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Research Article

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Stability result for two coefficients in a coupled hyperbolic-parabolic system

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Abstract: This work is concerned with the study of the inverse problem of determining two coefficients in a hyperbolic-parabolic system using the following observation data: an interior measurement of only one component and data of two components at a fixed time over the whole spatial domain. A Lipschitz stability result is proved using Carleman estimates.

Keywords: Inverse problems, Carleman estimates, coupled systems

MSC 2010: 35R30, 35M30, 35Q92

1 Introduction

Let Ω be a simply connected bounded domain of $\mathbb{R}^N$ with C^2 boundary $\partial\Omega$. We denote Γ the boundary of Ω which consists of an inflow part $\Gamma_- = \{x \in \partial\Omega : A(x) \cdot \nu(x) < 0\}$, an outflow part $\Gamma_+ = \{x \in \partial\Omega : A(x) \cdot \nu(x) > 0\}$ and a solid wall $\Gamma_0 = \{x \in \partial\Omega : A(x) \cdot \nu(x) = 0\}$, where $A(x)$ is a given vector field in $\mathbb{R}^N$ and $\nu(x)$ is the outer normal to $\partial\Omega$ at $x \in \partial\Omega$. Let $\omega \subset \Omega$ be a nonempty subdomain such that $\partial\omega \supset \Gamma_+$.

We shall use the following notations: for any fixed $T > 0$, we set

$$\Omega_T = \Omega \times (0, T), \quad \Sigma_T = \partial\Omega \times (0, T), \quad \Sigma_T^- = \Gamma_- \times (0, T), \quad \omega_T = \omega \times (0, T)$$

and we consider the following hyperbolic-parabolic system:

$$\begin{cases} \partial_t u + A(x) \cdot \nabla u = F(u, v) & \text{in } \Omega_T, \\ \partial_t v - \Delta v = G(u, v) & \text{in } \Omega_T, \\ u(x, t) = h(x, t) & \text{on } \Sigma_T^-, \\ v(x, t) = g(x, t) & \text{on } \Sigma_T, \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x) & \text{in } \Omega. \end{cases} \tag{1.1}$$

System (1.1) is a modified version of the system studied in [17] which arises from mathematical biology. It describes the process of tumour-induced angiogenesis. This process allows the tumour to progress from the avascular (lacking blood vessels) to the vascular (possessing a blood supply) state and is initiated and controlled by a diffusive chemical compound, known as tumour-angiogenesis factor (TAF) which is released by the tumour cells into the surrounding tissue.

We consider here a simplified of tumour-induced angiogenesis developed by Chaplain–Stuart [9]. Here $u(x, t)$ represents the cells density of the blood vessels and $v(x, t)$ is the TAF concentration. The reaction

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terms F and G are given by

$$F(u,v) = \mu(x)v - \gamma(x)u, \quad G(u,v) = \delta(x)u - k(x)v,$$

where μ, γ, δ, k are time-independent coefficients.

The cells grow by feeding on nutrient (TAF). The nutrients are consumed at a rate k. The coefficient μ denotes the influence of TAF on cell division. Cells grow rate is δ and linear loss of cells with rate γ is assumed. The aim of this work is to reconstruct the two coefficients μ and δ from an interior measurement of only one component and data of two components at a fixed time $\theta \in (0, T)$, that is, $v|_{\omega\times(0,T)}$ and $(u,v)|_{\Omega\times\{\theta\}}$. More precisely, see Theorem 5.1.

Note that it would be possible to recover all four coefficients if we repeat the observations. Nevertheless, recover the coefficients from boundary data with the observation of only one component is impossible with this method. Indeed, we could use Lemmata 4.1, 4.2 and 4.4 to obtain a Carleman estimate for our system with the observation of the two components q_1 and q_2. But there is no way to explain the L^2-norm of the normal derivative of q_1 on Γ_+ in terms of he L^2-norm of the normal derivative of q_2 on Γ_+.

The key ingredient to obtain such a result is Carleman estimates. The use of these estimates to achieve uniqueness and stability results in inverse problems is now well-established. They have been introduced by Bukhgeim and Klibanov in [8], Klibanov in [21, 22] and Fursikov and Imanuvilov in [15]. We can cite recent survey papers about Carleman estimates of Yamamoto [30] and Klibanov [23].

For parabolic equations, we refer to some works. Benabdallah, Gaitan and Le Rousseau in [7] consider the heat equation with a discontinuous diffusion coefficient and give uniqueness and stability results for both diffusion coefficient and initial condition from a measurement of the solution on an arbitrary part of the boundary and at some arbitrary positive time. Imanuvilov and Yamamoto in [19] prove global Lipschitz stability for a source term of a parabolic equation with Fourier boundary conditions using observations on an arbitrarily small sub-domain. Yuan and Yamamoto in [31] determine some coefficients of the principal part of a parabolic equation by boundary observations.

For transport equations, we can cite Machida and Yamamoto, in [27], the authors give a Lipschitz stability result on determining a time independent scattering coefficient by boundary data. Klibanov and Pamyatnykh in [24, 25], prove the Lipschitz stability estimate for the non-stationary single-speed transport equation with lateral boundary data. Gaitan and Ouzzane in [16] prove a stability result for an absorption coefficient with only one observation on a part of the boundary.

Furthermore, there are some papers devoted to inverse problems for coupled parabolic systems, we can refer to Cristofol, Gaitan and Ramoul [11], where the authors give a simultaneous stability result for one coefficient and for the initial conditions with a single observation acting on a subdomain. In [12], Cristofol, Gaitan, Ramoul and Yamamoto consider a nonlinear parabolic system with two components and prove a Lipschitz stability estimate to determine two coefficients of the system by data of only one component. Benabdallah, Cristofol, Gaitan and Yamamoto [6] give a Lipschitz stability result on determining some of the coefficients in a 2×2 and a 3×3 reaction-diffusion-convection systems.

For hyperbolic-parabolic systems, for example arising in the thermoelasticity, we can refer to Bellassoued and Yamamoto in [5]. Wu and Liu in [29], Albano and Tataru in [1].

For elasticity, we refer to Isakov and Kim in [20] and Imanuvilov, Isakov and Yamamoto in [18].

However, to our knowledge there are no results on hyperbolic-parabolic systems where the hyperbolic equation is a first order PDE. This kind of model is of interest in a lot of models arising in mathematical biology. This paper is the first step in the study of inverse problems linked to angiogenesis process.

In this work, we first establish Carleman inequalities for the system with regular weight functions. The choice of such weight functions is imposed by the transport equation. Next, we prove the stability result including energy estimates that will require a Carleman estimate for the backward system.

The outline of this paper is as follows: In Section 2, we recall some existence, uniqueness and regularity results for system (1.1). In Section 3, we give the Carleman estimates for the forward and backward system with suitable weight functions. These Carleman estimates are proved in Section 4. In Section 5, we establish the stability result through several steps.

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2 Direct problem

In this section we give some existence, uniqueness and regularity results for solutions of system (1.1).

2.1 Existence and regularity

Let us first introduce the following spaces:

$$\mathcal{W}_2 = \{u \in L^2(\Omega \times (0,T)) : \partial_t u + A \cdot \nabla u \in L^2(\Omega \times (0,T)),\ u(\cdot\,,0) \in L^2(\Omega),\ u|_{\Gamma_- \times (0,T)} \in L^2(\Gamma_- \times (0,T))\}$$

and

$$H^{2,1}(\Omega_T) = L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega)),$$
$$H^{\frac{3}{2},\frac{3}{4}}(\Sigma_T) = L^2(0,T;H^{\frac{3}{2}}(\Gamma)) \cap H^{\frac{3}{4}}(0,T;L^2(\Gamma)).$$

The first regularity result we prove is the following:

Theorem 2.1. *We assume that*
(i) $A \in (W^{1,\infty}(\Omega))^N$,
(ii) $u_0 \in L^2(\Omega)$, $v_0 \in H^1(\Omega)$,
(iii) $h \in L^2(\Sigma_T^-)$, $g \in H^{\frac{3}{2},\frac{3}{4}}(\Sigma_T)$,
(iv) $k, \delta, \gamma, \mu \in L^\infty(\Omega)$,
where the compatibility condition $g(x,0) = v_{0|\Gamma}$ *is checked. Then* (1.1) *admits an unique solution* (u,v) *such that*

$$u \in \mathcal{W}_2 \quad \text{and} \quad u \in C([0,T];L^2(\Omega)), \quad v \in H^{2,1}(\Omega_T).$$

For the proof of the theorem see [28].

We need to improve the regularity of the solutions of (1.1). For this, we consider the following.

Assumption 2.2. Assume that
(i) $A \in (W^{1,\infty}(\Omega))^N \cap (H^2(\Omega))^N$,
(ii) $k, \delta \in H^5(\Omega) \cap L^\infty(\Omega)$, $\gamma, \mu \in H^3(\Omega) \cap L^\infty(\Omega)$,
(iii) $u_0 \in H^5(\Omega)$, $v_0 \in H^7(\Omega)$,
(iv) $\partial_t^3 h \in L^2(\Sigma_T^-)$, $\partial_t^3 g \in H^{\frac{3}{2},\frac{3}{4}}(\Sigma_T)$.

Compatibility Conditions 2.3. Assume that
(i) $\partial_t h|_{t=0} + A \cdot \nabla u_0 + \gamma u_0 - \mu v_0 = 0$ on $\Gamma_- \times \{t=0\}$,
(ii) $\partial_t g|_{t=0} - \Delta v_0 + k v_0 - \delta u_0 = 0$ on $\Gamma \times \{t=0\}$,
(iii) $\partial_t h|_{t=0} + A \cdot \nabla(-A \cdot \nabla u_0 - \gamma u_0 + \mu v_0) + \gamma(-A \cdot \nabla u_0 - \gamma u_0 + \mu v_0) - \mu(\Delta v_0 - k v_0 + \delta u_0) = 0$ on $\Gamma_- \times \{t=0\}$,
(iv) $\partial_t g|_{t=0} - \Delta(\Delta v_0 - k v_0 + \delta u_0) + k(\Delta v_0 - k v_0 + \delta u_0) - \delta(-A \cdot \nabla u_0 - \gamma u_0 + \mu v_0) = 0$ on $\Gamma \times \{t=0\}$,
(v) $\partial_t h|_{t=0} + A \cdot \nabla \zeta_1 + \gamma \zeta_1 - \mu \zeta_2 = 0$ on $\Gamma_- \times \{t=0\}$,
(vi) $\partial_t g|_{t=0} - \Delta \zeta_2 + k \zeta_2 - \delta \zeta_1 = 0$ on $\Gamma \times \{t=0\}$,
where

$$\zeta_1 := -A \cdot \nabla(-A \cdot \nabla u_0 - \gamma u_0 + \mu v_0) - \gamma(-A \cdot \nabla u_0 - \gamma u_0 + \mu v_0) + \mu(\Delta v_0 - k v_0 + \delta u_0)$$

and

$$\zeta_2 := \Delta(\Delta v_0 - k v_0 + \delta u_0) - k(\Delta v_0 - k v_0 + \delta u_0) + \delta(-A \cdot \nabla u_0 - \gamma u_0 + \mu v_0).$$

Indeed, by a mixture of parabolic and transport results that can be found in [2, 13, 14], and by means of an adapted Banach fixed point approach and the Gronwall Lemma, we can prove that under Assumption 2.2 and Compatibility Conditions 2.3, the solutions u, v are such that

$$u, \partial_t u, \partial_t^2 u, \partial_t^3 u \in \mathcal{W}_2, \quad v, \partial_t v, \partial_t^2 v, \partial_t^3 v \in H^{2,1}(\Omega_T).$$

The details of the proof of these regularity results for the solutions are given in [28].

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2.2 Positivity of the solution

Assumption 2.4. Assume that
(i) $u_0 \geq 0$, $h \geq 0$ and $F(0, \eta) \geq 0$ for all $(0, \eta) \in V$, $\eta \geq 0$,
(ii) $v_0 \geq 0$, $g \geq 0$ and $G(\xi, 0) \geq 0$ for all $(\xi, 0) \in V$, $\xi \geq 0$,
where V is an open set in $\mathbb{R}^2$ such that $\{(u_0(x), v_0(x)) : x \in \overline{\Omega}\} \subset V$.

Lemma 2.5. *Under the hypothesis of Theorem 2.1, suppose that additional Assumptions 2.4 are fulfilled. Then the solutions u, v are such that $u \geq 0$ and $v \geq 0$ in $\overline{\Omega} \times (0, T)$.*

Note that these assumptions lead to sign conditions for the coefficients in the reaction terms F and G. For the proof we refer to [17].

3 Carleman estimates

In this section, we give the Carleman estimates for system (1.1). For this purpose, we shall first introduce suitable weight functions.

Assumption 3.1. Let ψ be a $C^2(\overline{\Omega_T})$ function that verifies the following properties:
(i) $\psi(x, t) > 0$ for all $(x, t) \in \Omega \times (0, T)$,
(ii) $|\nabla\psi| \geq c > 0$ for all $x \in \overline{\Omega}$,
(iii) $\partial_\nu \psi < 0$ on $\Gamma_- \times (0, T)$,
(iv) $\partial_t \psi + A(x) \cdot \nabla\psi < 0$ for all $(x, t) \in \Omega \times (0, T)$,
(v) $\partial_t \psi(x, t) < 0$ on $\Omega \times (0, T)$.

An example of such a function ψ is

$$\psi(x, t) = \alpha(x) - \beta(t) + M, \tag{3.1}$$

where M is a positive constant such that $\psi > 0$ in $\Omega \times (0, T)$ and ψ verifies

$$\sup_{x\in\Omega} \alpha(x) < \inf_{t\in(0,T)} \beta(t).$$

Note that this assumption leads to a geometrical condition. For example, if $A(x) = x$, $\alpha(x) = |x - x_0|^2$ with $x_0 \in \mathbb{R}^N \setminus \overline{\Omega}$, then we can take $\beta(t) = e^{d(t-\frac{T}{2})}$, $d > 0$ and the geometrical condition is

$$T > \frac{2}{d} \sup_{x\in\Omega} \ln |x - x_0|^2.$$

Let $\lambda > 0$ be a parameter; we then define the weight function $\varphi(x, t)$ by

$$\varphi(x, t) = e^{\lambda\psi(x,t)}. \tag{3.2}$$

We point out that the choice of such a weight function leads to an observation acting on a part Γ_+ of the boundary Γ on the right-hand side of the estimate. We then derive estimates with observations in a subdomain ω of Ω such that $\partial\omega \supset \Gamma_+$.

Throughout this paper, we shall use the following notations:

$$I_1(q, \Omega_T) = \int_{\Omega_T} s\lambda^2\varphi|q|^2 e^{2s\varphi}\, dx\, dt,$$

$$I_2(q, \Omega_T) = \int_{\Omega_T} (s\varphi)^{-1}(|\partial_t q|^2 + |\Delta q|^2)e^{2s\varphi}\, dx\, dt + \int_{\Omega_T} s\lambda^2\varphi|\nabla q|^2 e^{2s\varphi}\, dx\, dt + \int_{\Omega_T} s^3\lambda^4\varphi^3|q|^2 e^{2s\varphi}\, dx\, dt.$$

Now, let us state our Carleman estimates for both forward and backward system.

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Theorem 3.2. *Let ψ and φ be defined by* (3.1) *and* (3.2), *respectively. There exist s_0, λ_0 and a positive constant $C = C(s_0, \lambda_0, \Omega, T, \omega)$ such that*

$$I_1(q_1, \Omega_T) + I_2(q_2, \Omega_T) \le C \int_{\Omega_T} |Lq_1|^2 e^{2s\varphi}\, dx\, dt + C \int_{\Omega_T} |Pq_2|^2 e^{2s\varphi}\, dx\, dt + Cs^4\lambda^4 \int_{\omega_T} \varphi^4 |q_2|^2 e^{2s\varphi}\, dx\, dt + C \int_{\omega_T} |\partial_t q_2|^2 e^{2s\varphi}\, dx\, dt$$

for all $s > s_0$, $\lambda > \lambda_0$ and all q_1, q_2 satisfying

$$\begin{cases} Lq_1 := \partial_t q_1 + A(x)\cdot\nabla q_1 \in L^2(\Omega\times(0,T)), & q_1 \in L^2(\Omega\times(0,T)),\ \ q_1|_{\Gamma_-} = 0, \quad q_1(\cdot\,,0) = q_1(\cdot\,,T) = 0, \\ Pq_2 := \partial_t q_2 - \Delta q_2 \in L^2(\Omega\times(0,T)), & q_2 \in L^2((0,T); H_0^1(\Omega)\cap H^2(\Omega)), \quad q_2(\cdot\,,0) = q_2(\cdot\,,T) = 0. \end{cases}$$

Theorem 3.3. *Let ψ and φ defined by* (3.1) *and* (3.2), *respectively. There exist s_0, λ_0 and a positive constant $C = C(s_0, \lambda_0, \Omega, T, \omega)$ such that*

$$\begin{aligned} I_1(q_1, \Omega_T) + I_2(q_2, \Omega_T) \le{}& C \int_{\Omega_T} |L_{\mathrm{bw}}\, q_1|^2 e^{2s\varphi}\, dx\, dt + C \int_{\Omega_T} |P_{\mathrm{bw}}\, q_2|^2 e^{2s\varphi}\, dx\, dt \\ &+ Cs^4\lambda^4 \int_{\omega_T} \varphi^4 |q_2|^2 e^{2s\varphi}\, dx\, dt + C \int_{\omega_T} |\partial_t q_2|^2 e^{2s\varphi}\, dx\, dt \\ &+ Cs^2\lambda^2 \int_{\omega_T} \varphi^2 |q_1|^2 e^{2s\varphi}\, dx\, dt + C \int_{\omega_T} \varphi |A(x)\cdot\nabla q_1|^2 e^{2s\varphi}\, dx\, dt, \end{aligned}$$

for all $s > s_0$, $\lambda > \lambda_0$ and all q_1, q_2 satisfying

$$\begin{cases} L_{\mathrm{bw}} q_1 := -\partial_t q_1 + A(x)\cdot\nabla q_1 \in L^2(\Omega\times(0,T)), & q_1 \in L^2(\Omega\times(0,T)),\ \ q_1|_{\Gamma_-} = 0, \quad q_1(\cdot\,,0) = q_1(\cdot\,,T) = 0, \\ P_{\mathrm{bw}} q_2 := -\partial_t q_2 - \Delta q_2 \in L^2(\Omega\times(0,T)), & q_2 \in L^2((0,T); H_0^1(\Omega)\cap H^2(\Omega)), \quad q_2(\cdot\,,0) = q_2(\cdot\,,T) = 0. \end{cases}$$

4 Proof of Theorem 3.2 and Theorem 3.3

To prove Theorem 3.2, we first derive two Carleman estimates, one associated to the transport operator and the other one to the parabolic operator using the same weight function. Note that we obtain Carleman estimates with observations acting on Γ_+. Then we derive from the previous inequalities Carleman estimates with localized observations on a subdomain ω. These two previous estimates allow us to obtain a Carleman inequality for the system with the observation of only one component on ω. The proof of Theorem 3.3 is similar to the proof of Theorem 3.2.

4.1 Carleman inequalities associated to the transport operator

In the two following lemmata, we state Carleman estimates for both forward and backward transport operators.

Lemma 4.1. *Let ψ and φ be defined by* (3.1) *and* (3.2), *respectively. There exist s_0, λ_0 and a positive constant $C = C(s_0, \lambda_0, \Omega, T, \Gamma)$ such that*

$$I_1(q_1, \Omega_T) \le C \int_{\Omega_T} |Lq_1|^2 e^{2s\varphi}\, dx\, dt$$

for all $s > s_0$, $\lambda > \lambda_0$ and all q_1 satisfying

$$Lq_1 := \partial_t q_1 + A(x)\cdot\nabla q_1 \in L^2(\Omega\times(0,T)), \quad q_1 \in L^2(\Omega\times(0,T)),\ q_1|_{\Gamma_-} = 0,\ q_1(\cdot\,,0) = q_1(\cdot\,,T) = 0.$$

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Lemma 4.2. *Let ψ and φ be defined by* (3.1) *and* (3.2)*, respectively. There exist s_0, λ_0 and a positive constant $C = C(s_0, \lambda_0, \Omega, T, \Gamma)$ such that*

$$I_1(q_1, \Omega_T) \le C \int_{\Omega_T} |L_{\mathrm{bw}} q_1|^2 e^{2s\varphi}\, dx\, dt + Cs\lambda \int_0^T \int_{\Gamma_+} \varphi |q_1|^2 A(x) \cdot \nu e^{2s\varphi}\, d\sigma\, dt \tag{4.1}$$

for all $s > s_0$, $\lambda > \lambda_0$ and all q_1 satisfying

$$L_{\mathrm{bw}} q_1 := -\partial_t q_1 + A(x) \cdot \nabla q_1 \in L^2(\Omega \times (0, T)), \quad q_1 \in L^2(\Omega \times (0, T)),\ q_1|_{\Gamma_-} = 0,\ q_1(\cdot, 0) = q_1(\cdot, T) = 0.$$

For the proof of these lemmata, we use the same ideas as in [16].

Next, using Lemma 4.2, we prove for the backward operator a Carleman estimate with a single observation acting on a subdomain ω such that $\partial\omega \supset \Gamma_+$.

Lemma 4.3. *Let ψ and φ be defined by* (3.1) *and* (3.2)*, respectively. There exist s_0, λ_0 and a positive constant $C = C(s_0, \lambda_0, \Omega, T, \omega)$ such that*

$$I_1(q_1, \Omega_T) \le C \int_{\Omega_T} |L_{\mathrm{bw}}\, q_1|^2 e^{2s\varphi}\, dx\, dt + Cs^2\lambda^2 \int_{\omega_T} \varphi^2 |q_1|^2 e^{2s\varphi}\, dx\, dt + C \int_{\omega_T} \varphi |A(x) \cdot \nabla q_1|^2 e^{2s\varphi}\, dx\, dt$$

for all $s > s_0$, $\lambda > \lambda_0$ and all q_1 satisfying

$$L_{\mathrm{bw}}\, q_1 := -\partial_t q_1 + A(x) \cdot \nabla q_1 \in L^2(\Omega \times (0, T)), \quad q_1 \in L^2(\Omega \times (0, T)),\ q_1|_{\Gamma_-} = 0,\ q_1(\cdot, 0) = q_1(\cdot, T) = 0.$$

Proof of Lemma 4.3. We choose a function $\rho \in C^2(\overline{\Omega}; [0, 1])$ satisfying

$$\begin{cases} \rho(x) = 1, & x \in \omega', \\ \rho(x) = 0, & x \in \overline{\Omega} \setminus \omega, \end{cases} \tag{4.2}$$

where $\omega' \subset \omega$ and $\partial\omega' \supset \Gamma_+$. An integration by parts gives

$$\begin{aligned} s\lambda \int_0^T \int_{\Gamma_+} \rho\varphi |q_1|^2 A(x) \cdot \nu e^{2s\varphi}\, d\sigma\, dt &= s\lambda \int_{\omega_T} \rho\varphi A(x) \cdot \nabla(|q_1|^2) e^{2s\varphi}\, dx\, dt + s\lambda \int_{\omega_T} \nabla \cdot (A(x)\rho\varphi e^{2s\varphi}) |q_1|^2\, dx\, dt \\ &=: Q_1 + Q_2. \end{aligned}$$

Using (4.2), an integration by parts and Young's inequality, we obtain

$$Q_2 \le Cs^2\lambda^2 \int_{\omega_T} \varphi\rho |q_1|^2 e^{2s\varphi}\, dx\, dt + C \int_{\omega_T} \varphi\rho |A(x) \cdot \nabla q_1|^2 e^{2s\varphi}\, dx\, dt, \tag{4.3}$$

and

$$\begin{aligned} Q_1 &= s\lambda \int_{\omega_T} (\lambda\rho\varphi\nabla\psi \cdot A(x) + \varphi\nabla\rho \cdot A(x) + \rho\varphi\nabla \cdot A(x)) |q_1|^2 e^{2s\varphi}\, dx\, dt \\ &\quad + 2s^2\lambda^2 \int_{\omega_T} \varphi^2 A(x) \cdot \nabla\psi |q_1|^2 e^{2s\varphi}\, dx\, dt \\ &\le Cs^2\lambda^2 \int_{\omega_T} \varphi^2 |q_1|^2 e^{2s\varphi}\, dx\, dt. \end{aligned} \tag{4.4}$$

Therefore, from (4.3) and (4.4), it follows

$$s\lambda \int_0^T \int_{\Gamma_+} \varphi |q_1|^2 A(x) \cdot \nu e^{2s\varphi}\, d\sigma\, dt \le Cs^2\lambda^2 \int_{\omega_T} \varphi^2 |q_1|^2 e^{2s\varphi}\, dx\, dt + C \int_{\omega_T} \varphi |A(x) \cdot \nabla q_1|^2 e^{2s\varphi}\, dx\, dt.$$

Using this last inequality in (4.1) and the definition of Lq_1, the proof of Lemma 4.3 is completed. □

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4.2 Carleman inequalities associated to the parabolic operator

In this subsection, we recall the general form of the Carleman estimate associated to the operator

$$Pq = \pm\partial_t q - \Delta q,$$

see [30, 31].

Lemma 4.4. *Let ψ and φ be defined by* (3.1) *and* (3.2)*, respectively. There exist s_0, λ_0 and a positive constant $C = C(s_0, \lambda_0, \Omega, T, \Gamma)$ such that*

$$I_2(q_2, \Omega_T) \le C \int_{\Omega_T} |Pq_2|^2 e^{2s\varphi}\, dx\, dt + C \int_0^T \int_{\Gamma_+} s\lambda\varphi |\partial_\nu q_2|^2 e^{2s\varphi}\, d\sigma\, dt \tag{4.5}$$

for all $s > s_0$, $\lambda > \lambda_0$ and all q_2 satisfying

$$Pq_2 = \pm\partial_t q_2 - \Delta q_2 \in L^2(\Omega \times (0, T)), \quad q_2 \in L^2((0, T); H_0^1(\Omega) \cap H^2(\Omega)),\ q_2(\cdot, 0) = q_2(\cdot, T) = 0.$$

Now, using the previous lemma, we prove a Carleman estimate with a single observation acting on a subdomain ω, such that $\partial\omega \supset \Gamma_+$.

Lemma 4.5. *Let ψ and φ be defined by* (3.1) *and* (3.2)*, respectively. There exist s_0, λ_0 and a positive constant $C = C(s_0, \lambda_0, \Omega, T, \omega)$ such that*

$$I_2(q_2, \Omega_T) \le C \int_{\Omega_T} |P\, q_2|^2 e^{2s\varphi}\, dx\, dt + C \int_{\omega_T} |\Delta q_2|^2 e^{2s\varphi}\, dx\, dt + Cs^2\lambda^2 \int_{\omega_T} \varphi^2 |\nabla q_2|^2 e^{2s\varphi}\, dx\, dt$$

for all $s > s_0$, $\lambda > \lambda_0$ and all q_2 satisfying

$$P\, q_2 = \pm\partial_t q_2 - \Delta q_2 \in L^2(\Omega \times (0, T)), \quad q_2 \in L^2((0, T); H_0^1(\Omega) \cap H^2(\Omega)),\ q_2(\cdot, 0) = q_2(\cdot, T) = 0.$$

The idea of the proof is taken from [29].

Proof of Lemma 4.5. Let us consider the function $g_0 \in C^1(\overline{\Omega}; \mathbb{R}^n)$ such that

$$g_0(x) = \nu(x) \quad \text{on } \Gamma_+ \tag{4.6}$$

and the function ρ defined in (4.2). We have

$$\begin{aligned} s\lambda \int_0^T \int_{\Gamma_0} \varphi\rho |\partial_\nu q_2|^2 e^{2s\varphi}\, d\sigma\, dt &= s\lambda \int_{\omega_T} \varphi\rho(\nabla q_2 \cdot g_0) e^{2s\varphi} \Delta q_2\ dx\, dt + s\lambda \int_{\omega_T} \nabla(\varphi\rho(\nabla q_2 \cdot g_0) e^{2s\varphi}) \cdot \nabla q_2\, dx\, dt \\ &=: Q_1 + Q_2. \end{aligned}$$

Using the properties of the cut-off function ρ (see (4.2)), the definition (4.6) of g_0 with Young's inequality, we estimate Q_1 and Q_2. Note that we have

$$|s\lambda\rho\varphi(\nabla q_2 \cdot g_0) e^{s\varphi}| \cdot |e^{s\varphi}\Delta q_2| \le Cs^2\lambda^2\varphi^2 |\nabla q_2|^2 e^{2s\varphi} + C|\Delta q_2|^2 e^{2s\varphi}.$$

Then

$$Q_1 \le Cs^2\lambda^2 \int_{\omega_T} \varphi^2 |\nabla q_2|^2 e^{2s\varphi}\, dx\, dt + C \int_{\omega_T} |\Delta q_2|^2 e^{2s\varphi}\, dx\, dt \tag{4.7}$$

and

$$\begin{aligned} Q_2 &= s\lambda \int_{\omega_T} (\nabla q_2 \cdot g_0)(\rho\lambda\varphi\nabla\psi + \varphi\nabla\rho + 2s\lambda\rho\varphi^2\nabla\psi) \cdot \nabla q_2 e^{2s\varphi}\, dx\, dt \\ &\quad + s\lambda \int_{\omega_T} \varphi\rho\nabla(\nabla q_2 \cdot g_0) \cdot \nabla q_2 e^{2s\varphi}\, dx\, dt \\ &\le Cs^2\lambda^2 \int_{\omega_T} \varphi^2 |\nabla q_2|^2 e^{2s\varphi}\, dx\, dt. \end{aligned} \tag{4.8}$$

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Thus, from (4.7) and (4.8), we obtain

$$s\lambda \int_0^T \int_{\Gamma_+} \varphi |\partial_\nu q_2|^2 e^{2s\varphi}\, d\sigma\, dt \le Cs^2\lambda^2 \int_{\omega_T} \varphi^2 |\nabla q_2|^2 e^{2s\varphi}\, dx\, dt + C \int_{\omega_T} |\Delta q_2|^2 e^{2s\varphi}\, dx\, dt.$$

So, using this last inequality in (4.5), we conclude the proof. □

The result of Lemma 4.5 can be improved.

Lemma 4.6. *Let ψ and φ be defined by* (3.1) *and* (3.2)*, respectively. There exist s_0, λ_0 and a positive constant $C = C(s_0, \lambda_0, \Omega, T, \omega)$ such that*

$$I_2(q_2, \Omega_T) \le C \int_{\Omega_T} |P q_2|^2 e^{2s\varphi}\, dx\, dt + C \int_{\omega_T} |\partial_t q_2|^2 e^{2s\varphi}\, dx\, dt + Cs^4\lambda^4 \int_{\omega_T} \varphi^4 |q_2|^2 e^{2s\varphi}\, dx\, dt$$

for all $s > s_0$, $\lambda > \lambda_0$ and all q_2 satisfying

$$P q_2 = \pm\partial_t q_2 - \Delta q_2 \in L^2(\Omega \times (0, T)), \quad q_2 \in L^2((0, T); H_0^1(\Omega) \cap H^2(\Omega)),\ q_2(\cdot, 0) = q_2(\cdot, T) = 0.$$

Proof of Lemma 4.6. The idea are the same as those given in [15]. The argument is a local regularity of the parabolic equation. Explicit computation using integrations by parts leads to the following estimates:

$$s^2\lambda^2 \int_{\omega_T} \varphi^2 |\nabla q_2|^2 e^{2s\varphi}\, dx\, dt \le C \int_{\Omega_T} |P q_2|^2 e^{2s\varphi}\, dx\, dt + Cs^4\lambda^4 \int_{\omega_T} \varphi^4 |q_2|^2 e^{2s\varphi}\, dx\, dt$$

and

$$\int_{\omega_T} \varphi^2 |\Delta q_2|^2 e^{2s\varphi}\, dx\, dt \le C \int_{\Omega_T} |P q_2|^2 e^{2s\varphi}\, dx\, dt + C \int_{\omega_T} |\partial_t q_2|^2 e^{2s\varphi}\, dx\, dt.$$

Then by Lemma 4.5 and the two previous estimates, the proof is completed. □

Finally, to prove Theorem 3.2 (respectively Theorem 3.3), we add up the estimates of Lemma 4.1 and Lemma 4.6 (respectively the estimates of Lemma 4.3 and 4.6).

5 The stability result

In this section, we apply the Carleman inequality of Theorem 3.2 in order to prove the following stability result.

Theorem 5.1. *Let ω be a subdomain in Ω satisfying $\partial\omega \supset \Gamma_+$. We assume that Assumption 2.2, Compatibility Conditions 2.3, Assumption 2.4 and Assumption 3.1 are checked. Let (u, v) (respectively $(\tilde{u}, \tilde{v})$) be a solution of system* (1.1) *associated to $(\mu, \gamma, \delta, k, u_0, v_0, h, g)$ (respectively $(\tilde{\mu}, \gamma, \tilde{\delta}, k, u_0, v_0, h, g)$). Then, there exists a positive constant $C = C(s_0, \lambda_0, \Omega, T, \omega)$ such that*

$$\begin{aligned}\|\mu - \tilde{\mu}\|^2_{L^2(\Omega)} + \|\delta - \tilde{\delta}\|^2_{L^2(\Omega)} \le{}& C\|v - \tilde{v}\|^2_{H^2((0,T);L^2(\omega))} + C\big(\|\Delta(v - \tilde{v})(x, \theta)\|^2_{L^2(\Omega)} + \|(v - \tilde{v})(x, \theta)\|^2_{L^2(\Omega)} \\ &+ \|(u - \tilde{u})(x, \theta)\|^2_{L^2(\Omega)} + \|A(x) \cdot \nabla(u - \tilde{u})(x, \theta)\|^2_{L^2(\Omega)} + \|\partial_t(u - \tilde{u})(x, \theta)\|^2_{L^2(\omega)}\big).\end{aligned}$$

Proof. The proof will be done in several steps.

Step 1: Linearization and differentiation in time. Let us set $U = u - \tilde{u}$, $V = v - \tilde{v}$, $Y = \partial_t U$ and $Z = \partial_t V$. Then Y and Z are solutions of

$$\begin{cases} \partial_t Y + A(x) \cdot \nabla Y = (\mu - \tilde{\mu})\partial_t \tilde{v} + \mu Z - \gamma Y & \text{in } \Omega_T, \\ \partial_t Z - \Delta Z = (\delta - \tilde{\delta})\partial_t \tilde{u} + \delta Y - kZ & \text{in } \Omega_T, \\ Y(x, t) = 0 & \text{on } \Sigma_T^-, \\ Z(x, t) = 0 & \text{on } \Sigma_T, \\ Y(x, 0) = (\mu - \tilde{\mu})v_0(x), \quad Z(x, 0) = (\delta - \tilde{\delta})u_0(x) & \text{in } \Omega. \end{cases} \tag{5.1}$$

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Step 2: Application of the Carleman estimate of Theorem 3.2. We choose $\eta \in (0, T)$ and we introduce a cut-off function $\chi \in C_c^\infty(\mathbb{R})$ such that $0 \le \chi \le 1$. we define

$$\chi(t) = \begin{cases} 1 & \text{if } \eta \le t \le T - \eta, \\ 0 & \text{if } t \le 0 \text{ or } t \ge T. \end{cases}$$

We set $\widetilde{Y} = \chi Y$ and $\widetilde{Z} = \chi Z$ in $\Omega \times (0, T)$. Thus, $\widetilde{Y}$ and $\widetilde{Z}$ satisfy the hypothesis

$$\widetilde{Y}(\cdot, 0) = \widetilde{Y}(\cdot, T) = \widetilde{Z}(\cdot, 0) = \widetilde{Z}(\cdot, T) = 0 \quad \text{in } \Omega.$$

Therefore, we can apply the Carleman inequality of Theorem 3.2 to $\widetilde{Y}$ and $\widetilde{Z}$; it follows that

$$\begin{aligned} I_1(\widetilde{Y}, \Omega_T) + I_2(\widetilde{Z}, \Omega_T) \le{}& C \int_{\Omega_T} |L\widetilde{Y}|^2 e^{2s\varphi}\, dx\, dt + C \int_{\Omega_T} |P\widetilde{Z}|^2 e^{2s\varphi}\, dx\, dt \\ &+ Cs^4\lambda^4 \int_{\omega_T} \varphi^4 |\widetilde{Z}|^2 e^{2s\varphi}\, dx\, dt + C \int_{\omega_T} |\partial_t Z|^2 e^{2s\varphi}\, dx\, dt, \end{aligned} \tag{5.2}$$

where

$$L\widetilde{Y} = \chi LY + Y\partial_t\chi \quad \text{and} \quad P\widetilde{Z} = \chi PZ + Z\partial_t\chi,$$

in $\Omega \times (0, T)$.

We set $\Omega_\eta = \Omega \times (\eta, T - \eta)$. Since $\partial_t\chi$ has a compact support in $(0, \eta) \cup (T - \eta, T)$, we deduce from (5.2) the following inequality:

$$\begin{aligned} I_1(Y, \Omega_\eta) + I_2(Z, \Omega_\eta) \le{}& C \int_{\Omega_T} |LY|^2 e^{2s\varphi}\, dx\, dt + C \int_{\Omega_T} |PZ|^2 e^{2s\varphi}\, dx\, dt + C \int_0^\eta \int_\Omega |Y|^2 e^{2s\varphi}\, dx\, dt \\ &+ C \int_{T-\eta}^T \int_\Omega |Y|^2 e^{2s\varphi}\, dx\, dt + C \int_0^\eta \int_\Omega |Z|^2 e^{2s\varphi}\, dx\, dt + C \int_{T-\eta}^T \int_\Omega |Z|^2 e^{2s\varphi}\, dx\, dt \\ &+ Cs^4\lambda^4 \int_{\omega_T} \varphi^4 |Z|^2 e^{2s\varphi}\, dx\, dt + C \int_{\omega_T} |\partial_t Z|^2 e^{2s\varphi}\, dx\, dt. \end{aligned} \tag{5.3}$$

In the sequel, we fix $\lambda = \lambda_0$ and use the fact that φ is bounded from below by 1 and from above by a constant depending on λ. We use the following notations:

$$J_1(q, \Omega_T) = \int_{\Omega_T} s|q|^2 e^{2s\varphi}\, dx\, dt,$$

$$J_2(q, \Omega_T) = \int_{\Omega_T} (s)^{-1}(|\partial_t q|^2 + |\Delta q|^2) e^{2s\varphi}\, dx\, dt + \int_{\Omega_T} s|\nabla q|^2 e^{2s\varphi}\, dx\, dt + \int_{\Omega_T} s^3 |q|^2 e^{2s\varphi}\, dx\, dt.$$

Then, from (5.3), we obtain

$$\begin{aligned} J_1(Y, \Omega_\eta) + J_2(Z, \Omega_\eta) \le{}& C \int_{\Omega_T} |LY|^2 e^{2s\varphi}\, dx\, dt + C \int_{\Omega_T} |PZ|^2 e^{2s\varphi}\, dx\, dt + Cs^4 \int_{\omega_T} |Z|^2 e^{2s\varphi}\, dx\, dt \\ &+ C \int_{\omega_T} |\partial_t Z|^2 e^{2s\varphi}\, dx\, dt + C \int_0^\eta \int_\Omega |Y|^2 e^{2s\varphi}\, dx\, dt + C \int_{T-\eta}^T \int_\Omega |Y|^2 e^{2s\varphi}\, dx\, dt \\ &+ C \int_0^\eta \int_\Omega |Z|^2 e^{2s\varphi}\, dx\, dt + C \int_{T-\eta}^T \int_\Omega |Z|^2 e^{2s\varphi}\, dx\, dt. \end{aligned} \tag{5.4}$$

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Step 3: Energy estimates. We want to give an estimation of the last fourth integrals of the right-hand side of (5.4). Let us denote

$$\begin{aligned} \mathcal{M}_1 &:= \int_0^\eta \int_\Omega |Y|^2 e^{2s\varphi}\,dx\,dt, \quad \mathcal{M}_2 := \int_{T-\eta}^T \int_\Omega |Y|^2 e^{2s\varphi}\,dx\,dt, \\ \mathcal{M}_3 &:= \int_0^\eta \int_\Omega |Z|^2 e^{2s\varphi}\,dx\,dt, \quad \mathcal{M}_4 := \int_{T-\eta}^T \int_\Omega |Z|^2 e^{2s\varphi}\,dx\,dt. \end{aligned} \tag{5.5}$$

The aim is to absorb $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$ and $\mathcal{M}_4$ by the terms of the left-hand side in inequality (5.4). For this purpose, we introduce the following weighted energies:

$$\begin{aligned} E_1(t) &= \frac{1}{2}\int_\Omega |Y|^2 e^{2s\varphi}\,dx, \\ E_2(t) &= \frac{1}{2}\int_\Omega |Z|^2 e^{2s\varphi}\,dx. \end{aligned}$$

The following lemma gives an estimation of $\mathcal{M}_2$ and $\mathcal{M}_4$:

Lemma 5.2. *Let $\mathcal{M}_2$ and $\mathcal{M}_4$ be defined by* (5.5)*. Then, we have the following estimates:*

$$\begin{aligned} \mathcal{M}_2 &\le C \int_\eta^{T-\eta} \int_\Omega |Y|^2 e^{2s\varphi}\,dx\,dt + \frac{C}{s}\int_0^T \int_\Omega |LY|^2 e^{2s\varphi}\,dx\,dt, \\ \mathcal{M}_4 &\le C \int_\eta^{T-\eta} \int_\Omega |Z|^2 e^{2s\varphi}\,dx\,dt + \frac{C}{s}\int_0^T \int_\Omega |PZ|^2 e^{2s\varphi}\,dx\,dt. \end{aligned}$$

Proof. The proof is based on weighted energy estimates. Such estimates have been introduced in [3] for the wave equation in a bounded domain. It is given in Appendix A. The main tools are integration by parts, the Gronwall Lemma and Young's inequality. □

Let $t \in (0, \eta)$. We make the change of variables $t \to T - t$ and we introduce

$$\begin{aligned} Y_{\mathrm{bw}}(x, t) &= Y(x, T - t), \\ Z_{\mathrm{bw}}(x, t) &= Z(x, T - t). \end{aligned}$$

Note that Y_{bw} and Z_{bw} satisfy the backward system associated to (5.1), where

$$\begin{aligned} L_{\mathrm{bw}} Y_{\mathrm{bw}} &:= -\partial_t Y_{\mathrm{bw}} + A(x)\cdot\nabla Y_{\mathrm{bw}} \\ P_{\mathrm{bw}} Z_{\mathrm{bw}} &:= -\partial_t Z_{\mathrm{bw}} - \Delta Z_{\mathrm{bw}}. \end{aligned}$$

Lemma 5.3. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be defined by* (5.5)*. Then, we have the following estimates:*

$$\begin{aligned} \mathcal{M}_1 &\le C \int_\eta^{T-\eta} \int_\Omega |Y_{\mathrm{bw}}|^2 e^{2s\varphi}\,dx\,dt + \frac{C}{s}\int_{\Omega_T} |L_{\mathrm{bw}} Y_{\mathrm{bw}}|^2 e^{2s\varphi}\,dx\,dt, \\ \mathcal{M}_3 &\le C \int_\eta^{T-\eta} \int_\Omega |Z_{\mathrm{bw}}|^2 e^{2s\varphi}\,dx\,dt + \frac{C}{s}\int_{\Omega_T} |P_{\mathrm{bw}} Z_{\mathrm{bw}}|^2 e^{2s\varphi}\,dx\,dt. \end{aligned}$$

Proof. We set

$$\begin{aligned} \widetilde{Y}_{\mathrm{bw}} &= \chi Y_{\mathrm{bw}}, \\ \widetilde{Z}_{\mathrm{bw}} &= \chi Z_{\mathrm{bw}} \end{aligned}$$

in $\Omega \times (0, T)$ and we apply the Carleman estimate of Theorem 3.3 to $\widetilde{Y}_{\mathrm{bw}}$ and $\widetilde{Z}_{\mathrm{bw}}$. As for Y and Z, we will

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obtain the following estimate:

$$\begin{aligned}
J_1(Y_{\mathrm{bw}}, \Omega_\eta) + J_2(Z_{\mathrm{bw}}, \Omega_\eta) \le{}& C \int_{\Omega_T} |L_{\mathrm{bw}} Y_{\mathrm{bw}}|^2 e^{2s\varphi}\, dx\, dt + C \int_{\Omega_T} |P_{\mathrm{bw}} Z_{\mathrm{bw}}|^2 e^{2s\varphi}\, dx\, dt + C \int_0^\eta \int_\Omega |Y_{\mathrm{bw}}|^2 e^{2s\varphi}\, dx\, dt \\
&+ C \int_{T-\eta}^{T} \int_\Omega |Y_{\mathrm{bw}}|^2 e^{2s\varphi}\, dx\, dt + C \int_0^\eta \int_\Omega |Z_{\mathrm{bw}}|^2 e^{2s\varphi}\, dx\, dt \\
&+ C \int_{T-\eta}^{T} \int_\Omega |Z_{\mathrm{bw}}|^2 e^{2s\varphi}\, dx\, dt + Cs^4 \int_{\omega_T} |Z_{\mathrm{bw}}|^2 e^{2s\varphi}\, dx\, dt \\
&+ C \int_{\omega_T} |\partial_t Z_{\mathrm{bw}}|^2 e^{2s\varphi}\, dx\, dt + Cs^2 \int_{\omega_T} |Y_{\mathrm{bw}}|^2 e^{2s\varphi}\, dx\, dt \\
&+ C \int_{\omega_T} |A(x) \cdot \nabla Y_{\mathrm{bw}}|^2 e^{2s\varphi}\, dx\, dt. \qquad (5.6)
\end{aligned}$$

Let us define $\mathcal{M}_{1,\mathrm{bw}}$, $\mathcal{M}_{2,\mathrm{bw}}$, $\mathcal{M}_{3,\mathrm{bw}}$, $\mathcal{M}_{4,\mathrm{bw}}$ as follows:

$$\mathcal{M}_{1,\mathrm{bw}} = \int_0^\eta \int_\Omega |Y_{\mathrm{bw}}|^2 e^{2s\varphi}\, dx\, dt, \quad \mathcal{M}_{2,\mathrm{bw}} = \int_{T-\eta}^{T} \int_\Omega |Y_{\mathrm{bw}}|^2 e^{2s\varphi}\, dx\, dt,$$

$$\mathcal{M}_{3,\mathrm{bw}} = \int_0^\eta \int_\Omega |Z_{\mathrm{bw}}|^2 e^{2s\varphi}\, dx\, dt, \quad \mathcal{M}_{4,\mathrm{bw}} = \int_{T-\eta}^{T} \int_\Omega |Z_{\mathrm{bw}}|^2 e^{2s\varphi}\, dx\, dt.$$

If we set $E_{1,\mathrm{bw}}(t) = \frac{1}{2} \int_\Omega |Y_{\mathrm{bw}}|^2 e^{2s\varphi}\, dx$, then as for E_1 we find

$$\begin{aligned}
\frac{dE_{1,\mathrm{bw}}}{dt} - s \int_\Omega (\partial_t \varphi + \nabla\varphi \cdot A(x)) |Y_{\mathrm{bw}}|^2 e^{2s\varphi}\, dx + \frac{1}{2} \int_{\Gamma_+} A(x) \cdot \nu |Y_{\mathrm{bw}}|^2 e^{2s\varphi}\, d\sigma \\
= \int_\Omega Y_{\mathrm{bw}} L_{\mathrm{bw}} Y_{\mathrm{bw}} e^{2s\varphi}\, dx + \frac{1}{2} \int_\Omega \nabla \cdot A(x) |Y_{\mathrm{bw}}|^2 e^{2s\varphi}\, dx.
\end{aligned}$$

From (iv) of Assumption 3.1, for all $s > 0$ large enough, we obtain

$$\frac{dE_{1,\mathrm{bw}}}{dt} + sc \int_\Omega |Y_{\mathrm{bw}}|^2 e^{2s\varphi}\, dx \le \int_\Omega Y_{\mathrm{bw}} L_{\mathrm{bw}} Y_{\mathrm{bw}} e^{2s\varphi}\, dx.$$

Remark 5.4. Note that, in fact, the change of variables $t \to T - t$ requires to do all the estimations with $\varphi(\cdot, T - t)$.

First, we estimate $\mathcal{M}_{2,\mathrm{bw}}$ (resp. $\mathcal{M}_{4,\mathrm{bw}}$) in the same way as for $\mathcal{M}_2$ (resp. $\mathcal{M}_4$) and we find

$$\mathcal{M}_{2,\mathrm{bw}} \le C \int_\eta^{T-\eta} \int_\Omega |Y_{\mathrm{bw}}|^2 e^{2s\varphi}\, dx\, dt + \frac{C}{s} \int_{\Omega_T} |L_{\mathrm{bw}}\, Y_{\mathrm{bw}}|^2 e^{2s\varphi}\, dx\, dt,$$

$$\mathcal{M}_{4,\mathrm{bw}} \le C \int_\eta^{T-\eta} \int_\Omega |Z_{\mathrm{bw}}|^2 e^{2s\varphi}\, dx\, dt + \frac{C}{s} \int_{\Omega_T} |P_{\mathrm{bw}}\, Z_{\mathrm{bw}}|^2 e^{2s\varphi}\, dx\, dt.$$

Note that $\mathcal{M}_1 = \mathcal{M}_{2,\mathrm{bw}}$ and $\mathcal{M}_2 = \mathcal{M}_{1,\mathrm{bw}}$, so we deduce the following estimates:

$$\mathcal{M}_1 \le C \int_\eta^{T-\eta} \int_\Omega |Y_{\mathrm{bw}}|^2 e^{2s\varphi}\, dx\, dt + \frac{C}{s} \int_{\Omega_T} |L_{\mathrm{bw}} Y_{\mathrm{bw}}|^2 e^{2s\varphi}\, dx\, dt,$$

$$\mathcal{M}_3 \le C \int_\eta^{T-\eta} \int_\Omega |Z_{\mathrm{bw}}|^2 e^{2s\varphi}\, dx\, dt + \frac{C}{s} \int_{\Omega_T} |P_{\mathrm{bw}} Z_{\mathrm{bw}}|^2 e^{2s\varphi}\, dx\, dt. \qquad \square$$

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Step 4: Carleman estimate with two observations. In this step, we will prove the following Carleman estimate with two observations by means of Lemma 5.2 and Lemma 5.3.

Proposition 5.5. *Let ω be a subdomain of Ω such that $\partial\omega \supset \Gamma_+$. Suppose that Assumptions 2.2, 3.1 and Compatibility Conditions 2.3 are checked. Then there exist s_0 and a positive constant $C = C(s_0, \lambda_0, \Omega, T, \omega)$ such that for all $s > s_0$,*

$$\begin{aligned} s\int_{\Omega_T} |Y|^2 e^{2s\varphi}\,dx\,dt + s\int_{\Omega_T} |Z|^2 e^{2s\varphi}\,dx\,dt \le{}& C\int_{\Omega_T} |\mu-\widetilde{\mu}|^2|\partial_t\widetilde{v}|^2 e^{2s\varphi}\,dx\,dt + C\int_{\Omega_T} |\delta-\widetilde{\delta}|^2|\partial_t\widetilde{u}|^2 e^{2s\varphi}\,dx\,dt \\ &+ Cs^4\int_{\omega_T} |Z|^2 e^{2s\varphi}\,dx\,dt + C\int_{\omega_T} |\partial_t Z|^2 e^{2s\varphi}\,dx\,dt \\ &+ Cs^2\int_{\omega_T} |Y|^2 e^{2s\varphi}\,dx\,dt + C\int_{\omega_T} |A(x)\cdot\nabla Y|^2 e^{2s\varphi}\,dx\,dt, \end{aligned} \tag{5.7}$$

where Y and Z are solutions of system (5.1).

Proof. Thanks to (5.4) and (5.6), for all $s > 0$ large enough, we obtain

$$\begin{aligned} &J_1(Y,\Omega_\eta) + J_1(Y_{\mathrm{bw}},\Omega_\eta) + J_2(Z,\Omega_\eta) + J_2(Z_{\mathrm{bw}},\Omega_\eta) \\ &\quad\le C\int_{\Omega_T} |LY|^2 e^{2s\varphi}\,dx\,dt + C\int_{\Omega_T} |L_{\mathrm{bw}}Y_{\mathrm{bw}}|^2 e^{2s\varphi}\,dx\,dt + C\int_{\Omega_T} |PZ|^2 e^{2s\varphi}\,dx\,dt + C\int_{\Omega_T} |P_{\mathrm{bw}}Z_{\mathrm{bw}}|^2 e^{2s\varphi}\,dx\,dt \\ &\qquad + Cs^4\int_{\omega_T} |Z|^2 e^{2s\varphi}\,dx\,dt + C\int_{\omega_T} |\partial_t Z|^2 e^{2s\varphi}\,dx\,dt + C\int_{\omega_T} |A(x)\cdot\nabla Y|^2 e^{2s\varphi}\,dx\,dt \\ &\qquad + Cs^2\int_{\omega_T} |Y|^2 e^{2s\varphi}\,dx\,dt + Cs^4\int_{\omega_T} |Z_{\mathrm{bw}}|^2 e^{2s\varphi}\,dx\,dt + C\int_{\omega_T} |\partial_t Z_{\mathrm{bw}}|^2 e^{2s\varphi}\,dx\,dt \\ &\qquad + Cs^2\int_{\omega_T} |Y_{\mathrm{bw}}|^2 e^{2s\varphi}\,dx\,dt + C\int_{\omega_T} |A(x)\cdot\nabla Y_{\mathrm{bw}}|^2 e^{2s\varphi}\,dx\,dt. \end{aligned}$$

Since $Y_{\mathrm{bw}}(x,t) = Y(x, T-t)$ for all $t \in (0,T)$ and $\varphi^{-1} \le C\varphi$, we deduce the following inequality:

$$\begin{aligned} J_1(Y,\Omega_\eta) + J_2(Z,\Omega_\eta) \le{}& C\int_{\Omega_T} |LY|^2 e^{2s\varphi}\,dx\,dt + C\int_{\Omega_T} |PZ|^2 e^{2s\varphi}\,dx\,dt + Cs^4\int_{\omega_T} |Z|^2 e^{2s\varphi}\,dx\,dt \\ &+ C\int_{\omega_T} |\partial_t Z|^2 e^{2s\varphi}\,dx\,dt + Cs^2\int_{\omega_T} |Y|^2 e^{2s\varphi}\,dx\,dt + C\int_{\omega_T} |A(x)\cdot\nabla Y|^2 e^{2s\varphi}\,dx\,dt. \end{aligned}$$

Using the estimations of $\mathcal{M}_i$ and $\mathcal{M}_{i,bw}$ for $i = 1,\dots,4$, we obtain the following Carleman estimate:

$$\begin{aligned} s\int_{\Omega_T} |Y|^2 e^{2s\varphi}\,dx\,dt + s\int_{\Omega_T} |Z|^2 e^{2s\varphi}\,dx\,dt \le{}& C\int_{\Omega_T} |LY|^2 e^{2s\varphi}\,dx\,dt + C\int_{\Omega_T} |PZ|^2 e^{2s\varphi}\,dx\,dt \\ &+ Cs^4\int_{\omega_T} |Z|^2 e^{2s\varphi}\,dx\,dt + C\int_{\omega_T} |\partial_t Z|^2 e^{2s\varphi}\,dx\,dt \\ &+ Cs^2\int_{\omega_T} |Y|^2 e^{2s\varphi}\,dx\,dt + C\int_{\omega_T} |A(x)\cdot\nabla Y|^2 e^{2s\varphi}\,dx\,dt. \end{aligned} \tag{5.8}$$

Note that we have

$$\int_{\Omega_T} |LY|^2 e^{2s\varphi}\,dx\,dt \le C\int_{\Omega_T} |\mu-\widetilde{\mu}|^2|\partial_t\widetilde{v}|^2 e^{2s\varphi}\,dx\,dt + C\int_{\Omega_T} |Z|^2 e^{2s\varphi}\,dx\,dt + C\int_{\Omega_T} |Y|^2 e^{2s\varphi}\,dx\,dt, \tag{5.9}$$

$$\int_{\Omega_T} |PZ|^2 e^{2s\varphi}\,dx\,dt \le C\int_{\Omega_T} |\delta-\widetilde{\delta}|^2|\partial_t\widetilde{u}|^2 e^{2s\varphi}\,dx\,dt + C\int_{\Omega_T} |Y|^2 e^{2s\varphi}\,dx\,dt + C\int_{\Omega_T} |Z|^2 e^{2s\varphi}\,dx\,dt. \tag{5.10}$$

Substituting inequalities (5.9) and (5.10) into (5.8), we conclude the proof of Proposition 5.5 for $s > 0$ large enough. □

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Remark 5.6. In (5.7), we obtain a lower bound of $\|e^{s\varphi}L_1Y\|^2_{L^2(\Omega\times(\eta,T-\eta))}$ but we can obtain it in $(\Omega\times(0,T))$. In fact, we have

$$\begin{aligned} s\int_{\Omega_T}|Y|^2e^{2s\varphi}\,dx\,dt &+ \int_{\Omega_T}|L_1Y|^2e^{2s\varphi}\,dx\,dt + s\int_{\Omega_T}|Z|^2e^{2s\varphi}\,dx\,dt \\ &\le C\int_{\Omega_T}|\mu-\widetilde{\mu}|^2|\partial_t\widetilde{v}|^2e^{2s\varphi}\,dx\,dt + C\int_{\Omega_T}|\delta-\widetilde{\delta}|^2|\partial_t\widetilde{u}|^2e^{2s\varphi}\,dx\,dt + Cs^4\int_{\omega_T}|Z|^2e^{2s\varphi}\,dx\,dt \\ &\quad + C\int_{\omega_T}|\partial_tZ|^2e^{2s\varphi}\,dx\,dt + Cs^2\int_{\omega_T}|Y|^2e^{2s\varphi}\,dx\,dt + C\int_{\omega_T}|A(x)\cdot\nabla Y|^2e^{2s\varphi}\,dx\,dt. \end{aligned} \tag{5.11}$$

Step 5: Carleman estimate with one observation. In this step, we will derive a Carleman estimate with only one observation of Z acting on ω and the data of Y at a fixed time $\theta\in(0,T)$. We will need the following lemma:

Lemma 5.7. *For $q\in L^2(\Omega\times(0,T))$ and $s>0$ we have*

$$\int_{\Omega_T}\left|\int_\theta^t q(x,\tau)\,d\tau\right|^2e^{2s\varphi}\,dx\,dt \le \frac{C}{s^2}\int_{\Omega_T}|q(x,t)|^2e^{2s\varphi}\,dx\,dt.$$

The proof is given in [10, 26].

Proposition 5.8. *Let ω be a subdomain of Ω such that $\partial\omega\supset\Gamma_+$. Suppose that Assumptions 2.2, 3.1 and Compatibility Conditions 2.3 are checked. There exist s_0 and a positive constant $C=C(s_0,\Omega,T,\omega)$ such that for all $s>s_0$,*

$$\begin{aligned} s\int_{\Omega_T}|Y|^2e^{2s\varphi}\,dx\,dt &+ \int_{\Omega_T}|L_1Y|^2e^{2s\varphi}\,dx\,dt + s\int_{\Omega}|Z|^2e^{2s\varphi}\,dx\,dt \\ &\le C\int_{\Omega_T}|\mu-\widetilde{\mu}|^2|\partial_t\widetilde{v}|^2e^{2s\varphi}\,dx\,dt + C\int_{\Omega_T}|\delta-\widetilde{\delta}|^2(|\partial_t\widetilde{u}|^2+|\partial_t^2\widetilde{u}|^2)e^{2s\varphi}\,dx\,dt \\ &\quad + C\int_{\omega_T}(s^4|Z|^2+|\partial_t^2Z|^2+|\partial_tZ|^2)e^{2s\varphi}\,dx\,dt + Cs^2\int_{\omega_T}|Y(x,\theta)|^2e^{2s\varphi}\,dx\,dt, \end{aligned} \tag{5.12}$$

where Y and Z are solutions of (5.1).

Proof. We set

$$\mathcal{K}=s^2\int_{\omega_T}|Y|^2e^{2s\varphi}\,dx\,dt \quad\text{and}\quad \mathcal{K}'=\int_{\omega_T}|A(x)\cdot\nabla Y|^2e^{2s\varphi}\,dx\,dt.$$

The aim is to estimate $\mathcal{K}$ and $\mathcal{K}'$ in terms of distributed observations of Z on $\omega\times(0,T)$. On the one hand, applying Lemma 5.7 to ∂_tY in $\omega\times(0,T)$, we find

$$\int_{\omega_T}\left|\int_\theta^t\partial_tY(x,\tau)\,d\tau\right|^2e^{2s\varphi}\,dx\,dt \le \frac{C}{s^2}\int_{\omega_T}|\partial_tY(x,t)|^2e^{2s\varphi}\,dx\,dt,$$

which gives

$$\int_{\omega_T}|Y(x,t)|^2e^{2s\varphi}\,dx\,dt \le \frac{C}{s^2}\int_{\omega_T}|\partial_tY(x,t)|^2e^{2s\varphi}\,dx\,dt + \int_{\omega_T}|Y(x,\theta)|^2e^{2s\varphi}\,dx\,dt.$$

Hence, we get

$$s^2\int_{\omega_T}|Y|^2e^{2s\varphi}\,dx\,dt \le C\int_{\omega_T}|\partial_tY|^2e^{2s\varphi}\,dx\,dt + Cs^2\int_{\omega_T}|Y(x,\theta)|^2e^{2s\varphi}\,dx\,dt.$$

On the other hand, we have

$$\delta\partial_tY=\partial_t^2Z-\Delta(\partial_tZ)-(\delta-\widetilde{\delta})\partial_t^2\widetilde{u}+k\partial_tZ.$$

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Therefore,

$$\int_{\omega_T} |\partial_t Y|^2 e^{2s\varphi}\,dx\,dt \le C\int_{\omega_T} |\partial_t^2 Z|^2 e^{2s\varphi}\,dx\,dt + C\int_{\omega_T} |\partial_t Z|^2 e^{2s\varphi}\,dx\,dt + C\int_{\omega_T} |\delta-\widetilde{\delta}|^2|\partial_t^2\widetilde{u}|^2 e^{2s\varphi}\,dx\,dt.$$

Consequently,

$$\begin{aligned}\mathcal{K} \le{}& C\int_{\omega_T} |\partial_t^2 Z|^2 e^{2s\varphi}\,dx\,dt + C\int_{\omega_T} |\partial_t Z|^2 e^{2s\varphi}\,dx\,dt + C\int_{\omega_T} |\delta-\widetilde{\delta}|^2|\partial_t^2\widetilde{u}|^2 e^{2s\varphi}\,dx\,dt\\ &+ Cs^2\int_{\omega_T} |Y(x,\theta)|^2 e^{2s\varphi}\,dx\,dt. \end{aligned}\tag{5.13}$$

Moreover, we have $A(x)\cdot\nabla Y = (\mu-\widetilde{\mu})\partial_t\widetilde{v} + \mu Z - \gamma Y - \partial_t Y$. Using Young's inequality, we obtain

$$\begin{aligned}\mathcal{K}' ={}& \int_{\omega_T} |(\mu-\widetilde{\mu})\partial_t\widetilde{v} + \mu Z - \gamma Y - \partial_t Y|^2 e^{2s\varphi}\,dx\,dt\\ \le{}& C\int_{\omega_T} |\partial_t Y|^2 e^{2s\varphi}\,dx\,dt + C\int_{\omega_T} |\mu-\widetilde{\mu}|^2|\partial_t\widetilde{v}|^2 e^{2s\varphi}\,dx\,dt + C\int_{\omega_T} |Z|^2 e^{2s\varphi}\,dx\,dt + C\int_{\omega_T} |Y|^2 e^{2s\varphi}\,dx\,dt. \end{aligned}\tag{5.14}$$

Finally, thanks to (5.13) and (5.14), we find

$$\begin{aligned}\mathcal{K}+\mathcal{K}' \le{}& Cs^2\int_{\omega_T} |Y(x,\theta)|^2 e^{2s\varphi}\,dx\,dt + C\int_{\omega_T} |\partial_t^2 Z|^2 e^{2s\varphi}\,dx\,dt + C\int_{\omega_T} |\partial_t Z|^2 e^{2s\varphi}\,dx\,dt\\ &+ Cs^4\int_{\omega_T} |Z|^2 e^{2s\varphi}\,dx\,dt + C\int_{\omega_T} |\delta-\widetilde{\delta}|^2|\partial_t^2\widetilde{u}|^2 e^{2s\varphi}\,dx\,dt + C\int_{\omega_T} |\mu-\widetilde{\mu}|^2|\partial_t\widetilde{v}|^2 e^{2s\varphi}\,dx\,dt.\end{aligned}$$

Substituting this into inequality (5.11), we conclude the proof of Proposition 5.8. □

Step 6: Stability result. This can be done in two parts. In Part 1, we prove a stability inequality for μ using the method introduced in [4]. In Part 2, we establish a stability inequality for δ using the method introduced in [31]. Finally, we will combine these two inequalities to obtain our stability result.

Part 1. Let θ in $(\eta, T-\eta)$ and $W = e^{s\varphi}\widetilde{Y}$. We define L_1 as

$$L_1 W = \partial_t W + A(x)\cdot\nabla W$$

and consider the integral

$$\mathcal{I} = \int_0^\theta\int_\Omega L_1 W.W\,dx\,dt.$$

We give an upper bound of $\mathcal{I}$ using the Carleman estimate (5.12). We have

$$|\mathcal{I}| = \left|\int_0^\theta\int_\Omega L_1 W.W\,dx\,dt\right| \le s^{-\frac{1}{2}}\left(\int_0^\theta\int_\Omega |L_1 W|^2\,dx\,dt\right)^{\frac{1}{2}}\left(s\int_0^\theta\int_\Omega |Y|^2 e^{2s\varphi}\,dx\,dt\right)^{\frac{1}{2}}.$$

Applying Young's inequality, we find

$$|\mathcal{I}| \le s^{-\frac{1}{2}}\left(\int_{\Omega_T} |L_1 W|^2\,dx\,dt + s\int_{\Omega_T} |Y|^2 e^{2s\varphi}\,dx\,dt\right).$$

Using (5.12), we obtain

$$\begin{aligned}|\mathcal{I}| \le{}& Cs^{-\frac{1}{2}}\Bigg(\int_{\Omega_T} |\mu-\widetilde{\mu}|^2|\partial_t\widetilde{v}|^2 e^{2s\varphi}\,dx\,dt + \int_{\Omega_T} |\delta-\widetilde{\delta}|^2(|\partial_t\widetilde{u}|^2 + |\partial_t^2\widetilde{u}|^2)e^{2s\varphi}\,dx\,dt\\ &+ \int_{\omega_T} (s^4|Z|^2 + |\partial_t^2 Z|^2 + |\partial_t Z|^2)e^{2s\varphi}\,dx\,dt + s^2\int_{\omega_T} |Y(x,\theta)|^2 e^{2s\varphi}\,dx\,dt\Bigg).\end{aligned}$$

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Now, let us compute $\mathcal{I}$. An integration by parts leads to

$$\int_0^\theta \int_{\Gamma_+} A(x)\cdot\nu|\widetilde{Y}|^2 e^{2s\varphi}\,d\sigma\,dt + \frac{1}{2}\int_\Omega |Y(x,\theta)|^2 e^{2s\varphi(x,\theta)}\,dx = \mathcal{I} + \frac{1}{2}\int_0^\theta\int_\Omega \nabla\cdot A(x)|\widetilde{Y}|^2 e^{2s\varphi}\,dx\,dt.$$

Since $A(x)\cdot\nu > 0$ on Γ_+, we have

$$\int_\Omega |Y(x,\theta)|^2 e^{2s\varphi(x,\theta)}\,dx \le \mathcal{I} + C\int_{\Omega_T} |Y|^2 e^{2s\varphi}\,dx\,dt.$$

Using again (5.12) and the estimation of $\mathcal{I}$, we find

$$\begin{aligned}\int_\Omega |Y(x,\theta)|^2 e^{2s\varphi(x,\theta)}\,dx \le s^2\int_{\omega_T} |Y(x,\theta)|^2 e^{2s\varphi}\,dx\,dt + C(s^{-\frac{1}{2}}+s^{-1})\Bigg(&\int_{\Omega_T} |\mu-\widetilde{\mu}|^2|\partial_t\widetilde{v}|^2 e^{2s\varphi}\,dx\,dt\\ &+\int_{\Omega_T} |\delta-\widetilde{\delta}|^2(|\partial_t\widetilde{u}|^2+|\partial_t^2\widetilde{u}|^2)e^{2s\varphi}\,dx\,dt\\ &+\int_{\omega_T}(s^4|Z|^2+|\partial_t^2 Z|^2+|\partial_t Z|^2)e^{2s\varphi}\,dx\,dt\Bigg).\end{aligned}$$

Moreover, we have

$$Y(x,\theta) = \partial_t U(x,\theta) = -A(x)\cdot\nabla U(x,\theta) + (\mu-\widetilde{\mu})(x)\widetilde{v}(x,\theta) + \mu(x)V(x,\theta) - \gamma(x)U(x,\theta).$$

Substituting Y into the last inequality, we obtain

$$\begin{aligned}&\int_\Omega |(\mu-\widetilde{\mu})\widetilde{v}(x,\theta)|^2 e^{2s\varphi(x,\theta)}\,dx\\ &\quad\le C(s^{-\frac{1}{2}}+s^{-1})\Bigg(\int_{\Omega_T}|\mu-\widetilde{\mu}|^2|\partial_t\widetilde{v}|^2 e^{2s\varphi}\,dx\,dt + \int_{\Omega_T}|\delta-\widetilde{\delta}|^2(|\partial_t\widetilde{u}|^2+|\partial_t^2\widetilde{u}|^2)e^{2s\varphi}\,dx\,dt\\ &\qquad+\int_{\omega_T}(s^4|Z|^2+|\partial_t^2 Z|^2+|\partial_t Z|^2)e^{2s\varphi}\,dx\,dt + s^2\int_{\omega_T}|Y(x,\theta)|^2 e^{2s\varphi}\,dx\,dt\Bigg)\\ &\qquad+C\int_\Omega |A(x)\cdot\nabla U(x,\theta)|^2 e^{2s\varphi(x,\theta)}\,dx + C\int_\Omega |U(x,\theta)|^2 e^{2s\varphi(x,\theta)}\,dx + C\int_\Omega |V(x,\theta)|^2 e^{2s\varphi(x,\theta)}\,dx. \quad (5.15)\end{aligned}$$

Part 2. Let θ in $(\eta, T-\eta)$ and

$$\mathcal{J} = \int_\Omega |Z(x,\theta)|^2 e^{2s\varphi(x,\theta)}\,dx.$$

We have

$$\begin{aligned}\int_\Omega |Z(x,\theta)|^2 e^{2s\varphi(x,\theta)}\,dx &= \int_\Omega |\chi(\theta)Z(x,\theta)|^2 e^{2s\varphi(x,\theta)}\,dx\\ &= \int_0^\theta \partial_t\Bigg(\int_\Omega |\chi(t)Z(x,t)|^2 e^{2s\varphi(x,t)}\,dx\Bigg)\,dt\\ &\le \int_{\Omega_T} 2\chi\partial_t\chi|Z|^2 e^{2s\varphi}\,dx\,dt + \int_{\Omega_T} 2|\chi|^2 Z\partial_t Z e^{2s\varphi}\,dx\,dt + \int_{\Omega_T} 2s\partial_t\varphi|\chi Z|^2 e^{2s\varphi}\,dx\,dt\\ &\le C\int_{\Omega_T}|\chi\partial_t Z|^2 e^{2s\varphi}\,dx\,dt + C(1+s)\int_{\Omega_T}|\chi Z|^2 e^{2s\varphi}\,dx\,dt. \quad (5.16)\end{aligned}$$

The following lemma gives Carleman estimates for χZ and $\chi\partial_t Z$.

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Lemma 5.9. *Let Z be the solution of system* (5.1). *We have the following Carleman estimates:*

$$\int_{\Omega_T} |\chi\partial_t Z|^2 e^{2s\varphi}\,dx\,dt \le Cs^{-3}\int_{\Omega_T} |(\mu-\widetilde{\mu})\partial_t^2\widetilde{v}|^2 e^{2s\varphi}\,dx\,dt + Cs^{-3}\int_{\Omega_T} |(\delta-\widetilde{\delta})\partial_t^2\widetilde{u}|^2 e^{2s\varphi}\,dx\,dt$$
$$+ Cs\int_{\omega_T} |(\chi\partial_t Z)|^2 e^{2s\varphi}\,dx\,dt + Cs^{-3}\int_{\omega_T} |(\chi\partial_t^2 Z)|^2 e^{2s\varphi}\,dx\,dt$$

and

$$s\int_{\Omega_T} |\chi Z|^2 e^{2s\varphi}\,dx\,dt \le Cs^{-2}\int_{\Omega_T} |(\mu-\widetilde{\mu})\partial_t\widetilde{v}|^2 e^{2s\varphi}\,dx\,dt + Cs^{-2}\int_{\Omega_T} |(\delta-\widetilde{\delta})\partial_t\widetilde{u}|^2 e^{2s\varphi}\,dx\,dt$$
$$+ Cs^2\int_{\omega_T} |(\chi Z)|^2 e^{2s\varphi}\,dx\,dt + Cs^{-2}\int_{\omega_T} |(\chi\partial_t Z)|^2 e^{2s\varphi}\,dx\,dt.$$

Proof. Firstly, we have

$$L(\chi\partial_t Y) = \chi(\mu-\widetilde{\mu})\partial_t^2\widetilde{v} + \mu\chi\partial_t Z - \gamma\chi\partial_t Y + \partial_t\chi\partial_t Y,$$
$$P(\chi\partial_t Z) = \chi P(\partial_t Z) + \partial_t\chi\partial_t Z = \chi(\delta-\widetilde{\delta})\partial_t^2\widetilde{u} + \chi\delta\partial_t Y - \chi k\partial_t Z + \partial_t\chi\partial_t Z.$$

Moreover, we have

$$\chi\partial_t Y(x,0) = \chi\partial_t Y(x,T) = \chi\partial_t Z(x,0) = \chi\partial_t Z(x,T) = 0 \quad \text{in } \Omega.$$

Thus, we can apply Theorem 3.2 to $\chi\partial_t Y$ and $\chi\partial_t Z$, we find

$$I_1(\chi\partial_t Y,\Omega_T) + I_2(\chi\partial_t Z,\Omega_T) \le C\int_{\Omega_T} |L(\chi\partial_t Y)|^2 e^{2s\varphi}\,dx\,dt + C\int_{\Omega_T} |P(\chi\partial_t Z)|^2 e^{2s\varphi}\,dx\,dt$$
$$+ Cs^4\lambda^4\int_{\omega_T} \varphi^4|(\chi\partial_t Z)|^2 e^{2s\varphi}\,dx\,dt + C\int_{\omega_T} |(\chi\partial_t^2 Z)|^2 e^{2s\varphi}\,dx\,dt.$$

We fix $\lambda = \lambda_0$ and we bound φ from below and from above, we obtain

$$\int_{\Omega_T} s|\chi\partial_t Y|^2 e^{2s\varphi}\,dx\,dt + \int_{\Omega_T} s^3|(\chi\partial_t Z)|^2 e^{2s\varphi}\,dx\,dt$$
$$\le C\int_{\Omega_T} |L(\chi\partial_t Y)|^2 e^{2s\varphi}\,dx\,dt + C\int_{\Omega_T} |P(\chi\partial_t Z)|^2 e^{2s\varphi}\,dx\,dt + Cs^4\int_{\omega_T} |(\chi\partial_t Z)|^2 e^{2s\varphi}\,dx\,dt$$
$$+ C\int_{\omega_T} |(\chi\partial_t^2 Z)|^2 e^{2s\varphi}\,dx\,dt.$$

Then, for all s large enough, we obtain

$$s\int_{\Omega_T} |\chi\partial_t Y|^2 e^{2s\varphi}\,dx\,dt + s^3\int_{\Omega_T} |\chi\partial_t Z|^2 e^{2s\varphi}\,dx\,dt$$
$$\le C\int_{\Omega_T} |(\mu-\widetilde{\mu})\partial_t^2\widetilde{v}|^2 e^{2s\varphi}\,dx\,dt + C\int_{\Omega_T} |(\delta-\widetilde{\delta})\partial_t^2\widetilde{u}|^2 e^{2s\varphi}\,dx\,dt + Cs^4\int_{\omega_T} |(\chi\partial_t Z)|^2 e^{2s\varphi}\,dx\,dt$$
$$+ C\int_{\omega_T} |(\chi\partial_t^2 Z)|^2 e^{2s\varphi}\,dx\,dt. \tag{5.17}$$

Now, let us establish a Carleman inequality for χZ. Similarly to $\chi\partial_t Z$, we obtain

$$s\int_{\Omega_T} |\chi Y|^2 e^{2s\varphi}\,dx\,dt + s^3\int_{\Omega_T} |\chi Z|^2 e^{2s\varphi}\,dx\,dt$$
$$\le C\int_{\Omega_T} |(\mu-\widetilde{\mu})\partial_t\widetilde{v}|^2 e^{2s\varphi}\,dx\,dt + C\int_{\Omega_T} |(\delta-\widetilde{\delta})\partial_t\widetilde{u}|^2 e^{2s\varphi}\,dx\,dt + Cs^4\int_{\omega_T} |(\chi Z)|^2 e^{2s\varphi}\,dx\,dt$$
$$+ C\int_{\omega_T} |(\chi\partial_t Z)|^2 e^{2s\varphi}\,dx\,dt. \tag{5.18}$$

From (5.17) and (5.18), we conclude the proof of Lemma 5.9. □

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Thanks to Lemma 5.9, inequality (5.16) becomes

$$\begin{aligned}\int_\Omega |Z(x,\theta)|^2 e^{2s\varphi(x,\theta)}\,dx \le{}& Cs^{-3}\int_{\Omega_T} |(\mu-\widetilde{\mu})\partial_t^2\widetilde{v}|^2 e^{2s\varphi}\,dx\,dt + Cs^{-3}\int_{\Omega_T} |(\delta-\widetilde{\delta})\partial_t^2\widetilde{u}|^2 e^{2s\varphi}\,dx\,dt\\ &+ Cs^{-2}\int_{\Omega_T} |(\mu-\widetilde{\mu})\partial_t\widetilde{v}|^2 e^{2s\varphi}\,dx\,dt + Cs^{-2}\int_{\Omega_T} |(\delta-\widetilde{\delta})\partial_t\widetilde{u}|^2 e^{2s\varphi}\,dx\,dt\\ &+ Cs\int_{\omega_T} |(\chi\partial_t Z)|^2 e^{2s\varphi}\,dx\,dt + Cs^{-3}\int_{\omega_T} |(\chi\partial_t^2 Z)|^2 e^{2s\varphi}\,dx\,dt\\ &+ Cs^2\int_{\omega_T} |(\chi Z)|^2 e^{2s\varphi}\,dx\,dt + Cs^{-2}\int_{\omega_T} |(\chi\partial_t Z)|^2 e^{2s\varphi}\,dx\,dt.\end{aligned}$$

Moreover, we have

$$Z(x,\theta) = \partial_t V(x,\theta) = \Delta V(x,\theta) + (\delta-\widetilde{\delta})\widetilde{u}(x,\theta) + \delta(x)U(x,\theta) - k(x)V(x,\theta).$$

Substituting Z into the last inequality, we find

$$\begin{aligned}\int_\Omega |(\delta-\widetilde{\delta})\widetilde{u}(x,\theta)|^2 e^{2s\varphi(x,\theta)}\,dx \le{}& Cs^{-3}\int_{\Omega_T} |(\mu-\widetilde{\mu})\partial_t^2\widetilde{v}|^2 e^{2s\varphi}\,dx\,dt + Cs^{-3}\int_{\Omega_T} |(\delta-\widetilde{\delta})\partial_t^2\widetilde{u}|^2 e^{2s\varphi}\,dx\,dt\\ &+ Cs^{-2}\int_{\Omega_T} |(\mu-\widetilde{\mu})\partial_t\widetilde{v}|^2 e^{2s\varphi}\,dx\,dt + Cs^{-2}\int_{\Omega_T} |(\delta-\widetilde{\delta})\partial_t\widetilde{u}|^2 e^{2s\varphi}\,dx\,dt\\ &+ Cs\int_{\omega_T} |(\chi\partial_t Z)|^2 e^{2s\varphi}\,dx\,dt + Cs^{-3}\int_{\omega_T} |(\chi\partial_t^2 Z)|^2 e^{2s\varphi}\,dx\,dt\\ &+ Cs^2\int_{\omega_T} |(\chi Z)|^2 e^{2s\varphi}\,dx\,dt + Cs^{-2}\int_{\omega_T} |(\chi Z)|^2 e^{2s\varphi}\,dx\,dt\\ &+ C\int_\Omega |\Delta V(x,\theta)|^2 e^{2s\varphi(x,\theta)}\,dx + C\int_\Omega |U(x,\theta)|^2 e^{2s\varphi(x,\theta)}\,dx\\ &+ C\int_\Omega |V(x,\theta)|^2 e^{2s\varphi(x,\theta)}\,dx. \qquad (5.19)\end{aligned}$$

By gathering (5.15) and (5.19), we obtain

$$\begin{aligned}&\int_\Omega |\mu-\widetilde{\mu}|^2|\widetilde{v}(x,\theta)|^2 e^{2s\varphi(x,\theta)}\,dx + \int_\Omega |\delta-\widetilde{\delta}|^2|\widetilde{u}(x,\theta)|^2 e^{2s\varphi(x,\theta)}\,dx\\ &\quad\le C(s^{-\frac{1}{2}}+s^{-1})\Big(\int_{\Omega_T} |\mu-\widetilde{\mu}|^2|\partial_t\widetilde{v}|^2 e^{2s\varphi}\,dx\,dt + \int_{\Omega_T} |\delta-\widetilde{\delta}|^2(|\partial_t\widetilde{u}|^2+|\partial_t^2\widetilde{u}|^2)e^{2s\varphi}\,dx\,dt\\ &\qquad+\int_{\omega_T}\big(s^2|Z|^2+|\partial_t^2 Z|^2+|\partial_t Z|^2\big)e^{2s\varphi}\,dx\,dt + s^2\int_{\omega_T}|Y(x,\theta)|^2 e^{2s\varphi}\,dx\,dt\Big)\\ &\qquad+ Cs^{-3}\int_{\Omega_T} |(\mu-\widetilde{\mu})\partial_t^2\widetilde{v}|^2 e^{2s\varphi}\,dx\,dt + Cs^{-3}\int_{\Omega_T} |(\delta-\widetilde{\delta})\partial_t^2\widetilde{u}|^2 e^{2s\varphi}\,dx\,dt\\ &\qquad+ Cs^{-2}\int_{\Omega_T} |(\mu-\widetilde{\mu})\partial_t\widetilde{v}|^2 e^{2s\varphi}\,dx\,dt + Cs^{-2}\int_{\Omega_T} |(\delta-\widetilde{\delta})\partial_t\widetilde{u}|^2 e^{2s\varphi}\,dx\,dt\\ &\qquad+ C\int_\Omega |\Delta V(x,\theta)|^2 e^{2s\varphi(x,\theta)}\,dx + C\int_\Omega |A(x)\cdot\nabla U(x,\theta)|^2 e^{2s\varphi(x,\theta)}\,dx\\ &\qquad+ C\int_\Omega |U(x,\theta)|^2 e^{2s\varphi(x,\theta)}\,dx + C\int_\Omega |V(x,\theta)|^2 e^{2s\varphi(x,\theta)}\,dx.\end{aligned}$$

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Since $\nabla(\chi Z) = \chi\nabla Z$ and $\Delta(\chi\partial_t Z) = \chi\Delta\partial_t Z$, we obtain for all s large enough that

$$\begin{aligned}
&\int_\Omega |\mu-\widetilde{\mu}|^2|\widetilde{v}(x,\theta)|^2 e^{2s\varphi(x,\theta)}\,dx + \int_\Omega |\delta-\widetilde{\delta}|^2|\widetilde{u}(x,\theta)|^2 e^{2s\varphi(x,\theta)}\,dx \\
&\quad\le Cs^{-\frac{1}{2}}\Big(\int_{\Omega_T} |\mu-\widetilde{\mu}|^2|\partial_t\widetilde{v}|^2 e^{2s\varphi}\,dx\,dt + \int_{\Omega_T} |\delta-\widetilde{\delta}|^2(|\partial_t\widetilde{u}|^2 e^{2s\varphi}\,dx\,dt\Big) \\
&\qquad + Cs^{-\frac{1}{2}}\int_{\Omega_T} |\delta-\widetilde{\delta}|^2|\partial_t^2\widetilde{u}|^2 e^{2s\varphi}\,dx\,dt + Cs^{-3}\int_{\Omega_T} |\mu-\widetilde{\mu}|^2|\partial_t^2\widetilde{v}|^2 e^{2s\varphi}\,dx\,dt \\
&\qquad + Cs^{-\frac{1}{2}}\Big(\int_{\omega_T} (s^2|Z|^2 + |\partial_t^2 Z|^2 + |\partial_t Z|^2)e^{2s\varphi}\,dx\,dt\Big) + Cs^2\int_{\omega_T} |Z|^2 e^{2s\varphi}\,dx\,dt \\
&\qquad + Cs^{-1}\Big(\int_{\omega_T} (|Z|^2 + s^2|(\partial_t Z)|^2)e^{2s\varphi}\,dx\,dt\Big) + C\int_\Omega |\Delta V(x,\theta)|^2 e^{2s\varphi(x,\theta)}\,dx \\
&\qquad + C\int_\Omega |V(x,\theta)|^2 e^{2s\varphi(x,\theta)}\,dx + C\int_\Omega |A(x)\cdot\nabla U(x,\theta)|^2 e^{2s\varphi(x,\theta)}\,dx \\
&\qquad + C\int_\Omega |U(x,\theta)|^2 e^{2s\varphi(x,\theta)}\,dx + Cs^{\frac{3}{2}}\int_{\omega_T} |Y(x,\theta)|^2 e^{2s\varphi}\,dx\,dt. \qquad (5.20)
\end{aligned}$$

Remark 5.10. For $R \in W^{1,2}(0,T;L^\infty(\Omega))$ and $|R(x,\theta)| \ge r_0 > 0$ a.e. in $\overline{\Omega}$, there exists $g_0 \in L^2(0,T)$ such that $|\partial_t R(x,t)| \le g_0(t)|R(x,\theta)|$ for all $x \in \Omega$ and $t \in (0,T)$ (see [4, 30]).

The previous remark applied to (5.20) leads to

$$\begin{aligned}
&\int_\Omega |\mu-\widetilde{\mu}|^2|\widetilde{v}(x,\theta)|^2 e^{2s\varphi(x,\theta)}\,dx + \int_\Omega |\delta-\widetilde{\delta}|^2|\widetilde{u}(x,\theta)|^2 e^{2s\varphi(x,\theta)}\,dx \\
&\quad\le Cs^{-\frac{1}{2}}\Big(\int_\Omega |\mu-\widetilde{\mu}|^2|\widetilde{v}(x,\theta)|^2 e^{2s\varphi(x,\theta)}\,dx + \int_\Omega |\delta-\widetilde{\delta}|^2|\widetilde{u}(x,\theta)|^2 e^{2s\varphi(x,\theta)}\,dx\Big) \\
&\qquad + Cs^{-\frac{1}{2}}\int_{\omega_T} (|\partial_t^2 Z|^2 + |\partial_t Z|^2)e^{2s\varphi}\,dx\,dt + Cs^2\int_{\omega_T} |Z|^2 e^{2s\varphi}\,dx\,dt \\
&\qquad + C\int_\Omega (|\Delta V(x,\theta)|^2 + |V(x,\theta)|^2 + |A(x)\cdot\nabla U(x,\theta)|^2)e^{2s\varphi(x,\theta)}\,dx \\
&\qquad + C\int_\Omega |U(x,\theta)|^2 e^{2s\varphi(x,\theta)}\,dx + Cs^{\frac{3}{2}}\int_{\omega_T} |Y(x,\theta)|^2 e^{2s\varphi}\,dx\,dt.
\end{aligned}$$

Finally, for all large s, we deduce the following result:

$$\begin{aligned}
&\int_\Omega |\mu-\widetilde{\mu}|^2|\widetilde{v}(x,\theta)|^2 e^{2s\varphi(x,\theta)}\,dx + \int_\Omega |\delta-\widetilde{\delta}|^2|\widetilde{u}(x,\theta)|^2 e^{2s\varphi(x,\theta)}\,dx \\
&\quad\le Cs^{-\frac{1}{2}}\int_{\omega_T} (|(\partial_t Z)|^2 + |\partial_t^2 Z|^2)e^{2s\varphi}\,dx\,dt + Cs^2\int_{\omega_T} |Z|^2 e^{2s\varphi}\,dx\,dt \\
&\qquad + C\int_\Omega (|\Delta V(x,\theta)|^2 + |V(x,\theta)|^2 + |A(x)\cdot\nabla U(x,\theta)|^2)e^{2s\varphi(x,\theta)}\,dx \\
&\qquad + C\int_\Omega |U(x,\theta)|^2 e^{2s\varphi(x,\theta)}\,dx + Cs^{\frac{3}{2}}\int_{\omega_T} |Y(x,\theta)|^2 e^{2s\varphi}\,dx\,dt.
\end{aligned}$$

From Assumption 2.4, there exists θ such that $|\widetilde{u}(x,\theta)| \ge r_1 > 0$ and $|\widetilde{v}(x,\theta)| \ge r_2 > 0$. Since $e^{2s\varphi(x,\theta)}$ is bounded in $\Omega \times (0,T)$ and $e^{2s\varphi(x,\theta)} \ge c > 0$ for all $x \in \Omega$, we conclude the proof. □

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A Appendix

Proof of Lemma 5.2. Let us start with the estimation of $\mathcal{M}_2$. We have

$$\begin{aligned}\frac{dE_1}{dt} &= s\int_\Omega \partial_t\varphi|Y|^2e^{2s\varphi}\,dx + \int_\Omega |Y|\partial_t Ye^{2s\varphi}\,dx\\ &= s\int_\Omega \partial_t\varphi|Y|^2e^{2s\varphi}\,dx + \int_\Omega (LY - A(x)\cdot\nabla Y)Ye^{2s\varphi}\,dx.\end{aligned}$$

Then, we obtain

$$\frac{dE_1}{dt} - s\int_\Omega \partial_t\varphi|Y|^2e^{2s\varphi}\,dx + \frac{1}{2}\int_\Omega e^{2s\varphi}A(x)\cdot\nabla(|Y|^2)\,dx = \int_\Omega YLYe^{2s\varphi}\,dx.$$

After an integration by parts, we get

$$\begin{aligned}\frac{dE_1}{dt} - s\int_\Omega (\partial_t\varphi + \nabla\varphi\cdot A(x))|Y|^2e^{2s\varphi}\,dx + \frac{1}{2}\int_{\Gamma_+} A(x)\cdot\nu|Y|^2e^{2s\varphi}\,d\sigma\\ = \int_\Omega YLYe^{2s\varphi}\,dx + \frac{1}{2}\int_\Omega \nabla\cdot A(x)|Y|^2e^{2s\varphi}\,dx.\end{aligned} \tag{A.1}$$

Moreover, for all $s > 0$ large enough, from Assumption 3.1 (iv) we obtain

$$\frac{dE_1}{dt} + sC\int_\Omega |Y|^2e^{2s\varphi}\,dx \le \int_\Omega YLYe^{2s\varphi}\,dx. \tag{A.2}$$

Using the formula $2ab \le \varepsilon a^2 + \frac{b^2}{\varepsilon}$ with $\varepsilon = sC$, we estimate the right-hand side as follows:

$$\left|\int_\Omega YLYe^{2s\varphi}\,dx\right| \le \frac{1}{2}sC\int_\Omega |Y|^2e^{2s\varphi}\,dx + \frac{1}{2sC}\int_\Omega |LY|^2e^{2s\varphi}\,dx.$$

Substituting this estimate into (A.2), we have

$$\frac{dE_1}{dt} + sCE_1(t) \le \frac{1}{2sC}\int_\Omega |LY|^2e^{2s\varphi}\,dx.$$

On the other hand, for $t \in (T-\eta, T)$, using the Gronwall Lemma, we obtain

$$\begin{aligned}E_1(t) &\le e^{-sC(t-(T-\eta))}E_1(T-\eta) + \frac{e^{sC(T-t-\eta)}}{2sc}\int_{T-\eta}^{t}\int_\Omega e^{2s\varphi(\tau)}|LY(\tau)|^2\,dx\,d\tau\\ &\le e^{-sC(t-(T-\eta))}E_1(T-\eta) + \frac{1}{2sC}\int_{T-\eta}^{T}\int_\Omega e^{2s\varphi(\tau)}|LY(\tau)|^2\,dx\,d\tau.\end{aligned}$$

Integrating this inequality for t between $T-\eta$ et T, we get

$$\begin{aligned}\int_{T-\eta}^{T} E_1(t)\,dt &\le E_1(T-\eta)\int_{T-\eta}^{T} e^{-sC(t-(T-\eta))}\,dt + \int_{T-\eta}^{T}\frac{1}{2sC}\int_{T-\eta}^{T}\int_\Omega e^{2s\varphi(\tau)}|LY(\tau)|^2\,dx\,d\tau\,dt\\ &\le E_1(T-\eta)\int_{T}^{T-\eta} e^{-sc(t-(T-\eta))}\,dt + \frac{\eta}{2sc}\int_0^T\int_\Omega e^{2s\varphi}|LY|^2\,dx\,dt.\end{aligned}$$

Finally, we have

$$\int_{T-\eta}^{T} E_1(t)\,dt \le \frac{C}{s}E_1(T-\eta) + \frac{C}{s}\int_0^T\int_\Omega e^{2s\varphi}|LY|^2\,dx\,dt. \tag{A.3}$$

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Now, we want to estimate $E_1(T-\eta)$ by $E_1(\tau)$ for $\tau \in (\eta, T-\eta)$. We use (A.1) and we integrate between τ and $T-\eta$ to obtain

$$\int_\tau^{T-\eta} \frac{dE_1}{dt} dt + \frac{1}{2} \int_\tau^{T-\eta} \int_{\Gamma_+} A(x)\cdot \nu |Y|^2 e^{2s\varphi}\, d\sigma\, dt = \int_\tau^{T-\eta} s \int_\Omega (\partial_t \varphi + \nabla\varphi \cdot A(x))|Y|^2 e^{2s\varphi}\, dx\, dt$$
$$+ \frac{1}{2}\int_\Omega \nabla\cdot A(x)|Y|^2 e^{2s\varphi}\, dx\, dt + \int_\tau^{T-\eta}\int_\Omega YLYe^{2s\varphi}\, dx\, dt.$$

Thus we have

$$\int_\tau^{T-\eta} \frac{dE_1}{dt} dt \le Cs \int_\tau^{T-\eta}\int_\Omega |Y|^2 e^{2s\varphi}\, dx\, dt + \frac{C}{s}\int_\tau^{T-\eta}\int_\Omega |LY|^2 e^{2s\varphi}\, dx\, dt,$$

which gives

$$E_1(T-\eta) - E_1(\tau) \le Cs \int_\eta^{T-\eta}\int_\Omega |Y|^2 e^{2s\varphi}\, dx\, dt + \frac{C}{s}\int_0^T\int_\Omega |LY|^2 e^{2s\varphi}\, dx\, dt.$$

Integrating between η and $T-\eta$, we obtain, for $s>0$ sufficiently large,

$$E_1(T-\eta) \le Cs \int_\eta^{T-\eta} E_1(t) dt + \frac{C}{s}\int_0^T\int_\Omega |LY|^2 e^{2s\varphi}\, dx\, dt. \tag{A.4}$$

Finally, thanks to (A.4) and (A.3), we obtain

$$\int_{T-\eta}^{T} E_1(t)\, dt \le C\int_\eta^{T-\eta} E_1(t) dt + \frac{C}{s}\int_0^T\int_\Omega e^{2s\varphi}|LY|^2\, dx\, dt.$$

That is

$$\mathcal{M}_2 \le C\int_\eta^{T-\eta}\int_\Omega |Y|^2\, dx\, dt + \frac{C}{s}\int_0^T\int_\Omega e^{2s\varphi}|LY|^2\, dx\, dt.$$

Next, we will estimate $\mathcal{M}_4$. We will need the following auxiliary lemma (we refer to [3, 10, 26] for the proof):

Lemma A.1. *Let $\varphi \in C^2(\Omega)$ such that $1 > |\nabla\varphi| \ge \delta > 0$. There exist $s_0 > 0$ and $C > 0$ such that, for all $s \ge s_0$ and all $Z \in H_0^1(\Omega)$,*

$$s^2 \int_\Omega e^{2s\varphi}|Z|^2\, dx \le C\int_\Omega e^{2s\varphi}|\nabla Z|^2\, dx.$$

We have

$$\frac{dE_2}{dt} = s\int_\Omega \partial_t\varphi |Z|^2 e^{2s\varphi}\, dx + \int_\Omega |Z|\partial_t Z e^{2s\varphi}\, dx$$
$$= s\int_\Omega \partial_t\varphi |Z|^2 e^{2s\varphi}\, dx + \int_\Omega (PZ + \Delta Z) Z e^{2s\varphi}\, dx.$$

Then

$$\frac{dE_2}{dt} - s\int_\Omega \partial_t\varphi |Z|^2 e^{2s\varphi}\, dx - \int_\Omega Z\Delta Z e^{2s\varphi}\, dx = \int_\Omega ZPZe^{2s\varphi}\, dx.$$

An integration by parts leads to

$$\frac{dE_2}{dt} - s\int_\Omega \partial_t\varphi |Z|^2 e^{2s\varphi}\, dx + \int_\Omega |\nabla Z|^2 e^{2s\varphi}\, dx + 2s\int_\Omega e^{2s\varphi} Z\nabla Z\cdot\nabla\varphi\, dx = \int_\Omega ZPZe^{2s\varphi}\, dx.$$

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On the other hand, we have

$$2s\int_\Omega e^{2s\varphi}Z\nabla Z\cdot\nabla\varphi\,dx = s\int_\Omega e^{2s\varphi}\nabla\varphi\nabla|Z|^2\,dx = -s\int_\Omega e^{2s\varphi}(2s|\nabla\varphi|^2+\Delta\varphi)|Z|^2\,dx.$$

Thus

$$\frac{dE_2}{dt} - Cs^2\int_\Omega |\nabla\varphi|^2|Z|^2e^{2s\varphi}\,dx + \int_\Omega |\nabla Z|^2e^{2s\varphi}\,dx \le Cs\int_\Omega (|\partial_t\varphi|+|\Delta\varphi|)|Z|^2e^{2s\varphi}\,dx + \int_\Omega ZPZe^{2s\varphi}\,dx.$$

Applying Lemma A.1 to the last inequality, we obtain

$$\frac{dE_2}{dt} - Cs^2\int_\Omega |\nabla\varphi|^2e^{2s\varphi}|Z|^2\,dx + Cs^2\int_\Omega |Z|^2e^{2s\varphi}\,dx \le \int_\Omega ZPZe^{2s\varphi}\,dx + Cs\int_\Omega |Z|^2e^{2s\varphi}\,dx.$$

For s large enough, the last term of the right hand side is absorbed by the last term of the left hand side, so we have

$$\frac{dE_2}{dt} - Cs^2\int_\Omega |\nabla\varphi|^2e^{2s\varphi}|Z|^2\,dx + Cs^2\int_\Omega |Z|^2e^{2s\varphi}\,dx \le \int_\Omega ZPZe^{2s\varphi}\,dx.$$

This leads, for s large, to the following inequality:

$$\frac{dE_2}{dt} + Cs^2\int_\Omega (1-|\nabla\varphi|^2))e^{2s\varphi}|Z|^2\,dx \le \int_\Omega ZPZe^{2s\varphi}\,dx.$$

According to the assumption of Lemma 5.2 we obtain

$$\frac{dE_2}{dt} + Cs\int_\Omega e^{2s\varphi}|Z|^2\,dx \le \int_\Omega ZPZe^{2s\varphi}\,dx.$$

In the same way as for E_1, we obtain

$$\mathcal{M}_4 \le C\int_\eta^{T-\eta}\int_\Omega |Z|^2e^{2s\varphi}\,dx\,dt + \frac{C}{s}\int_0^T\int_\Omega |PZ|^2e^{2s\varphi}\,dx\,dt. \qquad \square$$

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