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A simple proof of the Wirsching-Goodwin representation of integers connected to 1 in the 3x + 1 problem.

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Summary. A proof of the Wirsching-Goodwin representation of integers connected to 1 in the 3x + 1 problem (see [3] and [2]) with elementary mathematics.

1 Basic elements

1.1 Definitions

Let $n \in \mathbb{N}$.

Direct algorithm

$$T(n) = \begin{cases} 3n+1 & if \quad n \equiv 1 \mod 2\\ n/2 & if \quad n \equiv 0 \mod 2 \end{cases}$$

Inverse algorithm

$$U(n) = \left\{ 2n \quad and \quad \frac{n-1}{3} \quad if \quad n \equiv 4 \mod 6 \right\}$$

${\bf Graph}\ G(n)$

Let $(n_1, n_2) \in \mathbb{N}^2$. n_1 and n_2 are connected by an edge if $n_1 = T(n_2)$ or $n_2 = T(n_1)$. G(n) is the subset of the integers connected to n.

Conjecture "3x + 1"

 $\forall n \in \mathbb{N}, \exists k \in \mathbb{N} : T^k(n) = 1$. An equivalent assertion is $G(1) = \mathbb{N}$.

2 Restriction to odd integers

2.1 f and h

If the "3x + 1" conjecture is true for the odd integers it is also true for the even ones by definition of T. The expressions of T and U restricted to odd terms are the following with n odd:

- T becomes f: $f(n) = (3n+1)2^{-j(3n+1)}$ with j(3n+1) is the power of 2 in the prime factors decomposition of 3n + 1. f is often called the "Syracuse function".
- U becomes h, see[1]:

$$h(n) = \begin{cases} \emptyset & \text{if } n \equiv 0 \mod 3\\ \frac{n2^k - 1}{3}, k = 2, 4, 6... & \text{if } n \equiv 1 \mod 3\\ \frac{n2^k - 1}{3}, k = 1, 3, 5... & \text{if } n \equiv 2 \mod 3 \end{cases}$$

The proof of the expression of h needs the following lemma.

Lemma 1.

$$2^k \stackrel{\text{mod } 3}{\equiv} \left\{ \begin{array}{ccc} 2 & if & k & odd \\ 1 & if & k & even \end{array} \right.$$

Proof. $(2^k \equiv 1 \mod 3) \Rightarrow (2^k = 3x + 1 \text{ with } x \in \mathbb{N}) \Rightarrow 2^{k+1} = 3.2x + 2 \Rightarrow (2^{k+1} \equiv 1 \mod 3).$

 $(2^k \equiv 2 \mod 3) \Rightarrow (2^k = 3x + 2 \text{ with } x \in \mathbb{N}) \Rightarrow 2^{k+1} = 3(2x+1) + 1 \Rightarrow (2^{k+1} \equiv 2 \mod 3).$

 $(2^0 \equiv 1 \mod 3) \Rightarrow (2^{2k} \equiv 1 \mod 3) \text{ and } (2^{2k+1} \equiv 2 \mod 3)$

The expression of h comes from the following:

$$f(n) = (3n+1)2^{-j(3n+1)} \Rightarrow n = \frac{f(n)2^{j(3n+1)} - 1}{3} \in \mathbb{N}$$
(1)

There are 3 cases

- $(f(n) \equiv 0 \mod 3)$ (1) is impossible,
- $(f(n) \equiv 1 \mod 3) \Rightarrow f(n) = 3x + 1$ with $x \in \mathbb{N}$

$$\begin{array}{rcl} \displaystyle \frac{f(n)2^{j(3n+1)}-1}{3} & = & \displaystyle \frac{(3x+1)2^{j(3n+1)}-1}{3} \\ & = & \displaystyle \frac{3x.2^{j(3n+1)}+2^{j(3n+1)}-1}{3} \\ & \in & \mathbb{N} \quad if \quad j(3n+1) \; even \quad (lemma1) \end{array}$$

• $(f(n) \equiv 2 \mod 3) \Rightarrow f(n) = 3x + 2$ with $x \in \mathbb{N}$

$$\begin{array}{rcl} \displaystyle \frac{f(n)2^{j(3n+1)}-1}{3} & = & \displaystyle \frac{(3x+2)2^{j(3n+1)}-1}{3} \\ & = & \displaystyle \frac{3x \cdot 2^{j(3n+1)}+2^{j(3n+1)+1}-1}{3} \\ & \in & \mathbb{N} \quad if \quad j(3n+1) \; odd \quad (lemma1) \end{array}$$

2.2 Graph g(n)

Let (n_1, n_2) be odd integers. n_1 and n_2 are connected by an edge if $n_1 = f(n_2)$ or $n_1 = f(n_2)$. g(n) is the subset of the odd integers connected to n.

3 Properties of q(1)

3.1 Expression of $n \in g(1)$ as a sum of fractions

Theorem 1. Let $n \in g(1)$. $\exists (b, a > u_1 > u_2, ... > u_b = 0) \in \mathbb{N}^{b+2}$:

$$n = \frac{2^a}{3^b} - \sum_{i=1,b} \frac{2^{u_i}}{3^{b-i+1}}.$$

Note that $\frac{2^a}{3^b} \ge 1 \Rightarrow a \ge b \frac{\log 3}{\log 2}$.

Proof. $n \in g(1) \Leftrightarrow \exists b : n \in h^{(b)}(1)$. The proof use induction with b. Theorem 1 is true for b = 1 because $h(1) = \left\{\frac{2^k - 1}{3}, k = 2, 4, 6..., \right\}$, and for b = 2 because

$$h^{(2)}(1) \subset \left\{ \frac{1}{3} \left(\frac{2^{k_1} - 1}{3} 2^{k_2} - 1 \right), k_1 = 2, 4, 6...; k_2 \in \mathbb{N} \right\}$$
$$\subset \left\{ \frac{2^{k_1 + k_2}}{3^2} - \frac{2^{k_2}}{3^2} - \frac{2^0}{3}, k_1 = 2, 4, 6...; k_2 \in \mathbb{N} \right\}$$

Assume that theorem 1 is true for $l \leq b - 1$.

$$h^{(b)}(1) \subset \left\{ \frac{1}{3} \left[\left(\frac{2^a}{3^{b-1}} - \sum_{i=1,b-1} \frac{2^{u_i}}{3^{b-i}} \right) 2^k - 1 \right], (a > u1 > \dots u_{b-1} = 0, k) \in \mathbb{N}^{b+1} \right\}$$
$$\subset \left\{ \frac{2^{a+k}}{3^b} - \sum_{i=1,b-1} \frac{2^{u_i+k}}{3^{b-i+1}} - \frac{2^0}{3}, (a > u1 > \dots u_{b-1} = 0, k) \in \mathbb{N}^{b+1} \right\}$$

The last expression has the form claimed in theorem 1.

Note that k_1 is even but k_l may be odd or even for l > 1. Thus $a - u_1$ is even. If $a - u_1 = 2$, h(1) = 1, so the first "interesting" value is $a - u_1 = 4$.

Proof. An alternative proof of theorem 1 using $f : n \in g(1) \Leftrightarrow \exists b \in \mathbb{N} : f^b(n) = 1$. b is the number of odd integers (excluding 1) in the sequence from n to 1.

Induction with *b*:

Let b = 1. $3n + 1 = 2^{j(3n+1)}x$ a partial prime factors decomposition of 3n + 1.

$$f(n) = (3n+1)2^{-j(3n+1)}$$

= $2^{j(3n+1)}x2^{-j(3n+1)}$
= x

 $b = 1 \Rightarrow f(n) = 1 \Rightarrow x = 1 \Rightarrow 3n + 1 = 2^{j(3n+1)} \Rightarrow n = \frac{2^{j(3n+1)}}{3} - \frac{1}{3}.$ Let b = 2. $b = 2 \Rightarrow f(f(n)) = 1 \Rightarrow f(n) = \frac{2^{j(3f(n)+1)}}{3} - \frac{1}{3}.$

$$f(n) = (3n+1)2^{-j(3n+1)} \Rightarrow (3n+1)2^{-j(3n+1)} = \frac{2^{j(3f(n)+1)}}{3} - \frac{1}{3}$$

Therefore

$$n = \frac{2^{j(3n+1)+j(3f(n)+1)}}{3^2} - \frac{2^{j(3n+1)}}{3^2} - \frac{1}{3}.$$

Assuming that the theorem is true till b-1 we have to prove that it is true for b.

$$f(n) = (3n+1)2^{-j(3n+1)} = \frac{2^a}{3^{b-1}} - \sum_{i=1,b-1} \frac{2^{u_i}}{3^{b-i}}.$$

Therefore

$$n = \frac{2^{a+j(3n+1)}}{3^b} - \sum_{i=1,b-1} \frac{2^{u_i+j(3n+1)}}{3^{b-i+1}} - \frac{1}{3}.$$

Note that the general form of u_i is thus $u_{b-i} = \sum_{l=1,i} j[3f^{(l-1)}(n)+1]$, with $f^{(0)} = Id$, and $a = \sum_{l=1,b} j[3f^{(l-1)}(n)+1]$.

3.2 Admissible tuple $(b, a > u_1 > u_2, ... > u_b = 0)$

Only some values of $(b, a > u_1 > u_2, ... > u_b = 0)$ give an integer n in theorem 1, most of them do not.

Definition 1. A tuple $(b, a \ge b \frac{log_3}{log_2}, a > u_1 > u_2, ... > u_b = 0)$ of b + 1 integers is admissible if $\frac{2^a}{3^b} - \sum_{i=1,b} \frac{2^{u_i}}{3^{b-i+1}} \in \mathbb{N}$.

The admissible parity of $u_i - u_{i+1}$ is determined by the remainder modulo 3 of the integer obtained at step *i* (see the definition of *h* in section 2.1).

Lemma 2. Let
$$\frac{n2^{k}-1}{3} \in \mathbb{N}$$
 and $\frac{n2^{k}-1}{3} \equiv v \mod 3$. Then $\frac{n2^{k+2}-1}{3} \equiv v+1 \mod 3$.

The lemma indicates that v is a periodic function of k with period 6: $\frac{n2^k-1}{3} \equiv \frac{n2^{k+6}-1}{3}$ $\mod 3.$

Proof.
$$\frac{n2^{k}-1}{3} \equiv v \mod 3 \Rightarrow \frac{n2^{k}-1}{3} = 3x + v,$$

$$\frac{n2^{k+2}-1}{3} = 4\frac{n2^{k}-1}{3} + 1$$
$$= 4(3x + v) + 1$$
$$\equiv v + 1 \mod 3$$

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Lemma 3. Let $n \in \mathbb{N}$ and $n_1 = \frac{n2^k - 1}{2} \notin \mathbb{N}$. Then $\forall l \in \mathbb{N}$, $\frac{n_1 2^l - 1}{2} \notin \mathbb{N}$.

The lemma indicates that if $(b, a > u_1 > u_2, ... > u_b = 0)$ is admissible and k has not the correct parity, the tuple $(b + 1, a + k > u_1 + k > u_2 + k, ... > u_b + k, u_{b+1} = 0)$ is not admissible and all tuples based on it are also not admissible. Conversely, if $(b, a > u_1 > u_1)$ $u_2, \dots > u_b = 0$ is admissible, $(b-1, a-u_{b-1} > u_1 - u_{b-1} > u_2 - u_{b-1}, \dots > u_{b-1} - u_{b-1} = u_{b-1}$ 0) is also admissible and all such successive reduced tuples till $(1, a - u_1 > u_1 - u_1 = 0)$

Proof. $n_1 = \frac{n2^k - 1}{3} = \frac{p}{3}$, with p and 3 relatively prime. $\frac{n_12^l - 1}{3} = \frac{\frac{p}{3}2^l - 1}{3} = \frac{p2^l - 3}{9}$. Suppose that $\frac{p2^l - 3}{9} = x \in \mathbb{N}$. Then $p2^l = 9x + 3$ that is impossible because p and 3 are relatively prime.

3.3 Tree structure of q(1)

Lemma 4. g(1) is a tree with an additionnal loop in its root 1.

Proof. Let h^* be a modified version of h: $h^*(1) = \frac{n2^k - 1}{3}, k = 4, 6, 8, ...,$ $g^*(1) = \{1 \cup h^*(1) \cup h[h^*(1)], ... \cup h^{(l)}[h^*(1)]..\}$. The case $n \in h(n_1) \cap h(n_2)$ with $n_1 \neq n_2$, is impossible because there is only one f(n). Thus $g^*(1)$ is a tree because any $n \in g^*(1)$ cannot have two different parents. g(1) is equal to $g^*(1)$ with a supplementary loop at node 1.

The following definition 2 and proposition 1 are not nessessary for the proof of theorem 3 and may be skipped.

Definition 2. $g^*(1)[t,s] \subset g^*(1)$ is the graph generated by the admissible tuples with $b \leq t, 4 \leq a - u_1 \leq 2 + 6s \text{ and } u_i - u_{i+1} \leq 6s.$

Proposition 1. $|g^*(1)[t,s]| = 1 + \frac{3s[(2s)^t - 1]}{2s - 1}$, with |A| is the cardinal of the set A.

Proof. Lemma 2 implies that for each node of the tree there are 3s admissible children of which 2s have children.

Note that with s = 1 one obtains that the ratio of integers pertaining to $g^*(1)[t, 1]$ and less than $\max(g^*(1)[t,1]) \simeq \frac{2^{2+6t}}{3^t}$ is greater than $\frac{3}{4} \left(\frac{2^5}{3}\right)^t$.

Lemma 5. $2^{2.3^k} - 1 \equiv 0 \mod 3^{k+1}$

Proof. By induction. The lemma is true for k = 0 and k = 1 because $2^2 - 1 = 3$ and $2^6 - 1 = 63 = 3^2 7$. Assume that lemma is true till k - 1, with $(2^{2 \cdot 3^{k-1}} - 1) = 3^k x$ with $x \in \mathbb{N}$.

$$2^{2.3^{k}} - 1 = (2^{2.3^{k-1}})^{3} - 1$$

= $(2^{2.3^{k-1}} - 1) [(2^{2.3^{k-1}})^{2} + (2^{2.3^{k-1}}) + 1]$
= $3^{k}x [(2^{2.3^{k-1}})^{2} - 1 + (2^{2.3^{k-1}} - 1) + 3]$
= $3^{k}x [(2^{2.3^{k-1}} - 1) (2^{2.3^{k-1}} + 1) + 3^{k}x + 3]$
= $3^{k}x [3^{k}x (2^{2.3^{k-1}} + 1) + 3^{k}x + 3]$
= $3^{k+1}x [3^{k-1}x (2^{2.3^{k-1}} + 2) + 1]$

Lemma 6. Let $(b, u_0 > u_1 > u_2, ... > u_b = 0)$ an admissible tuple. Let j < b and $u'_i = u_i + 2.3^{b-j-1}$ if $i \leq j$ and $u'_i = u_i$ if i > j. Then the tuple $(b, u'_0 > u'_1 > u'_2, ... > u'_b = 0)$ is admissible.

Proof. Let
$$n = \frac{2^{u_0}}{3^b} - \sum_{i=1,b} \frac{2^{u_i}}{3^{b-i+1}}$$
, and $x = \frac{2^{u'_0}}{3^b} - \sum_{i=1,b} \frac{2^{u'_i}}{3^{b-i+1}}$.

$$x - n = \left(\frac{2^{u_0}}{3^b} - \sum_{i=1,j} \frac{2^{u_i}}{3^{b-i+1}}\right) \left(2^{2\cdot 3^{b-j-1}} - 1\right)$$

$$= \left(\frac{2^{u_0}}{3^b} - \sum_{i=1,j} \frac{2^{u_i}}{3^{b-i+1}}\right) \left(q3^{b-j}\right)$$

$$= q\left(\frac{2^{u_0}}{3^j} - \sum_{i=1,j} \frac{2^{u_i}}{3^{j-i+1}}\right)$$

Lemma 5 implies that $q \in \mathbb{N}$ and lemma 3 implies that the second term is integer, therefore x is integer.

We introduce an alternative notation for the tuple $(b, u_0 > u_1 > u_2, ... > u_b = 0)$.

Let $v_i = u_{i-1} - u_i$, i = 1, ...b. The tuple $(b, \sum_{i=1,b} v_i, \sum_{i=2,b} v_i, ...v_b)$ is equal to the tuple $(b, u_0 > u_1 > u_2, ... > u_b = 0)$. The alternative notation for this tuple is $(b, v_1, v_2, ..., v_b)$.

Note that $v_i = j(3f^{(b-i)}(n) + 1)$, with n given by theorem 1, see the second proof of theorem 1.

Theorem 2. Let $v_i \in \mathbb{N}$, i = 2, ...b with $1 \leq v_i \leq 2.3^{b-i}$ and b > 1. For each tuple $(v_2, v_3, ...v_b) \exists v_1 \text{ even with } 4 \leq v_1 \leq 2.3^{b-1}$ such that $(b, \sum_{i=1,b} v_i, \sum_{i=2,b} v_i, ...v_b)$ is admissible.

Proof. On one side the cardinal number of $\{v_2, ..., v_b\}$ is

$$\{v_2, \dots v_b\} | = \prod_{i=2,b} 2.3^{b-i}$$

= $2^{b-1} 3^{\sum_{i=2,b}(b-i)}$
= $2^{b-1} 3^{\sum_{k=0,b-2}k}$
= $2^{b-1} 3^{\frac{(b-2)(b-1)}{2}}$

On the other side the number of admissible $\{v_1v_2, ...v_b\}$ nodes of $g^*(1)$ is equal to the product of the number of admissible nodes with children at each step excepted the last one with sterile nodes taken into account. At the first step v_1 , this number is $\frac{2}{3}3^{b-1}$, at the step v_2 it is equal to $\frac{2}{3}3^{b-2}$, and so on till the last step with one admissible node (with or without child for this last step). The product is equal to $2^{b-1}3^{\frac{(b-2)(b-1)}{2}}$. The admissible nodes are different from each other because $g^*(1)$ is a tree, therefore there is a one to one function between the admissible nodes obtained with *b* steps and the tuple $\{v_1^*, v_2, ... v_b\}$ defined in the theorem.

Example.

 F												
$v1^*$	4	4	8	8	10	10	14	14	16	16	20	20
v2	3	5	2	6	1	5	4	6	1	3	2	4
v3	2	1	1	2	1	2	2	1	2	1	2	1
n	17	35	75	2417	151	4849						1242755

Table 1: The 12 admissible tuples with b = 3

Among the possible sequences $(v_2, ..., v_b)$ allowed by the Theorem 2 some are specially interesting such as the strictly ascending sequence $(f^{(i)}(n), i = 1, b - 1)$ (see n = 151 in the above table as an example), given in the following corollary.

Corollary. $\forall b \in \mathbb{N}, \ \exists n \in \mathbb{N} \ : \forall i \in (1, b - 1), \ f^{(i)}(n) > f^{(i-1)}(n).$

Proof. n is obtained with $v_i = 1$, i = 2 : b, and v_1 given by theorem 2 and lemma 6. \Box

The Wirsching-Goodwin representation of the nodes of g(1) obtained with b steps (see [2]) may be now stated in the following theorem. Let $g^*(1,b) = \{n \in g(1) : f^{(b)} = 1 \text{ and } f^{(b-1)} \neq 1\}$ and v_1^* the value of v_1 whose existence is proven in theorem 2.

Theorem 3. There is a one to one relation between $g^*(1,b)$ with b > 1 and the set of the tuples $(b, v'_1, v'_2, ..., v'_b)$ with $v'_i = v_i + 2.3^{b-i}c_i$, $c_i \in \mathbb{N}^*$, $v_i \in \mathbb{N}$, i = 2, ...b with $1 \le v_i \le 2.3^{b-i}$ and $4 \le v_1 = v_1^* \le 2.3^{b-1}$.

Proof. Direct from theorem2 and lemma 6.

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