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A simple proof of the Wirsching-Goodwin representation of integers connected to 1 in the $3x + 1$ problem.

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Summary. *This paper gives a simple proof of the Wirsching-Goodwin representation of integers connected to 1 in the $3x+1$ problem (see [5] and [2]). This representation permits to compute all the ascending sequences $(f^{(i)}(n), i = 1, b-1)$ with a last value $f^{(b)}(n) = 1$.*

1 Basic elements

In the presentation of the book "The Ultimate Challenge: The $3x+1$ Problem", [3], J.C. Lagarias write *The $3x + 1$ problem, or Collatz problem, concerns the following seemingly innocent arithmetic procedure applied to integers: If an integer x is odd then "multiply by three and add one", while if it is even then "divide by two". The $3x + 1$ problem asks whether, starting from any positive integer, repeating this procedure over and over will eventually reach the number 1. Despite its simple appearance, this problem is unsolved.* We refer to this book and other papers from the same author for a good review of the context and the references.

1.1 Definitions

Let $n \in \mathbb{N}$.

Direct algorithm

$$T(n) = \begin{cases} 3n + 1 & \text{if } n \equiv 1 \pmod{2} \\ n/2 & \text{if } n \equiv 0 \pmod{2} \end{cases}$$

Inverse algorithm

$$U(n) = \left\{ 2n \quad \text{and} \quad \frac{n-1}{3} \quad \text{if } n \equiv 4 \pmod{6} \right\}$$

Graph $G(n)$

Let $(n_1, n_2) \in \mathbb{N}^2$. n_1 and n_2 are connected by an edge if $n_1 = T(n_2)$ or $n_2 = T(n_1)$. $G(n)$ is the subset of the integers connected to n .

Conjecture " $3x + 1$ "

$\forall n \in \mathbb{N}, \exists k \in \mathbb{N} : T^k(n) = 1$. An equivalent assertion is $G(1) = \mathbb{N}^*$.

2 Restriction to odd integers

2.1 f and h

If the " $3x + 1$ " conjecture is true for the odd integers it is also true for the even ones by definition of T . The expressions of T and U restricted to odd terms are the following with n odd:

- T becomes f : $f(n) = (3n + 1)2^{-j(3n+1)}$ with $j(3n + 1)$ the power of 2 in the prime factors decomposition of $3n + 1$. f is often called the "Syracuse function".
- U becomes h , see[1]:

$$h(n) = \begin{cases} \emptyset & \text{if } n \equiv 0 \pmod{3} \\ \frac{n2^k-1}{3}, k = 2, 4, 6... & \text{if } n \equiv 1 \pmod{3} \\ \frac{n2^k-1}{3}, k = 1, 3, 5... & \text{if } n \equiv 2 \pmod{3} \end{cases}$$

The expression of h comes from the following:

$$f(n) = (3n + 1)2^{-j(3n+1)} \Rightarrow n = \frac{f(n)2^{j(3n+1)} - 1}{3} \in \mathbb{N} \quad (1)$$

There are 3 cases

- $(f(n) \equiv 0 \pmod{3})$ (1) is impossible,
- $(f(n) \equiv 1 \pmod{3}) \Rightarrow f(n) = 3x + 1$ with $x \in \mathbb{N}$

$$\begin{aligned} \frac{f(n)2^{j(3n+1)} - 1}{3} &= \frac{(3x + 1)2^{j(3n+1)} - 1}{3} \\ &= \frac{3x \cdot 2^{j(3n+1)} + 2^{j(3n+1)} - 1}{3} \\ &\in \mathbb{N} \quad \text{if } j(3n + 1) \text{ even} \quad (\text{see Lemma 5 on page 10}) \end{aligned}$$

- $(f(n) \equiv 2 \pmod{3}) \Rightarrow f(n) = 3x + 2$ with $x \in \mathbb{N}$

$$\begin{aligned} \frac{f(n)2^{j(3n+1)} - 1}{3} &= \frac{(3x + 2)2^{j(3n+1)} - 1}{3} \\ &= \frac{3x \cdot 2^{j(3n+1)} + 2^{j(3n+1)+1} - 1}{3} \\ &\in \mathbb{N} \quad \text{if } j(3n + 1) \text{ odd} \quad (\text{see Lemma 5 on page 10}) \end{aligned}$$

2.2 Graph $g(n)$

Let (n_1, n_2) be odd integers. n_1 and n_2 are connected by an edge if $n_1 = f(n_2)$ or $n_1 = f(n_2)$. $g(n)$ is the subset of the odd integers connected to n .

3 Properties of $g(1)$

Many authors have tried to give a representation of $g(1)$. Goodwin [2] give a short history of their work and provide some more results. This paper follow the same line of research and the theorems 1, 2 and 3 are not new. However we tried to give a simple and clear presentation of the results and the proofs. The theorems 4 and 5 are new as long as we know.

3.1 Expression of $n \in g(1)$ as a sum of fractions

Theorem 1. Let $n \in g(1)$. $\exists(b, a > u_1 > u_2, \dots > u_b = 0) \in \mathbb{N}^{b+2}$:

$$n = \frac{2^a}{3^b} - \sum_{i=1, b} \frac{2^{u_i}}{3^{b-i+1}}.$$

Note that $\frac{2^a}{3^b} \geq 1 \Rightarrow a \geq b \frac{\log 3}{\log 2}$.

Proof. $n \in g(1) \Leftrightarrow \exists b : n \in h^{(b)}(1)$. The proof use induction with b . Theorem 1 is true for $b = 1$ because $h(1) = \left\{ \frac{2^k-1}{3}, k = 2, 4, 6, \dots, \right\}$, and for $b = 2$ because

$$\begin{aligned} h^{(2)}(1) &\subset \left\{ \frac{1}{3} \left(\frac{2^{k_1}-1}{3} 2^{k_2} - 1 \right), k_1 = 2, 4, 6, \dots; k_2 \in \mathbb{N} \right\} \\ &\subset \left\{ \frac{2^{k_1+k_2}}{3^2} - \frac{2^{k_2}}{3^2} - \frac{2^0}{3}, k_1 = 2, 4, 6, \dots; k_2 \in \mathbb{N} \right\} \end{aligned}$$

Assume that theorem 1 is true for $l \leq b-1$.

$$\begin{aligned} h^{(b)}(1) &\subset \left\{ \frac{1}{3} \left[\left(\frac{2^a}{3^{b-1}} - \sum_{i=1, b-1} \frac{2^{u_i}}{3^{b-i}} \right) 2^k - 1 \right], (a > u_1 > \dots u_{b-1} = 0, k) \in \mathbb{N}^{b+1} \right\} \\ &\subset \left\{ \frac{2^{a+k}}{3^b} - \sum_{i=1, b-1} \frac{2^{u_i+k}}{3^{b-i+1}} - \frac{2^0}{3}, (a > u_1 > \dots u_{b-1} = 0, k) \in \mathbb{N}^{b+1} \right\} \end{aligned}$$

The last expression has the form claimed in theorem 1. □

Note that k_1 is even but k_l may be odd or even for $l > 1$. Thus $a - u_1$ is even. If $a - u_1 = 2$, $h(1) = 1$, so the first "interesting" value is $a - u_1 = 4$.

Proof. An alternative proof of theorem 1 on the previous page using $f : n \in g(1) \Leftrightarrow \exists b \in \mathbb{N} : f^b(n) = 1$. b is the number of odd integers (excluding 1) in the sequence from n to 1.

Induction with b :

Let $b = 1$ and $3n + 1 = 2^{j(3n+1)}x$ be a partial prime factors decomposition of $3n + 1$.

$$\begin{aligned} f(n) &= (3n + 1)2^{-j(3n+1)} \\ &= 2^{j(3n+1)}x2^{-j(3n+1)} \\ &= x \end{aligned}$$

$$b = 1 \Rightarrow f(n) = 1 \Rightarrow x = 1 \Rightarrow 3n + 1 = 2^{j(3n+1)} \Rightarrow n = \frac{2^{j(3n+1)}}{3} - \frac{1}{3}.$$

$$\text{Let } b = 2. \quad b = 2 \Rightarrow f(f(n)) = 1 \Rightarrow f(n) = \frac{2^{j(3f(n)+1)}}{3} - \frac{1}{3}.$$

$$f(n) = (3n + 1)2^{-j(3n+1)} \Rightarrow (3n + 1)2^{-j(3n+1)} = \frac{2^{j(3f(n)+1)}}{3} - \frac{1}{3}.$$

Therefore

$$n = \frac{2^{j(3n+1)+j(3f(n)+1)}}{3^2} - \frac{2^{j(3n+1)}}{3^2} - \frac{1}{3}.$$

Assuming that the theorem is true till $b - 1$ we have to prove that it is true for b .

$$f(n) = (3n + 1)2^{-j(3n+1)} = \frac{2^a}{3^{b-1}} - \sum_{i=1, b-1} \frac{2^{u_i}}{3^{b-i}}.$$

Therefore

$$n = \frac{2^{a+j(3n+1)}}{3^b} - \sum_{i=1, b-1} \frac{2^{u_i+j(3n+1)}}{3^{b-i+1}} - \frac{1}{3}.$$

□

Note that the general form of u_i is thus $u_{b-i} = \sum_{l=1, i} j[3f^{(l-1)}(n) + 1]$, with $f^{(0)} = Id$, and $a = \sum_{l=1, b} j[3f^{(l-1)}(n) + 1]$.

3.2 Admissible tuple $(b, a > u_1 > u_2, \dots > u_b = 0)$

Only some values of $(b, a > u_1 > u_2, \dots > u_b = 0)$ give an integer n in theorem 1, most of them do not.

Definition 1. A tuple $(b, a \geq b \frac{\log 3}{\log 2}, a > u_1 > u_2, \dots > u_b = 0)$ of $b + 1$ integers is admissible if $\frac{2^a}{3^b} - \sum_{i=1, b} \frac{2^{u_i}}{3^{b-i+1}} \in \mathbb{N}$.

The admissible parity of $u_i - u_{i+1}$ is determined by the remainder modulo 3 of the integer obtained at step i (see the definition of h in section 2.1 on page 2).

Lemma 1. Let $\frac{n2^k-1}{3} \in \mathbb{N}$ and $\frac{n2^k-1}{3} \equiv v \pmod{3}$. Then $\frac{n2^{k+2}-1}{3} \equiv v + 1 \pmod{3}$.

The lemma indicates that v is a periodic function of k with period 6: $\frac{n2^k-1}{3} \equiv \frac{n2^{k+6}-1}{3} \pmod{3}$.

Proof. $\frac{n2^k-1}{3} \equiv v \pmod{3} \Rightarrow \frac{n2^k-1}{3} = 3x + v$,

$$\begin{aligned} \frac{n2^{k+2}-1}{3} &= 4\frac{n2^k-1}{3} + 1 \\ &= 4(3x + v) + 1 \\ &\equiv v + 1 \pmod{3} \end{aligned}$$

□

Lemma 2. *Let $n \in \mathbb{N}$ and $n_1 = \frac{n2^k-1}{3} \notin \mathbb{N}$. Then $\forall l \in \mathbb{N}, \quad \frac{n_1 2^l - 1}{3} \notin \mathbb{N}$.*

The lemma indicates that if $(b, a > u_1 > u_2, \dots > u_b = 0)$ is admissible and k has not the correct parity, the tuple $(b+1, a+k > u_1+k > u_2+k, \dots > u_b+k, u_{b+1} = 0)$ is not admissible and all tuples based on it are also not admissible. Conversely, if $(b, a > u_1 > u_2, \dots > u_b = 0)$ is admissible, $(b-1, a-u_{b-1} > u_1-u_{b-1} > u_2-u_{b-1}, \dots > u_{b-1}-u_{b-1} = 0)$ is also admissible and all such successive reduced tuples till $(1, a-u_1 > u_1-u_1 = 0)$

Proof. $n_1 = \frac{n2^k-1}{3} = \frac{p}{3}$, with p and 3 relatively prime. $\frac{n_1 2^l - 1}{3} = \frac{\frac{p}{3} 2^l - 1}{3} = \frac{p2^l - 3}{9}$. Suppose that $\frac{p2^l - 3}{9} = x \in \mathbb{N}$. Then $p2^l = 9x + 3$ that is impossible because p and 3 are relatively prime. □

3.3 Structure of $g(1)$

Lemma 3. *$g(1)$ is a tree with an additionnal loop in its root 1.*

Proof. Let h^* be a modified version of h : $h^*(1) = \frac{2^k-1}{3}, k = 4, 6, 8, \dots$,

$g^*(1) = \{1 \cup h^*(1) \cup h[h^*(1)], \dots \cup h^{(l)}[h^*(1)] \dots\}$. The case $n \in h(n_1) \cap h(n_2)$ with $n_1 \neq n_2$, is impossible because there is only one $f(n)$. Thus $g^*(1)$ is a tree because any $n \in g^*(1)$ cannot have two different parents. $g(1)$ is equal to $g^*(1)$ with a supplementary loop at node 1. □

The following definition 2 and proposition 1 are not necessary for the proof of theorem 3 on page 7 and may be skipped.

Definition 2. $g^*(1)[t, s] \subset g^*(1)$ is the graph generated by the admissible tuples with $b \leq t, 4 \leq a - u_1 \leq 2 + 6s$ and $u_i - u_{i+1} \leq 6s$.

Proposition 1. $|g^*(1)[t, s]| = 1 + \frac{3s[(2s)^t - 1]}{2s - 1}$, with $|A|$ is the cardinal of the set A .

Proof. Lemma 1 on the preceding page implies that for each node of the tree there are $3s$ admissible children of which $2s$ have children. □

Note that with $s = 1$ one obtains that the ratio of integers pertaining to $g^*(1)[t, 1]$ and less than $\max(g^*(1)[t, 1]) \simeq \frac{2^{2+6t}}{3^t}$ is greater than $\frac{3}{4} \left(\frac{3}{2^5}\right)^t$.

Lemma 4. *Let $(b, u_0 > u_1 > u_2, \dots > u_b = 0)$ an admissible tuple. Let $j < b$ and $u'_i = u_i + 2 \cdot 3^{b-j-1}$ if $i \leq j$ and $u'_i = u_i$ if $i > j$. Then the tuple $(b, u'_0 > u'_1 > u'_2, \dots > u'_b = 0)$ is admissible.*

Proof. Let $n = \frac{2^{u_0}}{3^b} - \sum_{i=1, b} \frac{2^{u_i}}{3^{b-i+1}}$, and $x = \frac{2^{u'_0}}{3^b} - \sum_{i=1, b} \frac{2^{u'_i}}{3^{b-i+1}}$.

$$\begin{aligned} x - n &= \left(\frac{2^{u_0}}{3^b} - \sum_{i=1, j} \frac{2^{u_i}}{3^{b-i+1}} \right) (2^{2 \cdot 3^{b-j-1}} - 1) \\ &= \left(\frac{2^{u_0}}{3^b} - \sum_{i=1, j} \frac{2^{u_i}}{3^{b-i+1}} \right) (q 3^{b-j}) \\ &= q \left(\frac{2^{u_0}}{3^j} - \sum_{i=1, j} \frac{2^{u_i}}{3^{j-i+1}} \right) \end{aligned}$$

Lemma 7 on page 11 implies that $q \in \mathbb{N}$ and lemma 2 on the preceding page implies that the second term is integer, therefore x is integer. \square

We introduce an alternative notation for the tuple $(b, u_0 > u_1 > u_2, \dots > u_b = 0)$.

Let $v_i = u_{i-1} - u_i$, $i = 1, \dots, b$. The tuple $(b, \sum_{i=1, b} v_i, \sum_{i=2, b} v_i, \dots, v_b)$ is equal to the tuple $(b, u_0 > u_1 > u_2, \dots > u_b = 0)$. The alternative notation for this tuple is $(b, v_1, v_2, \dots, v_b)$.

Note that $v_i = j(3^{f(b-i)}(n) + 1)$, with n given by theorem 1 on page 3, see the second proof of theorem 1 on page 3.

Theorem 2. *Let $v_i \in \mathbb{N}$, $i = 2, \dots, b$ with $1 \leq v_i \leq 2 \cdot 3^{b-i}$ and $b > 1$. For each tuple $(v_2, v_3, \dots, v_b) \exists v_1$ even with $4 \leq v_1 \leq 2 \cdot 3^{b-1}$ such that $(b, \sum_{i=1, b} v_i, \sum_{i=2, b} v_i, \dots, v_b)$ is admissible.*

Proof. The cardinal number of $F = \{v_2, \dots, v_b\}$ is

$$\begin{aligned} |\{v_2, \dots, v_b\}| &= \prod_{i=2, b} 2 \cdot 3^{b-i} \\ &= 2^{b-1} 3^{\sum_{i=2, b} (b-i)} \\ &= 2^{b-1} 3^{\sum_{k=0, b-2} k} \\ &= 2^{b-1} 3^{\frac{(b-2)(b-1)}{2}} \end{aligned}$$

Let E be the set of the admissible $\{v_1, v_2, \dots, v_b\}$. $\#E$ is equal to the product of the number of admissible nodes with children at each step excepted the last one with sterile nodes taken into account. At the first step v_1 , this number is $\frac{2}{3} 3^{b-1}$. Then for each v_1 there are 3^{b-2} possible admissible values for v_2 . From these values only $\frac{2}{3} 3^{b-2}$ have children, and so on till the last step with one admissible node (with or without child for this last step). The product is equal to $2^{b-1} 3^{\frac{(b-2)(b-1)}{2}}$.

Let $t : E \mapsto F$ with $t(v_1, v_2, \dots, v_b) = (v_2, \dots, v_b)$. t is injective because $t(v'_1, v'_2, \dots, v'_b) = t(v_1, v_2, \dots, v_b) \Rightarrow (v'_2, \dots, v'_b) = (v_2, \dots, v_b)$. $n = \frac{2^a}{3^b} - \sum_{i=1}^b \frac{2^{u_i}}{3^{b-i+1}} \in \mathbb{N}$ and $n' = \frac{2^{a'}}{3^b} - \sum_{i=1}^b \frac{2^{u_i}}{3^{b-i+1}} \in \mathbb{N}$. Therefore $n' - n = \frac{2^{a'}}{3^b} - \frac{2^a}{3^b} = 2^a \frac{2^{a'-a}-1}{3^b} = 2^a \frac{2^{v'_1-v_1}-1}{3^b} \in \mathbb{N}$ and thus $v'_1 - v_1 = p \cdot 2 \cdot 3^{b-1}$ with $p \geq 1$. Therefore $v'_1 = v_1$ and t is injective. $\#E = \#F$ and t injective imply that t is bijective and that only one $v_1 \leq 2 \cdot 3^{b-1}$ correspond to a t -uple (v_2, \dots, v_b) .

Note that the number of admissible $\{v_2, \dots, v_b\}$ corresponding to one v_1 is

$$\frac{2^{b-1} 3^{\frac{(b-2)(b-1)}{2}}}{\frac{2}{3} 3^{b-1}} = 2^{b-2} 3^{\frac{(b-3)(b-2)}{2}}.$$

□

v1*	4	4	8	8	10	10	14	14	16	16	20	20
v2	3	5	2	6	1	5	4	6	1	3	2	4
v3	2	1	1	2	1	2	2	1	2	1	2	1
n	17	35	75	2417	151	4849	...					1242755

Table 1: The 12 admissible tuples with $b = 3$

Among the possible sequences (v_2, \dots, v_b) allowed by the Theorem 2 on the previous page some are specially interesting such as the strictly ascending sequence $(f^{(i)}(n))$, $i = 1, b-1$ (see $n = 151$ in the above table as an example), given in the following corollary.

Corollary. $\forall b \in \mathbb{N}, \exists n \in \mathbb{N} : \forall i \in (1, b-1), f^{(i)}(n) > f^{(i-1)}(n)$.

Proof. n is obtained with $v_i = 1$, $i = 2 : b$, and v_1 given by theorem 2 on the preceding page and lemma 4 on the previous page. □

The Wirsching-Goodwin representation of the nodes of $g(1)$ obtained with b steps (see [2]) may be now stated in the following theorem. Let $g^*(1, b) = \{n \in g(1) : f^{(b)} = 1 \text{ and } f^{(b-1)} \neq 1\}$ and v_1^* the value of v_1 whose existence is proven in theorem 2 on the preceding page.

Theorem 3. *There is a one to one relation between $g^*(1, b)$ with $b > 1$ and the set of the tuples $(b, v'_1, v'_2, \dots, v'_b)$ with $v'_i = v_i + 2 \cdot 3^{b-i} c_i$, $c_i \in \mathbb{N}^*$, $v_i \in \mathbb{N}$, $i = 2, \dots, b$ with $1 \leq v_i \leq 2 \cdot 3^{b-i}$ and $4 \leq v_1 = v_1^* \leq 2 \cdot 3^{b-1}$.*

Proof. Direct from theorem 2 on the previous page and lemma 4 on the preceding page. □

For each b and (v_2, \dots, v_b) there is a unique $v_1 \in (4, 2 \cdot 3^{b-1} + 2)$. The theorem 4 give its value.

Theorem 4. $v_1^* = a - \sum_{i=2,b} v_i$ with

$$a \equiv \log_2 \left(\sum_{i=1,b} 2^{u_i} 3^{i-1} \mod 3^b \right) \mod (2 \cdot 3^{b-1}).$$

Proof.

$$n = \frac{2^a}{3^b} - \sum_{i=1,b} \frac{2^{u_i}}{3^{b-i+1}} \Rightarrow 2^a = n3^b + \sum_{i=1,b} 2^{u_i} 3^{i-1},$$

The group $(\mathbb{Z}/3^b \mathbb{Z})^*$ is cyclic and 2 is a primitive root modulo 3^b (see [4]). Thus

$$\begin{aligned} F : (\mathbb{Z}/3^b \mathbb{Z})^* &\mapsto \mathbb{Z}/2 \cdot 3^{b-1} \mathbb{Z} \\ i &\mapsto 2^i \mod 3^b. \end{aligned}$$

is bijective and F^{-1} is the reciprocal function,

$$\begin{aligned} F^{-1} : \mathbb{Z}/2 \cdot 3^{b-1} \mathbb{Z} &\mapsto (\mathbb{Z}/3^b \mathbb{Z})^* \\ p &\mapsto \log_2 \left(p \mod 3^b \right) \mod (2 \cdot 3^{b-1}) \end{aligned}$$

$$2^a \equiv \sum_{i=1,b} 2^{u_i} 3^{i-1} \mod 3^b \text{ imply}$$

$$a \equiv F^{-1} \left(\sum_{i=1,b} 2^{u_i} 3^{i-1} \right) = \log_2 \left(\sum_{i=1,b} 2^{u_i} 3^{i-1} \mod 3^b \right) \mod (2 \cdot 3^{b-1})$$

□

3.4 Ascending Collatz sequences excepted the last term

It is possible to give explicitly a and v_1 in some particular cases such as $(v_2 = v_3 = \dots = v_b = 1)$ and any b . The following theorem defines all the strictly ascending sequence $(f^{(i)}(n), i = 1, b-1)$ with a last value $f^{(b)}(n) = 1$.

Theorem 5. *Let $(v_2 = v_3 = \dots = v_b = 1)$ then $v_1^* = 3^{b-1} + 1$, with the corresponding*

$$n = \frac{2^{b-1} 2^{3^{b-1}+1}}{3^b} - \sum_{i=1}^b \frac{2^{b-i}}{3^{b-i+1}}.$$

Proof. With induction with b . The theorem is true for $b = 2$ and $b = 3$ (see the above table). Assume that it is true till $b-1$. Let $n_j = \frac{2^{j-1} 2^{3^{b-1}+1}}{3^j} - \sum_{i=1}^j \frac{2^{j-i}}{3^{j-i+1}}$, $j = 1, b$, the values obtained at step j with b total steps, and $m_j = \frac{2^{j-1} 2^{3^{b-2}+1}}{3^j} - \sum_{i=1}^j \frac{2^{j-i}}{3^{j-i+1}}$, $j =$

v_1^*	v_2	v_3	v_4	v_5	v_6	n
4	1					3
10	1	1				151
28	1	1	1			26512143
82	1	1	1	1		318400215865581346424671
244	1	1	1	1	1	...

Table 2: v_1^* for $b = 2, \dots, 6$ and $v_i = 1, i = 2 \dots b$

$1, b-1$, the values obtained at step j with $b-1$ total steps. The m_j are integers by the induction hypothesis. Note that

$$\begin{aligned} 3^{b-1} + 1 &= 3^{b-2} + 1 + 2 \cdot 3^{b-2} \\ &= v_1^*(b-1) + 2 \cdot 3^{b-2} \end{aligned}$$

Therefore the lemma 4 on page 6 implies that n_1, \dots, n_{b-1} are integers. We have to prove that $n_b \in \mathbb{N}$. This is true if $n_{b-1} \equiv 2 \pmod{3}$.

$$\begin{aligned} n_{b-1} - m_{b-1} &= \frac{2^{b-2}2^{3^{b-1}+1}}{3^{b-1}} - \sum_{i=1}^{b-1} \frac{2^{b-1-i}}{3^{b-i}} - \frac{2^{b-2}2^{3^{b-2}+1}}{3^{b-1}} + \sum_{i=1}^{b-1} \frac{2^{b-1-i}}{3^{b-i}} \\ &= \frac{2^{b-2}2^{3^{b-1}+1}}{3^{b-1}} - \frac{2^{b-2}2^{3^{b-2}+1}}{3^{b-1}} \\ &= 2^{b-1} \frac{2^{3^{b-1}} - 2^{3^{b-2}}}{3^{b-1}} \\ &= 2^{b-1} 2^{3^{b-2}} \frac{2^{2 \cdot 3^{b-2}} - 1}{3^{b-1}} \\ &= 2^{b-1} 2^{3^{b-2}} \frac{3^{b-1}x}{3^{b-1}} \\ &= x \cdot 2^{3^{b-2}+b-1} \end{aligned}$$

$3^{b-2} + b - 1 \equiv b \pmod{2}$, and $x \equiv 1 \pmod{3}$ imply that

$$x2^{3^{b-2}+b-1} \equiv \begin{cases} 1 \pmod{3} & \text{if } b \equiv 0 \pmod{2} \\ 2 \pmod{3} & \text{if } b \equiv 1 \pmod{2} \end{cases}$$

$n_{b-1} = m_{b-1} + x \cdot 2^{3^{b-2}+b-1}$, and the lemma 8 on page 11 implies that

$$n_{b-1} = \begin{cases} 3p + 1 + 3q + 1 \equiv 2 \pmod{3} & \text{if } b \equiv 0 \pmod{2} \\ 3p + 3q + 2 \equiv 2 \pmod{3} & \text{if } b \equiv 1 \pmod{2} \end{cases}$$

□

A simpler (but not directly related to theorem 3) proof of theorem 5 is the following: a Collatz sequence (n_0, n_1, \dots, n_b) is growing if $\forall i, n_{i+1} = (3n_i + 1)/2$. Therefore $\exists k \in \mathbb{N}^*$, $n_0 = k2^b - 1$ and $n_b = k3^b - 1$ and $n_i = k3^i 2^{b-i} - 1$. If the following term of the sequence is 1, then $\exists l \in \mathbb{N}^*$, $1 = (k3^{b+1} - 2)/2^l$. Thus $k3^{b+1} = 2 + 2^l$ and $2^l \equiv -2 \pmod{3^{b+1}}$, which is equivalent to $l - 1 \equiv 3^b \pmod{2 \times 3^b}$. The first term of this Collatz sequence is $n_0 = (1 + 2^{(2p+1)3^b})(2/3)^{b+1} - 1$ with $p \in \mathbb{N}$.

References

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A Proof of Lemmas

Lemma 5.

$$2^k \equiv \begin{cases} 2 & \text{if } k \text{ odd} \\ 1 & \text{if } k \text{ even} \end{cases} \pmod{3}$$

Proof. $(2^k \equiv 1 \pmod{3}) \Rightarrow (2^k = 3x + 1 \text{ with } x \in \mathbb{N}) \Rightarrow 2^{k+1} = 3 \cdot 2x + 2 \Rightarrow (2^{k+1} \equiv 1 \pmod{3})$.

$(2^k \equiv 2 \pmod{3}) \Rightarrow (2^k = 3x + 2 \text{ with } x \in \mathbb{N}) \Rightarrow 2^{k+1} = 3(2x + 1) + 1 \Rightarrow (2^{k+1} \equiv 2 \pmod{3})$.

$(2^0 \equiv 1 \pmod{3}) \Rightarrow (2^{2^k} \equiv 1 \pmod{3})$ and $(2^{2^{k+1}} \equiv 2 \pmod{3})$ □

Lemma 6. $\frac{2^{3^k} + 1}{3^{k+1}} \in \mathbb{N}$ and $\equiv 1 \pmod{3}$

Proof. By induction. The lemma is true for $k = 0$ and $k = 1$.

Assume that the lemma is true till $k - 1$, which implies that $2^{3^{k-1}} + 1 = 3^k x$ with

$$x \equiv 1 \pmod{3}.$$

$$\begin{aligned}
2^{3^k} + 1 &= (2^{3^{k-1}})^3 + 1 \\
&= (2^{3^{k-1}} + 1) \left[(2^{3^{k-1}})^2 - (2^{3^{k-1}}) + 1 \right] \\
&= 3^k x \left[(2^{3^{k-1}})^2 - 1 - (2^{3^{k-1}} + 1) + 3 \right] \\
&= 3^k x \left[(2^{3^{k-1}} - 1) (2^{3^{k-1}} + 1) + 3^k x + 3 \right] \\
&= 3^k x \left[3^k x (2^{3^{k-1}} - 1) + 3^k x + 3 \right] \\
&= 3^{k+1} x \left[3^{k-1} x \cdot 2^{3^{k-1}} + 1 \right]
\end{aligned}$$

$$\frac{2^{3^k} + 1}{3^{k+1}} = 3^{k-1} x^2 \cdot 2^{3^{k-1}} + x \equiv x \pmod{3}.$$

□

Lemma 7. $\frac{2^{2 \cdot 3^k} - 1}{3^{k+1}} \equiv 1 \pmod{3}$

Proof.

$$\begin{aligned}
\frac{2^{2 \cdot 3^k} - 1}{3^{k+1}} &= (2^{3^k} + 1)(2^{3^k} - 1) \\
&= x(2^{3^k} - 1), \text{ see Lemma 6 on the previous page} \\
&= 3y(2^{3^k} - 1) + (2^{3^k} - 1) \\
&\equiv 1 \pmod{3}, \text{ see Lemma 5 on the preceding page}
\end{aligned}$$

□

Lemma 8. Let n_b given by theorem 5 on page 8.

$$n_b \equiv \begin{cases} 0 \pmod{3} & \text{if } b \equiv 0 \pmod{2} \\ 1 \pmod{3} & \text{if } b \equiv 1 \pmod{2} \end{cases}$$

Proof. By induction. The lemma is true for $b = 2$ and $b = 3$ because $n_2 = 3$ and $n_3 = 151$. Assume that the lemma is true till $b - 1$.

$$\begin{aligned}
n_b - m_{b-1} &= \frac{2^{b-1} 2^{3^{b-1}+1}}{3^b} - \sum_{i=1}^b \frac{2^{b-i}}{3^{b-i+1}} - \frac{2^{b-2} 2^{3^{b-2}+1}}{3^{b-1}} + \sum_{i=1}^{b-1} \frac{2^{b-1-i}}{3^{b-i}} \\
&= \frac{2^{b-1} 2^{3^{b-1}+1}}{3^b} - \frac{2^{b-1}}{3^b} - \frac{2^{b-2} 2^{3^{b-2}+1}}{3^{b-1}} \\
&= \frac{2^{b-1}}{3^{b-1}} \left(\frac{2^{3^{b-1}+1} - 1}{3} - 2^{3^{b-2}} \right) \\
&= 2^{b-1} \left(\frac{2^{3^{b-2}} + 1}{3^{b-1}} \right) \left(\frac{2 \cdot 2^{2 \cdot 3^{b-2}} - 2 \cdot 2^{3^{b-2}} - 1}{3} \right)
\end{aligned}$$

Lemma 6 on the preceding page implies that

$$\begin{aligned}
n_b - m_{b-1} &= 2^{b-1} \left(3^{b-3} x^2 2^{3^{b-3}} + x \right) \left(\frac{2 \cdot 2^{2 \cdot 3^{b-2}} - 2 \cdot 2^{3^{b-2}} - 1}{3} \right) \\
&= 2^{b-1} \left(3^{b-3} x^2 2^{3^{b-3}} + x \right) \left(\frac{2(2^{2 \cdot 3^{b-2}} - 1) - 2(2^{3^{b-2}} + 1) + 3}{3} \right) \\
&= 2^{b-1} \left(3^{b-3} x^2 2^{3^{b-3}} + x \right) \left(\frac{2 \cdot 3^{b-1} y - 2 \cdot 3^{b-1} x + 3}{3} \right) \\
&= 2^{b-1} \left(3^{b-3} x^2 2^{3^{b-3}} + x \right) (2 \cdot 3^{b-2} y - 2 \cdot 3^{b-2} x + 1) \\
&\equiv 2^{b-1} \pmod{3} \\
n_b &= m_{b-1} + 2^{b-1} \left(3^{b-3} x^2 2^{3^{b-3}} + x \right) (2 \cdot 3^{b-2} y - 2 \cdot 3^{b-2} x + 1). \text{ Therefore}
\end{aligned}$$

$$n_b = \begin{cases} 3p + 1 + 3q + 2 \equiv 0 \pmod{3} & \text{if } b \equiv 0 \pmod{2} \\ 3p + 3q + 1 \equiv 1 \pmod{3} & \text{if } b \equiv 1 \pmod{2} \end{cases}$$

□