On the existence of vector fields with nonnegative divergence in rearrangement-invariant spaces

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Abstract

We investigate the existence of solutions of

$$\operatorname{div} F = \mu, \text{ on } \mathbf{R}^d. \tag{*}$$

Here, $\mu \geq 0$ is a Radon measure, and we look for a solution $F \in X(\mathbf{R}^d \to \mathbf{R}^d)$, where X is a rearrangement-invariant space.

We first prove the equivalence of the following assertions:
(i) (*) has a solution for some nontrivial μ;
(ii) the function $x \mapsto |x|^{1-d} 1_{B^c}(x)$ belongs to X.

Here, B is the unit ball in $\mathbf{R}^d$.

We next investigate the solvability of (*) when μ is fixed. A *sufficient* condition is that $I_1\mu \in X$, where $I_1\mu$ is the 1-Riesz potential of μ. This condition turns out to be also *necessary* when the Boyd indexes of X belong to $(0, 1)$.

Our analysis generalizes the one of Phuc and Torres (2008) when $X = L^p$.

We will study the existence of solutions in different function spaces for the divergence equation

$$\operatorname{div} F = \mu, \quad \text{on } \mathbf{R}^d, \tag{1}$$

where μ is a nonnegative Radon measure. Here, $d \geq 2$.

Our work is motivated by the following result of Phuc and Torres:

Theorem 1. (Theorem 3.1 in [4]) *Let $1 \leq p \leq d/(d-1)$ and let μ be a nonnegative Radon measure on $\mathbf{R}^d$. If there exists a vector field $F \in L^p(\mathbf{R}^d \to \mathbf{R}^d)$ such that $\operatorname{div} F = \mu$ on $\mathbf{R}^d$, then we have that $\mu \equiv 0$.*

The proof given in [4] uses the Calderón-Zygmund theory. More specifically, assume that (1) has a solution F in L^p. It is shown first that the 1-Riesz potential $I_1\mu$ of μ, defined by the formula

$$I_1\mu(x) = \int_{\mathbf{R}^d} |x-y|^{1-d} d\mu(y),$$

satisfies the relation

$$I_1\mu(x) = (1-d) \lim_{\varepsilon \to 0} \int_{|\mathbf{x}-\mathbf{y}|>\boldsymbol{\varepsilon}} F(y) \frac{x-y}{|x-y|^{d+1}} d\mu(y) = c_d \sum_{j=1}^{d} R_j F_j(x), \quad \text{a.e. on } \mathbf{R}^d,$$

where R_j are the Riesz transforms and c_d is a constant only depending on d. Now, since $F \in L^p$, the Calderón-Zygmund theory ensures that $I_1\mu \in L^p$, whenever $p > 1$, and that $I_1\mu \in L^{1,\infty}$, if $p = 1$. However, since we have the trivial inequality

$$I_1\mu(x) \geq \frac{\mu(B(0,R))}{(|x|+R)^{d-1}}, \text{ for any } R > 0,$$

we must have $\mu(B(0,R)) = 0$ for all $R > 0$. Indeed, the function $(|x|+R)^{1-d}$ is never in $L^{1,\infty}$ or an L^p space for $p \in (1, d/(d-1)]$. Therefore $\mu \equiv 0$.

Also, using functional analytical methods, in [4] is proved (this follows easily from Theorem 3.2 and Theorem 3.3 from [4]) that the constant $d/(d-1)$ in the above theorem is sharp, in the sense that if $d/(d-1) < p \leq \infty$ then there exists an $F \in L^p$ such that $\operatorname{div} F = 1_B m$. Here, m is the Lebesgue measure and 1_B is the characteristic function of the unit ball.

Rewriting the condition on p in an integral form, we can express these facts by saying that if the divergence equation has a solution in L^p, then the measure μ is forced to be trivial if and only if the function $|x|^{1-d}1_{B^c}$ is not in L^p (here, 1_{B^c} is the characteristic function of the complement of the unit ball). As we will see, this phenomenon still occurs in a more general context where instead of the L^p spaces we consider rearrangement-invariant spaces (r.i. spaces for short). Our proof is quite elementary and does not use tools like the Calderón-Zygmund theory. It only makes use of basic properties of r.i. spaces whose definition is recalled below.

Following the presentation in [1] (Chapters 1 and 2) we define first the notion of the *Banach function space.* Consider a measured space (Y, ν) and the set

$$M^+ := \{f : Y \to [0,\infty] \mid f \text{ is } \nu\text{-measurable}\}.$$

We call *function norm* a mapping $\rho : M^+ \to [0,\infty]$ with the following properties:

(P1) $\rho(f) = 0$ iff $f = 0$ $\nu - a.e.$, $\rho(af) = a\rho(f)$ and $\rho(f+g) \leq \rho(f) + \rho(g)$;
(P2) $0 \leq g \leq f$ $\nu - a.e.$ implies $\rho(g) \leq \rho(f)$;
(P3) $0 \leq f_n \uparrow f$ $\nu - a.e.$ implies $\rho(f_n) \uparrow \rho(f)$;
(This condition has an immediate important consequence called the *Fatou property: if* f_n, f are nonnegative measurable functions and $f_n \to f$ $\nu - a.e.$, then $\rho(f) \leq \varliminf_{n\to\infty} \rho(f_n)$.)
(P4) $\nu(E) < \infty$ implies $\rho(1_E) < \infty$;
(P5) $\nu(E) < \infty$ implies $\int_E f d\nu \leq C_E \rho(f)$

whenever $f, f_n, g \in M^+$, $a \geq 0$ and E is a measurable subset of Y. Here, $C_E > 0$ is a constant only depending on E.

The set of measurable functions $f : Y \to \mathbf{R}$ for which $\rho(|f|) < \infty$ is called the *Banach function space* associated to ρ. It turns out that this space (in which we consider two functions equal when they are equal $\nu - a.e.$) with the norm $\|\cdot\| = \rho(|\cdot|)$ is a complete normed vector space (Theorem 1.6 in [1], p. 5).

A r.i. *space* is a Banach function space associated to a function norm ρ with the property that $\rho(|f|) = \rho(|g|)$ for every pair of measurable functions on Y with the same distribution function $\lambda_f = \lambda_g$ (we recall that $\lambda_f(t) = \nu(\{x \in Y \mid |f(x)| > t\})$ for all $t \geq 0$). Here are few examples: the Lebesgue spaces L^p, the Lorentz spaces $L^{p,q}$, the Orlicz spaces $\Phi(L)$.

In our case we will always have $Y = \mathbf{R}^d$ and $\nu = m$ will be the Lebesgue measure on $\mathbf{R}^d$. In this setting, we will use the following version of Theorem 4.8 in [1], p. 61:

Lemma 1. *Let m be the Lebesgue measure on $\mathbf{R}^d$ and let $(E_j)_{j\geq 1}$ be a sequence of measurable pairwise disjoint subsets of $\mathbf{R}^d$, each with finite positive measure. Let $E = \mathbf{R}^d \backslash \bigcup_j E_j$. For each measurable nonnegative function f on $\mathbf{R}^d$, we define*

$$Af = f 1_E + \sum_{j=1}^{\infty} \left(\frac{1}{m(E_j)} \int_{E_j} f dx \right) 1_{E_j}.$$

Then A is a contraction on each r.i. space X over $(\mathbf{R}^d, m)$, that is,

$$\|Af\|_X \leq \|f\|_X\,, \text{ for all } f \in X.$$

We can now state the first result:

Theorem 2. *Let μ be a nonnegative Radon measure on $\mathbf{R}^d$, and X a r.i. space of functions on $\mathbf{R}^d$ such that $|x|^{1-d} 1_{B^c}$ does not belong to X. If the equation $\operatorname{div} F = \mu$ has a solution $F \in X(\mathbf{R}^d \to \mathbf{R}^d)$, then $\mu \equiv 0$.*

Proof. For each integer $j \geq 0$ we consider the set $U_j = B(0, 2^{j+1}) \backslash\, B(0, 2^j)$ and the function $\varphi_j \in C_c^{0,1}(\mathbf{R}^d)$ defined by $\varphi_j(x) = 1$ if $|x| \in [0, 2^j)$, $\varphi_j(x) = -2^{-j}|x| + 2$ if $|x| \in [2^j, 2^{j+1})$ and $\varphi_j(x) = 0$ if $|x| \geq 2^{j+1}$. We consider also the weights $g_j := \mu(B(0, 2^j))$.

Supposing that the equation $\operatorname{div} F = \mu$ has a solution in the space X, we estimate the weights g_j as follows:

$$g_j \leq \int_{\mathbf{R}^d} \varphi_j d\mu = -\int_{\mathbf{R}^d} F \cdot \nabla \varphi_j dx \leq \frac{1}{2^j} \int_{U_j} |F| dx \quad \text{for all } j \geq 0$$

so that

$$\frac{g_j}{2^{j(d-1)}} \leq c \frac{1}{m(U_j)} \int_{U_j} |F| dx \quad \text{for all } j \geq 0, \tag{2}$$

where c is a positive constant depending on d. Now if A is the operator defined in Lemma 1 corresponding to the sequence of sets U_0, U_1,..., we have

$$A|F| = |F| 1_B + \sum_{j=0}^{\infty} \left(\frac{1}{m(U_j)} \int_{U_j} |F| dx \right) 1_{U_j}$$

and, by Lemma 1 and axiom (P2), we obtain that

$$\left\| \sum_{j=0}^{\infty} \left(\frac{1}{m(U_j)} \int_{U_j} |F| dx \right) 1_{U_j} \right\|_X \leq 2 \|F\|_X < \infty. \tag{3}$$

Of course we always have $g_j \geq g_0$ and we can use (P2), (2) and (3) to write

$$g_0 \rho(|x|^{1-d} 1_{B^c}) \leq \rho\left(\sum_{j=0}^{\infty} \frac{g_0}{2^{j(d-1)}} 1_{U_j} \right) \leq \rho\left(\sum_{j=0}^{\infty} \frac{g_j}{2^{j(d-1)}} 1_{U_j} \right) < \infty,$$

where ρ is the norm function which defines the norm on X.

However, since $\rho(|x|^{1-d} 1_{B^c}) = \infty$, the quantity g_0 must be zero. By a translation argument, the measure μ must be trivial. $\square$

We saw that the condition

$$|x|^{1-d} 1_{B^c} \notin X \tag{4}$$

was used for proving the nonexistence of a solution F when $\mu \not\equiv 0$.

In order to obtain existence results we assume that condition (4) does not hold, that is

$$|x|^{1-d} 1_{B^c} \in X. \tag{5}$$

In this case we will prove the following

Proposition 1. *Assume (5) and let μ be a measure such that $\mu = \phi m$ for a nonnegative function $\phi \in L_c^\infty(\mathbf{R}^d)$. Then (1) has a solution F in $X(\mathbf{R}^d \to \mathbf{R}^d)$.*

The above result is an immediate consequence of the following two statements (which do not require (5)):

Proposition 2. *Let $\phi \in L_c^\infty(\mathbf{R}^d)$ be such that $\phi \geq 0$ and let μ be the measure defined by $\mu = \phi m$. Then there exists a constant $C > 0$ only depending on μ and d such that*

$$I_1\mu(x) \leq C(1_B(x) + |x|^{1-d} 1_{B^c}(x)), \quad \text{on } \mathbf{R}^d.$$

(The proof of Proposition 2 is immediate.)

Proposition 3. Let *μ be a nonnegative Radon measure on $\mathbf{R}^d$ and let X be a r.i. space of functions on $\mathbf{R}^d$. If $I_1\mu \in X$, then there exists a vector field $F \in X(\mathbf{R}^d \to \mathbf{R}^d)$ such that* $\operatorname{div} F = \mu$ *in the distributional sense.*

Proof of Proposition 3. If ρ is the norm function defining the norm on X, we have (using the property (P5)) that, for any $x_0 \in \mathbf{R}^d$, there exist a constant C_{x_0} such that

$$\int_{B(x_0,1)} I_1\mu dx \leq C_{x_0} \rho(I_1\mu) = C_{x_0} \|I_1\mu\|_X < \infty.$$

It follows that $I_1\mu$ must be a finite quantity a.e. on $\mathbf{R}^d$. Now we can fix a point $x_1 \in \mathbf{R}^d$, such that $I_1\mu(x_1) < \infty$. Using this property of x_1, we find that:

$$\begin{aligned} \int_{\mathbf{R}^d} \frac{d\mu(y)}{\langle y \rangle^{d-1}} &\leq C_1 \left(\int_{|x_1-y|<1} \frac{d\mu(y)}{\langle y \rangle^{d-1}} + \int_{|x_1-y|\geq 1} \frac{d\mu(y)}{|x_1-y|^{d-1}} \right) \\ &\leq C_2 \left(\mu(B(x_1,1)) + I_1\mu(x_1) \right) < \infty \end{aligned} \tag{6}$$

for some positive constants C_1 and C_2. Here, $\langle y \rangle = (1+|y|^2)^{1/2}$.

If E is the standard fundamental solution of the Laplacian on $\mathbf{R}^d$, we define the vector field $F : \mathbf{R}^d \to \mathbf{R}^d$ by the formula

$$F_j(x) = \int_{\mathbf{R}^d} \partial_j E\,(x-y)\, d\mu(y), \;\; j \in \{1,..., d\}.$$

We can easily see that F is a.e. well-defined. Indeed there exist a constant $C_3 > 0$ such that

$$\int_{\mathbf{R}^d} |\partial_j E\,(x-y)|\, d\mu(y) \leq C_3 \int_{\mathbf{R}^d} \left| \frac{x_j - y_j}{|x-y|^d} \right| d\mu(y) \leq C_3 I_1\mu(x),$$

and thus $|F| \leq C_4 I_1\mu < \infty$ a.e. In addition, since $I_1\mu$ is already in X, using the monotonicity of ρ we get $F \in X$. Choosing a test function $\varphi \in C_c^\infty(\mathbf{R}^d)$ and using (6), we get that

$$\int_{\mathbf{R}^d} \int_{\mathbf{R}^d} \left| \frac{x_j - y_j}{|x-y|^d} \right| d\mu(y) |\partial_j \varphi(x)| dx \leq \int_{\mathbf{R}^d} I_1 |\partial_j \varphi| d\mu \leq C_\varphi \int_{\mathbf{R}^d} \frac{d\mu(y)}{\langle y \rangle^{d-1}} < \infty.$$

Here, we have used the straightforward estimate

$$I_1|\partial_j\varphi|(y) \leq C_\varphi \frac{1}{\langle y\rangle^{d-1}}.$$

We can now prove, using Fubini's theorem, that F solves (1):

$$-\sum_j < F_j, \partial_j\varphi >= -\sum_j \int_{\mathbf{R}^d}\int_{\mathbf{R}^d} \partial_j E\,(x-y)\,\partial_j\varphi(x)dxd\mu(y) = \sum_j \int_{\mathbf{R}^d} (\partial_j E) * (\partial_j\varphi)d\mu$$

$$= \sum_j \int_{\mathbf{R}^d} \partial_j^2(E*\varphi)d\mu = \int_{\mathbf{R}^d} \varphi d\mu =< \varphi, \mu > .$$

So $\operatorname{div} F = \mu$ in the distributional sense on $\mathbf{R}^d$. $\square$

Remark. The above proof does not extend to the case of signed Radon measures. The existence problem in this case is more difficult and seems to be unsolved even in the L^p setting (see [4], p. 1575 and the references therein).

As we saw, in the L^p case, the proof sketched after the statement of Theorem 1 gives a stronger conclusion when $1 < p < \infty$, namely if we can find a solution $F \in L^p$ of equation (1) then, not only that the condition (5) is satisfied, but the 1-Riesz potential of μ must be in L^p too. In what follows we prove that we have a similar situation in the case of r.i. spaces, giving a sufficient condition in terms of the Boyd indexes of the considered space. We recall some basic facts which will be useful and the definition of these indexes, again following the presentation in [1]:

Let X be a r.i. space over $\mathbf{R}^d$ whose function norm is ρ. We can define the *associate norm* ρ' of ρ by:

$$\rho'(g) = \sup\left\{\int_{\mathbf{R}^d} fgdx \mid\ f \in M^+,\, \rho(f) \leq 1\right\}, \text{ for } g \in M^+.$$

It is known (Theorem 2.2 in [1], p. 8) that ρ' is a norm function whose corresponding Banach function space, which is also an r.i space, will be denoted by X'. The following Hölder type inequality is a direct consequence of the definition:

$$\int_{\mathbf{R}^d} |fg|dx \leq \|f\|_X \|g\|_{X'}, \text{ when } f \in X,\, g \in X'.$$

Let g^* denote the nonincreasing rearrangement of a measurable function $g : \mathbf{R}^d \to \mathbf{R}$:

$$g^*(s) := \inf\{t > 0|\ \lambda_g(t) \leq s\},\ s > 0.$$

We also recall the following inequality of Hardy and Littlewood (Theorem 2.2 in [1], p. 44) that will be useful later. We have that

$$\int_{\mathbf{R}^d} |fg|dx \leq \int_0^\infty f^*(s)g^*(s)ds$$

for all measurable functions f, g on $\mathbf{R}^d$.

The Luxemburg representation theorem (Theorem 4.10 in [1], p. 62) provides a unique rearrangement-invariant function norm $\overline{\rho}$ defined on the nonnegative measurable functions on $(0,\infty)$, defined by

$$\overline{\rho}(h) = \sup\left\{\int_0^\infty h^* g^* dx \mid\ g \in M^+,\, \rho'(g) \leq 1\right\},$$

with the property that $\rho(f) = \overline{\rho}(f^*)$. The corresponding r.i. space of $\overline{\rho}$ will be denoted by $\overline{X}$. For any $\theta > 0$ we can define the dilation operator $E_\theta : \overline{X} \to \overline{X}$ by the formula $E_\theta f(s) = f(\theta s)$ for all $f \in \overline{X}$. One may prove that each E_θ is a bounded operator. The *lower Boyd index* and *the upper Boyd index* are given by

$$\underline{\alpha}_X := \sup_{0<t<1} \frac{\log \left\| E_{1/t} \right\|}{\log t}, \qquad \overline{\alpha}_X := \inf_{1<t} \frac{\log \left\| E_{1/t} \right\|}{\log t}$$

respectively. Here $\left\| E_{1/t} \right\|$ is the norm of the operator $E_{1/t}$. It turns out (Proposition 5.13 in [1], p. 149) that we can actually take limits in the definition:

$$\underline{\alpha}_X = \lim_{t\to 0} \frac{\log \left\| E_{1/t} \right\|}{\log t}, \qquad \overline{\alpha}_X = \lim_{t\to\infty} \frac{\log \left\| E_{1/t} \right\|}{\log t},$$

and that always $0 \leq \underline{\alpha}_X \leq \overline{\alpha}_X \leq 1$. As an important example consider the spaces L^p. In this case both indexes are equal to $1/p$. For the Lorentz spaces $L^{p,q}$ ($1 \leq p < \infty$, $1 \leq q \leq \infty$) the indexes are again both equal to $1/p$.

In order to obtain the necessity of the condition $I_1\mu \in X$, we adapt the proof of Theorem 1. To do so, we will need the following lemma which is just a rephrasing of some ideas presented in [1] and [3] (see, more specifically, the results of Calderón and Stein in section 3 in [3]).

Recall that a singular integral operator is an operator K of the form

$$Kf(x) = \lim_{\varepsilon\to 0^+} \int_{|y|>\varepsilon} k(y) f(x-y)dy.$$

The kernel k is *odd* if k is a function of the form $k(r\omega) = r^{-d}\Omega(\omega)$ for all $r > 0$ and all $\omega \in \mathbf{S}^{d-1}$, where $\Omega \in L^1(\mathbf{S}^{d-1})$ is odd.

Lemma 2. *Let X be a r.i. space of functions on $\mathbf{R}^d$ such that $0 < \underline{\alpha}_X \leq \overline{\alpha}_X < 1$. Then any singular integral operator with odd kernel is well-defined and bounded from $L^2 \cap X$ into X. In particular, the Riesz transforms R_1,..., $R_d : L^2 \cap X \to X$ are well-defined and bounded.*

Proof. Let K, k be as above. It is well-known that the operator K is well-defined and continuous on $L^p(\mathbf{R}^d)$ for $1 < p < \infty$. According to Theorem 3 in [3] p. 193, if $f \in L^2(\mathbf{R}^d)$ then, for all $s > 0$, we have that

$$(Kf)^*(s) \leq \frac{1}{s}\int_0^s (Kf)^*(t)dt \leq \|\Omega\|_{L^1(S(0,1))} \left(\frac{1}{s}\int_0^\infty f^*(t)\sinh^{-1}\left(\frac{s}{t}\right)dt \right).$$

Introducing the two operators

$$Pg(s) = \frac{1}{s}\int_0^s g(t)dt \quad \text{and} \quad Qg(s) = \int_s^\infty g(t)\frac{dt}{t},$$

for g measurable nonnegative, and integrating by parts, we can write for all $s > 0$,

$$\begin{aligned} \frac{1}{s}\int_0^\infty f^*(t)\sinh^{-1}\left(\frac{s}{t}\right)dt &= \int_0^\infty \frac{Pf^*(t)}{\sqrt{s^2+t^2}}dt = \int_0^s \frac{Pf^*(t)}{\sqrt{s^2+t^2}}dt + \int_s^\infty \frac{Pf^*(t)}{\sqrt{s^2+t^2}}dt \\ &\leq P^2 f^*(s) + QPf^*(s), \end{aligned}$$

concluding that there exist a constant $C_k > 0$ such that for all $s > 0$ we have

$$(Kf)^*(s) \leq C_k \left(P^2 + QP\right) f^*(s). \tag{7}$$

Theorem 5.15 in [1] guarantees that the operators P,Q are well-defined and continuous from $\overline{X}$ into $\overline{X}$ in the case where the Boyd indexes of X are in the interval $(0,1)$. Under this assumption, the inequality (7) implies that there exist a constant $C_{k,X}>0$ only depending on k and X such that, for all $f\in L^2\cap X(\mathbf{R}^d)$ we have

$$\|Kf\|_X \le C_{k,X}\|f\|_X\,,$$

and we obtain the conclusion. ☐

Theorem 3. *Let X be a r.i. space of functions on $\mathbf{R}^d$ such that $0<\underline{\alpha}_X\le\overline{\alpha}_X<1$. If (1) has a solution $F\in X(\mathbf{R}^d\to\mathbf{R}^d)$, then $I_1\mu\in X$. Moreover, there exists a constant $C_X>0$ only depending on X such that $\|I_1\mu\|_X\le C_X\|F\|_X$.*

Remark 1. In particular, Theorem 3 applies to all L^p and, more generally, $L^{p,q}$ spaces with $1<p<\infty$, $1\le q\le\infty$. Also the theorem applies to all reflexive Orlicz spaces.

Proof. Firstly we observe that, using Fubini's theorem and the monotone convergence theorem, we can rewrite the 1-Riesz potential of a Radon measure ν on $\mathbf{R}^d$, for which $I_1|\nu|<\infty$ a.e.:

$$\begin{aligned} I_1\nu(x) &= \lim_{\delta\to0^+}\int_{\mathbf{R}^d}\min(|x-y|^{1-d},\delta^{1-d})d\nu(y) = (d-1)\lim_{\delta\to0^+}\int_{\mathbf{R}^d}\left(\int_\delta^\infty \frac{1_{B(x,r)}(y)}{r^d}dr\right)d\nu(y)\\ &= (d-1)\lim_{\delta\to0^+}\int_\delta^\infty\frac{\nu(B(x,r))}{r^d}dr = (d-1)\int_0^\infty\frac{\nu(B(x,r))}{r^d}dr. \end{aligned} \tag{8}$$

Suppose (1) has a solution $F\in X$. Consider a standard radial bump function $\varphi\in C_c^\infty(\mathbf{R}^d)$ with $0\le\varphi\le1$, *supp* $\varphi\subseteq\overline{B}(0,1)$, $\|\varphi\|_{L^1(\mathbf{R}^d)}=1$ and some $\phi\in C_c^\infty(\mathbf{R}^d)$ with $0\le\phi\le1$, $\phi=1$ on $\overline{B}(0,1)$. For any $\theta,\varepsilon>0$ we define φ_ε and ϕ_θ on $\mathbf{R}^d$ by the formula $\varphi_\varepsilon(x)=\varepsilon^{-d}\varphi(x/\varepsilon)$ and $\phi_\theta(x)=\phi(\theta x)$. Fixing an $\varepsilon>0$, the smooth functions $F_\varepsilon:=F*\varphi_\varepsilon$ and $\mu_{\varepsilon,\theta}:=(\mu*\varphi_\varepsilon)\,\phi_\theta+F_\varepsilon\cdot\nabla\phi_\theta$, clearly satisfy $\operatorname{div}(\phi_\theta F_\varepsilon)=\mu_{\varepsilon,\theta}$. As in [4], we can now use the Gauss-Ostrogradskii theorem and (8) to compute $I_1\mu_{\varepsilon,\theta}(x)$ for all x in $\mathbf{R}^d$, in terms of $\phi_\theta F_\varepsilon$:

$$\begin{aligned} I_1\mu_{\varepsilon,\theta}(x) &= (d-1)\lim_{\delta\to0^+}\int_\delta^\infty\frac{1}{r^d}\int_{S(x,r)}(\phi_\theta F_\varepsilon)\cdot n\,d\sigma dr\\ &= (d-1)\lim_{\delta\to0^+}\int_\delta^\infty\int_{S(x,r)}(\phi_\theta F_\varepsilon)(y)\cdot\frac{x-y}{|x-y|^{d+1}}d\sigma(y)dr\\ &= (d-1)\lim_{\delta\to0^+}\int_{|x-y|>\delta}(\phi_\theta F_\varepsilon)(y)\cdot\frac{x-y}{|x-y|^{d+1}}dy. \end{aligned}$$

The last expression equals $c_d\sum_j R_j(\phi_\theta F_{\varepsilon,j})(x)$ a.e. in x. Thanks to Lemma 2 and noticing that $\phi_\theta F_{\varepsilon,j}\in C_c^\infty(\mathbf{R}^d)\subset L^2$, we have that there exists a constant $C_X>0$, only depending on X, with

$$\left\|I_1\mu_{\varepsilon,\theta}\right\|_X\le c_d\left\|\sum_j R_j(\phi_\theta F_{\varepsilon,j})\right\|_X\le C_X\sum_j\|\phi_\theta F_{\varepsilon,j}\|_X\le C_X\sum_j\|F_{\varepsilon,j}\|_X\,,\quad\text{for all }\varepsilon>0. \tag{9}$$

It is not hard to see that, if $f\in X$, we have

$$\begin{aligned} \rho(|f*\varphi_\varepsilon|) &= \rho\left(\left|\int_{\mathbf{R}^d}f(\cdot-\varepsilon y)\varphi(y)dy\right|\right)\le\int_{\mathbf{R}^d}\rho\left(|f(\cdot-\varepsilon y)|\,\varphi(y)\right)dy\\ &= \int_{\mathbf{R}^d}\rho\left(|f(\cdot-\varepsilon y)|\right)\varphi(y)dy \end{aligned}$$

(we just consider an increasing sequence of nonnegative continuous functions converging pointwise to the function $|f|$ and then we apply (P3) to reduce the problem to the case of continuous functions, case which can be handled using Riemann sums and the property (P3) as before). Since f and $|f(\cdot - \varepsilon y)|$ have the same distribution function and X is a r.i. space, we get that $\int_{\mathbf{R}^d} f(\cdot - \varepsilon y)\varphi(y)dy$ belongs to X and its norm is bounded by $\|f\|_X$. This fact combined with (9) gives us

$$\left\|I_1\mu_{\varepsilon,\theta}\right\|_X \leq C_X \|F\|_X < \infty, \text{ for all } \theta, \varepsilon > 0. \tag{10}$$

It remains to show that this implies $I_1\mu \in X$ and the expected estimate. We have that $I_1\mu_{\varepsilon,\theta} = I_1(\mu_\varepsilon\phi_\theta) + I_1(F_\varepsilon \cdot \nabla\phi_\theta)$, where $\mu_\varepsilon := \mu * \varphi_\varepsilon$. When $\theta \to 0$, for the second term we can write for each $x \in \mathbf{R}^d$,

$$\begin{aligned}
|I_1(F_\varepsilon \cdot \nabla\phi_\theta)(x)| &\leq \theta \int_{\mathbf{R}^d} \frac{|F_\varepsilon(y) \cdot \nabla\phi(\theta y)|}{|x-y|^{d-1}} dy = \int_{\mathbf{R}^d} \frac{|F_\varepsilon(y/\theta) \cdot \nabla\phi(y)|}{|\theta x - y|^{d-1}} dy \\
&\leq \left\|E_{1/\theta}F_\varepsilon\right\|_X \left\|\frac{\nabla\phi}{|\theta x - \cdot|^{d-1}}\right\|_{X'} \leq \left\|E_{1/\theta}\right\| \|F_\varepsilon\|_X \left\|\frac{\nabla\phi}{(1-|\theta x|)^{d-1}}\right\|_{X'} \\
&\leq 2\left\|E_{1/\theta}\right\| \|F_\varepsilon\|_X \|\nabla\phi\|_{X'} \leq 2\theta^{\underline{\alpha}_X/2} \|F_\varepsilon\|_X \|\nabla\phi\|_{X'} \to 0.
\end{aligned}$$

The dominated convergence theorem gives for the first term that $I_1(\phi_\theta\mu_\varepsilon) \to I_1\mu_\varepsilon$ pointwise when $\theta \to 0$. From these two observations, (10) and the Fatou property of ρ (which follows from (P3)) we conclude:

$$\|I_1\mu_\varepsilon\|_X \leq \varliminf_{\varepsilon\to 0} \left\|I_1\mu_{\varepsilon,\theta}\right\|_X \leq C_X \|F\|_X < \infty, \text{ for all } \varepsilon > 0. \tag{11}$$

We now let $\varepsilon \to 0$ in (11). For each $x \in \mathbf{R}^d$ and $r > 0$ we can write:

$$\mu_\varepsilon(B(x,r)) = \int_{B(x,r)} \int_{\mathbf{R}^d} \varphi_\varepsilon(z-y)d\mu(y)dz = \int_{\mathbf{R}^d} \varphi_\varepsilon * 1_{B(x,r)}(y)d\mu(y).$$

It is not hard to see that, taking $\varepsilon \to 0$, $\varphi_\varepsilon * 1_{B(x,r)}(y) \to 1$ when $y \in B(x,r)$, $\varphi_\varepsilon * 1_{B(x,r)}(y) \to 0$ when $y \notin \overline{B}(x,r)$ and $\varphi_\varepsilon * 1_{B(x,r)}(y) \to 1/2$ when $y \in \partial B(x,r)$. Moreover the function $\varphi_\varepsilon * 1_{B(x,r)}$ is bounded by 1 and has its support contained in $B(x, r+1)$ when ε is small. The dominated convergence theorem yields

$$\mu_\varepsilon(B(x,r)) \to \mu(B(x,r)) + \frac{1}{2}\mu(\partial B(x,r)), \text{ when } \varepsilon \to 0$$

and hence, for any $l \geq 1$,

$$\begin{aligned}
\int_{1/l}^{l} \frac{\mu(B(x,r))}{r^d} dr &\leq \int_{1/l}^{l} \frac{\mu(B(x,r))}{r^d} dr + \frac{1}{2}\int_{1/l}^{l} \frac{\mu(\partial B(x,r))}{r^d} dr \\
&= \lim_{\varepsilon\to 0} \int_{1/l}^{l} \frac{\mu_\varepsilon(B(x,r))}{r^d} dr.
\end{aligned}$$

The inequality from (11) and the Fatou property of ρ will give

$$\begin{aligned}
(d-1)\left\|\int_{1/l}^{l} \frac{\mu(B(\cdot,r))}{r^d} dr\right\|_X &\leq (d-1)\varliminf_{\varepsilon\to 0} \left\|\int_{1/l}^{l} \frac{\mu_\varepsilon(B(\cdot,r))}{r^d} dr\right\|_X \\
&\leq \varliminf_{\varepsilon\to 0} \|I_1\mu_\varepsilon\|_X \leq C_X \|F\|_X
\end{aligned}$$

and we can finish the proof by using the Fatou property and (8), taking $l \to \infty$. $\square$

The above result covers the case of L^p spaces when $1 < p < \infty$. However, even in the L^p setting, the fact that the equation (1) has a solution in X does not imply that the 1-Riesz potential of the measure belongs to X. More specifically, we have the following classical result (see [4]):

Theorem 4. (Theorem 3.3 in [4]) *Let μ be a nonnegative Radon measure on $\mathbf{R}^d$. Then the equation (1) has a solution $F \in L^\infty(\mathbf{R}^d \to \mathbf{R}^d)$ if and only if the measure μ is $(d-1)$ −Frostman, i.e., there exist a constant M only depending on μ such that*

$$\mu(B(x,r)) \leq M r^{d-1}, \text{ for all } x \in \mathbf{R}^d \text{ and } r > 0.$$

In order to prove Theorem 4, let μ be a Radon measure. The fact that there exists a solution $F \in L^\infty$ for the equation $\operatorname{div} F = \mu$ is equivalent to the fact that μ belongs to the dual of the space $w^{1,1}$. Here, $w^{1,1}$ is the closure of $C_c^\infty(\mathbf{R}^d)$ under the norm $\|\cdot\|_{w^{1,1}}$, where $\|u\|_{w^{1,1}} := \|\nabla u\|_{L^1}$, $u \in C_c^\infty(\mathbf{R}^d)$. Now, for nonnegative measures this condition is equivalent to the fact that μ is $(d-1)$ −Frostman. This result is due to Meyers and Ziemer (originally appearing in [2]; see also Lemma 4.9.1 in [5], p. 209 for a proof of a more general statement).

Clearly, there exist $(d-1)$ −Frostman nonnegative measures whose 1-Riesz potential is unbounded. Take for example the measure μ defined by $\mu(E) = m_{d-1}(E \cap \{x_1 = 0\})$ for all Borel sets $E \subset \mathbf{R}^d$. Here, m_{d-1} is the $(d-1)$ −dimensional Lebesgue measure on the hyperplane $\{x_1 = 0\}$. For this μ, the quantity $\int_1^\infty r^{-d}\mu(B(x,r))dr$ is infinite for all $x \in \mathbf{R}^d$ and then, by (8), $I_1\mu$ is infinite everywhere.

Note that the Boyd indexes of L^∞ are 0 and thus the example obtained in the previous paragraph does not contradict Theorem 3.

The case of the space L^1 is also a pathological one, the Boyd indexes being equal to 1. However we cannot find a counterexample for the assertion of the Theorem 3 in the case of nonnegative measures. Indeed, by Theorem 1, the measure μ will be trivial and then $I_1\mu \equiv 0 \in L^1$. Nevertheless, we can give a simple example of a vector field $F \in L^1$ and of a *signed* Radon measure μ such that $\operatorname{div} F = \mu$, but the 1-Riesz potential, $I_1\mu$, does not belong to L^1. The construction of F and μ relies on the following observation. Consider $\psi \in C_c^\infty(B(0,1))$. When $|x|$ is large we can write, for $r = |x|$, $\omega = x/|x|$, that

$$\begin{aligned} I_1\psi(x) &= \frac{1}{r^{d-1}} \int_{B(0,1)} \frac{\psi(y)}{|\omega - y/r|^{d-1}} dy = \frac{1}{r^{d-1}} \int_{B(0,1)} \frac{\psi(y)}{|1 - 2y\cdot\omega/r + |y|^2/r^2|^{(d-1)/2}} dy \\ &= \frac{1}{r^{d-1}} \int_{B(0,1)} \psi(y)dy + \frac{d-1}{r^d}\omega \cdot \left(\int_{B(0,1)} y\psi(y)dy \right) + \frac{1}{r^{d+1}} \int_{B(0,1)} \psi(y)h(r,y)dy, \end{aligned}$$

where h is a smooth bounded function on $(1,\infty) \times \mathbf{R}^d$. Thus,

$$I_1\psi(x) = \frac{A}{|x|^{d-1}} + \frac{b\cdot x}{|x|^{d+1}} + O\left(\frac{1}{|x|^{d+1}}\right) \quad \text{as } |x| \to \infty, \tag{12}$$

where

$$A := \int_{B(0,1)} \psi(y)dy \quad \text{and} \quad b := \int_{B(0,1)} y\psi(y)dy.$$

The right hand side of (12) belongs to L^1 if and only if $A = 0$ and $b = 0$. In conclusion, $\psi \in C_c^\infty(B(0,1))$ has the property that $I_1\psi \in L^1$ if and only if

$$\int_{B(0,1)} \psi(y)dy = \int_{B(0,1)} y_j\psi(y)dy = 0 \text{ for all } j \in \{1,...,d\}.$$

We can now construct our example. Let $\varphi \in C_c^\infty(B(0,1))$ be such that $\int_{B(0,1)} \varphi(y)dy \neq 0$, and set $F = (\varphi, 0, ..., 0) \in L^1$ and $\mu = (\partial_1\varphi)\, m$. Clearly, we have

$$\int_{B(0,1)} y_1\partial_1\varphi(y)dy = -\int_{B(0,1)} \varphi(y)dy \neq 0$$

and, by the above observation (with $\psi = \partial_1\varphi$), $I_1\mu$ does not belong to L^1.

These examples show that, at least in the case where the measure is signed, we cannot expect for the pathological L^p spaces, namely L^1 and L^∞, to have the property stated in the above Theorem 3. This is also the case in the more general context of r.i. spaces: as long as at least one of the Boyd indexes of the space X is equal to 0 or 1, we can always find a signed Radon measure which is the divergence of a field F belonging to X, but whose 1-Riesz potential does not have the norm in X controlled by the norm of F. It is not hard to observe that, after minor modifications in the proof, the conclusion of Theorem 3 remains true in the case of signed Radon measures. With this in mind, Proposition 4 below is a sort of converse.

Proposition 4. *Let X be a r.i. space of functions on $\mathbf{R}^d$ with the property that whenever μ is a signed Radon measure on $\mathbf{R}^d$ with $\mu = \operatorname{div} F$ for a vector field $F \in X(\mathbf{R}^d \to \mathbf{R}^d)$, we have that* $I_1\mu^+$, $I_1\mu^-$ are finite a.e., $I_1\mu \in X$ and $\|I_1\mu\|_X \leq C_X \|F\|_X$ *for a positive constant C_X. Then* $0 <\underline{\alpha}_X \leq \overline{\alpha}_X < 1$.

Proposition 4 is a consequence of Lemma 3 below, which is a $d-$dimensional version of Proposition 4.10 p. 140 in [1], with essentialy the same proof. To state Lemma 3, consider the operators P and Q defined in the proof of Lemma 2 and let S be the Calderón operator defined by the formula:

$$Sf(s) = Pf(s) + Qf(s) = \int_0^s \frac{f(t)}{s}dt + \int_s^\infty \frac{f(t)}{t}dt = \int_0^\infty f(t) \min\left(\frac{1}{t}, \frac{1}{s}\right) dt, \quad s > 0,$$

initially for nonnegative measurable functions f on $(0, \infty)$ (see [1], p 133 and 142).

Lemma 3. *Let X be a r.i. space of functions on $\mathbf{R}^d$ and $f \in \overline{X}$. Consider the sets*

$$C^+ := (0, \infty)^d, \quad C^- := (-\infty, 0)^d$$

and the function $G : \mathbf{R}^d \to [0, \infty]$, $G(x) = f^*(v_d|x|^d)1_{C^-}(x)$, where $v_d := m(B(0,1))/2^d$. Then G and f are equimeasurable functions, the Riesz transforms R_1G,..., R_dG of G are well-defined as functions on C^+ and there exist a constant $c_d > 0$ such that:

$$((R_1G + ... + R_dG)1_{C^+})^* (s) \geq c_d S(f^*)(s), \text{ for all } s > 0. \tag{13}$$

Moreover, if $(R_1G + ... + R_dG)1_{C^+} \in X$ for all $f \in \overline{X}$ then $0 <\underline{\alpha}_X \leq \overline{\alpha}_X < 1$. In particular, the same conclusion holds if the sum of the Riesz transforms is a well-defined operator from X into X .

Proof of Lemma 3. Consider for simplicity the function $g : (0, \infty) \to [0, \infty]$ with $g(s) = f^*(v_d s^d)$. Note that g is nonincreasing, and thus $g^* = g$. It is easy to see that, since $G(x) = g(|x|)1_{C^-}(x)$ and since g is nonincreasing, we have $\lambda_G(t) = v_d\lambda_g^d(t)$ for all $t > 0$. Hence, a simple computation gives us $G^*(s) = g^*(v_d^{-1/d}s^{1/d}) = g(v_d^{-1/d}s^{1/d}) = f^*(s)$ for all $s > 0$, which shows the equimeasurability of G and f. Taking now a $j \in \{1, ..., d\}$ we can write, for $x \in C^+$:

$$\begin{aligned}
R_jG(x) &= c_d^1 \int_{C^+} \frac{x_j + y_j}{|x+y|^{d+1}} g(|y|)dy = c_d^1 \int_{C^+} \frac{x_j}{|x+y|^{d+1}} g(|y|)dy + c_d^1 \int_{C^+} \frac{y_j}{|x+y|^{d+1}} g(|y|)dy \\
&\geq c_d^2 x_j \int_{C^+} \min\left(\frac{1}{|x|^{d+1}}, \frac{1}{|y|^{d+1}}\right) g(|y|)dy + c_d^2 \int_{C^+} y_j \min\left(\frac{1}{|x|^{d+1}}, \frac{1}{|y|^{d+1}}\right) g(|y|)dy \\
&\geq c_d^3 \frac{x_j}{|x|^{d+1}} \int_{B(0,|x|)\cap C^+} g(|y|)dy + c_d^3 \int_{B^c(0,|x|)\cap C^+} \frac{y_j}{|y|^{d+1}} g(|y|)dy.
\end{aligned}$$

Summing up these inequalities yields

$$\begin{aligned}
\sum_{j=1}^d R_jG(x) &\geq c_d^3 \frac{x_1 + ... + x_d}{|x|^{d+1}} \int_{B(0,|x|)\cap C^+} g(|y|)dy + c_d^3 \int_{B^c(0,|x|)\cap C^+} \frac{y_1 + ... + y_d}{|y|^{d+1}} g(|y|)dy \\
&\geq c_d^4 \frac{1}{|x|^d} \int_{B(0,|x|)\cap C^+} g(|y|)dy + c_d^4 \int_{B^c(0,|x|)\cap C^+} \frac{1}{|y|^d} g(|y|)dy \\
&\geq c_d^5 \frac{1}{|x|^d} \int_0^{|x|} r^{d-1} g(r)dr + c_d^5 \int_{|x|}^\infty \frac{g(r)}{r} dr \\
&\geq c_d^6 \frac{1}{|x|^d} \int_0^{v_d|x|^d} f^*(t)dt + c_d^6 \int_{v_d|x|^d}^\infty \frac{f^*(t)}{t} dt \geq c_d S(f^*)(v_d|x|^d).
\end{aligned}$$

Since $S(f^*)$ is a nonincreasing function, we can see as above that the nonincreasing rearrangement of the function $x \to S(f^*)(v_d|x|^d)1_{C^+}(x)$ computed in $s > 0$ is equal to $S(f^*)(s)$. Hence, we have proved the inequality (13).

To prove the next claim, observe that the inequality (13) under the assumption that $(R_1G + ... + R_dG)1_{C^+} \in X$ gives us that, if $f \in \overline{X}$, then we must have $S(f^*) \in \overline{X}$.

Let us note that we have $S(|f|) \leq S(f^*)$. This can be easily seen by applying the Hardy-Littlewood inequality to $|f|$ and the nonincreasing function $t \to \min(1/s, 1/t)$ when $s > 0$ is fixed:

$$S(|f|)(s) = \int_0^\infty |f(t)| \min\left(\frac{1}{s}, \frac{1}{t}\right) dt \leq \int_0^\infty f^*(t) \min\left(\frac{1}{s}, \frac{1}{t}\right) dt = S(f^*)(s).$$

Up to now we have that, if $f \in \overline{X}$ and $(R_1G + ... + R_dG)1_{C^+} \in X$, then $S(|f|) \in \overline{X}$. As in the proof of Theorem 1.8, p. 7, [1] we suppose by contradiction that the operator $S : \overline{X} \to \overline{X}$ is not continuous. Then we can find a sequence $(f_n)_{n\geq 1}$ of nonnegative functions in $\overline{X}$ with $\|f_n\|_{\overline{X}} = 1$ and such that $\|S(f_n)\|_{\overline{X}} \geq n^3$ for all $n \geq 1$. The series $\sum_{n\geq 1} f_n/n^2$ being absolutely convergent in $\overline{X}$, it defines a function $f \in \overline{X}$, hence $S(f) \in \overline{X}$. However since all the functions f_n are nonnegative, we have $f \geq f_n/n^2$ and consequently $S(f) \geq S(f_n)/n^2$ which implies $\|S(f)\|_{\overline{X}} \geq \|S(f_n)\|_{\overline{X}}/n^2 \geq n$ for all $n \geq 1$, obtaining a contradiction.

Having that S is a continuous operator, we can use Theorem 5.15, p. 150, [1] to obtain the statement about the Boyd indexes of X. □

Proof of Proposition 4. To prove the Proposition 4, consider a function $f \in \overline{X}$ and the field $F = (G, ..., G)$, with G constructed from f as in Lemma 3. Suppose first that f^* is compactly supported. Note that $\mu := \operatorname{div} F$ is not always a Radon measure (we can compute explicitly μ^+ and μ^- to see that these measures are not always locally finite), but is of course a distribution. With the notation from the proof of Theorem 3, we have that $\mu * \varphi_\varepsilon$ and $F * \varphi_\varepsilon$ are smooth compactly supported functions. In particular, $\mu * \varphi_\varepsilon$ is a compactly supported signed Radon measure on $\mathbf{R}^d$.

Since $\mu * \varphi_\varepsilon = \operatorname{div}(F * \varphi_\varepsilon)$ and, as in the proof of Theorem 3, $\|F * \varphi_\varepsilon\|_X \leq \|F\|_X$, we must have then, that $\|I_1(\mu * \varphi_\varepsilon)\|_X \leq C_X \|F * \varphi_\varepsilon\|_X \leq C_X \|F\|_X$ for all $\varepsilon > 0$. As above, the formula (8) and the Gauss-Ostrogradskii theorem give us that:

$$I_1(\mu * \varphi_\varepsilon)(x) = (d-1) \lim_{\delta \to 0^+} \int_{|x-y|>\delta} F * \varphi_\varepsilon(y) \cdot \frac{x-y}{|x-y|^{d+1}} dy \text{ for all } x \in \mathbf{R}^d. \tag{14}$$

Fix $r > 0$ and take $\varepsilon \in (0, r)$. The support of F is contained in the closure of the set C^-. Hence $F * \varphi_\varepsilon$ is supported in the closure of $C^- + \varepsilon B$. We can write, for all $x \in C_r^+ := (r, ..., r) + C^+$, that

$$\begin{aligned} \lim_{\delta \to 0^+} \int_{|x-y|>\delta} F * \varphi_\varepsilon(y) \cdot \frac{x-y}{|x-y|^{d+1}} dy &= \int_{C^- + \varepsilon B} F * \varphi_\varepsilon(y) \cdot \frac{x-y}{|x-y|^{d+1}} dy \\ &\geq \int_{C^-} F * \varphi_\varepsilon(y) \cdot \frac{x-y}{|x-y|^{d+1}} dy. \end{aligned} \tag{15}$$

Since the integrand of the last term is nonnegative, one can use Tonelli's theorem to change the order of integration, and find that

$$\begin{aligned} \int_{C^-} F * \varphi_\varepsilon(y) \cdot \frac{x-y}{|x-y|^{d+1}} dy &= \int_{\mathbf{R}^d} \left(\int_{C^-} F(y - \varepsilon\xi) \cdot \frac{x-y}{|x-y|^{d+1}} dy \right) \varphi(\xi) d\xi \\ &\geq \int_{C^+} \left(\int_{C^-} g(|y - \varepsilon\xi|) \cdot \frac{x_1 - y_1 + ... + x_d - y_d}{|x-y|^{d+1}} dy \right) \varphi(\xi) d\xi. \end{aligned}$$

The fact that the function g is nonincreasing enables us to see that for a fixed $\xi \in C^+$ we have $g(|y - \varepsilon\xi|) \leq g(|y - \varepsilon'\xi|)$ for all $y \in C^-$ and for all $0 < \varepsilon' < \varepsilon$. Hence the monotone convergence theorem gives us

$$\int_{C^-} g(|y - \varepsilon\xi|) \cdot \frac{x_1 - y_1 + ... + x_d - y_d}{|x-y|^{d+1}} dy \to \int_{C^-} g(|y|) \cdot \frac{x_1 - y_1 + ... + x_d - y_d}{|x-y|^{d+1}} dy,$$

when $\varepsilon \to 0$. With the help of Fatou's lemma and the above calculation we have

$$\varliminf_{\varepsilon \to 0} \int_{C^-} F * \varphi_\varepsilon(y) \cdot \frac{x-y}{|x-y|^{d+1}} dy \geq A \int_{C^-} g(|y|) \cdot \frac{x_1 - y_1 + ... + x_d - y_d}{|x-y|^{d+1}} dy,$$

where $A = \int_{C^+} \varphi(\xi) d\xi > 0$. Using (14), (15) we obtain

$$\varliminf_{\varepsilon \to 0} I_1(\mu * \varphi_\varepsilon)(x) \geq (d-1) A \int_{C^-} g(|y|) \cdot \frac{x_1 - y_1 + ... + x_d - y_d}{|x-y|^{d+1}} dy \geq 0,$$

for all $x \in C_r^+$ and all $r > 0$. Since $(d-1) A$ do not depend on r, this inequality can be rewriten as

$$\varliminf_{\varepsilon \to 0} (I_1(\mu * \varphi_\varepsilon)) 1_{C^+} \geq c_d (R_1 G + ... + R_d G) 1_{C^+} \geq 0,$$

and consequently by the Fatou property of X, we get

$$\|(R_1 G + ... + R_d G) 1_{C^+}\|_X \leq C_X^1 \varliminf_{\varepsilon \to 0} \|I_1(\mu * \varphi_\varepsilon)\|_X$$

which implies $\|(R_1 G + ... + R_d G) 1_{C^+}\|_X \leq C_X^2 \|F\|_X$, and by using inequality (13) from Lemma 3, we get that

$$\|S(f^*)\|_{\overline{X}} \leq C_X^3 \|f^*\|_{\overline{X}} \tag{16}$$

whenever $f \in \overline{X}$ and f^* is compactly supported. Now, if $f \in \overline{X}$ and f^* is not necessary compactly supported, from (16) we have $\left\|S(f^* 1_{(0,n)})\right\|_{\overline{X}} \leq C_X^3 \left\|f^* 1_{(0,n)}\right\|_{\overline{X}}$ for all $n \geq 1$. By the monotone convergence theorem and the Fatou property of $\overline{X}$ we get, taking $n \to \infty$, that (16) is true whenever $f \in \overline{X}$. Since as in the proof of Lemma 3 we have $S(|f|) \leq S(f^*)$, we get now that $\|S(|f|)\|_{\overline{X}} \leq C_X^3 \|f\|_{\overline{X}}$ for all $f \in \overline{X}$, obtaining that S is bounded from $\overline{X}$ into $\overline{X}$.

The conclusion follows now as in Lemma 3. □

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