal{H}))$  and, if  $\mathbf{M}(X|\mathcal{H})$  is a.s. bounded, then also  $\mathbf{M}(Y|\mathcal{H}) = \text{cone}(\mathbf{M}(X|\mathcal{H}))$ .*

*Proof.* By definition,  $\mathbf{m}(Y|\mathcal{H}) \supset \text{cone}(\mathbf{m}(X|\mathcal{H}))$ , since the latter set is  $\mathcal{H}$ -measurable. If  $(\xi_0, \xi)$  is an  $\mathcal{H}$ -measurable selection of  $\mathbf{m}(Y|\mathcal{H})$ , then  $\eta = \xi/\xi_0$  a.s. belongs to  $X$  and is  $\mathcal{H}$ -measurable, whence  $\eta \in \mathbf{m}(X|\mathcal{H})$ , and  $(\xi_0, \xi) = \xi_0(1, \eta)$ .

Obviously,  $\mathbf{M}(Y|\mathcal{H}) \subset \text{cone}(\mathbf{M}(X|\mathcal{H}))$ . We show that  $\mathbf{E}(Y|\mathcal{H}) = \text{cone}(\mathbf{M}(X|\mathcal{H}))$ . The support function of  $Y$  is given by

$$\begin{aligned} h_Y((u_0, u)) &= \sup\{tu_0 + t\langle u, x \rangle : t \geq 0, x \in X\} \\ &= \begin{cases} 0, & \text{if } u_0 + h_X(u) \leq 0, \\ \infty & \text{otherwise.} \end{cases} \end{aligned}$$

Thus,  $\mathbf{E}h_Y((u_0, u)) = 0$  if  $u_0 + h_X(u) \leq 0$  a.s. and is infinite otherwise. It suffices to note that  $u_0 + h_X(u) \leq 0$  a.s. if and only if  $u_0 \leq -\text{ess sup}_{\mathcal{H}} h_X(u)$  and refer to Theorem 5.2.  $\square$

*Example 7.3* (Random cone in  $\mathbb{R}^2$ ). If finance, the random segment  $X = [S^b, S^a] \subset \mathbb{R}_+$  models the bid-ask spread, and the positive dual cone to  $Y = \text{cone}(X)$  is called the solvency cone, see [13]. Proposition 7.2 shows that  $\mathbf{m}(Y|\mathcal{H})$  is the cone generated by  $[\text{ess sup}_{\mathcal{H}} S^b, \text{ess inf}_{\mathcal{H}} S^a]$ , while  $\mathbf{M}(Y|\mathcal{H})$  is generated by  $[\text{ess inf}_{\mathcal{H}} S^b, \text{ess sup}_{\mathcal{H}} S^a]$ .

## Appendix A. CONDITIONAL ESSENTIAL SUPREMUM

Let  $\Xi \subset \mathcal{L}^0(\mathbb{R}, \mathcal{F})$  be a (possibly uncountable) family of real-valued  $\mathcal{F}$ -measurable random variables and let  $\mathcal{H}$  be a sub- $\sigma$ -algebra of  $\mathcal{F}$ . The following result is well known, see e.g. [6, Appendix A.5] and further refinements in [2] and [12].

**Theorem A.1.** *For any family  $\Xi$  of random variables, there exists a unique  $\hat{\xi} \in \mathcal{L}^0((-\infty, +\infty], \mathcal{H})$ , denoted by  $\text{ess sup}_{\mathcal{H}} \Xi$  and called the  $\mathcal{H}$ -conditional supremum of  $\Xi$ , such that  $\hat{\xi} \geq \xi$  a.s. for all  $\xi \in \Xi$ , and  $\eta \geq \hat{\xi}$  a.s. for an  $\eta \in \mathcal{L}^0((-\infty, +\infty], \mathcal{H})$  and all  $\xi \in \Xi$  implies  $\eta \geq \hat{\xi}$  a.s.*

It is easy to verify the tower property

$$\text{ess sup}_{\mathcal{H}'} \text{ess sup}_{\mathcal{H}} \Xi = \text{ess sup}_{\mathcal{H}'} \xi$$

if  $\mathcal{H}' \subset \mathcal{H}$ .

Let  $\mathcal{Q}$  be the set of all probability measures  $\mathbb{Q}$  absolutely continuous with respect to  $\mathbb{P}$ . In the following,  $\mathbb{E}_{\mathbb{Q}}$  designates the expectation under  $\mathbb{Q}$ .

**Theorem A.2.** *Let  $\Xi \subset \mathcal{L}^0(\mathbb{R}, \mathcal{F})$  and let  $\mathcal{H}$  be a sub- $\sigma$ -algebra of  $\mathcal{F}$ . If each  $\xi \in \Xi$  is a.s. non-negative or its generalised conditional expectation  $\mathbb{E}_{\mathbb{Q}}^g(\xi|\mathcal{H})$  exists for all  $\mathbb{Q} \in \mathcal{Q}$ , then*

$$\text{ess sup}_{\mathcal{H}} \Xi = \text{ess sup}_{\mathcal{F}} \{\mathbb{E}_{\mathbb{Q}}^g(\xi|\mathcal{H}), \xi \in \Xi, \mathbb{Q} \in \mathcal{Q}\}.$$

Moreover, if  $\Xi = \{\xi\}$  is a singleton, the family  $\{\mathbb{E}_{\mathbb{Q}}^g(\xi|\mathcal{H}), \mathbb{Q} \in \mathcal{Q}\}$  is directed upward, and there exists a sequence  $\mathbb{Q}_n \in \mathcal{Q}$ ,  $n \geq 1$ , such that  $\mathbb{E}_{\mathbb{Q}_n}(\xi|\mathcal{H}) \uparrow \text{ess sup}_{\mathcal{H}} \Xi$  everywhere on  $\Omega$ .

*Proof.* Since  $\text{ess sup}_{\mathcal{H}} \Xi \geq \xi$  for all  $\xi \in \Xi$  and  $\text{ess sup}_{\mathcal{H}} \Xi$  is  $\mathcal{H}$ -measurable,  $\text{ess sup}_{\mathcal{H}} \Xi \geq \mathbb{E}_{\mathbb{Q}}^g(\xi|\mathcal{H})$  for all  $\mathbb{Q} \in \mathcal{Q}$  and  $\xi \in \Xi$ . Therefore,

$$\text{ess sup}_{\mathcal{H}} \Xi \geq \text{ess sup}_{\mathcal{F}} \{\mathbb{E}_{\mathbb{Q}}(\xi|\mathcal{H}), \xi \in \Xi, \mathbb{Q} \in \mathcal{Q}\} = \tilde{\gamma}.$$

It remains to show that  $\tilde{\gamma} \geq \xi$  a.s. for all  $\xi \in \Xi$ . This is trivial on the set  $\{\tilde{\gamma} = \infty\}$ . Therefore, we may assume without loss of generality that  $\tilde{\gamma} < \infty$  a.s. Assume that there exist a  $\xi \in \Xi$  and a non-null set  $A \in \mathcal{F}$  such that  $\tilde{\gamma} < \xi$  on  $A$ . Then  $\tilde{\gamma}\mathbf{1}_A + \xi\mathbf{1}_{A^c} \leq \xi$  and the inequality is strict on  $A$ . Let  $d\mathbb{Q} = \alpha\mathbf{1}_A d\mathbb{P}$  for  $\alpha = \mathbb{P}(A)^{-1}$ , so that  $\mathbb{Q} \in \mathcal{Q}$ . By definition of  $\tilde{\gamma}$ ,

$$\mathbb{E}_{\mathbb{Q}}^g(\xi|\mathcal{H})\mathbf{1}_A + \xi\mathbf{1}_{A^c} \leq \xi,$$

and the inequality is strict on  $A$ . Taking the conditional expectation yields that

$$\mathbb{E}_{\mathbb{Q}}^g(\xi|\mathcal{H})\mathbb{E}_{\mathbb{Q}}^g(\mathbf{1}_A|\mathcal{H}) + \mathbb{E}_{\mathbb{Q}}^g(\xi\mathbf{1}_{A^c}|\mathcal{H}) \leq \mathbb{E}_{\mathbb{Q}}(\xi|\mathcal{H}),$$

whence

$$\mathbb{E}_{\mathbb{Q}}^g(\xi\mathbf{1}_{A^c}|\mathcal{H}) \leq \mathbb{E}_{\mathbb{Q}}^g(\xi|\mathcal{H})\mathbb{E}_{\mathbb{Q}}^g(\mathbf{1}_{A^c}|\mathcal{H}),$$

and the inequality is strict on  $A$ . Indeed, the random variables in the inequality above take their values in  $\mathbb{R}$  by assumption. We then obtain a contradiction, since  $\mathbb{Q}(A^c) = 0$ .

To show that the family  $\{\mathbb{E}_{\mathbb{Q}}^g(\xi|\mathcal{H}), \mathbb{Q} \in \mathcal{Q}\}$  is directed upward, consider  $\mathbb{Q}_1, \mathbb{Q}_2 \in \mathcal{Q}$  such that  $d\mathbb{Q}_i = \alpha_i d\mathbb{P}$ ,  $i = 1, 2$ . Define  $\mathbb{Q} \in \mathcal{Q}$  by letting  $d\mathbb{Q} = \text{cad}\mathbb{P}$  with

$$\alpha = \alpha_1 \mathbf{1}_{\mathbb{E}_{\mathbb{Q}_1}(\xi|\mathcal{H}) \geq \mathbb{E}_{\mathbb{Q}_2}(\xi|\mathcal{H})} + \alpha_2 \mathbf{1}_{\mathbb{E}_{\mathbb{Q}_1}(\xi|\mathcal{H}) < \mathbb{E}_{\mathbb{Q}_2}(\xi|\mathcal{H})}$$

and  $c > 0$  chosen such that  $\mathbf{Q}(\Omega) = 1$ . Then

$$\mathbf{E}_{\mathbf{Q}}^g(\xi \mathbf{1}_H) = \mathbf{E}_{\mathbf{P}}^g(\alpha \xi \mathbf{1}_H) \geq \mathbf{E}_{\mathbf{Q}}^g(\mathbf{E}_{\mathbf{Q}_i}^g(\xi | \mathcal{H}) \mathbf{1}_H) \quad \text{a.s.}$$

for every  $H \in \mathcal{H}$ , whence  $\mathbf{E}_{\mathbf{Q}}^g(\xi | \mathcal{H}) \geq \mathbf{E}_{\mathbf{Q}_i}^g(\xi | \mathcal{H})$  a.s. for  $i = 1, 2$ . The conclusion follows.  $\square$

Similar definitions and results hold for the conditional essential infimum.

## Appendix B. GENERALISED CONDITIONAL EXPECTATION

**Definition B.1.** Let  $\mathcal{H}$  be a sub- $\sigma$ -algebra of  $\mathcal{F}$ . We say that the generalised conditional expectation of  $\xi \in \mathcal{L}^0(\mathfrak{X}, \mathcal{F})$  exists if there exists an  $\mathcal{H}$ -measurable partition  $\{A_i, i \geq 1\}$  such that  $\xi \mathbf{1}_{A_i}$  is integrable for all  $i \geq 1$ . In this case, we say that  $\xi \in \mathcal{L}_{\mathcal{H}}^1(\mathfrak{X}, \mathcal{F})$  and let

$$(B.1) \quad \mathbf{E}^g(\xi | \mathcal{H}) = \sum_{i=1}^{\infty} \mathbf{E}(\xi \mathbf{1}_{A_i} | \mathcal{H}) \mathbf{1}_{A_i}.$$

It is easy to see that the generalised conditional expectation does not depend on the chosen partition.

**Theorem B.2.** *We have  $\xi \in \mathcal{L}_{\mathcal{H}}^1(\mathfrak{X}, \mathcal{F})$  if and only if  $\xi \in \mathcal{L}^0(\mathfrak{X}, \mathcal{F})$  with  $\mathbf{E}(\|\xi\| | \mathcal{H}) < \infty$  a.s. For random variables  $\xi \in \mathcal{L}_{\mathcal{H}}^1(\mathbb{R}, \mathcal{F})$ , we have  $\mathbf{E}^g(\xi | \mathcal{H}) = \mathbf{E}(\xi^+ | \mathcal{H}) - \mathbf{E}(\xi^- | \mathcal{H})$ , where  $\xi^+ = \xi \vee 0$  and  $\xi^- = -(\xi \wedge 0)$ .*

*Proof.* If  $\mathbf{E}(\|\xi\| | \mathcal{H}) < \infty$  a.s., define an  $\mathcal{H}$ -measurable partition by letting

$$A_n = \{\omega : \mathbf{E}(\|\xi\| | \mathcal{H}) \in [n, n+1)\} \in \mathcal{H}, \quad n \geq 0.$$

Since

$$\mathbf{E}(\|\xi\| \mathbf{1}_{A_n}) = \mathbf{E}(\mathbf{E}(\|\xi\| | \mathcal{H}) \mathbf{1}_{A_n}) \leq n+1,$$

we have  $\xi \mathbf{1}_{A_n}$  is integrable, and

$$\mathbf{E}^g(\xi | \mathcal{H}) = \sum_n \mathbf{E}(\xi \mathbf{1}_{A_n} | \mathcal{H}) \mathbf{1}_{A_n}$$

If  $\xi$  is a random variable, then  $\mathbf{E}(\xi^+ | \mathcal{H}) < \infty$  and  $\mathbf{E}(\xi^- | \mathcal{H}) < \infty$  a.s. and

$$\begin{aligned} \mathbf{E}^g(\xi | \mathcal{H}) &= \sum_n (\mathbf{E}(\xi^+ | \mathcal{H}) - \mathbf{E}(\xi^- | \mathcal{H})) \mathbf{1}_{A_n} \\ &= \mathbf{E}(\xi^+ | \mathcal{H}) - \mathbf{E}(\xi^- | \mathcal{H}). \end{aligned}$$

Reciprocally, let  $\xi \in \mathcal{L}_{\mathcal{H}}^1(\mathfrak{X}, \mathcal{F})$ , i.e. there exists an  $\mathcal{H}$ -measurable partition  $\{A_i, i \geq 1\}$  such that  $\|\xi\| \mathbf{1}_{A_i}$  is integrable for all  $i \geq 1$ . Then

$$\mathbf{E}(\|\xi\| | \mathcal{H}) = \sum_n \mathbf{E}(\|\xi\| \mathbf{1}_{A_n} | \mathcal{H}) \mathbf{1}_{A_n}.$$

Since  $\|\xi\|\mathbf{1}_{A_n}$  is integrable,  $\mathbf{E}(\mathbf{E}(\|\xi\|\mathbf{1}_{A_n}|\mathcal{H})) = \mathbf{E}(\|\xi\|\mathbf{1}_{A_n}) < \infty$ , so that  $\mathbf{E}(\|\xi\|\mathbf{1}_{A_n}|\mathcal{H}) < \infty$  a.s. and  $\mathbf{E}(\|\xi\||\mathcal{H}) < \infty$  a.s.  $\square$

**Lemma B.3.** *Random element  $\xi \in \mathcal{L}^0(\mathfrak{X}, \mathcal{F})$  admits a generalised conditional expectation if and only if  $\xi = \gamma\tilde{\xi}$ , where  $\gamma \in \mathcal{L}^0([1, \infty), \mathcal{H})$  and  $\tilde{\xi}$  is integrable.*

*Proof.* Let  $\xi$  admit a generalised conditional expectation. Define  $\gamma = (1 + \mathbf{E}(\|\xi\||\mathcal{H}))$ , so that  $\mathbf{E}(\|\tilde{\xi}\||\mathcal{H}) \leq 1$ . Hence,  $0 \leq \mathbf{E}(\|\tilde{\xi}\|) \leq 1$  and  $\tilde{\xi}$  is integrable. Reciprocally, if  $\xi = \gamma\tilde{\xi}$  with  $\gamma \in \mathcal{L}^0([1, \infty), \mathcal{H})$  and integrable  $\tilde{\xi}$ , then  $\xi\mathbf{1}_{A_i}$  is integrable for  $A_i = \{\gamma \in [i, i+1)\}$ ,  $i \geq 1$ .  $\square$

**Lemma B.4.** *If  $\xi \in \mathcal{L}_{\mathcal{H}}^1(\mathfrak{X}, \mathcal{F})$  and  $\zeta \in \mathcal{L}^0(\mathfrak{X}^*, \mathcal{H})$ , then*

$$\mathbf{E}^g(\langle \zeta, \xi \rangle | \mathcal{H}) = \langle \zeta, \mathbf{E}^g(\xi | \mathcal{H}) \rangle.$$

*Proof.* If  $\{A_n, n \geq 1\}$  is the partition from (B.1), then the statement follows by partitioning  $\Omega$  with  $A_{nm} = A_n \cap \{\|\zeta\| \in [m, m+1)\}$  and using the linearity property of the conditional expectation.  $\square$

In view of Lemma B.2, let  $\mathcal{L}_{\mathcal{H}}^p(\mathfrak{X}, \mathcal{F})$  with  $p \in [1, \infty]$  be the family of  $\xi \in \mathcal{L}^0(\mathfrak{X}, \mathcal{F})$  such that  $\mathbf{E}(\|\xi\|^p | \mathcal{H}) < \infty$  a.s. if  $p \in [1, \infty)$  and  $\text{ess sup}_{\mathcal{H}} \|\xi\| < \infty$  a.s. if  $p = \infty$ .

**Lemma B.5** (Dominated convergence). *Let  $\{\xi_n, n \geq 1\}$  be a sequence from  $\mathcal{L}_{\mathcal{H}}^p(\mathfrak{X}, \mathcal{F})$  with  $p \in [1, \infty)$  which converges a.s. to  $\xi \in \mathcal{L}^0(\mathfrak{X}, \mathcal{F})$ . If  $\|\xi_n\| \leq \gamma$  a.s. for some  $\gamma \in \mathcal{L}_{\mathcal{H}}^p(\mathbb{R}_+, \mathcal{F})$  and all  $n$ , then  $\xi_n \rightarrow \xi$  in  $\mathcal{L}_{\mathcal{H}}^p(\mathfrak{X}, \mathcal{F})$ .*

*Proof.* Consider a partition  $\{A_i, i \geq 1\}$  of elements from  $\mathcal{H}$  such that  $\gamma\mathbf{1}_{A_i} \in \mathcal{L}^p(\mathfrak{X}, \mathcal{F})$  for all  $i \geq 1$ . Observe that  $\|\xi\| \leq \gamma$  a.s. Applying the conditional dominated convergence theorem for integrable random variables, we obtain that  $\mathbf{E}(\|\xi_n - \xi\|^p \mathbf{1}_{A_i} | \mathcal{H}) \rightarrow 0$  for all  $i \geq 1$ . Hence,  $\mathbf{E}(\|\xi_n - \xi\|^p | \mathcal{H}) \rightarrow 0$  as  $n \rightarrow \infty$ .  $\square$

**Lemma B.6.** *The set  $\mathcal{L}^p(\mathfrak{X}, \mathcal{F})$  is dense in  $\mathcal{L}_{\mathcal{H}}^p(\mathfrak{X}, \mathcal{F})$  for all  $p \in [1, \infty]$ .*

*Proof.* Let  $p \in [1, \infty)$ . Consider  $\xi \in \mathcal{L}_{\mathcal{H}}^p(\mathfrak{X}, \mathcal{F})$ . By Lemma B.3,  $\xi = \gamma\tilde{\xi}$  where  $\gamma \in \mathcal{L}^0([1, \infty), \mathcal{H})$  and  $\tilde{\xi} \in \mathcal{L}^p(\mathfrak{X}, \mathcal{F})$ . Define  $\xi_n = \gamma\mathbf{1}_{\|\gamma\| \leq n}\tilde{\xi} \in \mathcal{L}^p(\mathfrak{X}, \mathcal{F})$ . By Lemma B.5,  $\xi_n \rightarrow \xi$  in  $\mathcal{L}_{\mathcal{H}}^p(\mathfrak{X}, \mathcal{F})$ . For  $p = \infty$ , a similar argument applies with  $\xi_n = \xi$  if  $\text{ess sup}_{\mathcal{H}} \|\xi\| \leq n$  and  $\xi_n = 0$  otherwise.  $\square$

**Lemma B.7.** *Let  $\{\xi_n, n \geq 1\}$  be a sequence from  $\mathcal{L}_{\mathcal{H}}^p(\mathfrak{X}, \mathcal{F})$  which converges to  $\xi$  in  $\mathcal{L}_{\mathcal{H}}^p(\mathfrak{X}, \mathcal{F})$ . Then there exists a random  $\mathcal{H}$ -measurable sequence  $\{n_k, k \geq 1\}$  of  $\mathcal{H}$ -measurable natural numbers, such that  $\xi_{n_k} \in \mathcal{L}_{\mathcal{H}}^p(\mathfrak{X}, \mathcal{F})$  and  $\xi_{n_k} \rightarrow \xi$  a.s.*

*Proof.* Since  $\xi_n \rightarrow \xi$  in  $\mathcal{L}_{\mathcal{H}}^p(\mathfrak{X}, \mathcal{F})$ , we deduce that  $\mathbf{E}(\|\xi_m - \xi\|^p | \mathcal{H}) \rightarrow 0$  a.s. as  $m \rightarrow \infty$ . Define the  $\mathcal{H}$ -measurable sequence  $\{n_k, k \geq 1\}$  of natural numbers by

$$n_1 = \inf\{n : \mathbf{E}(\|\xi_i - \xi\|^p | \mathcal{H}) \leq 2^{-p}, \text{ for all } i \geq n\},$$

$$n_{k+1} = \inf\{n > n_k : \mathbf{E}(\|\xi_i - \xi\|^p | \mathcal{H}) \leq 2^{-p(k+1)}, \text{ for all } i \geq n\}.$$

Since  $\mathbf{E}(\|\xi_{n_k} - \xi\|^p | \mathcal{H}) \leq 2^{-pk}$ , we deduce that  $\mathbf{E}(\|\xi_{n_k} - \xi\|^p) \leq 2^{-pk}$ . Therefore,  $(\xi_{n_k} - \xi) \rightarrow 0$  in  $\mathcal{L}^p(\mathfrak{X}, \mathcal{F})$  and almost surely for a subsequence. Observe that

$$\mathbf{E}(\|\xi_{n_k}\| | \mathcal{H}) = \sum_{j \geq k} \mathbf{E}(\|\xi_j\| | \mathcal{H}) \mathbf{1}_{n_k=j} < \infty,$$

so that  $\xi_{n_k} \in \mathcal{L}_{\mathcal{H}}^p(\mathfrak{X}, \mathcal{F})$ . The conclusion follows.  $\square$

#### REFERENCES

- [1] C. D. Aliprantis and K. C. Border. *Infinite Dimensional Analysis*. Springer, Berlin, 2006.
- [2] E. N. Barron, P. Cardaliaguet, and R. Jensen. Conditional essential suprema with applications. *Appl. Math. Optim.*, 48:229–253, 2003.
- [3] P. Cheridito, M. Kupper, and N. Vogelpoth. Conditional analysis on  $\mathbb{R}^d$ . In A. Hamel, F. Heyde, A. Löhne, B. Rudloff, and C. Schrage, editors, *Set Optimization and Applications in Finance - The State of the Art*. Springer, Berlin, 2015.
- [4] E. N. Dancer and B. Sims. Weak-star separability. *Bull. Austral. Math. Soc.*, 20:253–257, 1979.
- [5] F. Delbaen. *Monetary Utility Functions*. Osaka University Press, Osaka, 2012.
- [6] H. Föllmer and A. Schied. *Stochastic Finance. An Introduction in Discrete Time*. De Gruyter, Berlin, 2 edition, 2004.
- [7] A. H. Hamel and F. Heyde. Duality for set-valued measures of risk. *SIAM J. Finan. Engineering*, 1:66–95, 2010.
- [8] A. H. Hamel, B. Rudloff, and M. Yankova. Set-valued average value at risk and its computation. *Math. Finan. Economics*, 7:229–246, 2013.
- [9] C. Hess. Set-valued integration and set-valued probability theory: An overview. In E. Pap, editor, *Handbook of Measure Theory*, chapter 14, pages 617–673. Elsevier, 2002.
- [10] C. Hess, R. Seri, and C. Choirat. Essential intersection and approximation results for robust optimization. *J. Nonlinear Convex Anal.*, 15:979–1002, 2014.
- [11] F. Hiai and H. Umegaki. Integrals, conditional expectations, and martingales of multivalued functions. *J. Multiv. Anal.*, 7:149–182, 1977.
- [12] Y. Kabanov and E. Lépinette. Essential supremum with respect to a random partial order. *J. Math. Econom.*, 49(6):478–487, 2013.
- [13] Y. M. Kabanov and M. Safarian. *Markets with Transaction Costs. Mathematical Theory*. Springer, Berlin, 2009.
- [14] T.-W. Ma. *Classical Analysis on Normed Spaces*. World Scientific Publishing Co., Inc., River Edge, NJ, 1995.
- [15] I. Molchanov. *Theory of Random Sets*. Springer, London, 2005.

- [16] I. Molchanov and I. Cascos. Multivariate risk measures: a constructive approach based on selections. *Math. Finance*, 26:867–900, 2016.

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