Initial trace of positive solutions to fractional diffusion equation with absorption

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Abstract

In this paper, we prove the existence of an initial trace $\mathcal{T}_u$ for any positive solution u to the semilinear fractional diffusion equation (H)

$$\partial_t u + (-\Delta)^s u + f(t,x,u) = 0 \quad \text{in} \quad (0,+\infty)\times\mathbb{R}^N,$$

where $N \geq 1$, the operator $(-\Delta)^s$ with $s \in (0,1)$ is the fractional Laplacian, $f : \mathbb{R}_+ \times \mathbb{R}^N \times \mathbb{R}_+ \to \mathbb{R}$ is a Caratheodory function satisfying $f(t,x,u)u \geq 0$ for all $(t,x,u) \in \mathbb{R}_+ \times \mathbb{R}^N \times \mathbb{R}_+$ and $\mathbb{R}_+ = [0,+\infty)$. We define the regular set of the trace $\mathcal{T}_u$ as an open subset of $\mathcal{R}_u \subset \mathbb{R}^N$ carrying a nonnegative Radon measure ν_u such that

$$\lim_{t\to 0}\int_{\mathcal{R}_u} u(t,x)\zeta(x)\,dx = \int_{\mathcal{R}_u}\zeta\,d\nu_u, \quad \forall\,\zeta \in C_0^2(\mathcal{R}_u),$$

and the singular set $\mathcal{S}_u = \mathbb{R}^N \setminus \mathcal{R}_u$ as the set points a such that

$$\limsup_{t\to 0}\int_{B_\rho(a)} u(t,x)\,dx = +\infty \quad \text{for any } \rho > 0.$$

We also study the reverse problem of constructing a positive solution to (H) with a given initial trace $(\mathcal{S},\nu)$, where $\mathcal{S} \subset \mathbb{R}^N$ is a closed set and ν is a positive Radon measure on $\mathcal{R} = \mathbb{R}^N \setminus \mathcal{S}$ and develop the case $f(t,x,u) = t^\beta u^p$ with $\beta > -1$ and $p > 1$.

Key words: Fractional heat equation, Initial trace, Singularities

MSC2010: 35K55, 35R11, 35K99

Contents

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1 Introduction

The first aim of this paper is to study the existence of an *initial trace* of positive solutions to the semilinear fractional diffusion equation

$$\partial_t u + (-\Delta)^s u + f(t,x,u) = 0 \quad \text{in} \quad Q_\infty := \mathbb{R}^*_+ \times \mathbb{R}^N, \tag{1.1}$$

where $f : \mathbb{R}^*_+ \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function satisfying

$$f(t,x,u)u \geq 0, \quad \forall\, (t,x,u) \in \mathbb{R}^*_+ \times \mathbb{R}^N \times \mathbb{R}, \tag{1.2}$$

and $\mathbb{R}^*_+ = (0,+\infty)$. The fractional Laplacian$(-\Delta)^s$ with $s \in (0,1)$ is defined in the principal value sense that

$$(-\Delta)^s u(x) = \lim_{\varepsilon \to 0^+} (-\Delta)^s_\varepsilon u(x),$$

where

$$(-\Delta)^s_\varepsilon u(x) := -a_{N,s} \int_{\mathbb{R}^N} \frac{u(z)-u(x)}{|z-x|^{N+2s}} \chi_\varepsilon(|x-z|) dz\ , \quad a_{N,s} = \frac{\Gamma(\frac{N}{2}+s)}{\pi^{\frac{N}{2}}\Gamma(2-s)} s(1-s), \tag{1.3}$$

for $\varepsilon > 0$ and

$$\chi_\varepsilon(r) = \begin{cases} 0 & \text{if} \quad r \in [0,\varepsilon] \\ 1 & \text{if} \quad r > \varepsilon. \end{cases}$$

The solutions of (1.1) are intended in the classical sense and, in order $(-\Delta)^s u(t,x)$ to be well-defined, we always assume that $u(t,.) \in \mathcal{L}^s(\mathbb{R}^N)$ for any $t > 0$, where

$$\mathcal{L}^s(\mathbb{R}^N) = \left\{ \phi \in L^1_{loc}(\mathbb{R}^N) \ \text{ s.t. } \ \|\phi\|_{\mathcal{L}^s} := \int_{\mathbb{R}^N} \frac{|\phi(x)|\, dx}{1+|x|^{N+2s}} < +\infty \right\}. \tag{1.4}$$

Notice that the constant functions belong to $\mathcal{L}^s(\mathbb{R}^N)$. If $\omega \subset \mathbb{R}^N$ and $0 < T \leq +\infty$, we set $Q_T^\omega = (0,T) \times \omega$, $Q_T^{\mathbb{R}^N} = Q_T$, $Q_\infty = \mathbb{R}_+^* \times \mathbb{R}^N$ and denote by $B_\rho(z)$ (resp. $K_\rho(z)$) the open ball (resp. open cube with sides parallel to the axis) with center $z \in \mathbb{R}^N$ and radius (side length) $\rho > 0$. We define the *regular set* of the initial trace of a positive solution u of (1.1) by

$$\mathcal{R}_u = \left\{ z \in \mathbb{R}^N : \exists\, \rho > 0 \text{ s.t.} \iint_{Q_1^{B_\rho(z)}} f(t,x,u)\, dxdt < +\infty \right\}. \tag{1.5}$$

Clearly $\mathcal{R}_u$ is open. The *conditional singular set* $\tilde{\mathcal{S}}_u$ is $\mathbb{R}^N \setminus \mathcal{R}_u$ and *the conditional initial trace* is the couple $Tr_c(u) := (\tilde{\mathcal{S}}_u, \nu)$. Our first result is the following statement which is the starting point of our work.

Theorem A *Let u be a nonnegative classical solution of (1.1) and the regular set $\mathcal{R}_u$ of u is given in (1.5), then there exists a nonnegative Radon measure ν_u on $\mathcal{R}_u$ such that*

$$\lim_{t \to 0} \int_{\mathcal{R}_u} u(t,x)\zeta(x)\, dx = \int_{\mathcal{R}_u} \zeta\, d\nu_u, \quad \forall\, \zeta \in C_0^2(\mathcal{R}_u). \tag{1.6}$$

The problem of the initial trace of nonnegative solutions for semilinear heat equations was initiated by Marcus and Véron in [37] with equation

$$\partial_t u - \Delta u + u^p = 0 \quad \text{in } Q_\infty, \tag{1.7}$$

for $p > 1$. They showed the existence of an initial trace $Tr(u)$ represented by a closed subset $\mathcal{S}_u$ of $\mathbb{R}^N$ and a nonnegative Radon measure ν_u on $\mathcal{R}_u = \mathbb{R}^N \setminus \mathcal{S}_u$. On $\mathcal{R}_u$ the initial trace is achieved as in (1.6). On $\mathcal{S}_u$ they proved that for any $z \in \mathcal{S}_u$,

$$\lim_{t \to 0} \int_{B_\rho(z)} u(t,x)dx = +\infty \quad \text{for any } \rho > 0. \tag{1.8}$$

They also highlighted the existence of a critical exponent $p_c = 1 + \frac{2}{N}$, which plays a crucial role in the fine analysis of the initial trace. For example they obtained that if p is *subcritical*, i.e. $1 < p < p_c$, (1.6) can be sharpened in the form

$$c_2(p,N) \leq \liminf_{t \to 0} t^{\frac{1}{p-1}} u(z,t) \leq \limsup_{t \to 0} t^{\frac{1}{p-1}} u(z,t) \leq c_1(p), \tag{1.9}$$

for some positive constants $c_1(p) > c_2(p,N)$. Furthermore they proved that for any couple $(\mathcal{S}, \nu)$, where $\mathcal{S}$ is a closed subset of $\mathbb{R}^N$ and ν a nonnegative Radon measure on $\mathcal{R} = \mathbb{R}^N \setminus \mathcal{S}$, there exists a unique nonnegative solution u of (1.7) with the initial trace $Tr(u) = (\mathcal{S}, \nu)$. The *supercritical* case $p \geq p_c$ turned out to be much more delicate and was finally elucidated in a series of works by Marcus and Véron [41] and Gkikas and Véron [32] following some deep ideas introduced by Marcus and Véron in [40] and Marcus [36] for solving similar questions dealing with semilinear elliptic equations. Al Sayed and Véron in [4] extended the subcritical analysis performed in [37] to the non-autonomous equation

$$\partial_t u - \Delta u + t^\beta u^p = 0 \quad \text{in } Q_\infty, \tag{1.10}$$

with $\beta > -1$ and $p > 1$. Note that the choice $\beta > -1$ is natural otherwise the initial trace would be essentialy zero as it can be verified with the equation without absorption.

The main difficulty to extend some of the previous results dealing with (1.7) and (1.10) comes from the fact that the fractional Laplacian is a non-local operator. A more precise characterization of the conditional singular set needs additional assumptions on u or on f. We define the *singular set* $\mathcal{S}_u$ of u by

$$\mathcal{S}_u = \left\{ z \in \mathbb{R}^N : \limsup_{t \to 0} \int_{B_\rho(z)} u(t,x)dx = +\infty \text{ for any } \rho > 0 \right\}. \tag{1.11}$$

This set is closed and it follows from Theorem A that $\mathcal{S}_u \subset \tilde{\mathcal{S}}_u$. The *initial trace* is the couple $Tr(u) := (\mathcal{S}_u, \nu)$. This initial trace can also be seen as an outer regular Borel measure with regular part (or Radon part) ν and singular part $\mathcal{S}_u$. When $s = 1$ then $Tr(u) = Tr_c(u)$ because the set $\mathcal{S}_u$ is also characterized as the set of $z \in \mathbb{R}^N$ where

$$\int_0^{\frac{T}{2}} \int_{B_\rho(z)} f(t,x,u)dxdt = \infty \text{ for any } \rho > 0. \tag{1.12}$$

When $0 < s < 1$ and no extra assumption on f are made, $Tr(u)$ could be different from $Tr_c(u)$.

Theorem B *Assume that u is a nonnegative solution of (1.1). If $u \in L^1(0,T;\mathcal{L}^s(\mathbb{R}^N))$, then $\mathcal{S}_u = \tilde{\mathcal{S}}_u$ and more precisely for any $z \in \mathcal{S}_u$,*

$$\lim_{t \to 0} \int_{B_\rho(z)} u(t,x)\,dx = +\infty \text{ for any } \rho > 0. \tag{1.13}$$

The above assumption on u can be verified when the absorption is strong and the singular set is compact. Another type of characterization of the singular set needs the following assumptions on f: $f(t,x,u)$ satisfies $f(t,x,0) = 0$ and

$$0 \le f(t,x,u) \le t^\beta g(u) \quad \forall\, (t,x,u) \in \mathbb{R}_+ \times \mathbb{R}^N \times \mathbb{R}_+, \tag{1.14}$$

where $\mathbb{R}_+ = [0, +\infty)$, $\beta > -1$, g is nondecreasing, continuous and verifies the subcritical growth assumption,

$$\int_1^\infty g(s) s^{-1-p^*_\beta} ds < +\infty, \tag{1.15}$$

with

$$p^*_\beta = 1 + \frac{2s(1+\beta)}{N}. \tag{1.16}$$

The role of the subcritical growth assumption (1.1) has been highlighted in [28] as the natural condition to solve the initial value problem with a bounded positive Radon measure for equation (1.1) (see Section 2.2).

Theorem C *Assume (1.14) and either (1.15) holds if $-1 < \beta \le 0$, or*

$$\int_1^\infty g(s) s^{-2-\frac{2s}{N}} ds < +\infty, \tag{1.17}$$

if $\beta > 0$. If u is a nonnegative solution of (1.1) with initial trace $(\mathcal{S}_u, \nu_u)$. If $\mathcal{S}_u \neq \emptyset$ and $z \in \mathcal{S}_u$, then (1.13) holds. More precisely $u \geq u_{z,\infty}$ where $u_{z,\infty} = \lim_{k\to\infty} u_{k\delta_z}$ and $u_{k\delta_z}$ is the solution of

$$\begin{aligned} \partial_t u + (-\Delta)^s u + t^\beta g(u) = 0 \quad \text{in} \quad Q_\infty \\ u(0,.) = k\delta_z. \end{aligned} \tag{1.18}$$

The existence and uniqueness of solutions to (1.18) follow from [28, Th 1.1]. If $g : \mathbb{R} \to \mathbb{R}_+$ is nondecreasing and satisfies that

$$\mathcal{G}(t) := \int_t^\infty \frac{ds}{g(s)} < +\infty \quad \text{for } t > 0, \tag{1.19}$$

and if $\beta > -1$, denote $U(t) = \mathcal{G}^{-1}(\frac{t^{\beta+1}}{\beta+1})$, where $\mathcal{G}^{-1}$ is the inverse function of $\mathcal{G}$, then the function U verifies that

$$\int_{U(t)}^\infty \frac{ds}{g(s)} = \frac{t^{\beta+1}}{\beta+1}, \tag{1.20}$$

and defines as the maximal solution of the ODE

$$\partial_t U + t^\beta g(U) = 0 \quad \text{on } \mathbb{R}_+^* \quad \text{satisfying } U(0) = +\infty. \tag{1.21}$$

Theorem D *Assume that $f(t,x,r) \geq t^\beta g(r)$, where $\beta > -1$ and g satisfies (1.19). If u is a nonnegative solution of (1.1) belonging to $L^1_{loc}(0,T;\mathcal{L}^s(\mathbb{R}^N))$, then*

$$u(t,x) \leq U(t), \quad \forall\, (t,x) \in Q_\infty. \tag{1.22}$$

Furthermore, if g satisfies

$$\int_1^\infty \frac{sds}{g(s)\left(\int_s^\infty \frac{d\tau}{g(\tau)}\right)^{\frac{\beta}{\beta+1}}} < +\infty, \tag{1.23}$$

then $\mathcal{S}_u = \tilde{\mathcal{S}}_u$ and (1.13) holds for any $z \in \mathcal{S}_u$.

Theorem E *Assume that $f(t,x,s) = t^\beta g(s)$, where $\beta > -1$ and g satisfies (1.19), is nondecreasing and is locally Lipschitz continuous. If u is a nonnegative solution of (1.1) belonging to $L^1_{loc}(0,T;\mathcal{L}^s(\mathbb{R}^N))$, then either*

$$u(t,x) < U(t), \quad \forall\, (t,x) \in Q_\infty, \tag{1.24}$$

or

$$u(t,x) = U(t), \quad \forall\, (t,x) \in Q_\infty. \tag{1.25}$$

In the second part of this paper we study in detail the initial trace problem for the equation

$$\partial_t u + (-\Delta)^s u + t^\beta u^p = 0 \quad \text{in} \quad Q_\infty, \tag{1.26}$$

when $s \in (0,1)$, $\beta > -1$ and $p \in (1, p^*_\beta)$. A second critical value of p appears

$$p^{**}_\beta = 1 + \frac{2s(1+\beta)}{N+2s}. \tag{1.27}$$

Actually, if $u_k := u_{k\delta_0}$ is unique solution to

$$\begin{aligned} \partial_t u + (-\Delta)^s u + t^\beta u^p = 0 \qquad & \text{in} \quad Q_\infty, \\ u(0,\cdot) = k\delta_0 \qquad & \text{in} \quad \mathbb{R}^N, \end{aligned} \tag{1.28}$$

it is proved in [28] that $u_\infty = \lim_{k\to\infty} u_k$ is very different according $1 < p < p^{**}_\beta$ or $p^{**}_\beta < p < p^*_\beta$. Precisely, (i) if $1 < p < p^{**}_\beta$, then

$$u_\infty(t,x) = U_{p,\beta}(t) := \left(\frac{1+\beta}{p-1}\right)^{\frac{1}{p-1}} t^{-\frac{1+\beta}{p-1}}. \tag{1.29}$$

The absorption is dominant, as if $s = 0$.
(ii) If $p^{**}_\beta < p < p^*_\beta$, then

$$u_\infty(t,x) = V(t,x) := t^{-\frac{1+\beta}{p-1}} v\left(\frac{x}{t^{\frac{1}{2s}}}\right), \tag{1.30}$$

where v is the minimal positive solution of

$$\begin{aligned} (-\Delta)^s v - \frac{1}{2s}\nabla v \cdot \eta - \frac{1+\beta}{p-1} v + v^p = 0 \quad \text{in} \quad \mathbb{R}^N, \\ \lim_{|\eta|\to\infty} |\eta|^{\frac{2s(1+\beta)}{p-1}} v(\eta) = 0. \end{aligned} \tag{1.31}$$

The function V is called the very singular solution of (1.26). In this case *the diffusion is dominant*, as when $s = 1$.

We first prove the following result which complements Theorem C in the case where $\beta > 0$. The proof is delicate and uses a form of parabolic Harnack inequality valid for solutions of (1.26).

Theorem F *Assume $\beta > -1$, $1 < p < p^*_\beta$ and u is a nonnegative solution of (1.26) with initial trace $(\mathcal{S}_u, \nu_u)$. If $\mathcal{S}_u \neq \emptyset$ and $z \in \mathcal{S}_u$ then $u \geq u_{z,\infty}$.*

We observe that $\tilde{\mathcal{S}}_{u_\infty} = \mathcal{S}_{u_\infty} = \{0\}$ when $p^{**}_\beta < p \leq p^*_\beta$ and $\tilde{\mathcal{S}}_{u_\infty} = \mathcal{S}_{u_\infty} = \mathbb{R}^N$ when $1 < p < p^{**}_\beta$. Notice that the case $p = p^{**}_\beta$ remained unsolved in [28]. In this paper, we prove that $\mathcal{S}_{u_\infty} = \mathbb{R}^N$ also for $p = p^{**}_\beta$. Our main result concerning (1.26) is the following.

Theorem G *Let u be a positive solution of (1.26).*
(i) If $p \in (1, p^{**}_\beta]$ and $\mathcal{S}_u \neq \emptyset$. Then $\mathcal{S}_u = \mathbb{R}^N$ and $u \geq U_{p,\beta}$. If we assume moreover that $u \in L^1_{loc}(0,\infty; \mathcal{L}^s(\mathbb{R}^N))$, then $u = U_{p,\beta}$.
(ii) If there exists $\kappa \in [1,N] \cap \mathbb{N}$ such that $p \in (1, p^*_\beta) \cap \left(1, 1 + \frac{2s(1+\beta)}{\kappa+2s}\right]$ and $\mathcal{S}_u$ contains an affine plane $\mathcal{L}$ of codimension κ. Then the conclusions of (i) hold.

If $\kappa = N$, (ii) is just (i). Note that if $0 < s \leq \frac{1}{2}$ or if $\kappa \geq N-2$, then $(p^{**}_\beta, p^*_\beta) \cap \left(p^{**}_\beta, 1+\frac{2s(1+\beta)}{\kappa+2s}\right] = (p^{**}_\beta, p^*_\beta)$, while, if $\frac{1}{2} < s < 1$ and $\kappa = N-1$, then $(p^{**}_\beta, p^*_\beta) \cap \left(p^{**}_\beta, 1+\frac{2s(1+\beta)}{\kappa+2s}\right] = \left(p^{**}_\beta, 1+\frac{2s(1+\beta)}{N-1+2s}\right]$.

Conversely, given a closed set of $\mathcal{S} \subset \mathbb{R}^N$ and a nonnegative Radon measure on ν on $\mathcal{R} = \mathbb{R}^N \setminus \mathcal{S}$, we study the existence of solution of (1.26) with a given initial trace $Tr_c(u) = Tr(u) = (\mathcal{S}, \nu)$, that is a solution of the following problem

$$\begin{aligned} \partial_t u + (-\Delta)^s u + t^\beta u^p = 0 \qquad &\text{in} \quad Q_\infty \\ Tr(u) = (\mathcal{S}, \nu). \end{aligned} \tag{1.32}$$

This means that u is a classical solution of the equation in Q_∞ and that (1.6) and (1.22) hold. By Theorem G any closed set cannot be the singular part of the initial trace of a positive solution of (1.26) if p is too small (diffusion effect) or if p is too large. In the same sense any positive bounded Radon measure ν cannot be the regular part of the initial trace of a positive solution of (1.26) since condition (1.15) is equivalent to $p < p^*_\beta$. However this condition is restrictive and there exist several sufficient conditions linking ν, s, β and p. Hence we say that a nonnegative bounded measure ν is an *admissible measure* if the initial value problem

$$\begin{aligned} \partial_t u + (-\Delta)^s u + t^\beta u^p = 0 \quad &\text{in} \quad Q_\infty \\ u(0,.) = \nu, \end{aligned} \tag{1.33}$$

admits a solution u_ν, always unique, and it is a *good measure* if it is stable in the sense that if ν is replaced by $\nu * \rho_n$ for some sequence of mollifiers, then $u_{\nu*\rho_n}$ and $t^\beta u^p_{\nu*\rho_n}$ converges to u_ν and $t^\beta u^p_\nu$ respectively in $L^1(Q_T)$. We denote by H_s is the kernel in $\mathbb{R}^*_+ \times \mathbb{R}^N$ associated to $(-\Delta)^s$. It is expressed by

$$H_s(t,x) = \frac{1}{t^{\frac{N}{2s}}} \tilde{H}_s\left(\frac{x}{t^{\frac{1}{2s}}}\right) \quad \text{where} \quad \tilde{H}_s(x) = \int_{\mathbb{R}^N} e^{ix.\xi - |\xi|^{2s}} d\xi. \tag{1.34}$$

It is proved in [24], [15] that H_s satisfies the following two-side estimate,

$$\frac{c_3^{-1} t}{t^{1+\frac{N}{2s}} + |x|^{N+2s}} \leq H_s(t,x) \leq \frac{c_3 t}{t^{1+\frac{N}{2s}} + |x|^{N+2s}} \quad \forall\, (t,x) \in \mathbb{R}^*_+ \times \mathbb{R}^N. \tag{1.35}$$

The associated potential $\mathbb{H}_s[\nu]$ of a bounded Radon measure ν in $\mathbb{R}^N$ is defined by

$$\mathbb{H}_s[\nu](t,x) = \int_{\mathbb{R}^N} H_s(t, x-y) d\nu(y).$$

We first prove that a nonnegative bounded measure with Lebesgue decomposition $\nu = \nu_0 + \nu_s$, where $\nu_0 \in L^1(\mathbb{R}^N)$ and ν_s is singular with respect to the N-dim Lebesgue measure is a good measures if $t^\beta (\mathbb{H}_s[\nu])^p \in L^1(Q_1)$. The expression of admissibility for a Radon measure needs the introduction of Bessel capacities which are presented in Section 4.1. Our main existence result of solutions for (1.33) is the following.

Theorem H *Let $N \geq 1$, $p > 1$ and $-1 < \beta < p-1$. Then a nonnegative bounded measure ν in $\mathbb{R}^N$ is an admissible measure if and only if ν vanishes on Borel subsets of $\mathbb{R}^N$ with zero $\mathrm{cap}^{\mathbb{R}^N}_{\frac{2s(1+\beta)}{p},p'}$-Bessel capacity.*

Concerning problem (1.32) we have the following general result.

Theorem I *Assume that $N \geq 1$ and $p > 1 + \frac{2s(1+\beta)}{1+2s}$. If $\mathcal{S}$ is a closed subset of $\mathbb{R}^N$ such that $\mathcal{S} = \overline{\mathrm{int}\,\mathcal{S}}$ and ν is a nonnegative Radon measure on $\mathcal{R} = \mathcal{S}^c$ such that for any compact set $K \subset \mathcal{R}$, $\chi_K \nu$ is an admissible measure. Then problem (1.32) admits a solution.*

It is interesting to compare this result with [37, Th. 4.11] dealing with the case $s = 1$. It is proved there that for any closed set satisfying a non-thinness condition (expressed later on in terms of Bessel capacity [32]) but always fulfilled when $1 < p < 1 + \frac{2}{N}$ and any nonnegative admissible Radon measure, problem (1.32) admits a solution. There the condition $p > 1 + \frac{2s(1+\beta)}{1+2s}$ has no counterpart when $s = 1$. Theorem L shows that this condition is fundamental in order to have existence without condition at infinity, even in the case where $\mathcal{S} = \emptyset$ and ν is a mere $L^1_{loc}(\mathbb{R}^N)$ function. In some particular cases, the existence of a solution to (1.32) with no extra condition on $\mathcal{S}$ or ν can be proved as the next results show it.

Theorem J *Assume that $\beta > -1$, $p > 1$ and one of the following assumptions is fulfilled:*
(i) either $N = 1$ and $1 + \frac{2s(1+\beta)}{1+2s} < p < 1 + 2s(1+\beta)$,
(ii) or $N = 2$, $\frac{1}{2} < s < 1$ and $1 + \frac{2s(1+\beta)}{1+2s} < p < 1 + s(1+\beta)$.
Then for any closed set $\mathcal{S} \subset \mathbb{R}^N$ and any nonnegative measure ν in $\mathcal{R} = \mathcal{S}^c$, there exists a nonnegative solution u to (1.32).

As a consequence of the previous results we obtain existence with initial data measure in $\mathbb{R}^N$ of solutions without condition at infinity in the spirit of Brezis classical result [17].

Corollary K *Let $N \geq 1$, $\beta > -1$ and $p > 1$.*
(a) If $p > 1 + \frac{2s(1+\beta)}{1+2s}$ and $-1 < \beta < p-1$, then there exists a positive solution $u \in L^1_{loc}(0,\infty; \mathcal{L}^s(\mathbb{R}^N))$ to problem (1.33) for any nonnegative Radon measure ν in $\mathbb{R}^N$ if and only if for any $n \in \mathbb{N}^*$, $\chi_{B_n}\nu$ is an admissible measure.
(b) If one of the assumptions (i) or (ii) of Theorem J is fulfilled, then for any nonnegative Radon measure ν in $\mathbb{R}^N$ there exists a positive solution to problem (1.33).

Conditions (i) or (ii) of Theorem J are essentially necessary for unconditional existence, since we have the following result.

Theorem L *Assume that $\beta > -1$, $p > 1$ and $1 < p < 1 + \frac{2s(1+\beta)}{1+2s}$. If $\phi \in L^1_{loc}(\mathbb{R}^N)$ is a nonnegative function which satisfies*

$$\lim_{|x|\to\infty} \frac{\phi(x)}{|x|^{\frac{2s(\beta+1)}{\beta+2-p}}} = +\infty, \tag{1.36}$$

then the sequence of solutions $\{u_n\}$ of (1.33) with initial data $\nu = \nu_n = \inf\{\phi, \phi_n\}$, where $\phi_n = \inf\{\phi(z) : |z| \geq n\}$ is increasing and converges to $U(t)$.

This implies that there exists no solution of (1.33) with initial data ϕ. Notice that $\beta+2-p$ is positive for $p<1+\frac{2s(1+\beta)}{1+2s}$. For the mere heat equation a theory of maximal growth for admissibility growth of initial data has been developed in [51] and for the fractional heat equation in [15]. In both cases the representation formulas play an important role. For equations with potential a phenomenon of instantaneous blow-up is proved [6] for solution of

$$\partial_t u-\Delta u-V(x)u=0 \quad \text{in} \quad Q_\infty, \tag{1.37}$$

when $V\sim c|x|^{-2}$, for any nonnegative initial data. This phenomenon of instantaneous blow-up has been recently highlighted in [48] for the the semilinear equation

$$\partial_t u-\Delta u+u\left(\ln(u+1)\right)^\alpha=0 \quad \text{in} \quad Q_\infty, \tag{1.38}$$

when $1<\alpha<2$. It is shown there that the limit of a sequence of solutions with fast growing initial data is the maximal solution of $u'+u\left(\ln(u+1)\right)^\alpha=0$ on $(0,\infty)$.

Acknowledgements

H. Chen is supported by NNSF of China, No: 11726614, 11661045, by the Jiangxi Provincial Natural Science Foundation, No: 20161ACB20007. The authors are grateful to the referee for the careful reading of the manuscript which enabled several improvements in the presentation of this work.

2 Initial trace with general nonlinearity

2.1 Existence of an initial trace

Proof of Theorem A. For any bounded domain $\omega\subset\mathbb{R}^N$, we denote by $C_0^2(\overline{\omega})$ the space of functions $\xi:\mathbb{R}^N\to\mathbb{R}$ which are C^2 and have compact support in $\overline{\omega}$. We always assume that $N\geq 1$ and $0<s<1$. Let ϕ_ω be the first eigenfunction of $(-\Delta)^s$ in $H_0^s(\omega)$, with corresponding eigenvalue $\lambda_\omega>0$, i.e. the solution of

$$\begin{array}{rll}(-\Delta)^s\phi_\omega=\lambda_\omega\phi_\omega & \text{in} & \omega\\ \phi_\omega=0 & \text{in} & \omega^c.\end{array} \tag{2.1}$$

Existence and basic properties of the eigenfunctions can be found in [5], [16]. We normalize ϕ_ω by $\sup\phi_\omega=1$. We say that ω is of class **E. S. C.** if it satisfies the *exterior sphere condition*. It is known by [45, Prop 1.1] that $\phi_\omega(x)\leq c(\text{dist}\,(x,\partial\omega))^s$ in ω, and there exists $q>2$ such that $\phi_\omega^q\in C_0^2(\overline{\omega})$. We denote by $K_\rho(z)$ the open cube with sides parallel to the axis of center $z\in\mathbb{R}^N$ and length sides $\rho>0$, and $K_1:=K_1(0)$. Then

$$\phi_{K_\rho(z)}(x)=\phi_{K_1}\left(\frac{x-z}{\rho}\right) \quad \text{and} \quad \lambda_{K_\rho(z)}=\frac{\lambda_{K_1}}{\rho^{2s}}.$$

The next lemma is an improvement of [26, Lemma 2.3].

Lemma 2.1 *Let $q \in \mathbb{N} \cap [2, \infty)$ and $\zeta \in C_0^2(\overline{\omega})$, $\zeta \geq 0$, then*

$$\begin{aligned}(-\Delta)^s \zeta^q(x) &= q\zeta^{q-1}(x)(-\Delta)^s\zeta(x) - \frac{a_{N,s}}{q(q-1)}\int_{\mathbb{R}^N} \frac{\zeta^q(y) - \zeta^q(x) - q(\zeta(y)-\zeta(x))\zeta^{q-1}(y)}{|x-y|^{N+2s}}dy \\ &\geq q\zeta^{q-1}(x)(-\Delta)^s\zeta(x) - \frac{a_{N,s}}{q}\zeta^{q-2}(x)\int_{\mathbb{R}^N}\frac{(\zeta(y)-\zeta(x))^2}{|x-y|^{N+2s}}dy.\end{aligned} \tag{2.2}$$

Proof. From [26, Lemma 2.3], we know that

$$(-\Delta)^s\zeta^q(x) = q\zeta^{q-1}(x)(-\Delta)^s\zeta(x) - q(q-1)a_{N,s}\int_{\mathbb{R}^N}\left(\int_{\zeta(x)}^{\zeta(y)}(\zeta(y)-t)t^{q-2}dt\right)\frac{dy}{|x-y|^{N+2s}}.$$

By integration by parts, we obtain that

$$\begin{aligned}\int_{\zeta(x)}^{\zeta(y)}(\zeta(y)-t)t^{q-2}dt &= \frac{1}{q(q-1)}\left(\zeta^q(y) - \zeta^q(x) - q(\zeta(y)-\zeta(x))\zeta^{q-1}(x)\right) \\ &= \frac{\zeta(y)-\zeta(x)}{q(q-1)}\left[\zeta^{q-1}(y) + \zeta^{q-2}(y)\zeta(x) + ... + \zeta(y)\zeta^{q-2}(x) - (q-1)\zeta^{q-1}(x)\right].\end{aligned}$$

Since for any $a, b \geq 0$

$$\begin{aligned}&b^{q-1} + ab^{q-2} + ...a^{q-2}b - (q-1)a^{q-1} \\ &\quad= b^{q-1} - a^{q-1} + a(b^{q-2} - a^{q-2}) + a^2(b^{q-3} - a^{q-3}) + ... + a^{q-2}(b-a) \\ &\quad= (b-a)\left[\left(b^{q-2} + ab^{q-3} + ... + a^{q-2}\right) + a\left(b^{q-3} + ab^{q-4} + ... + a^{q-3}\right) + ... + a^{q-2}\right] \\ &\quad\geq (q-1)(b-a)a^{q-2},\end{aligned}$$

we obtain (2.2). □

Remark. By the mean value theorem, we see that there exists $m_\zeta \in \{z = \zeta(w) : w \in \mathbb{R}^N\}$ such that

$$\begin{aligned}L(\zeta^q) &:= \frac{a_{N,s}}{q(q-1)}\int_{\mathbb{R}^N}\frac{\zeta^q(y) - \zeta^q(x) - q(\zeta(y)-\zeta(x))\zeta^{q-1}(x)}{|x-y|^{N+2s}}dy \\ &= \frac{a_{N,s}}{2}m_\zeta^{q-2}\int_{\mathbb{R}^N}\frac{(\zeta(y)-\zeta(x))^2}{|x-y|^{N+2s}}dy.\end{aligned} \tag{2.3}$$

Proposition 2.2 *Assume that f satisfies (1.2) and u is a nonnegative solution of (1.1) such that $u(t,\cdot) \in \mathcal{L}^s(\mathbb{R}^N)$ for all $t \in (0,T)$. If $f(\cdot,\cdot,u) \in L^1(Q_\omega^T)$ for some bounded domain $\omega \subset \mathbb{R}^N$ of class* **E. S. C.** *and $T > 0$. Then there exists $\ell_\omega \geq 0$ such that*

$$\lim_{t\to 0}\int_\omega u(t,x)\phi_\omega^q(x)dx = \ell_\omega. \tag{2.4}$$

Furthermore, we have that

$$\begin{aligned}\ell_\omega + \frac{a_{N,s}}{q}\int_0^T\int_{\mathbb{R}^N}u(s,x)\phi_\omega^{q-2}(x)\left(\int_{\mathbb{R}^N}\frac{(\phi_\omega(y)-\phi_\omega(x))^2}{|x-y|^{N+2s}}dy\right)dx \\ \leq e^{q\lambda_\omega T}X(T) + \int_0^T\int_\omega f(s,x,u)\phi_\omega^q(x)e^{q\lambda_\omega s}dxdt.\end{aligned} \tag{2.5}$$

Proof. Since $\phi_\omega^q \in C_0^2(\overline{\omega})$, there holds

$$\frac{d}{dt}\int_\omega u(t,x)\phi_\omega^q(x)dx + \int_{\mathbb{R}^N} u(t,x)(-\Delta)^s\phi_\omega^q(x)dx + \int_\omega f(t,x,u)\phi_\omega^q(x)dx = 0. \tag{2.6}$$

Set

$$X(t) = \int_\omega u(t,x)\phi_\omega^q(x)dx,$$

then

$$\frac{d}{dt}\left(e^{q\lambda_\omega t}X(t) - \int_t^T\int_\omega f(s,x,u)\phi_\omega^q(x)e^{q\lambda_\omega s}dxds\right) = e^{q\lambda_\omega t}\int_{\mathbb{R}^N} L(\phi_\omega^q)(x)dx \geq 0. \tag{2.7}$$

This implies that $\lim_{t\to 0} X(t) = \ell_\omega$ exists and

$$\ell_\omega + \int_0^T\int_{\mathbb{R}^N} L(\phi_\omega^q)(x)e^{q\lambda_\omega s}u(s,x)dxds + \int_0^T\int_\omega f(s,x,u)\phi_\omega^q(x)e^{q\lambda_\omega s}dxds = e^{q\lambda_\omega T}X(T),$$

which implies (2.5) by Lemma 2.1. □

As an immediate consequence we have,

Corollary 2.3 *Under the assumptions of Theorem A, $u \in L^1(Q_G^T)$ for any compact set $G \subset \mathcal{R}_u$.*

The proof of Theorem A is completed by the following statement:

Proposition 2.4 *There exists a nonnegative Radon measure μ_u on $\mathcal{R}_u$ such that for any $\zeta \in C_0^2(\mathcal{R}_u)$, there holds*

$$\lim_{t\to 0}\int_{\mathcal{R}_u} u(t,x)\zeta(x)dx = \int_{\mathcal{R}_u}\zeta d\mu_u. \tag{2.8}$$

Proof. Let $\zeta \in C_0^2(\mathcal{R}_u)$ with support K and let G be an open subset containing K such that ∂G is smooth and $\overline{G}$ is a compact subset of $\mathcal{R}_u$ and assume $0 \leq \zeta \leq 1$. We put

$$Y(t) = \int_{\mathbb{R}^N} u(t,x)\zeta(x)dx = \int_G u(t,x)\zeta(x)dx,$$

and

$$\int_{\mathbb{R}^N} f(t,x,u)\zeta(x)dx = \int_G f(t,x,u)\zeta(x)dx,$$

then

$$Y'(t) + \int_{\mathbb{R}^N} u(t,x)(-\Delta)^s\zeta(x)dx + \int_G f(t,x,u)\zeta(x)dx = 0.$$

Since $\zeta \geq 0$, we have that

$$\begin{aligned}
&\frac{1}{a_{N,s}}\int_{\mathbb{R}^N} u(t,x)(-\Delta)^s\zeta(x)dx \\
&\qquad = \int_{\mathbb{R}^N} u(t,x)\int_G \frac{\zeta(x)-\zeta(y)}{|x-y|^{N+2s}}dydx + \int_G u(t,x)\zeta(x)\int_{G^c}\frac{dy}{|x-y|^{N+2s}}dx \\
&\qquad = \int_G u(t,x)\int_G \frac{\zeta(x)-\zeta(y)}{|x-y|^{N+2s}}dydx - \int_{G^c} u(t,x)\int_G \frac{\zeta(y)}{|x-y|^{N+2s}}dydx \\
&\qquad\qquad + \int_G u(t,x)\zeta(x)\int_{G^c}\frac{dy}{|x-y|^{N+2s}}dx \\
&\qquad \leq \int_G u(t,x)\int_G \frac{\zeta(x)-\zeta(y)}{|x-y|^{N+2s}}dydx + \int_G u(t,x)\zeta(x)\int_{G^c}\frac{dy}{|x-y|^{N+2s}}dx
\end{aligned} \tag{2.9}$$

If we define the regional fractional Laplacian of order s relative to G by

$$(-\Delta)^s_G\zeta(x) := a_{N,s}\int_G \frac{\zeta(x)-\zeta(y)}{|x-y|^{N+2s}}dy,$$

then the right-hand side of (2.9) is bounded from above by

$$\Lambda(t) = \left(\|(-\Delta)^s_G\zeta\|_{L^\infty} + \max_{x\in K}\int_{G^c}\frac{dy}{|x-y|^{N+2s}}\right)\int_G u(t,x)dx,$$

since ζ is C^2 with support in $K \subset G \subset \overline{G} \Subset \mathcal{R}_u$. By Corollary 2.3, $\Lambda \in L^1(0,T)$. Because

$$\frac{d}{dt}\left(Y(t) - \int_t^T\left(\Lambda(s) + \int_G f(t,x,u)\zeta(x)dx\right)ds\right) \geq 0, \tag{2.10}$$

and

$$\int_0^T\left(\Lambda(s) + \int_G f(t,x,u)\zeta(x)dx\right)ds < \infty \tag{2.11}$$

Combining (2.10) and (2.11) we infer that the following limit exists

$$\lim_{t\to 0} Y(t) = \lim_{t\to 0}\int_G u(t,x)\zeta(x)dx := \tilde{\mu}_u(\zeta). \tag{2.12}$$

By replacing ζ by $\|\zeta\|_{L^\infty}\zeta$ we can drop the condition $\zeta \leq 1$. with support in K, then

$$0 \leq \lim_{t\to 0}\int_G u(t,x)\zeta(x)dx = \tilde{\mu}_u(\zeta) \leq \ell_G\|\zeta\|_{L^\infty}. \tag{2.13}$$

Next we assume that $\zeta \in C_0(\mathcal{R}_u)$ is nonnegative, with support $K \subset G \subset \overline{G} \Subset \mathcal{R}_u$, then there exists an increasing sequences $\{\zeta_n\} \subset C_0^2(\mathcal{R}_u)$ of nonnegative functions smaller than ζ which converges to ζ uniformly (take for example $\zeta_n = (\zeta - n^{-1})_+ * \rho_n$ for some sequence of mollifiers $\{\rho_n\}$ with $\mathrm{supp}(\rho_n) \subset B_{n^{-2}}$). The sequence $\{\tilde{\mu}_u(\zeta_n)\}$ is increasing and bounded from above by $M\ell_G \sup_G \zeta$. Hence it is convergent and its limit, still denoted by $\tilde{\mu}_u(\zeta)$ is independent of the

sequence $\{\zeta_n\}$. We can also consider a uniform approximation of ζ from above in considering $\zeta_n' = (\sigma_n + \zeta) * \rho_n$, where $\sigma_n = n^{-1}\chi_{K_n}$ and $K_n = \{x \in \mathbb{R}^N : \mathrm{dist}\,(x, K) \leq n^{-1}\}$. Actually,

$$\tilde{\mu}_u(\zeta) = \sup\{\tilde{\mu}_u(\eta) : \eta \in C_0^2(\mathcal{R}_u), 0 \leq \eta \leq \zeta\} = \inf\{\tilde{\mu}_u(\eta') : \eta' \in C_0^2(\mathcal{R}_u), \zeta \leq \eta'\}. \tag{2.14}$$

This implies that for all η and η' belonging to $C_0^2(\mathcal{R}_u)$ such that $\eta \leq \zeta \leq \eta'$, we have that

$$\tilde{\mu}_u(\eta) \leq \liminf_{t\to 0} \int_{\mathcal{R}_u} u(t,x)\zeta(x)dx \leq \limsup_{t\to 0} \int_{\mathcal{R}_u} u(t,x)\zeta(x)dx \leq \tilde{\mu}_u(\eta'). \tag{2.15}$$

Combined with (2.14) we infer the existence of the limit and

$$\lim_{t\to 0} \int_{\mathcal{R}_u} u(t,x)\zeta(x)dx = \tilde{\mu}_u(\zeta). \tag{2.16}$$

Finally, if $\zeta \in C_0(\mathcal{R}_u)$ is a signed function we write $\zeta = \zeta_+ - \zeta_-$ and $\mu_u(\zeta) = \tilde{\mu}_u(\zeta_+) - \tilde{\mu}_u(\zeta_-)$. Hence μ_u is a positive Radon measure on $\mathcal{R}_u$, and (2.8) follows from (2.16) with ζ replaced by ζ_+ and ζ_-. □

Lemma 2.5 *Assume that $G \subset \mathbb{R}^N$ is a bounded smooth domain and $\eta \in C_0^2(G)$. Then there exists $c_5 > 0$ such that*

$$|(-\Delta)^s \eta(x)| \leq \frac{c_5 \|\eta\|_{C^2}}{1 + |x|^{N+2s}} \qquad \forall\, x \in \mathbb{R}^N. \tag{2.17}$$

Moreover, assume that $\eta \geq 0$ in G, then $(-\Delta)^s \eta \leq 0$ in G^c and for any $\delta > 0$ there exists $c_\delta > 1$ independent of η such that

$$\frac{\|\eta\|_{L^1}}{c_\delta(1 + |x|^{N+2s})} \leq -(-\Delta)^s \eta(x) \leq \frac{c_\delta \|\eta\|_{L^1}}{1 + |x|^{N+2s}}, \tag{2.18}$$

for $x \in \{z \in \mathbb{R}^N : \mathrm{dist}\,(z, G) \geq \delta\}$.

Proof. Let $x \in G^c$ and $y \in \mathbb{R}^N$, then $\eta(x) - \eta(y) \leq 0$ and hence $(-\Delta)^s \eta \leq 0$ in G^c. For $y \in G$ and $x \in G^c$ satisfying $\mathrm{dist}\,(x, G) > \delta$, there exists $c_6 > 1$ such that

$$c_6^{-1}(1 + |x|^{N+2s}) \leq |x - y|^{N+2s} \leq c_6(1 + |x|^{N+2s}).$$

Together with

$$(-\Delta)^s \eta(x) = -a_{N,s} \int_G \frac{\eta(y)}{|y - x|^{N+2s}} dy \qquad \forall\, x \in G^c,$$

one obtains the claim. □

Remark. Estimate (2.17) has essentially been already obtained in [11, Lemma 2.1] but we kept it for the sake of completeness. (2.17). Estimate (2.20) is new and will be useful in the sequel.

Proof of Theorem B. Let $\rho > \rho' > 0$ and $\zeta \in C_0^2(B_\rho(z))$ such that $0 \le \zeta \le 1$ and $\zeta = 1$ on $B_{\rho'}(z))$. Then there holds

$$\int_{B_\rho(z)} u(t,x)\zeta(x)dx = \int_{B_\rho(z)} u(T,x)\zeta(x)dx + \int_t^T \int_{B_\rho(z)} f(s,x,u)\zeta(x)dxds + \int_t^T \int_{\mathbb{R}^N} u(s,x)(-\Delta)^s\zeta(x)dxds.$$

The function ζ satisfies

$$|(-\Delta)^s\zeta(x)| \le \frac{c_5\|\zeta\|_{C^2}}{1+|x|^{N+2s}}, \quad \forall\, x \in \mathbb{R}^N.$$

Since $(t,x) \mapsto (1+|x|^{N+2s})^{-1}u((t,x) \in L^1(Q_T)$, we infer that

$$\lim_{t\to 0} \int_{B_\rho(z)} u(t,x)\zeta(x)dx = +\infty, \tag{2.19}$$

which implies the claim. □

2.2 Pointwise estimates

Proof of Theorem C. In what follows we characterize the singular set of the initial trace when the absorption reaction is subcritical, that is it satisfies (1.14), (1.15) and (1.16) hold. Under these two last assumptions for any bounded Radon measure in $\mathbb{R}^N$, it is proved in [28, Th 1.1] that there exists a unique weak solution $u := u_\mu$ to

$$\begin{aligned} \partial_t u + (-\Delta)^s u + t^\beta g(u) = 0 \qquad & \text{in } \ Q_\infty \\ u(0,\cdot) = \mu \qquad & \text{in } \ \mathbb{R}^N. \end{aligned} \tag{2.20}$$

We recall by a weak solution, we mean a function $u \in L^1(Q_T)$ such that $t^\beta g(u) \in L^1(Q_T)$ for all $T > 0$ satisfying

$$\int_0^T \int_{\mathbb{R}^N} \left[\left(-\partial_t\xi + (-\Delta)^s\xi\right) u + t^\beta g(u)\xi \right] dxdt = \int_{\mathbb{R}^N} \xi(0,x)d\mu(x) \qquad \forall\, \xi \in \mathbb{Y}_{s,T}, \tag{2.21}$$

where $\mathbb{Y}_{s,T}$ is the space of functions ξ defined in Q_∞ satisfying

(*i*) $\|\xi\|_{L^1(Q_T)} + \|\xi\|_{L^\infty(Q_T)} + \|\partial_t\xi\|_{L^\infty(Q_T)} + \|(-\Delta)^s\xi\|_{L^\infty(Q_T)} < +\infty$,

(*ii*) $\xi(T) = 0$ and for $0 < t < T$, there exist $M > 0$ and $\epsilon_0 > 0$ such that for $0 < \epsilon \le \epsilon_0$, $\|(-\Delta)^s_\epsilon\xi(t,\cdot)\|_{L^\infty(\mathbb{R}^N)} \le M$.

Furthermore, if μ_j converges to μ weakly in the sense of measures, then u_{μ_j} converges to u_μ locally uniformly in Q_∞. Up to translation we can assume that $z = 0$. Since (1.22) holds, for any $k > 0$ there exist two sequences $\{t_n\}$ and $\{\rho_n\}$ converging to 0 such that

$$\int_{B_{\rho_n}} u(t_n,x)dx = k. \tag{2.22}$$

Case 1: $\beta \leq 0$. For $R > 0$, let v_n^R be the solution of

$$\begin{aligned} \partial_t v + (-\Delta)^s v + t^\beta g(v) &= 0 && \text{in } (t_n, \infty) \times B_R \\ v(t,x) &= 0 && \text{in } (t_n, \infty) \times B_R^c \\ v(t_n, .) &= u(t_n, x)\chi_{B_{\rho_n}} && \text{in } B_R, \end{aligned} \tag{2.23}$$

where B_R denote the ball in $\mathbb{R}^N$ centered at origin with the radius R. By the comparison principle, $u \geq v_n^R$ in $[t_n, \infty) \times B_R$. We set $v_n^R(t,x) = v_n^R(\tau + t_n, x) = \tilde{v}_n^R(\tau, x)$. Since $\beta \leq 0$, there holds

$$\partial_t \tilde{v}_n^R + (-\Delta)^s \tilde{v}_n^R + \tau^\beta g(\tilde{v}_n^R) \geq 0 \quad \text{in } (0, \infty) \times B_R.$$

Hence $\tilde{v}_n^R \geq u_n^R$ where u_n^R is the solution of

$$\begin{aligned} \partial_t v + (-\Delta)^s v + t^\beta g(v) &= 0 && \text{in } (0, \infty) \times B_R \\ v(t,x) &= 0 && \text{in } (0, \infty) \times B_R^c \\ v(0, .) &= u(t_n, x)\chi_{B_{\rho_n}} && \text{in } B_R. \end{aligned} \tag{2.24}$$

Letting $R \to \infty$ we infer that u_n^R increases and converges to the solution u_n^∞ of

$$\begin{aligned} \partial_t v + (-\Delta)^s v + t^\beta g(v) &= 0 && \text{in } (0, \infty) \times \mathbb{R}^N \\ v(0, .) &= u(t_n, x)\chi_{B_{\rho_n}} && \text{in } \mathbb{R}^N, \end{aligned} \tag{2.25}$$

and there holds $u(t_n + \tau, x) \geq u_n^\infty(\tau, x)$ in $(0, \infty) \times \mathbb{R}^N$. Letting $n \to \infty$ and using the above mentioned stability result, we obtain that u_n^∞ converges to $u_{k\delta_0}$ and $u \geq u_{k\delta_0}$. Since it holds true for any k, the claim follows.

Case 2: $\beta > 0$. Clearly $u \geq v_n^\infty$ where v_n^∞ satisfies

$$\begin{aligned} \partial_t v + (-\Delta)^s v + t^\beta g(v) &= 0 && \text{in } (t_n, \infty) \times \mathbb{R}^N \\ v(t_n, .) &= u(t_n, x)\chi_{B_{\rho_n}} && \text{in } \mathbb{R}^N, \end{aligned} \tag{2.26}$$

Then $v_n^\infty(t,x) \leq \mathbb{H}_s[u(t_n, x)\chi_{B_{\rho_n}}](t - t_n, x)$. Since g satisfies (1.17) it follows from [28, Proof of Th 1.1] that the set of function $\left\{ g\left(\mathbb{H}_s[u(t_n, .)\chi_{B_{\rho_n}}](. - t_n, .) \right) \right\}$ is uniformly integrable in $(t_n, \infty) \times \mathbb{R}^N$ and it is the same with $\{g(v_n^\infty)\}$. Therefore, for any $T > 0$, $\left\{ t^\beta g(v_n^\infty) \right\}$ is uniformly integrable in $(t_n, T) \times \mathbb{R}^N$. Hence $\{v_n^\infty\}$ converges locally in $(0, \infty) \times \mathbb{R}^N$ to $u_{k\delta_0}$ and $u \geq u_{k\delta_0}$ as above.

□

Remark. We will see in Theorem F that if $g(u) = u^p$ the concentration result holds under the mere condition (1.15) whatever is the sign of β. The difficulty in the case $\beta > 0$ comes from the fact that the ball B_{ρ_n} may shrink very quickly with t_n and that a pointwise isolated singularity at (τ, z) with $\tau > 0$ can be removable for equation (2.20). In the power case we can control the rate of shrinking thanks to a Harnack-type inequality.

Proof of Theorem D. (i) *Proof of (1.22).* Let $\gamma \in C^2(\mathbb{R})$ be a convex nondecreasing function vanishing on $(-\infty, 0]$ such that $\gamma(r) \leq r_+$. For $\epsilon > 0$, let U_ϵ be the solution of

$$\begin{aligned} \partial_t U + t^\beta g(U) &= 0 \qquad \text{in } (\epsilon, +\infty) \\ U(\epsilon) &= +\infty. \end{aligned} \tag{2.27}$$

Indeed,

$$U_\epsilon(t) = \mathcal{G}^{-1}(\frac{t^{\beta+1} - \epsilon^{\beta+1}}{\beta+1}),$$

where $\mathcal{G}^{-1}$ is the inverse function of $\mathcal{G}$, see (1.20). Then there holds

$$\begin{aligned}(-\Delta)^s\gamma(u(t,.) - U_\epsilon(t))(x) &= \gamma'((u(t,x) - U_\epsilon(t))(-\Delta)^s u(x) \\ &\quad - a_{N,s}\frac{\gamma''(u(t,z_x) - U_\epsilon(t))}{2}\int_{\mathbb{R}^N}\frac{(u(t,y) - u(t,x))^2}{|x-y|^{N+2s}}dy.\end{aligned}$$

Notice that the integral is convergent if $t > \epsilon$, since $\gamma(u(t,\cdot) - U(t-\epsilon)) = \gamma(u(t,.) - U_\epsilon(t))$, where $0 \le u(t,\cdot) \le U_\epsilon(t)$ and u satisfies

$$\int_{\mathbb{R}^N}\frac{u(\cdot,x)dx}{1+|x|^{N+2s}} < +\infty \qquad \text{a.e. in } (0,T).$$

Then

$$\begin{aligned}&\partial_t\gamma(u(t,x) - U_\epsilon(t)) + (-\Delta)^s\gamma(u(t,.) - U_\epsilon(t))(x) \\ &\qquad \le \gamma'(u(t,x) - U_\epsilon(t)) \cdot (\partial_t u(t,x) - \partial_t U_\epsilon(t) + (-\Delta)^s u(x)) \\ &\qquad \le \gamma'(u(t,x) - U_\epsilon(t)) \cdot \big(t^\beta g(U_\epsilon(t)) - f(t,x,u(t,x))\big) \\ &\qquad \le 0.\end{aligned}$$

Therefore, $\gamma(u(\cdot,\cdot) - U_\epsilon(\cdot))$ is a subsolution. Let $\eta \in C_0^\infty(\mathbb{R}^N)$ be a nonnegative function. Using Lemma 2.5 we have that

$$\left|\int_{\mathbb{R}^N}\gamma(u(t,x) - U_\epsilon(t))(-\Delta)^s\eta dx\right| \le c_5\|\eta\|_{C^2}\int_{\mathbb{R}^N}\frac{u(t,x)dx}{1+|x|^{N+2s}}.$$

Since $u \in L^1_{loc}(0,T;\mathcal{L}^s(\mathbb{R}^N))$, for almost all s,t such that $\epsilon < s < t$, there holds

$$\begin{aligned}\int_{\mathbb{R}^N}\gamma(u(t,x) - U_\epsilon(t))\eta(x)dx + \int_s^t\int_{\mathbb{R}^N}\gamma(u(\tau,x) - U_\epsilon(t))(-\Delta)^s\eta(x)dxd\tau \\ \le \int_{\mathbb{R}^N}\gamma(u(s,x) - U_\epsilon(s))\eta(x)dx.\end{aligned}$$

Since $\gamma(u(s,x) - U_\epsilon(s))\eta(x) \le u(s,x)\eta(x)$ and $u(s,.)\eta \in L^1(\mathbb{R}^N)$, we get from the dominated convergence theorem that

$$\lim_{s\downarrow\epsilon}\int_{\mathbb{R}^N}\gamma(u(s,x) - U_\epsilon(s))\eta(x)dx = 0.$$

Hence, letting $s \to \epsilon$ and $\gamma(r) \uparrow r_+$, we get

$$\begin{aligned}\int_{\mathbb{R}^N}(u(t,x) - U_\epsilon(t))_+\eta(x)dx &\le \left|\int_\epsilon^t\int_{\mathbb{R}^N}(u(\tau,x) - U_\epsilon(t))_+(-\Delta)^s\eta(x)dxd\tau\right| \\ &\le c_5\|\eta\|_{C^2}\int_\epsilon^t\int_{\mathbb{R}^N}\frac{(u(\tau,x) - U_\epsilon(t))_+}{1+|x|^{N+2s}}dxd\tau.\end{aligned} \tag{2.28}$$

Next, for $n \geq 1$, we replace η by $\eta_n(x) = \eta(n^{-1}x)$, where $0 \leq \eta \leq 1$, $\eta(x) = 1$ on B_1 and $\mathrm{supp}(\eta) \subset B_2$. We can also assume that η is radially decreasing and $\eta(0) = 1$. Since $\|\eta_n\|_{C^2} \leq \|\eta\|_{C^2}$, we obtain from (2.28) and the monotone convergence theorem that the following holds for almost all $t \in (\epsilon, T)$

$$\int_{\mathbb{R}^N} (u(t,x) - U_\epsilon(t))_+ dx \leq c_5 \|\eta\|_{C^2} \int_\epsilon^t \int_{\mathbb{R}^N} \frac{(u(\tau,x) - U_\epsilon(\tau))_+}{1+|x|^{N+2s}} dx d\tau. \tag{2.29}$$

This inequality implies that $(u(t,.) - U_\epsilon(t))_+ \in L^1(\mathbb{R}^N)$ for almost all $t \in (\epsilon, T)$. We set

$$\Psi_\epsilon(t) = \int_\epsilon^t \int_{\mathbb{R}^N} \frac{(u(\tau,x) - U_\epsilon(\tau))_+}{1+|x|^{N+2s}} dx d\tau.$$

Then

$$\Psi'_\epsilon(t) = \int_{\mathbb{R}^N} \frac{(u(t,x) - U_\epsilon(t))_+}{1+|x|^{N+2s}} dx \leq \int_{\mathbb{R}^N} (u(t,x) - U_\epsilon(t))_+ dx \leq c_5 \|\eta\|_{C^2} \Psi_\epsilon(t).$$

Since $\Psi_\epsilon(\epsilon) = 0$ we obtain $\Psi_\epsilon(t) = 0$ on $(0,T)$, hence $u(t,x) \leq U_\epsilon(t)$ a.e. on $(\epsilon, T) \times \mathbb{R}^N$. Letting $\epsilon \to 0$, we get the claim.

(ii) *End of the proof.* Because of Theorem B it is sufficient to prove that if (1.23) holds true, then $U \in L^1(0,1)$. Indeed, we recall that

$$\mathcal{G}(s) = \int_s^\infty \frac{d\tau}{g(\tau)}.$$

Clearly, $\mathcal{G}$ is an decreasing diffeomorphism from $\mathbb{R}_+^*$ onto $(0, \Phi(0))$ and $U(t) = \mathcal{G}^{-1}\left(\frac{t^{\beta+1}}{\beta+1}\right)$. Set $U(t) = s$, then $t = ((\beta+1)\mathcal{G}(s))^{\frac{1}{\beta+1}}$ and we get

$$\begin{aligned} \int_0^1 U(t)dt &= \int_\infty^{U(1)} s\mathcal{G}'(s) \left((\beta+1)\mathcal{G}(s)\right)^{-\frac{\beta}{\beta+1}} ds \\ &= (\beta+1)^{-\frac{\beta}{\beta+1}} \int_{U(1)}^\infty \frac{sds}{g(s)\left(\int_s^\infty \frac{d\tau}{g(\tau)}\right)^{\frac{\beta}{\beta+1}}} < +\infty, \end{aligned}$$

which completes the proof. □

The following weight function plays an important role in the description of the initial trace problem for positive solutions of the fractional heat equation

$$\Phi(x) = \frac{1}{\left(1 + (|x|^2 - 1)_+^4\right)^{\frac{N+2s}{8}}}, \quad \forall\, x \in \mathbb{R}^N. \tag{2.30}$$

It has the remarkable property that

$$-c_6\Phi(x) \leq (-\Delta)^s \Phi(x) \leq c_6 \Phi(x), \quad \forall\, x \in \mathbb{R}^N, \tag{2.31}$$

for some constant $c_6 > 0$ (see [15], [11]). Furthermore, for some $c_7 > 1$,

$$\frac{1}{c_7(1+|x|^{N+2s})} \leq \Phi(x) \leq \frac{c_7}{1+|x|^{N+2s}}, \quad \forall\, x \in \mathbb{R}^N. \tag{2.32}$$

Lemma 2.6 *Let $f : \mathbb{R}^*_+ \times \mathbb{R}^N \times \mathbb{R}_+ \to \mathbb{R}_+$ be a Caratheodory function which satisfies (1.2) and is nondecreasing with respect to the variable u. For given $u_0 \in L^1(\mathbb{R}^N)$ is nonnegative, problem*

$$\begin{aligned} \partial_t u + (-\Delta)^s u + f(t,x,u) = 0 \qquad & \text{in } Q_\infty, \\ u(0,\cdot) = u_0 \qquad & \text{in } \mathbb{R}^N \end{aligned} \tag{2.33}$$

has a unique weak solution $u \in C(\mathbb{R}^+; L^1(\mathbb{R}^N))$ satisfying that

$$\int_{\mathbb{R}^N} u(t,x)\Phi(x)dx + \int_0^t \int_{\mathbb{R}^N} \left(u(s,x)(-\Delta)^s\Phi(x) + f(s,x,u)\Phi(x)\right) dxds = \int_{\mathbb{R}^N} u_0(x)\Phi(x)dx. \tag{2.34}$$

Proof. Since u is a weak solution of (2.33) and the function Φ satisfies the assumptions (i)-(ii) in [28, Def. 1.1], we get (2.34). □

Corollary 2.7 *Assume that f satisfies the assumptions of Lemma 2.6 and that inequalities (1.14)-(1.15)-(1.16) hold. Then for any nonnegative measure μ in $\mathbb{R}^N$ verifying*

$$\int_{\mathbb{R}^N} \Phi(x)\, d\mu(x) < +\infty, \tag{2.35}$$

there exists a weak solution $u \in C_b(\mathbb{R}^+; \mathcal{L}^s(\mathbb{R}^N)) \cap L^1(\mathbb{R}^+; \mathcal{L}^s(\mathbb{R}^N))$ of (2.33) in the sense that

$$\int_0^t \int_{\mathbb{R}^N} \left[-(\partial_t\xi + (-\Delta)^s\xi)u + \xi f(s,x,u)\right] dxds + \int_{\mathbb{R}^N} u(t,x)\xi(t,x)dx = \int_{\mathbb{R}^N} \xi(0,x)d\mu(x), \tag{2.36}$$

for any $\xi \in C_0^2(\overline{Q}_T)$ satisfying the assumptions (i)-(ii) in [28, Def. 1.1]. Furthermore,

$$\int_{\mathbb{R}^N} u(t,x)\Phi(x)dx + \int_0^t \int_{\mathbb{R}^N} \left[u(s,x)(-\Delta)^s\Phi(x) + f(s,x,u)\Phi(x)\right] dxds = \int_{\mathbb{R}^N} \Phi(x)d\mu(x). \tag{2.37}$$

Proof. By the assumptions on f and for any $n > 0$, it follows from [28, Th. 1.1] that

$$\begin{aligned} \partial_t u + (-\Delta)^s u + f(t,x,u) = 0 \qquad & \text{in } Q_\infty \\ u(0,\cdot) = \mu_n := \chi_{B_n} u_0 \qquad & \text{in } \mathbb{R}^N, \end{aligned} \tag{2.38}$$

has a unique solution $u_n \in L^1(Q_T)$ verifying $f(\cdot,\cdot,u_n) \in L^1(Q_T)$. If ρ_k is a sequence of mollifiers with compact supports and $\mu_{n,k} = (\chi_{B_n} u_0) * \rho_k$, the sequence $\{u_{n,k}\}$ of weak solutions of

$$\begin{aligned} \partial_t u + (-\Delta)^s u + f(t,x,u) = 0 \qquad & \text{in } Q_\infty \\ u(0,\cdot) = \mu_{n,k} \qquad & \text{in } \mathbb{R}^N, \end{aligned} \tag{2.39}$$

then $u_{n,k}$ satisfies that

$$\begin{aligned} \int_{\mathbb{R}^N} u_{n,k}(t,x)\Phi(x)dx + \int_0^t \int_{\mathbb{R}^N} \left[u_{n,k}(s,x)(-\Delta)^s\Phi(x) + f(s,x,u_{n,k})\Phi(x)\right] dxds \\ = \int_{\mathbb{R}^N} \mu_{n,k}(x)\Phi(x)dx. \end{aligned} \tag{2.40}$$

When $k \to \infty$, we know from the proof of [28, Th. 1.1] that, up to a subsequence, $\{u_{n,k}\}_k$ converges a.e. in Q_T to some function u_n, $\{f(\cdot,\cdot,u_{n,k})\}_k$ converges a.e. to $\{f(\cdot,\cdot,u_n)\}$ and that $\{u_{n,k}\}_k$ and $\{f(\cdot,\cdot,u_{n,k})\}_k$ are uniformly integrable in $L^1(Q_T)$. Furthermore $u_n \in C([0,T];L^1(\mathbb{R}^N))$ and for any $t \in (0,T]$, $\{u_{n,k}(t,\cdot)\}_k$ converges to $u_n(t,\cdot)$ in $L^1(\mathbb{R}^N)$. This implies that

$$\int_{\mathbb{R}^N} u_n(t,x)\Phi(x)dx + \int_0^t \int_{\mathbb{R}^N} \left[u_n(s,x)(-\Delta)^s\Phi(x) + f(s,x,u_n)\Phi(x)\right] dxds = \int_{\mathbb{R}^N} \Phi(x)d\mu_n(x). \tag{2.41}$$

Furthermore,

$$\int_0^t \int_{\mathbb{R}^N} \left[-(\partial_t\xi + (-\Delta)^s\xi)u_n + \xi f(s,x,u_n)\right] dxds + \int_{\mathbb{R}^N} u_n(t,x)\xi(t,x)dx = \int_{\mathbb{R}^N} \xi(0,x)d\mu_n(x), \tag{2.42}$$

for all $\xi \in C_0^2(\overline{Q}_T)$ satisfying the assumptions (i)-(ii) in [28, Def. 1.1]. When $n \to \infty$, $u_n \uparrow u$ and $f(s,x,u_n) \uparrow f(s,x,u)$. Using the monotone convergent theorem we see that u satisfies (2.37),and that the sequences $\{u_n\}_n$ and $\{f(\cdot,\cdot,u_n)\}_n$ converges to u and $f(\cdot,\cdot,u)$ in $L^1(0,T;\mathcal{L}^s(\mathbb{R}^N))$ respectively. Using estimate (2.17) we can let n to infinity in (2.42) and obtain (2.36). □

As it is pointed out in [15], the weight function Φ plays a role similar to an eigenfunction of $(-\Delta)^s$. We prove a backward-forward uniqueness result for solutions of (1.1) inspired from [15, Lemma 4.2].

Theorem 2.8 *Assume that $u \mapsto f(t,x,u)$ is locally Lipschitz continuous on $\mathbb{R}$, uniformly with respect to $x \in \mathbb{R}^N$ and locally uniformly with respect to $t \in \mathbb{R}_+^*$. If u_1 and u_2 belong to $L^1_{loc}(\mathbb{R}_+^*;\mathcal{L}^s(\mathbb{R}^N)) \cap L^\infty_{loc}(\mathbb{R}_+^*;L^\infty(\mathbb{R}^N))$ and are weak solutions of (1.1) in Q_T which coincide for $t = t_0 > 0$, then $u_1 = u_2$ in Q_T.*

Proof. For any $0 < \epsilon < t_0 < T < \infty$, u_1 and u_2 are uniformly bounded in $[\epsilon,T] \times \mathbb{R}^N$. Hence the function D defined by

$$D(t,x) = \begin{cases} \dfrac{f(t,x,u_1(t,x)) - f(t,x,u_2(t,x))}{u_1(t,x) - u_2(t,x)} & \text{if } u_1(t,x) \neq u_2(t,x) \\ 0 & \text{if } u_1(t,x) = u_2(t,x) \end{cases}$$

is bounded in $[\epsilon,T] \times \mathbb{R}^N$ by some constant $M = M(\epsilon,T) > 0$. Set $w = u_1 - u_2$, it satisfies

$$\partial_t w + (-\Delta)^s w + Dw = 0 \quad \text{in } Q_T,$$

and is uniformly bounded in $[\epsilon,T] \times \mathbb{R}^N$. Hence

$$\frac{d}{dt}\int_{\mathbb{R}^N} w(t,x)\Phi(x)dx + \int_{\mathbb{R}^N} w(t,x)(-\Delta)^s\Phi(x)dx + \int_{\mathbb{R}^N} D(t,x)w(t,x)\Phi(x)dx = 0.$$

Using (2.31) we get

$$-(c_5+M)\int_{\mathbb{R}^N} w(t,x)\Phi(x)dx \leq \frac{d}{dt}\int_{\mathbb{R}^N} w(t,x)\Phi(x)dx \leq (c_5+M)\int_{\mathbb{R}^N} w(t,x)\Phi(x)dx. \tag{2.43}$$

This implies

$$-(c_5+M)\int_{\mathbb{R}^N} w(t,x)\Phi(x)dx \le \frac{d}{dt}\int_{\mathbb{R}^N} w(t,x)\Phi(x)dx \le (c_5+M)\int_{\mathbb{R}^N} w(t,x)\Phi(x)dx, \quad (2.44)$$

and

$$\begin{aligned} &(i) \qquad \int_{\mathbb{R}^N} w(t,x)\Phi(x)dx \le e^{(c_5+M)(t-s)}\int_{\mathbb{R}^N} w(s,x)\Phi(x)dx, \\ &(ii) \qquad e^{(c_5+M)(s-t)}\int_{\mathbb{R}^N} w(s,x)\Phi(x)dx \le \int_{\mathbb{R}^N} w(t,x)\Phi(x)dx, \end{aligned} \quad (2.45)$$

for all $\epsilon \le s \le t \le T$. Taking $s = t_0$ in (i) and $t = t_0$ in (ii) yields $w \equiv 0$ in $[\epsilon, T] \times \mathbb{R}^N$. □

Proof of Theorem E. By Theorem D we know that $u \le U$. If there exists some $(t_0, x_0) \in Q_T$ such that $u((t_0, x_0)) = U(t_0)$, then either $u((t_0, x)) = U(t_0)$ for all $x \in \mathbb{R}^N$, or

$$(-\Delta)^s(u-U)(t_0,x_0) < 0 \qquad \forall\, x \in \mathbb{R}^N.$$

Since $f(t,x,u) - t^\beta g(U) \ge 0$ and $\partial_t(u-U)(t_0,x_0) = 0$ we infer that $u((t_0,.)) \equiv U(t_0)$. Since g is nondecreasing this situation is impossible, hence $u((t_0,.)) = U(t_0)$. Since g is locally Lipschitz continuous, this implies $u = U$ in Q_T by Theorem 2.8. □

A straightforward consequence of Theorems B, C and D is the next statement.

Corollary 2.9 *Let $f(t,x,r) = t^\beta g(r)$, where $\beta > -1$ and $g : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and nondecreasing and satisfies (1.15), (1.19) and (1.23). If u is a nonnegative of (1.1) in Q_T belonging to $L^1_{loc}(0,T;\mathcal{L}^s(\mathbb{R}^N))$ such that $\mathcal{S}_u \neq \emptyset$, there holds*

$$u(t,x) \ge u_{\infty,z}(t,x) = u_{\infty,0}(x-z,t) \qquad \forall\,(t,x) \in Q_T. \quad (2.46)$$

3 The case $f(t,x,u) = t^\beta u^p$

We denote by $(-\Delta)^s_{\mathbb{R}^\kappa}$ the fractional Laplacian in $\mathbb{R}^\kappa$ and $(-\Delta)^s_{\mathbb{R}^N} = (-\Delta)^s$. The following standard lemma will be useful in the sequel.

Lemma 3.1 *Let $1 \le \kappa \le N-1$ be an integer. If $u \in C^2(\mathbb{R}^\kappa) \cap \mathcal{L}^s(\mathbb{R}^\kappa)$ and $\tilde{u}(x_1,x') = u(x_1)$ for $(x_1,x') \in \mathbb{R}^\kappa \times \mathbb{R}^{N-\kappa}$, then*

$$(-\Delta)^s\tilde{u}(x_1,x') = (-\Delta)^s_{\mathbb{R}^\kappa} u(x_1). \quad (3.47)$$

Proof. This more or less well known lemma is based upon the explicit value of the constant $a_{N,s}$ in the definition of $(-\Delta)^s$. For the sake of completeness we give here the proof.

$$\begin{aligned} (-\Delta)^s\tilde{u}(x_1,x') &= a_{N,s}\int_{\mathbb{R}^\kappa}\int_{\mathbb{R}^{N-\kappa}} \frac{u(x_1)-u(y_1)}{((x_1-y_1)^2+|x'-y'|^2)^{\frac{N}{2}+s}}dy'dy_1 \\ &= a_{N,s}\int_{\mathbb{R}^\kappa}\left(\int_{\mathbb{R}^{N-\kappa}} \frac{dy'}{((x_1-y_1)^2+|y'|^2)^{\frac{N}{2}+s}}\right)(u(x_1)-u(y_1))\,dy_1 \\ &= a_{N,s}\left(\int_{\mathbb{R}^{N-\kappa}} \frac{dz'}{(1+|z'|^2)^{\frac{N}{2}+s}}\right)\int_{\mathbb{R}^\kappa}\frac{u(x_1)-u(y_1)}{|x_1-y_1|^{\kappa+2s}}dy_1 \\ &= \frac{a_{N,s}}{a_{\kappa,s}}\left(\left|S^{N-1-\kappa}\right|\int_0^\infty \frac{r^{N-\kappa-1}dr}{(1+r^2)^{\frac{N}{2}+s}}\right)(-\Delta)^s_{\mathbb{R}^\kappa}u(x_1). \end{aligned}$$

Since

$$\left|S^{N-1-\kappa}\right| = \frac{2\pi^{\frac{N-1-\kappa}{2}}}{\Gamma(\frac{N-1-\kappa}{2})},$$

and (see e.g. [46, p. 103])

$$\int_0^\infty \frac{r^{N-\kappa-1}dr}{(1+r^2)^{\frac{N}{2}+s}} = \frac{1}{2}B\left(\frac{N-\kappa}{2}, \frac{\kappa}{2}+s\right) = \frac{1}{2}\frac{\Gamma(\frac{\kappa}{2}+s)\Gamma(\frac{N-\kappa}{2})}{\Gamma(\frac{N}{2}+s)},$$

by Euler's formula, where B denotes beta function, we deduce that

$$\frac{a_{\kappa,s}}{a_{N,s}} = \left|S^{N-1-\kappa}\right| \int_0^\infty \frac{r^{N-\kappa-1}dr}{(1+r^2)^{\frac{N}{2}+s}},$$

which yields (3.47). □

The next statement is a straightforward consequence.

Corollary 3.2 *Assume that* $u(x) = u(x_1, ...x_N) = \sum_{j=1}^N u_j(x_j)$, *then*

$$(-\Delta)^s \tilde{u}(x) = \sum_{j=1}^N \left((-\Delta)^s_{\mathbb{R}} u_j\right)(x_j). \tag{3.48}$$

3.1 Proof of Theorem F

By Theorem E there holds $u(t,x) \le ct^{-\frac{1+\beta}{p-1}}$ for some $c_* = c_*(\beta, p) > 0$, hence u satisfies

$$\partial_t u + (-\Delta)^s u + \frac{c(t,x)}{t}u = 0, \tag{3.49}$$

where $0 \le c(t,x) = t^\beta u^{p-1}(t,x) \le c_*^{p-1} := C_*$. Let d_s be the fractional parabolic pseudo-distance (i.e. the triangle inequality holds up to a multiplicative constant if $s < \frac{1}{2}$) in $\mathbb{R}^N \times \mathbb{R}$,

$$d_s((t,x),(s,y)) = \sqrt{|x-y|^2 + |t-s|^{\frac{1}{s}}}.$$

Lemma 3.3 *If* $z \in \mathcal{S}_u$, *there holds*

$$\limsup_{d_s((t,x),(0,z))\to 0} t^{\frac{N}{2s}} u(t,x) = \infty. \tag{3.50}$$

Proof. If (3.50) does not hold there exists $m, \epsilon_0 > 0$ such that

$$u(t,x) \le mt^{-\frac{N}{2s}} \qquad \forall (t,x) \text{ s.t. } |x-z|^2 + t^{\frac{1}{s}} \le \epsilon_0^2.$$

Hence

$$\gamma(t,x) := t^\beta u^{p-1}(t,x) \le m^p t^{\beta - \frac{N(p-1)}{2s}} \qquad \forall (t,x) \in (0,t_1) \times B_{\epsilon_1}(z),$$

where $\epsilon_1 = \frac{\epsilon_0}{2}$ and $t_1 = \left(\frac{3}{4}\right)^{2s} \epsilon_0^s$. By assumption $p < p^*_\beta$, or equivalently $\beta - \frac{N(p-1)}{2s} > -1$. Hence $\tilde{\gamma}(t) := \|\gamma(.,t)\|_{L^\infty(B_{\epsilon_1}(z))} \in L^1(0,t_1)$. We write the equation satisfied by u in $B_{\epsilon_1}(z) \times (0,t_1)$ in the form

$$\partial_t u + (-\Delta)^s u + \gamma(t,x)u = 0, \tag{3.51}$$

and, as in the proof of Theorem A, we take for test function ϕ^q, where $q \geq 2$ and $\phi_\epsilon = \phi_{B_\epsilon(z)}$ is the first normalized eigenfunction of $(-\Delta)^s$ in $H^s_0(\phi_{B_\epsilon(z)})$ for some $0 < \epsilon < \epsilon_1$. If λ_ϵ is the corresponding eigenvalue, we obtain as in Proposition 2.2,

$$\frac{d}{dt} e^{q\lambda_\epsilon t} \int_{B_\epsilon(z)} u\phi_\epsilon^q dx + e^{q\lambda_\epsilon t} \tilde{\gamma}(t) \int_{B_\epsilon(z)} u\phi_\epsilon^q dx \geq 0.$$

If we put $X(t) = e^{q\lambda_\epsilon t} \int_{B_\epsilon(z)} u\phi_\epsilon^q dx$, then $X' + \tilde{\gamma}(t)X \geq 0$ on $(0,t_1)$, which implies that the function $t \mapsto e^{\int_0^t \tilde{\gamma}(s)ds} X(t)$ is increasing on $(0,t_1)$. Hence

$$\lim_{t\to 0} \int_{B_\epsilon(z)} u(.,t)\phi_\epsilon^q dx \leq e^{q\lambda_\epsilon t_1 + \int_0^{t_1} \tilde{\gamma}(s)ds} \int_{B_\epsilon(z)} u(.,t_1)\phi_\epsilon^q dx,$$

which implies that $z \in \mathcal{R}_u$, contradiction. □

Notice that the above lemma contains a result which is interesting in itself.

Corollary 3.4 *If γ is a measurable function in $\mathbb{R}^N \times (0,T)$ such that for any compact set $K \subset \mathbb{R}^N$ the function $\tilde{\gamma}_K(t) := \operatorname*{ess\,sup}_{x\in K} |\gamma(t,x)|$ is integrable on $(0,T)$, then any nonnegative function $u \in L^1_{loc}(0,T;\mathcal{L}^s(\mathbb{R}^N)$ satisfying (3.51) admits an initial trace ν which is a nonegative Radon measure in $\mathbb{R}^N$.*

The next result is an Harnack-type inequality valid for positive solutions of (1.26). For the mere fractional heat equation, two-sided Harnack inequalitis are proved in [11] and [15].

Lemma 3.5 *Let $\theta > 0$ and w be a nonnegative solution of (1.26) in Q_∞. Then for any $t > \frac{t}{2} \geq s > \frac{t}{4}$ and $x, y \in \mathbb{R}^N$ such that $|x-y| \leq \theta t^{\frac{1}{2s}}$, there holds*

$$w(t,y) \geq Mw(x,s), \tag{3.52}$$

where $M > 0$ depends on N, s, β, p and θ.

Proof. Since w satisfies

$$\partial_t w + (-\Delta)^s w + \frac{C_*}{t} w \geq 0, \tag{3.53}$$

$t^{C_*} w(t,.)$ is a supersolution of the fractional diffusion equation $\partial_t v + (-\Delta)^s v = 0$, hence

$$t^{C_*} w(t,x) \geq \left(\frac{t}{4}\right)^{C_*} \int_{\mathbb{R}^N} H_s(\tfrac{3t}{4}, x-z) w(\tfrac{t}{4}, z) dz,$$

which implies, thanks to identity (1.35),

$$w(t,x) \geq \frac{3t}{c_3 4^{C_*+1}} \int_{\mathbb{R}^N} \frac{w(\frac{t}{4},z)dz}{(\frac{3t}{4})^{1+\frac{N}{2s}} + |x-z|^{N+2s}}.$$

Since w is a subsolution of the fractional diffusion equation,

$$w(s,y) \leq \int_{\mathbb{R}^N} H_s(s-\tfrac{t}{4}, y-z) w(\tfrac{t}{4},z)dz \leq c_3(s-\tfrac{t}{4}) \int_{\mathbb{R}^N} \frac{w(\frac{t}{4},z)dz}{(s-\frac{t}{4})^{1+\frac{N}{2s}} + |y-z|^{N+2s}}.$$

Hence

$$w(s,y) \leq \frac{c_3^2 4^{C_*+1}(s-\frac{t}{4})}{3t} \left(\sup_{z\in\mathbb{R}^N} \frac{(\frac{3t}{4})^{1+\frac{N}{2s}} + |x-z|^{N+2s}}{(s-\frac{t}{4})^{1+\frac{N}{2s}} + |y-z|^{N+2s}} \right) w(t,x) \tag{3.54}$$

If we assume that $|x-y| \leq \theta t^{\frac{1}{2s}}$ for some $\theta > 0$, we obtain the claim. □

End of the proof of Theorem F. By Lemma 3.3 there exists a sequence $\{t_n, x_n\} \subset Q_\infty$ converging to $(0,z)$ such that

$$u(s_n, x_n) \geq n s_n^{-\frac{N}{2s}}. \tag{3.55}$$

By Lemma 3.5, there holds with $t_n = 2s_n$,

$$u(t_n,x) \geq M n t_n^{-\frac{N}{2s}} \geq cn H_s(t_n, x-x_n) \qquad \forall x \in \mathbb{R}^N \text{ s.t. } |x-x_n| \leq \theta t_n^{\frac{1}{2s}}. \tag{3.56}$$

for some $c > 0$ depending on M and θ. This implies

$$\int_{|x-x_n|\leq \theta t_n^{\frac{1}{2s}}} u(t_n,x)dx \geq cn \int_{|x-x_n|\leq \theta t_n^{\frac{1}{2s}}} H_s(t_n, x-x_n)dx \geq c'n. \tag{3.57}$$

Then for any $k \in (0,n)$ there exists $k_n \in (0,n)$ such that

$$ck_n \int_{|x-x_n|\leq \theta t_n^{\frac{1}{2s}}} H_s(t_n, x-x_n)dx = k. \tag{3.58}$$

Using (1.34)

$$\int_{|x-x_n|\leq \theta t_n^{\frac{1}{2s}}} H_s(t_n, x-x_n)dx = \int_{B_\theta} \tilde{H}_s(y)dy,$$

hence $k_n \leq ck$ for some $c > 0$ independent of n. Set $\upsilon_n(x) = ck_n H_s(t_n, .-x_n)\chi_{B_{\theta t_n^{\frac{1}{2s}}}(x_n)}$. Then u is bounded from below in $(t_n,\infty)\times\mathbb{R}^N$ by the function u_n which satisfies

$$\begin{aligned} \partial_t u_n + (-\Delta)^s u_n + t^\beta u_n^p = 0 \quad \text{in} \quad (t_n,\infty)\times\mathbb{R}^N \\ u_n(t_n,.) = \upsilon_n, \end{aligned} \tag{3.59}$$

which in turn, satisfies

$$u_n(t,x) \leq \mathbb{H}_s[\upsilon_n](t-t_n,x) \leq ck_n H_s(t,x-x_n) \quad \text{in} \quad (t_n,\infty)\times\mathbb{R}^N.$$

Since $p < p^*_\beta$, it is proved in [28, Th 1.1] that the set of functions $\{t^\beta (ck_n H_s(.,.-x_n))^p\}$ is uniformly integrable in Q_∞, and this property is shared by the set $\{t^\beta (u_n)^p\}$. Because $u_n(t_n,.) \to k\delta_z$ it follows that $u_n \to u_{k\delta_z}$ locally uniformly in Q_∞, and $u \geq u_{k\delta_z}$. Since k is arbitrary, the claim follows. □

3.2 Proof of Theorem G (i)

When $f(t,x,u) = t^\beta g(u) := t^\beta u^p$, conditions (1.19) and (1.23) are fulfilled when $p > 1$ and $p > \beta + 2$ respectively. Condition $1 < p \le p^{**}_\beta$ is not compatible with $p > \beta + 2$, and condition $p^{**}_\beta < p < p^*_\beta$ necessitates $\beta + 2 < 1 + \frac{2s(1+\beta)}{N}$, equivalently $\beta + 1 < \frac{2s}{N}$.

Step 1. The case $1 < p < 1 + \frac{2s(1+\beta)}{N+2s}$). Let $z \in \mathcal{S}_u$. Since $r \mapsto r^p$ satisfies (1.15) there holds $u \ge u_{z,\infty}$ by Theorem C. Since $u_{z,\infty} = U_{p,\beta}$ by (1.29), we obtaind that $u \ge U_{p,\beta}$. If we assume that $u \in L^1_{loc}(0,T;\mathcal{L}^s(\mathbb{R}^N))$ then $u = U_{p,\beta}$ by Theorem D.

Step 2. The case $p = 1 + \frac{2s(1+\beta)}{N+2s}$. We set $u_\infty = u_{0,\infty}$. From [28, Lemma 5.3] ,

$$u_\infty(t,x) \ge \frac{c_7 t^{-\frac{N+2s}{2s}}}{1 + |t^{-\frac{1}{2s}}x|^{N+2s}} \qquad \forall\, (t,x) \in (0,1) \times \mathbb{R}^N, \tag{3.60}$$

where $c_7 > 0$. Since

$$\int_{B_1} \frac{t^{-\frac{N+2s}{2s}} dx}{1 + |t^{-\frac{1}{2s}}x|^{N+2s}} = t^{-1} \int_0^1 \frac{r^{N-1} dr}{1 + r^{N+2s}},$$

it follows from (3.60) that

$$\int_0^1 \int_{B_1(0)} u_\infty(t,x) dx dt = +\infty. \tag{3.61}$$

By Proposition 6.1 in Appendix, $x \mapsto u_k(t,x)$ is radially symmetric and decreasing, so is u_∞. Therefore, if we are able to prove that there exists $x \in \mathbb{R}^N \setminus \{0\}$ such that $\lim_{t\to 0} u_\infty(t,x) = \infty$, it will imply

$$\lim_{t\to 0} u_\infty(t,z) = \infty \qquad \text{uniformly with respect to z in } \overline{B}_{|x|}(0).$$

Hence $\overline{B}_{|x|}(0) \subset \mathcal{S}_{u_\infty}$ and by Theorem C,

$$u_\infty(t,x) \ge u_{z,\infty}(t,x) = u_\infty(t,x-z) \qquad \forall\, z \in \overline{B}_{|x|}(0) \tag{3.62}$$

Because u_∞ is radially symmetric and decreasing, it implies that

$$u_\infty(t,x) = u_\infty(t,x-z) \qquad \forall\, z \in \overline{B}_{|x|}(0). \tag{3.63}$$

By iterating this process we infer that $u_\infty(t,x)$ is indeed independent of x and tends to ∞ when $t \to 0$. It coincides therefore to the maximal solution $U_{p,\beta}$ of (1.21) with $g(u) = u^p$.

Henceforth we are lead to prove that $\mathcal{S}_{u_\infty} \cap \mathbb{R}^N \setminus \{0\} \neq \emptyset$. We proceed by contradiction in supposing that it does not hold, and let $x_0 \in \mathcal{S}^c_{u_\infty} \cap \mathbb{R}^N \setminus \{0\}$. Then $\limsup_{t\to 0} u_\infty(t,x_0) < \infty$ and

$$\sup_{0<t\le 1} u_\infty(t,x) \le \sup_{0<t\le 1} u_\infty(t,x_0) := M < \infty \quad \text{uniformly with respect to } x \text{ in } \overline{B}^c_{|x_0|}(0). \tag{3.64}$$

By rescaling we can assume that $|x_0| = 1$. Let $\bar{x} \in \overline{B}^c_3$ and $\eta \in C^2_0(B_1(\bar{x}))$ such that $\eta \ge 0$ and $\eta = 1$ on $\overline{B}_{\frac{1}{2}}(\bar{x})$. We denote

$$X_1(t) = \int_{\mathbb{R}^N} u_\infty(t,x)\eta(x)dx, \;\; Y_1(t) = t^\beta \int_{\mathbb{R}^N} u^p_\infty(t,x)\eta(x)dx, \;\; Z_1(t) = \int_{\mathbb{R}^N} u_\infty(t,x)(-\Delta)^s\eta(x)dx,$$

and there holds

$$X_1'(t) + Z_1(t) + Y_1(t) = 0 \quad \text{on } (0,1]. \tag{3.65}$$

Since u_∞ is bounded in $(0,1] \times B_2(\bar{x})$ by (3.64), $X_1(t)$ and $Y_1(t)$ remains bounded on $(0,1]$.

$$Z_1(t) = \int_{B_1} u_\infty(t,x)(-\Delta)^s\eta(x)dx + \int_{B_1^c} u_\infty(t,x)(-\Delta)^s\eta(x)dx := Z_{1,1}(t) + Z_{1,2}(t).$$

Since η has its support in $B_1(\bar{x})$, there exists $c_8 > 0$ such that

$$(-\Delta)^s\eta(x) \le -c_8 \qquad \forall\, x \in B_1(0).$$

Using (3.61) we obtain that

$$\int_0^1 Z_{1,1}(s)ds = -\infty.$$

Using (2.17) in Lemma 2.5, we have

$$|Z_{1,2}(t)| \le c_5\|\eta\|_{C^2}\int_{B_1^c}\frac{u_\infty(t,x)dx}{1+|x|^{N+2s}} \le c_9\int_1^\infty \frac{r^{N-1}dr}{1+r^{N+2s}}.$$

Hence

$$\int_0^1 Z_1(s)ds = -\infty.$$

Integrating (3.65) it contradicts the boundedness of X_1 and Y_1. Hence, for any $z \in \mathbb{R}^N$,

$$\limsup_{t\to 0} u_\infty(t,z) = \infty. \tag{3.66}$$

Using again the fact that $x \mapsto u_\infty(t,x)$ is radial and decreasing with respect to $|x|$, we get

$$\limsup_{t\to 0}\int_{B_\rho(z)} u_\infty(t,x)dx = \infty \qquad \forall\, \rho > 0. \tag{3.67}$$

By Theorem C, we infer that $u_\infty(t,x) \ge u_{z,\infty}(t,x) = u_\infty(t,x-z)$. Interchanging 0 and z we conclude again that $u_\infty(t,x)$ depends only on t, hence it coincides with $U_{\beta,p}(t)$, and clearly $\mathcal{S}_{u_\infty} = \mathbb{R}^N$. □

3.3 Proof of Theorem G (ii)

We assume that $\kappa \ge 1$ and $\mathcal{L} = \{0_{\mathbb{R}^\kappa}\} \times \mathbb{R}^{N-\kappa}$. We set $x = (x_1, x') \in \mathbb{R}^\kappa \times \mathbb{R}^{N-\kappa}$. We use Theorem G (i) with N replace by $N-\kappa$ to prove the part (ii). If $x = (x_1, x')$, then $\bar{x} = (0, x') \in \mathcal{L}$, hence by [28, Th 1.3 (ii)]

$$u_\infty(t, x-\bar{x}) \ge \frac{c_{10}t^{-\frac{N+2s}{2s}}}{1+(t^{-\frac{1}{2s}}|x-\bar{x}|)^{N+2s}} = \frac{c_{10}t^{-\frac{N+2s}{2s}}}{1+(t^{-\frac{1}{2s}}|x_1|)^{N+2s}}.$$

By Theorem C, we obtain

$$u(t,x) \geq \frac{c_{10} t^{-\frac{N+2s}{2s}}}{1+(t^{-\frac{1}{2s}}|x_1|)^{N+2s}} \qquad \forall\, (t,x) := (t,x_1,x') \in \mathbb{R}^*_+ \times \mathbb{R}^\kappa \times \mathbb{R}^{N-\kappa}. \tag{3.68}$$

For $n \in \mathbb{N}^*$, set $t_n = n^{-2s}$, $\rho_0 = \int_{\mathbb{R}^{N-\kappa}} \frac{dy'}{1+|y'|^{N+2s}}$, $\lambda_0 = \rho_0^{\frac{1}{p-1}}$ and

$$f_n(x_1) = \frac{c_{10}\lambda_0 (\rho_0 t_n)^{-\frac{1+\beta}{p-1}}}{1+\left((\rho_0 t_n)^{-\frac{1}{2s}}|x_1|\right)^{N+2s}}.$$

Then

$$\int_{\mathbb{R}^\kappa} f_n(x_1)dx_1 = c_\kappa t_n^{\frac{\kappa}{2s}-\frac{1+\beta}{p-1}} \int_0^\infty \frac{r^{\kappa-1}dr}{1+r^{N+2s}}$$

for some $c_\kappa > 0$. Since $p < 1 + \frac{2s(1+\beta)}{\kappa}$ the above integral is finite for any n but tends to ∞ with n. Hence we fix $m > 0$, then for any $n \in \mathbb{N}^*$ there exists $\epsilon_{n,m} > 0$ such that

$$\int_{|x_1|\leq \epsilon_{n,m}} f_n(x_1)dx_1 = m = c_\kappa t_n^{\frac{\kappa}{2s}-\frac{1+\beta}{p-1}} \int_0^{\epsilon_{n,m} t_n^{-\frac{N+2s}{2s}}} \frac{r^{\kappa-1}dr}{1+r^{N+2s}}.$$

Hence $\epsilon_{n,m} \to 0$ when $n \to \infty$. This implies that for any $\zeta \in C_0^\infty(\mathbb{R}^\kappa)$,

$$\lim_{n\to\infty} \int_{|x_1|\leq \epsilon_{n,m}} f_n(x_1)dx_1 = m\zeta(0).$$

Equivalently $f_{n,m} := f_n \chi_{B_{\epsilon_{n,m}}} \to m\delta_0$ in the sense of measures in $\mathbb{R}^\kappa$. Let $w_{n,m}$ be the solution of

$$\begin{aligned} \partial_t u + (-\Delta)^s_{\mathbb{R}^\kappa} u + t^\beta u^p = 0 \qquad & \text{in } \mathbb{R}^*_+ \times \mathbb{R}^\kappa \\ u(0,\cdot) = f_{n,m} \qquad & \text{in } \mathbb{R}^\kappa, \end{aligned} \tag{3.69}$$

in which formula $(-\Delta)^s_{\mathbb{R}^\kappa}$ denotes the fractional Laplacian in $\mathbb{R}^\kappa$, an index omitted if $\kappa = N$. Then $\tilde{w}_{n,m}(t,x_1,x') = w_{n,m}(t,x_1)$ is a solution of

$$\begin{aligned} \partial_t u + (-\Delta)^s u + t^\beta u^p = 0 \qquad & \text{in } \mathbb{R}^*_+ \times \mathbb{R}^N \\ u(0,\cdot) = \tilde{f}_{n,m} \qquad & \text{in } \mathbb{R}^N, \end{aligned} \tag{3.70}$$

with $\tilde{f}_{n,m}(x_1,x') = f_{n,m}(x_1)$. Since $u(t_n,x) \geq \tilde{f}_n(x)$ in $\mathbb{R}^N$, we obtain by the comparison principle that $u(t+t_n,x) \geq \tilde{w}_{n,m}(t,x)$ in $\mathbb{R}^*_+ \times \mathbb{R}^N$. Hence, by letting successively $n \to \infty$ and $m \to \infty$,

$$u(t+t_n,x_1,x') \geq w_{n,m}(t,x_1) \Longrightarrow u(t,x_1,x') \geq u^\kappa_{m\delta_0}(t,x_1) \Longrightarrow u(t,x_1,x') \geq u^\kappa_\infty(t,x_1), \tag{3.71}$$

where we have denoted by $u^\kappa_{m\delta_0}$ and u^κ_∞ respectively the solution of the equation in (3.69) with $m\delta_0$ as initial data and the limit of this solution when $m \to \infty$. Since $1 < p \leq 1 + \frac{2s}{\kappa+2s}$, $u^\kappa_\infty = U_{p,\beta}$ by (i), which ends the proof. □

Remark. It appears interesting to investigate whether the fact that the singular set $\mathcal{S}_u$ contains a (N-κ)-dimensional plane can be replaced by $\mathcal{S}_u$ contains a (N-κ)-dimensional submanifold.

4 Solution with a given initial trace: the general case

4.1 Problem with initial data measure

If ν is a bounded Radon measure on an open set $\mathcal{R} \subset \mathbb{R}^N$, that we note $\nu \in \mathfrak{M}^b(\mathcal{R})$), we extend it by 0 in $\mathcal{S} = \mathbb{R}^N \setminus \mathcal{R}$ and the new measure still denoted by ν, belongs to the space $\mathfrak{M}^b(\mathbb{R}^N)$ of bounded Radon measures on $\mathbb{R}^N$. Conversely, if $\nu \in \mathfrak{M}^b(\mathbb{R}^N)$ vanishes on $\mathcal{S} = \mathbb{R}^N \setminus \mathcal{R}$ its restriction to $\mathcal{R}$ belongs to $\mathfrak{M}^b(\mathcal{R})$.

Definition 4.1 *A nonnegative bounded Radon measure ν in $\mathbb{R}^N$ is an admissible measure if there exists a function $u = u_\nu \in L^1(Q_T)$ with $t^\beta u^p \in L^1(Q_T)$ solution of*

$$\begin{array}{c} \partial_t u + (-\Delta)^s u + t^\beta u^p = 0 \qquad \text{in} \quad Q_\infty \\ u(0,.) = \nu. \end{array} \tag{4.1}$$

It is a good measure if the sequence u_{ν_n} of solutions of (4.1) with initial data $\nu_n = \nu * \rho_n$, where $\{\rho_n\}$ is a sequence of mollifiers, converges to u_ν in $L^1(Q_T)$ and if $t^\beta u^p_{\nu_n}$ converges to $t^\beta u^p \in L^1(Q_T)$.

Uniqueness of solutions is proved in [28] as a result of the choice of $\mathbb{Y}_{s,T}$ as space of test functions. Notice also that if $p < p^*_\beta$ any nonnegative bounded measure is good. The following result will be useful in the sequel.

Proposition 4.1 *Let $p > 1$ and $\beta > -1$. If $\nu, \mu \in \mathfrak{M}^b_+(\mathbb{R}^N)$ are good measures (resp. admissible measures), then $\nu + \mu$ is a good measure (resp. admissible measure).*

Proof. We set $\nu_n = \nu * \rho_n$ and $\mu_n = \mu * \rho_n$ and denote by u_{ν_n}, u_{μ_n} and $u_{\nu_n+\mu_n}$ the solutions of the initial value problem (4.1) with ν replaced by ν_n, μ_n and $\nu_n + \mu_n$ respectively. Since $p > 1$, $u_{\nu_n} + u_{\mu_n}$ is a supersolution of (1.26). Hence $u_{\nu_n+\mu_n} \le u_{\nu_n} + u_{\mu_n}$. When $n \to \infty$, $u_{\nu_n+\mu_n}$ converges a.e. to some function u (see [28]). Since u_{ν_n} and u_{μ_n} converges in $L^1(Q_T)$, the sequence $u_{\nu_n+\mu_n}$ is uniformly integrable in Q_T, it converges to some w (up to extraction of a subsequence). Furthermore,

$$(u_{\nu_n+\mu_n})^p \le (u_{\nu_n} + u_{\mu_n})^p \le 2^{p-1}\left(u^p_{\nu_n} + u^p_{\mu_n}\right).$$

Since $t^\beta u^p_{\nu_n}$ and $t^\beta u^p_{\mu_n}$ converges in $L^1(Q_T)$ to $t^\beta u^p_\nu$ and $t^\beta u^p_\mu$ respectively, they are uniformly integrable. Hence the sequence $\{t^\beta (u_{\nu_n+\mu_n})^p\}$ is uniformly integrable in Q_T and thus, up to extraction of a second subsequence, $t^\beta (u_{\nu_n+\mu_n})^p$ converges to $t^\beta w^p$ in $L^1(Q_T)$. Going to the limit in the formulation (2.21) of the fact $u_{\nu_n+\mu_n}$ is a weak solution of (4.1) with initial data $\nu_n + \mu_n$, it follows that w satisfies the same equation (4.1) but now with initial data $\mu + \nu$. By uniqueness (see [28, Th 1.1] and notice that therein uniqueness needs no more condition on h than monotonicity), $w = u_{\nu+\mu}$ and the whole sequence $\{u_{\nu_n+\mu_n}\}$ converges to $u_{\nu+\mu}$. The proof in the case of admissible measures is similar. □

Proposition 4.2 *Let $p > 1$ and $\beta > -1$. If $\{\nu_k\} \subset \mathfrak{M}^b_+(\mathbb{R}^N)$ is a nondecreasing sequence of admissible measures converging to $\nu \in \mathfrak{M}^b_+(\mathbb{R}^N)$, then ν is an admissible measure.*

Proof. The sequence $\{u_{\nu_k}\}$ is increasing. Furthermore,

$$u_{\nu_k} \le \mathbb{H}_s[\nu_k] \le \mathbb{H}_s[\nu].$$

Hence there exists some $u \in L^1(Q_T)$ for any $T > 0$, $u \ge 0$, such that $u_{\nu_n} \to u$ in $L^1(Q_T)$ and a.e. in Q_∞. By identity (3.25) in the proof of [28, Th. 1.1], we have for $\tau \ge T$,

$$\int_0^1 \int_{\mathbb{R}^N} \left(u_{\nu_k} + (\tau - t) t^\beta u_{\nu_k}^q \right) dxdt + (\tau - T) \int_{\mathbb{R}^N} u_{\nu_k}(T, x) dx = \tau \int_{\mathbb{R}^N} d\nu_k \le 2\tau \|\nu\|_{\mathfrak{M}_+^b}. \tag{4.2}$$

Hence $t^\beta u^q \in L^1(Q_T)$ and $t^\beta u_{\nu_k}^q \to t^\beta u^q$ in $L^1(Q_T)$. By (2.21) there holds

$$\int_0^1 \int_{\mathbb{R}^N} \left[u_{\nu_k} \left(-\partial_t \xi + (-\Delta)^s \xi \right) + t^\beta u_{\nu_k}^q \xi \right] dxdt = \int_{\mathbb{R}^N} \xi(0, x) d\nu_k(x),$$

for all $\xi \in \mathbb{Y}_{s,T}$, so it follows that $u = u_\nu$. Hence ν is an admissible measure. □

The whole description of the set of admissible measures necessitates the introduction of Bessel capacities as in the case $s = 1$, see [37], [41]. We have a first partial answer.

Lemma 4.3 *Let $p > 1$ and $\beta > -1$. If $\nu \in \mathfrak{M}_+^b(\mathbb{R}^N)$ satisfies $t^\beta (\mathbb{H}_s[\nu])^p \in L^1(Q_T)$, then ν is a good measure.*

Proof. Let $\nu_n = \nu * \rho_n$, by the maximum principle

$$u_{\nu_n} \le \mathbb{H}_s[\nu_n] = \mathbb{H}_s[\nu * \rho_n] = \mathbb{H}_s[\nu] * \rho_n.$$

Since $\mathbb{H}_s[\nu] \in L^1(Q_T)$, $\mathbb{H}_s[\nu] * \rho_n \to \mathbb{H}_s[\nu]$ in $L^1(Q_T)$. Similarly $t^\beta (\mathbb{H}_s[\nu] * \rho_n)^p \to t^\beta (\mathbb{H}_s[\nu])^p$ in $L^1(Q_T)$. Since $u_{\nu_n} \le \mathbb{H}_s[\nu] * \rho_n$, we conclude that the sequences $\{u_{\nu_n}\}$ and $\{t^\beta u_{\nu_n}^p\}$ are uniformly integrable in $L^1(Q_T)$, hence they are precompact by Vitali's convergence theorem. As in the proof of Proposition 4.1 any cluster point w in the $L^1(Q_T)$-topology of the sequence $\{u_{\nu_n}\}$ is a weak solution of (4.1) with initial data ν hence $w = u_\nu$ and $u_{\nu_n} \to u_\nu$ by uniqueness of the solution. □

Proposition 4.4 *Let $p > 1$ and $\beta > -1$. Assume that $\nu \in \mathfrak{M}_+^b(\mathbb{R}^N)$ with Lebesgue decomposition $\nu = \nu_0 + \nu_s$, where ν_0 and ν_s belong to $\mathfrak{M}_+^b(\mathbb{R}^N)$, $\nu_0 \in L^1(\mathbb{R}^N)$ and ν_s is singular with respect to the N-dim Lebesgue measure. If $t^\beta (\mathbb{H}_s[\nu_s])^p \in L^1(Q_T)$, then ν is a good measure.*

Proof. By [28, Lemma 3.2] there exists a unique solution u_{ν_0} (resp. ν_s) to problem (4.1) with ν replaced by ν_0. (resp. ν_s). By [28, Lemma 3.2] the sequences $\{u_{\nu_0 * \rho_n}\}$ and $\{t^\beta u_{\nu_0 * \rho_n}^p\}$ are Cauchy sequences in $L^1(Q_T)$. By Lemma 4.3, the sequences $\{u_{\nu_s * \rho_n}\}$ and $\{t^\beta u_{\nu_s * \rho_n}^p\}$ share the same property. Hence ν_0 and ν_s are good measures and we conclude with Proposition 4.1. □

We recall some classical results about Bessel potentials, capacities and interpolation. For $0 < \gamma < N$, the Bessel kernel J_γ is defined in $\mathbb{R}^N \setminus \{0\}$ by $J_\gamma(x) = \mathcal{F}^{-1}\left((1 + |\xi|^2)^{-\frac{\gamma}{2}} \right)$, where $\mathcal{F}$ is the Fourier transform in $\mathbb{R}^N$, and the Bessel potential of a positive measure is

$$\mathbf{J}_\gamma[\mu](x) = J_\gamma * \mu(x) = \int_{\mathbb{R}^N} J_\gamma(x - y) d\mu(y). \tag{4.3}$$

For $1 \leq r < \infty$, the Bessel capacity $\mathrm{cap}_{\gamma,r}^{\mathbb{R}^N}$ of a compact set is

$$\mathrm{cap}_{\gamma,r}^{\mathbb{R}^N}(K) = \inf\{\|\mathbf{J}_\gamma[\zeta]\|_{L^r(\mathbb{R}^N)}^r : \zeta \in \omega_K\}, \tag{4.4}$$

where ω_K is the subset of nonnegative function belonging to the Schwartz space $\mathcal{S}(\mathbb{R}^N)$, with value larger or equal to 1 on K. Furthermore

$$(-\Delta + I)^{\frac{\gamma}{2}}\phi = \zeta \Longleftrightarrow \mathbf{J}_\gamma[\zeta] = \phi. \tag{4.5}$$

If a linear m-accretive operator A in $L^r(\mathbb{R}^N)$ with domain $D(A)$ is the infinitesimal generator of an analytic semigroup of bounded linear operators $S^A(t)$, i.e.

$$u(t) = S_t^{-A}\upsilon \quad \forall t \geq 0 \Longleftrightarrow \frac{du}{dt} + Au = 0 \quad \text{on } \mathbb{R}_+^*\,,\ u(0) = \upsilon, \tag{4.6}$$

the real interpolation classes between $D(A)$ and $L^r(\mathbb{R}^N)$ can be obtained (see [49, p. 96]) by

$$\left[D(A), L^r(\mathbb{R}^N)\right]_{\theta,r} = \left\{\upsilon \in L^r(\mathbb{R}^N) : \int_0^1 \int_{\mathbb{R}^N} t^{(1-\theta)r}|AS_t^A\upsilon|^r dx\frac{dt}{t} < +\infty\right\}, \tag{4.7}$$

and

$$\|\upsilon\|_{[D(A),L^r]_{\theta,r}} \equiv \|\upsilon\|_{L^r} + \left(\int_0^1 \|t^{1-\theta}AS_t^A\upsilon\|_{L^r}^r \frac{dt}{t}\right)^{\frac{1}{r}}. \tag{4.8}$$

If $A = (-\Delta)^s + I$, its domain $D(A)$ in $L^r(\mathbb{R}^N)$ is the Bessel potential space $(I-\Delta)^{-s}(L^r(\mathbb{R}^N)) = L^{2s,r}(\mathbb{R}^N)$: the result is stated in [34, Th 1] but it is an easy consequence of [47, Chap. 4, Th 3] applied to the Fourier multipliers $(I+|\xi|^2)^s(1+|\xi|^{2s})^{-1}$ and $(I+|\xi|^2)^{-s}(1+|\xi|^{2s})$. By classical interpolation properties of Bessel potential spaces (see e.g. [49]),

$$\left[D(A), L^r(\mathbb{R}^N)\right]_{\theta,r} = L^{2\theta s,r}(\mathbb{R}^N) = (I-\Delta)^{-\theta s}(L^r(\mathbb{R}^N)). \tag{4.9}$$

Since A is coercive, [49, Sec. 1.14.5], (4.8) can be replaced by

$$\|\upsilon\|_{[D(A),L^r]_{\theta,r}} \equiv \left(\int_0^1 \|t^{1-\theta}AS_t^A\upsilon\|_{L^r}^r \frac{dt}{t}\right)^{\frac{1}{r}}. \tag{4.10}$$

Proposition 4.5 *Let $N \geq 1$, $p > 1$ and $-1 < \beta < p-1$. If problem (4.1) admits a positive solution u_ν for some $\nu \in \mathfrak{M}_b^+(\mathbb{R}^N)$, then ν vanishes on Borel subsets of $\mathbb{R}^N$ with zero Bessel capacity $\mathrm{cap}_{\frac{2s(1+\beta)}{p},p'}^{\mathbb{R}^N}$ where $p' = \frac{p}{p-1}$, i.e.*

$$\forall K \subset \mathbb{R}^N, K \text{ Borel}, \mathrm{cap}_{\frac{2s(1+\beta)}{p},p'}^{\mathbb{R}^N}(K) = 0 \Longrightarrow \nu(K) = 0. \tag{4.11}$$

Proof. Assume that $u := u_\nu$ is the solution of (4.1). Since $\mathrm{cap}_{\frac{2s(1+\beta)}{p},p'}^{\mathbb{R}^N}$ is a Choquet capacity, let $K \subset \mathbb{R}^N$ is compact and $\zeta \in \mathcal{S}(\mathbb{R}^N)$ be such that $0 \leq \zeta$ in $\mathbb{R}^N$ and $\zeta \geq 1$ on K. We set $\Phi = e^{-t}\mathbb{H}_s[\zeta]$ and take $\Phi^{p'}$ as a test function. Then

$$\int_{\mathbb{R}^N} u\Phi^{p'}(1,.)dx + \int_0^1 \int_{\mathbb{R}^N} \left[\left(-\partial_t\Phi^{p'} + (-\Delta)^s\Phi^{p'}\right)u + t^\beta u^p\Phi^{p'}\right]dxdt = \int_{\mathbb{R}^N} \Phi^{p'}d\nu. \tag{4.12}$$

Note that $(-\Delta)^s\Phi^{p'} \geq p'\Phi^{p'-1}(-\Delta)^s\Phi$ and $\partial_t\Phi + (-\Delta)^s\Phi + \Phi = 0$, hence

$$-\partial_t\Phi^{p'} + (-\Delta)^s\Phi^{p'} \geq 2p'\Phi^{p'-1}(-\Delta)^s\Phi.$$

Then we adapt the duality argument of [7] and get from Hölder's inequality

$$\int_{\mathbb{R}^N} u\Phi^{p'}(1,x)dx + 2p'\left(\int_0^1\int_{\mathbb{R}^N} t^{-\frac{p'\beta}{p}}\left|(-\Delta)^s\Phi + \Phi\right|^{p'}dxdt\right)^{\frac{1}{p'}}\left(\int_0^1\int_{\mathbb{R}^N} t^\beta u^p\Phi^{p'}dxdt\right)^{\frac{1}{p}} + \int_0^1\int_{\mathbb{R}^N} t^\beta u^p\Phi^{p'}dxdt \geq \nu(K).$$

Applying (4.7), (4.9) with $r = p'$, $\theta = \frac{1+\beta}{p}$, we obtain directly for some $c_{11} > 1$,

$$\frac{1}{c_{11}}\|\zeta\|_{L^{\frac{2s(1+\beta)}{p},p'}} \leq \left(\int_0^1\int_{\mathbb{R}^N} t^{-\frac{p'\beta}{p}}\left|(-\Delta)^s\Phi + \Phi\right|^{p'}dxdt\right)^{\frac{1}{p'}} \leq c_{11}\|\zeta\|_{L^{\frac{2s(1+\beta)}{p},p'}}. \tag{4.13}$$

If $\mathrm{cap}^{\mathbb{R}^N}_{\frac{2s(1+\beta)}{p},p'}(K) = 0$, there exists a sequence $\{\zeta_n\} \subset \omega_K$ such that $\|\zeta_n\|_{L^{\frac{2s(1+\beta)}{p},p'}} \to 0$ as $n \to \infty$. Furthermore it is possible to assume $\zeta_n \leq 1$ in $\mathbb{R}^N$ (see [3]). Hence, up to a subsequence, $\zeta_n \to 0$ a.e. in $\mathbb{R}^N$. This implies $\Phi_n \leq 1$ and $\Phi_n \to 0$ a.e. in Q_∞. Therefore,

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} u\Phi_n^{p'}(1,x)dx \quad \text{and} \quad \lim_{n\to\infty}\int_0^1\int_{\mathbb{R}^N} t^\beta u^p\Phi_n^{p'}dxdt = 0.$$

Combining the previous inequalities we infer that $\nu(K) = 0$. □

Proposition 4.5 is the necessary condition in Theorem H. The next result provides the sufficient condition.

Proposition 4.6 *Let $N \geq 1$, $p > 1$, $-1 < \beta < p-1$ and $\nu \in \mathfrak{M}_b^+(\mathbb{R}^N)$ which vanishes on Borel subsets of $\mathbb{R}^N$ with zero $\mathrm{cap}^{\mathbb{R}^N}_{\frac{2s(1+\beta)}{p},p'}$-Bessel capacity. Then ν is an admissible measure.*

Proof. If ν vanishes Borel subsets with zero $\mathrm{cap}^{\mathbb{R}^N}_{\frac{2s(1+\beta)}{p},p'}$, there exists an increasing sequence of nonnegative measures $\{\nu_n\} \subset \left(L^{\frac{2s(1+\beta)}{p-1},p'}(\mathbb{R}^N)\right)' = L^{-\frac{2s(1+\beta)}{p-1},p}(\mathbb{R}^N)$ such that $\nu_n \to \nu$ in the sense of measures. This results is classical in the integer case and a proof in the Bessel case (similar in fact) can be found in [50, Prop. 3.6].
Next let $\zeta \in \mathcal{S}(\mathbb{R}^N)$ and $\Phi = e^{-t}\mathbb{H}_s[\zeta]$, then

$$\int_{\mathbb{R}^N}\Phi\mathbb{H}_s[\nu_n](1,x)dx + \int_0^1\int_{\mathbb{R}^N}\mathbb{H}_s[\nu_n]\left(2(-\Delta)^s\Phi + \Phi\right)dxdt = \int_{\mathbb{R}^N}\zeta d\nu_n.$$

Hence

$$\int_0^1\int_{\mathbb{R}^N}\mathbb{H}_s[\nu_n]\left((-\Delta)^s\Phi + \Phi\right)dxdt \leq \|\nu_n\|_{L^{-\frac{2s(1+\beta)}{p-1},p}}\|\zeta\|_{L^{\frac{2s(1+\beta)}{p-1},p'}}.$$

Consider the mapping

$$\zeta \mapsto L(\zeta) = \int_0^1 \int_{\mathbb{R}^N} t^{\frac{\beta}{p}} \mathbb{H}_s[\nu_n] t^{-\frac{\beta}{p}} \left((-\Delta)^s \Phi + \delta\Phi\right) dxdt.$$

It satisfies

$$\begin{aligned} |L(\zeta)| &\leq \|\nu_n\|_{L^{-\frac{2s(1+\beta)}{p-1},p}} \|\zeta\|_{L^{\frac{2s(1+\beta)}{p-1},p'}} \\ &\leq c\|\nu_n\|_{L^{-\frac{2s(1+\beta)}{p-1},p}} \left(\int_0^1 \int_{\mathbb{R}^N} t^{-\frac{p'\beta}{p}} \left|(-\Delta)^s \Phi + \delta\Phi\right|^{p'} dxdt\right)^{\frac{1}{p'}}, \end{aligned} \tag{4.14}$$

by (4.13). Hence $t^{\frac{\beta}{p}} \mathbb{H}_s[\nu_n] \in L^p(Q_1)$ and

$$\left(\int_0^1 \int_{\mathbb{R}^N} t^\beta \left(\mathbb{H}_s[\nu_n]\right)^p dxdt\right)^{\frac{1}{p}} \leq c_{12} \|\nu_n\|_{L^{-\frac{2s(1+\beta)}{p-1},p}}. \tag{4.15}$$

Hence the ν_n are good measures by Lemma 4.3. Then by Proposition 4.2, ν is an admissible measure. □

Remark. When $s = 1$ and $\beta = 0$, it is proved in [7] that the admissibility condition for measures is strongly linked to the removability for Borel sets in the sense that if $K \subset \mathbb{R}^N$ is a Borel set with zero $\mathrm{cap}^{\mathbb{R}^N}_{\frac{2}{p},p'}$-capacity, any $u \in C(\overline{Q}_\infty \setminus \{(0,K)\}) \cap C^{1,2}(Q_\infty)$ solution of (1.14) in Q_∞ which vanishes on $(0,x)$ for any $x \in \mathbb{R}^N \setminus K$ is identically zero. The set K is said *removable*. Furthermore, the condition is also necessary. Now, for equation (1.14) it is clear that a compact set K with positive $\mathrm{cap}^{\mathbb{R}^N}_{\frac{2s(1+\beta)}{p},p'}$-capacity it is not removable since it is the support of the capacitary measure (a positive measure belonging to $L^{\frac{2s(1+\beta)}{p},p'}(\mathbb{R}^N)$, [1, Chap. 2]) which is a good measure by Lemma 4.3. *We conjecture that the condition* $\mathrm{cap}^{\mathbb{R}^N}_{\frac{2s(1+\beta)}{p},p'}(K) = 0$ *implies the removability of the compact set K for equation (1.26) in the sense given above.*

4.2 Barrier function for $N = 1$

We set

$$W(z) = \begin{cases} \dfrac{\ln(e+z^2)}{1+z^{1+2s}} & \text{if} \quad z \geq 0 \\ 1 & \text{if} \quad z < 0, \end{cases} \tag{4.16}$$

where e is Neper constant, and

$$w(t,x) = t^{-\frac{1+\beta}{p-1}} W(t^{-\frac{1}{2s}} x), \quad \forall\, (t,x) \in \mathbb{R}^*_+ \times \mathbb{R}. \tag{4.17}$$

When $t \to 0$, the function w satisfies

$$\begin{aligned} &(i) \qquad w(t,x) = \frac{2t^{\frac{1+2s}{2s} - \frac{1+\beta}{p-1}} \ln t}{|x|^{1+2s}} (1 + o(1)) \qquad \text{if} \quad x > 0, \\ &(ii) \qquad w(t,x) = t^{-\frac{1+\beta}{p-1}} \qquad\qquad\qquad\qquad\quad \text{if} \quad x \leq 0. \end{aligned} \tag{4.18}$$

Lemma 4.1 *Assume that $p > 1 + \frac{2s(1+\beta)}{1+2s}$. Then there exists $\lambda_0 > 0$ such that for $\lambda \geq \lambda_0$, the function $w_\lambda := \lambda w$ satisfies*

$$\begin{aligned} \partial_t w_\lambda + (-\Delta)^s_{\mathbb{R}} w_\lambda + t^\beta w_\lambda^p \geq 0 \qquad & \text{in} \quad \mathbb{R}^*_+ \times \mathbb{R} \\ \lim_{t\to 0} w(t,x) = 0 \qquad & \text{if} \quad x > 0 \\ \lim_{t\to 0} w(t,x) = \infty \qquad & \text{if} \quad x \leq 0. \end{aligned} \tag{4.19}$$

Proof. Clearly the assertions concerning the limit of $w(t,x)$ when $t \to 0$ are satisfied since $\frac{1+2s}{2s} - \frac{1+\beta}{p-1} > 0$ by assumption. Then

$$\partial_t w_\lambda(t,x) = -\frac{\lambda(1+\beta)}{p-1} t^{-\frac{1+\beta}{p-1}-1} w(z) - \frac{\lambda}{2s} t^{-\frac{1+\beta}{p-1}-1} w'(z) z,$$

with $z = t^{-\frac{1}{2s}} x$ and

$$(-\Delta)^s_1 w_\lambda(t,x) = \lambda t^{-\frac{1+\beta}{p-1}-1} (-\Delta)^s_1 w(z).$$

Hence

$$\begin{aligned} &\partial_t w_\lambda(t,x) + (-\Delta)^s_{\mathbb{R}} w_\lambda(t,x) + t^\beta w_\lambda^p(t,x) \\ &\quad = \lambda t^{-\frac{1+\beta}{p-1}-1} \left[(-\Delta)^s_{\mathbb{R}} w(z) - \frac{1}{2s} w'(z) z - \frac{1+\beta}{p-1} w(z) + \lambda^{p-1} w^p(z) \right]. \end{aligned} \tag{4.20}$$

If $z > 0$, we obtain that

$$-\frac{1}{2s} w'(z) z - \frac{1+\beta}{p-1} w(z) = \left[\frac{1+2s}{2s} \frac{z^{1+2s}}{1+z^{1+2s}} - \frac{1+\beta}{p-1} - \frac{z^2 (e+z^2)^{-1}}{s \ln(e+z^2)} \right] w(z).$$

Since $\frac{1+2s}{2s} > \frac{1+\beta}{p-1}$, $\lim_{z\to\infty} \frac{z^{1+2s}}{1+z^{1+2s}} = 1$ and $\lim_{z\to\infty} \frac{1}{\ln(e+z^2)} = 0$, then there exist $R_0 > 0$ and $\sigma_0 > 0$ such that

$$-\frac{1}{2s} w'(z) z - \frac{1+\beta}{p-1} w(z) \geq \sigma_0 w(z) \qquad \forall\, z \geq R_0. \tag{4.21}$$

Next we deal with $(-\Delta)^s_{\mathbb{R}} w(z)$ and put

$$\tilde{w}(z) = \frac{\ln(e+z^2)}{1+|z|^{1+2s}} \qquad \forall\, z \in \mathbb{R},$$

so that $(-\Delta)^s_{\mathbb{R}} w(z) = (-\Delta)^s_{\mathbb{R}} \tilde{w}(z) + (-\Delta)^s_1 (1 - \tilde{w}\chi_{\mathbb{R}_-})(z)$.
For $z > 2$, using the equivalent definition of fractional Laplacian, we have that

$$\begin{aligned} -(-\Delta)^s_{\mathbb{R}} \tilde{w}(z) &= \frac{a_{1,s}}{2} \int_{-\infty}^{\infty} \frac{\frac{\ln(e+|z+\tilde{y}|^2)}{1+|z+\tilde{y}|^{1+2s}} + \frac{\ln(e+|z-\tilde{y}|^2)}{1+|z-\tilde{y}|^{1+2s}} - \frac{2\ln(e+z^2)}{1+z^{1+2s}}}{|\tilde{y}|^{1+2s}} d\tilde{y} \\ &= \frac{a_{1,s} w(z)}{2z^{2s}} \int_{-\infty}^{\infty} \frac{I_z(y)}{|y|^{1+2s}} dy, \end{aligned} \tag{4.22}$$

where

$$I_z(y) = \frac{1+z^{N+2s}}{1+z^{1+2s}|1+y|^{1+2s}} \frac{\ln(e+z^2|1+y|^2)}{\ln(e+z^2)} + \frac{1+z^{1+2s}}{1+z^{1+2s}|1-y|^{1+2s}} \frac{\ln(e+z^2|1-y|^2)}{\ln(e+z^2)} - 2.$$

Step 1: There exists $c_{12} > 0$ such that

$$\int_{\frac{1}{2}\le|y|\le\frac{3}{2}} \frac{I_z(y)}{|y|^{1+2s}}dy \le \frac{c_{12}}{w(z)z}. \tag{4.23}$$

Actually, for $-\frac{3}{2} < y < -\frac{1}{2}$, there exists $c_{13} > 0$ such that

$$\frac{1+z^{1+2s}}{1+z^{1+2s}|1-y|^{1+2s}} \frac{\ln(e+z^2|1-y|^2)}{\ln(e+z^2)} \le c_{13}$$

and then

$$\begin{aligned}
\int_{-\frac{3}{2}}^{-\frac{1}{2}} \frac{I_z(y)}{|y|^{1+2s}}dy &\le 2\int_0^{\frac{1}{2}} \frac{1+z^{1+2s}}{1+(zr)^{1+2s}} \frac{\ln(e+z^2r^2)}{\ln(e+z^2)}dr + c_{14} \\
&\le \frac{2}{w(z)z}\int_0^{\infty} \frac{\ln(e+t^2)}{1+t^{1+2s}}dt + c_{14} \\
&\le \frac{c_{15}}{w(z)z},
\end{aligned}$$

where $c_{14}, c_{15} > 0$, and the last inequality holds since $w(z)z \to 0$ as $z \to +\infty$. Similarly,

$$\int_{\frac{1}{2}}^{\frac{3}{2}} \frac{I_z(y)}{y^{N+2s}}dy_1 \le \frac{c_{16}}{w(z)z}.$$

Step 2: There exists $c_{17} > 0$ such that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{I_z(y)}{|y|^{1+2s}}dy \le c_{17}. \tag{4.24}$$

Indeed, since function I_z is C^2 in $[-\frac{1}{2}, \frac{1}{2}]$ and satisfies

$$I_z(0) = 0 \ \text{ and } \ I_z(y) = I_z(-y),$$

then $I_z'(0) = 0$ and there exists $c_{18} > 0$ such that

$$|I_z''(y)| \le c_{18} \quad \text{for any } y \in [-\tfrac{1}{2}, \tfrac{1}{2}].$$

Then we have

$$|I_z(y)| \le \frac{c_{18}}{2}y^2 \quad \text{for any } y \in [-\tfrac{1}{2}, \tfrac{1}{2}],$$

which implies that

$$\left|\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{I_z(y)}{|y|^{1+2s}}dy\right| \le c_{19}.$$

Step 3: There exists $c_{20} > 0$ such that

$$\left|\int_A \frac{I_z(y)}{|y|^{1+2s}}dy\right| \le c_{20}, \tag{4.25}$$

where $A = (-\infty, -\frac{3}{2}) \cup (\frac{3}{2}, +\infty)$. In fact, for $y \in A$, we observe that there exists $c_{21} > 0$ such that $I_z(y) \le c_{21}$ and

$$\int_A \frac{I_z(y)}{|y|^{1+2s}} dy \le 2 \int_{\frac{3}{2}}^{+\infty} \frac{c_{21}}{|y|^{1+2s}} dy \le c_{22}$$

for some $c_{22} > 0$. Consequently, by (4.22)-(4.25), there exists $c_{23} > 0$ such that

$$(-\Delta)^s_{\mathbb{R}} \tilde{w}(z) \ge -\frac{c_{23}}{1 + z^{1+2s}} \qquad \forall\, z \ge 2.$$

Since $1 - \tilde{w}\chi_{\mathbb{R}_-} = 1$ in $\mathbb{R}_+$ and $1 - \tilde{w}\chi_{\mathbb{R}_-} \le 1$ in $\mathbb{R}_-$, we have also

$$(-\Delta)^s_{\mathbb{R}}(1 - \tilde{w}\chi_{\mathbb{R}_-})(z) \ge 0 \qquad \forall\, z > 0.$$

Therefore, we obtain that

$$(-\Delta)^s_{\mathbb{R}} w(z) \ge -\frac{c_{23}}{1 + z^{1+2s}} \qquad \forall\, z \ge 2. \tag{4.26}$$

Combining (4.21) and (4.26), we infer that there exists $R_1 \ge R_0 + 2$ such that for $z > R_1$,

$$\begin{aligned}(-\Delta)^s_{\mathbb{R}} w(z) - \frac{1}{2s} w'(z) z - \frac{1+\beta}{p-1} w(z) &\ge \sigma_0 w(z) - \frac{c_{23}}{1 + z^{1+2s}} \\ &= w(z) \left(\sigma_0 - \frac{c_{23}}{\ln(e + z^2)} \right) \\ &\ge 0.\end{aligned}$$

For $z \le R_1$, there exists $c_{24} > 0$ such that

$$(-\Delta)^s_{\mathbb{R}} w(z) - \frac{1}{2s} w'(z) z - \frac{1+\beta}{p-1} w(z) \ge -c_{24},$$

and there exists $c_{25} > 0$ dependent of R_1 such that

$$w(z) \ge c_{25}.$$

Therefore, one can find $\Lambda_0 > 0$ such that for $\lambda \ge \Lambda_0$,

$$(-\Delta)^w_{\mathbb{R}}(z) - \frac{1}{2s} w'(z) z - \frac{1+\beta}{p-1} w(z) + \lambda^{p-1} w^p(z) \ge 0 \qquad \forall\, z \in \mathbb{R}, \tag{4.27}$$

which, together with (4.20), implies that (4.19) holds true. This ends the proof. □

4.3 Solutions with initial trace $(\mathcal{S}, 0)$

Lemma 4.7 *Assume that $N \geq 1$ and $p > 1 + \frac{2s(1+\beta)}{1+2s}$. Then for any $R > 0$ there exists a positive function $u = u_{\infty,B_R}$ minimal among the solutions of (1.26) in Q_∞, which satisfy*

$$\begin{array}{lll} \lim_{t\to 0} u(t,x) = \infty & \text{uniformly in} & \overline{B}_R \\ \lim_{t\to 0} u(t,x) = 0 & \text{uniformly in} & B^c_{R'} \quad \text{for any } R' > R. \end{array} \tag{4.28}$$

Furthermore, the mapping $R \mapsto u_{\infty,B_R}$ is increasing.

Proof. By scaling we can assume that $R = 1$ and we fix $\lambda \geq \lambda_0$. We denote by $\mathbf{e}_1$ the point with coordinates $(1, 0, ..., 0)$ in $\mathbb{R}^N$. The function

$$(t,x) \mapsto w_{e_1}(t, x_1, x') = \lambda t^{-\frac{1+\beta}{p-1}} W(t^{-\frac{1}{2s}}(x_1 - 1)), \tag{4.29}$$

is a super solution of (1.26) in Q_∞, which satisfies

$$\begin{array}{lll} (i) & \lim_{t\to 0} w_{e_1}(t, x_1, x') = \infty & \text{uniformly in} \quad (-\infty, 1] \times \mathbb{R}^{N-1}, \\ (ii) & \lim_{t\to 0} w_{e_1}(t,x) = 0 & \text{uniformly in} \quad [1+\epsilon, \infty) \times \mathbb{R}^{N-1}. \end{array} \tag{4.30}$$

Since equation (1.26) is invariant under rotations and translations, for any $a \in \partial B_1$ there exists a rotation $\mathcal{R}_a$ with center 0 such that $\mathcal{R}_a(a) = e_1$. Therefore, the function $(t,x) \mapsto w_a(t,x) := w_{e_1}(t, \mathcal{R}_a(x))$ is a solution of (1.26) in Q_∞ and it satisfies

$$\begin{array}{lll} (i) & \lim_{t\to 0} w_a(t,x) = \infty & \text{uniformly in} \quad \{x \in \mathbb{R}^N : \langle x, a\rangle \leq 1\}, \\ (ii) & \lim_{t\to 0} w_{e_1}(t,x) = 0 & \text{uniformly in} \quad \{x \in \mathbb{R}^N : \langle x, a\rangle \geq 1 + ge\}. \end{array} \tag{4.31}$$

For $k \in \mathbb{N}^*$, let $u_{k\chi_{B_1}}$ be the solution of

$$\begin{array}{rll} \partial_t u + (-\Delta)^s u + t^\beta u^p = 0 & \text{in} & Q_\infty \\ u(0,.) = k\chi_{B_1} & \text{in} & \mathbb{R}^N. \end{array} \tag{4.32}$$

Then the sequence $\{u_{k\chi_{B_1}}\}_k$ is increasing. For any $a \in \partial B_1$, $u_{k\chi_{B_1}} \leq w_a$, the following limit exists,

$$u_{\infty,B_1} = \lim_{k\to\infty} u_{k\chi_{B_1}},$$

and there holds

$$u_{\infty,B_1} \leq \inf\{w_a : a \in \partial B_1\}.$$

This solution u is clearly minimal by construction and the monotonicity of the mapping $R \mapsto u_{\infty,B_R}$ follows. □

Remark. In the previous result, the ball B_R can be replaced by any closed convex set with a non-empty interior. If $a \in \partial K$, let H_a be an affine separation hyperplane, with outer normal vector $\mathbf{n}_a$ and

$$H_a^+ = \{x \in \mathbb{R}^N : \langle x - a, \mathbf{n}_a\rangle > 0\} \quad \text{and} \quad H_a^- = \{x \in \mathbb{R}^N : \langle x - a, \mathbf{n}_a\rangle < 0\}.$$

The supersolutions w_a are expressed by

$$(t,x)\mapsto w_a(t,x)=\lambda t^{-\frac{1+\beta}{p-1}}W(t^{-\frac{1}{2s}}\langle x-a,\mathbf{n}_a\rangle)$$

and have initial trace $(0,\overline{H_a}^-)$. Then we construct the minimal solution $u=u_{\infty,K}$ of (1.26) with initial trace $(0,K)$ such that

$$\begin{array}{lllll}(i) & \lim_{t\to 0}u(t,x)=\infty & \text{uniformly in} & K, & \\ (ii) & \lim_{t\to 0}u(t,x)=0 & \text{uniformly in} & \{x\in K^c:\operatorname{dist}(x,K)\geq\epsilon\} & \forall\epsilon>0.\end{array} \tag{4.33}$$

Furthermore, the mapping $K\mapsto u_{\infty,K}$ is nondecreasing.

Proposition 4.8 *Assume that $N\geq 1$ and $p>1+\frac{2s(1+\beta)}{1+2s}$. Then for any closed set $\mathcal{S}$ such that $\overline{\operatorname{int}(\mathcal{S})}=\mathcal{S}$ there exists a positive function $u=u_{\infty,S}$ minimal among the solutions of (1.26) in Q_∞ which satisfy*

$$\begin{array}{lllll}(i) & \lim_{t\to 0}u(t,x)=\infty & \text{locally uniformly in} & \mathcal{S}, & \\ (ii) & \lim_{t\to 0}u(t,x)=0 & \text{locally uniformly in} & \{x\in\mathcal{S}^c:\operatorname{dist}(x,\mathcal{S})\geq\epsilon\} & \forall\,\epsilon>0.\end{array} \tag{4.34}$$

In particular $Tr(u_{\infty,S})=(\mathcal{S},0)$. Furthermore,

$$u_{\mathcal{S},\infty}(t,x)\leq c_9t^{-\frac{1+\beta}{p-1}}\frac{\ln\left(e+t^{-\frac{1}{s}}(\operatorname{dist}(x,\mathcal{S}))^2\right)}{1+t^{-\frac{1+2s}{2s}}(\operatorname{dist}(x,\mathcal{S}))^{1+2s}}\qquad\forall\,(t,x)\in Q_\infty. \tag{4.35}$$

Proof. We first assume that $\mathcal{S}$ is compact, hence precompact, and for any $\delta>0$ there exists a finite number of points $\xi_j\in\mathcal{S}$, $1\leq j\leq n_\delta$ such that

$$\mathcal{S}\subset\bigcup_{j=I}^{n_\delta}\overline{B}_\delta(\xi_j):=\mathcal{S}_\delta.$$

Clearly the mapping $\delta\mapsto n_\delta$ is nondecreasing, furthermore we can choose the points ξ_j such that $\delta\mapsto\mathcal{S}_\delta$ is decreasing for the order relation of inclusion between sets. Since $p>1$, the function

$$w_{\mathcal{S}_\delta}:=\sum_{j=1}^{n_\delta}u_{\infty,\overline{B}_\delta(\xi_j)}, \tag{4.36}$$

is a supersolution of (1.26) in Q_∞ and by Lemma 4.7 it satisfies

$$\begin{array}{lllll}(i) & \lim_{t\to 0}w_{\mathcal{S}_\delta}(t,x)=\infty & \text{uniformly in} & \mathcal{S}_\delta, & \\ (ii) & \lim_{t\to 0}w_{\mathcal{S}_\delta}(t,x)=0 & \text{uniformly in} & \{x\in\mathcal{S}_\delta^c:\operatorname{dist}(x,\mathcal{S}_\delta)\geq\epsilon\} & \forall\epsilon>0.\end{array} \tag{4.37}$$

For $k\in\mathbb{N}^*$ let $u_{k\chi_{\mathcal{S}}}$ be the solution of (1.26) in Q_∞ with initial data $k\chi_{\mathcal{S}}$. It exists since $\mathcal{S}$ has a non-empty interior, and it coincides with the solution of (1.26) in Q_∞ with initial data $k\chi_{\operatorname{int}(\mathcal{S})}$. Clearly there holds $u_{k\chi_{\mathcal{S}}}\leq w_{\mathcal{S}_\delta}$ and the sequence $\{u_{k\chi_{\mathcal{S}}}\}_k$ is increasing, then there exists

$$u_{\infty,\mathcal{S}}=\lim_{k\to\infty}u_{k\chi_{\mathcal{S}}}.$$

It is a positive solution of (1.26) in Q_∞ which tends to infinity on $\mathcal{S}$, by construction, and satisfies $u_{\infty,\mathcal{S}} \le w_{\mathcal{S}_\delta}$. This implies in particular that for any $\epsilon > 0$,

$$\lim_{t\to 0} u_{\infty,\mathcal{S}}(t,x) = 0 \quad \text{uniformly in} \quad \{x \in \mathcal{S}_\delta^c : \text{dist}\,(x,\mathcal{S}_\delta) \ge \epsilon\}.$$

Since this holds for any $\delta, \epsilon > 0$, the second assertion in (4.34) follows.

If S is unbounded, for any $\rho > 0$ large enough, $\mathcal{S}^\rho := \mathcal{S} \cap \overline{B}_\rho$ is a nonempty compact set and $\mathcal{S}^\rho = \overline{\text{int}}(\mathcal{S}^\rho)$. Hence there exists a solution $u_{\infty,\mathcal{S}^\rho}$ of (1.26) in Q_∞ with initial trace $(0,\mathcal{S}^\rho)$. By construction $\rho \mapsto u_{\infty,\mathcal{S}^\rho}$ is nondecreasing and converges to a nonnegative solution $u_{\infty,\mathcal{S}}$ of (1.26) in Q_∞. Let $a = (a_1, ..., a_N) \in \mathcal{S}^c$ and $\tau > 0$ such that

$$Q_a^\tau = \{x = (x_1, ..., x_N) : |x_j - a_j| \le \tau\} \subset \mathcal{S}^c.$$

We put

$$W_j(t,x_j) = \lambda t^{-\frac{1+\beta}{p-1}} \left(W(t^{-\frac{1}{2s}}(x_j - a_j + \tau) + W(t^{-\frac{1}{2s}}(a_j + \tau - x_j) \right)$$

with $\lambda \ge \lambda_0$, then W_j is a supersolution of (1.26) in $\mathbb{R}_+ \times \mathbb{R}$ and it satisfies

$$\begin{array}{lll} (i) & \lim_{t\to 0} W_j(t,x) = 0 & \text{locally uniformly in} \quad (a_j - \tau, a_j + \tau), \\ (ii) & \lim_{t\to 0} W_j(t,x) = \infty & \text{uniformly in} \quad (-\infty, a_j - \tau] \bigcup [a_j + \tau, \infty). \end{array}$$

Hence $W_{Q_a^\tau}(t,x) = \sum_j W_j(t,x)$ is a supersolution of (1.26) in Q_∞ and it satisfies

$$\begin{array}{lll} (i) & \lim_{t\to 0} W_{Q_a^\tau}(t,x) = 0 & \text{locally uniformly in} \quad Q_a^\tau, \\ (ii) & \lim_{t\to 0} W_{Q_a^\tau}(t,x) = \infty & \text{uniformly in} \quad \mathbb{R}^N \setminus Q_a^\tau. \end{array}$$

By construction $u_{\infty,\mathcal{S}^\rho} \le W_{Q_a^\tau}$ which implies $u_{\infty,\mathcal{S}} \le W_{Q_a^\tau}$. Hence $u_{\infty,\mathcal{S}}$ satisfies (4.34). The estimate from above can be made more precise (it does not depend from the fact that $\mathcal{S} = \overline{\text{int}}\,\mathcal{S}$) using (4.16) since

$$W_{Q_a^\tau}(a) \le 2N\lambda t^{-\frac{1+\beta}{p-1}} \frac{\ln\left(e + t^{-\frac{1}{s}}\tau^2\right)}{1 + t^{-\frac{1+2s}{2s}}\tau^{1+2s}}. \tag{4.38}$$

If we take $\tau = \frac{\text{dist}\,(a,\mathcal{S})}{\sqrt{N}}$, we obtain (4.35). Furthermore $u_{\infty,\mathcal{S}}$ is clearly minimal as the limit of an increasing sequence of solutions with bounded initial data having compact support. □

4.4 Proof of Theorem I

If $K \subset \mathcal{R}$ is compact, then $\nu_K = \chi_K \nu \in \mathfrak{M}_+^b(\mathcal{R})$; we extend it by zero and still denote by $\nu_K \in \mathfrak{M}^b(\mathbb{R}^N)$ its extension. For $\rho > 0$, $\mathcal{S}^\rho := \mathcal{S} \cap \overline{B}_\rho$ and for $\ell \in \mathbb{N}^*$, $\ell\chi_{\mathcal{S}_\rho} dx$ is a good measure. Since ν_K is a good measure, $\nu_K + \ell\chi_{\mathcal{S}_\rho} dx$ is a good measure by Proposition 4.1. Then there exists a solution $u := u_{\nu_K + \ell\chi_{\mathcal{S}_\rho} dx}$ of (4.1) in Q_∞ with initial data $\nu_K + \ell\chi_{\mathcal{S}_\rho} dx$ and it satisfies

$$\sup\left\{u_{\nu_K}, u_{\ell\chi_{\mathcal{S}_\rho} dx}\right\} \le u_{\nu_K + \ell\chi_{\mathcal{S}_\rho} dx} \le u_{\nu_K} + u_{\ell\chi_{\mathcal{S}_\rho} dx} \le u_{\nu_K} + u_{\infty,\mathcal{S}}. \tag{4.39}$$

Since $(\ell,\rho)\mapsto u_{\ell\chi_{\mathcal{S}_\rho}dx}$ is increasing, we can let ℓ and ρ go to infinity succesively and obtain that $u_{\nu_K+\ell\chi_{\mathcal{S}_\rho}dx}$ converges to a positive solution $\tilde u_K$ of (1.26) in Q_∞ and that

$$\sup\{u_{\nu_K},u_{\infty,\mathcal{S}}\}\le\tilde u_K\le u_{\nu_K}+u_{\infty,\mathcal{S}}. \tag{4.40}$$

This estimate implies that $Tr(\tilde u_K)=(\mathcal{S},\nu_K)$. To end the proof we consider an increasing sequence $\{K_n\}$ of compact sets such that $\bigcup_n K_n=\mathcal{R}$. Then estimate (4.40) holds with K replaced by K_n. Furthermore the sequences $\{u_{\nu_{K_n}}\}$ and $\{\tilde u_{K_n}\}$ are increasing. In order to prove that the sequence $\{u_{\nu_{K_n}}\}$ converges to some solution $\tilde u_\nu$ of (1.26) in Q_∞ which admits ν as the regular part of its initial trace, for $R>0$ we write $\nu_{K_n}=\chi_{\overline{B}_R}\nu_{K_n}+\chi_{\overline{B_R}^c}\nu_{K_n}$ (both solutions exist since K_n is admissible). Then

$$u_{\nu_{K_n}}\le u_{\chi_{\overline{B}_R}\nu_{K_n}}+u_{\chi_{B_R^c}\nu_{K_n}}\le u_{\chi_{\overline{B}_R}\nu}+u_{\infty,B_R^c\cap K_n}\le u_{\chi_{\overline{B}_R}\nu}+u_{\infty,B_R^c\cap\overline{\mathcal{R}}}, \tag{4.41}$$

which implies that the following limit exists and satisfies the upper estimate for any $R>0$,

$$\lim_{n\to\infty}u_{\nu_{K_n}}:=\tilde u_\nu\le u_{\chi_{\overline{B}_R}\nu_K}+u_{\infty,B_R^c\cap\overline{\mathcal{R}}}. \tag{4.42}$$

In turns it implies

$$\sup\{\tilde u_\nu,u_{\infty,\mathcal{S}}\}\le\lim_{n\to\infty}u_{K_n}:=\tilde u\le\tilde u_\nu+u_{\infty,\mathcal{S}}. \tag{4.43}$$

Furthermore, since $R>0$ in inequality (4.42) we infer that ν is the regular part of the initial trace of $\tilde u_\nu$ (notice that the singular part is not empty since ν can be unbounded). Hence $Tr(\tilde u)=(\mathcal{S},\nu)$. □

4.5 Proof of Corollary K, part (a)

If ν vanishes on Borel sets with zero $\mathrm{cap}^{\mathbb{R}^N}_{\frac{2s(1+\beta)}{p},p'}$-capacity, for any compact set $K\subset\mathbb{R}^N$, $\nu_K:=\chi_K\nu$ vanishes also on the same Borel sets. Hence there exists a solution u_{ν_K} to (4.1) with initial data ν_K (instead of ν). Next we replace K by an increasing sequence $\{\overline{B}_n\}_{n\in\mathbb{N}^*}$, and set $\nu_n=\chi_{\overline{B}_n}\nu$. Estimate (4.41) holds in the form

$$u_{\nu_{K_n}}\le u_{\chi_{\overline{B}_R}\nu}+u_{\infty,B_R^c}\qquad\forall\,n,R\ge1.$$

This implies that $\tilde u_\nu$ satisfies the same estimate for any $R>0$, which in turn implies that

$$\lim_{t\to\infty}\int_{\mathbb{R}^N}\tilde u_\nu(t,x)\zeta(x)dx=\int_{\mathbb{R}^N}\zeta d\nu\qquad\forall\,\zeta\in C_0^2(\mathbb{R}^N).$$

Hence $Tr(\tilde u_\nu)=(\{\emptyset\},\nu)$. The fact that $\tilde u_\nu\in L^1_{loc}(0,T;\mathcal{L}^s(\mathbb{R}^N))$ follows from the upper estimate $0\le\tilde u_\nu\le U_{p,\beta}$.

Conversaly (and here we do not use the assumption $p>1+\frac{2s(1+\beta)}{1+2s}$), if $u\in L^1_{loc}(0,\infty;\mathcal{L}^s(\mathbb{R}^N))$ is a solution with initial trace $Tr(u)=(\{\emptyset\},\nu)$, then $u\le U_{p,\beta}$, by Theorem D. We proceed as in the proof of Proposition 4.5. Let $K\subset B_R\subset\mathbb{R}^N$ be compact and $\Theta\in C_0^\infty(B_{2R})$ such that $0\le\Theta\le1$ and $\Theta=1$ on $\overline{B}_R$. Since

$$\|\Theta\phi\|_{W^{k,p'}}\le c(k,p)\|\phi\|_{W^{k,p'}}\qquad\forall\,\phi\in C_0^\infty(R^N)$$

for all $k \in \mathbb{N}^*$ by Leibnitz formula, it follows by interpolation that

$$\|\Theta\phi\|_{L^{\frac{2s(1+\beta)}{p},p'}} \le c(s,p)\|\phi\|_{L^{\frac{2s(1+\beta)}{p},p'}} \qquad \forall\, \phi \in C_0^\infty(R^N). \tag{4.44}$$

If $T \in C^\infty(\mathbb{R}_+)$ satisfies

$$\sup_{t>0} |t^{j-1}T(j)(t)| \le L < \infty, \qquad \forall\, j = 0,1,...,\ell := \ \mathbb{E}\left[\frac{2s(1+\beta)}{p}\right] + 1,$$

then, by *the smooth truncation theorem* (see [1, Th. 3.3.3]),

$$\|T\left(\mathbf{J}_{\frac{2s(1+\beta)}{p}}[\phi]\right)\|_{L^{\frac{2s(1+\beta)}{p},p'}} \le AL\|\mathbf{J}_{\frac{2s(1+\beta}{p}}[\phi]\|_{L^{\frac{2s(1+\beta)}{p},p'}} := AL\|\phi\|_{L^{\frac{2s(1+\beta)}{p},p'}} \qquad \forall\, \phi \in C_0^\infty(R^N). \tag{4.45}$$

If we take in particular a function T with value 1 in $[1,\infty)$ we infer that if $\mathrm{cap}_{\frac{2s(1+\beta)}{p},p'}^{\mathbb{R}^N}(K) = 0$, there exists a sequence $\{\zeta_n\} \subset C_0^\ell(B_{2R})$ such that $0 \le \zeta_n \le 1$, $\zeta_n = 1$ on K and $\|\zeta_n\|_{L^{\frac{2s(1+\beta)}{p},p'}} \to 0$ as $n \to \infty$. We set $\Phi_n = e^{-\delta t}\mathbb{H}_s[\zeta_n]$ and take $\Phi_n^{p'}$ for test function. Then for any $\epsilon > 0$ we have

$$\int_{\mathbb{R}^N} (u\Phi_n^{p'})(1,x)dx + 2p'\left(\int_\epsilon^1 \int_{\mathbb{R}^N} t^{-\frac{p'\beta}{p}} |(-\Delta)^s\Phi_n + \delta\Phi_n|^{p'}\, dxdt\right)^{\frac{1}{p'}} \left(\int_\epsilon^1 \int_{\mathbb{R}^N} t^\beta u^p \Phi_n^{p'} dxdt\right)^{\frac{1}{p}} + \int_\epsilon^1 \int_{\mathbb{R}^N} t^\beta u^p \Phi_n^{p'} dxdt \ge \int_{B_{2R}} (u\Phi_n^{p'})(\epsilon,x)dx.$$

When $\epsilon \to 0$, the right-hand side of the above inequality converges to $\int_{B_{2R}} \zeta_n^{p'} d\nu \ge \nu(K)$. Furthermore, we have that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} (u\Phi_n^{p'})(1,x)dx = 0,$$

by the dominated convergence theorem since $u \le U_{p,\beta}$ and $\Phi_n^{p'}(1,x) \to 0$ for all $x \in \mathbb{R}^N$, and

$$\lim_{n\to\infty} \lim_{\epsilon\to 0} \int_\epsilon^1 \int_{\mathbb{R}^N} t^{-\frac{p'\beta}{p}} |(-\Delta)^s\Phi_n + \delta\Phi_n|^{p'}\, dxdt = 0,$$

as in the proof of Proposition **??**. For the last term, we have

$$\int_\epsilon^1 \int_{\mathbb{R}^N} t^\beta u^p \Phi_n^{p'} dxdt = \int_\epsilon^1 \int_{B_{2R}} t^\beta u^p \Phi_n^{p'} dxdt + \int_\epsilon^1 \int_{B_{2R}^c} t^\beta u^p \Phi_n^{p'} dxdt := I_{\epsilon,n} + J_{\epsilon,n}.$$

By assumption $t^\beta u^p \in L^1((0,1) \times B_{2R})$, then $\lim_{n\to\infty} \lim_{\epsilon\to 0} I_{\epsilon,n} = 0$ by the dominated convergence theorem. Finally, since $u \le U_{p,\beta}$ by Theorem H and $H_s(t,x) \le \dfrac{ct}{t^{1+\frac{N}{2s}} + |x|^{N+2s}}$ by (1.35), we

obtain with various $c>0$ independent of R

$$\begin{aligned}
J_{\epsilon,n} &\le c\int_\epsilon^1 \int_{B^c_{3R}} t^{(1-p')\beta}\left(\int_{B_{2R}} \frac{dy}{t^{1+\frac{N}{2s}}+|x-y|^{N+2s}}\right)^{p'} dxdt\\
&\le cR^{Np'}\int_\epsilon^1 t^{(1-p')\beta}\int_{B^c_{3R}} \frac{dx}{\left(t^{1+\frac{N}{2s}}+|x|^{N+2s}\right)^{p'}}\\
&\le cR^{Np'}\int_\epsilon^1 t^{(1-p')\beta+\frac{N}{2s}-\frac{(N+2s)p'}{2s}}\int_{3Rt^{-\frac{1}{2s}}}^\infty \frac{r^{N-1}}{(1+r^{N+2s})^{p'}}\\
&\le cR^{N-2sp'}\int_\epsilon^1 t^{(1-p')\beta}dt\\
&\le c\tfrac{R^{N-2sp'}}{(p'-1)\beta-1}.
\end{aligned}$$

(Note that the assumption $\beta<p-1$ is crucial). Hence $\lim\limits_{n\to\infty}\lim\limits_{\epsilon\to 0} I_{\epsilon,n}=0$, always by the dominated convergence theorem. This implies that $\nu(K)=0$. □

5 The subcritical case

For equation (1.26), the subcritical case corresponds to the fact that

$$u_\infty(t,x)=V(t,x)=t^{-\frac{1+\beta}{p-1}}v(t^{-\frac{1}{2s}}x)\qquad \forall\,(t,x)\in Q_\infty,$$

where v is the minimal positive solution of (1.31).

5.1 Proof of Theorem J

Proposition 5.1 *Assume that* $1+\frac{2s(1+\beta)}{N+2s}<p<1+\frac{2s(1+\beta)}{N}$ *and* u *is a nonnegative solution of (1.32) where* $\mathcal{S}\neq\emptyset$. *Then*

$$u(t,x)\ge \frac{c_{10}t^{-\frac{1+\beta}{p-1}}}{1+(t^{-\frac{1}{2s}}d(x,\mathcal{S}))^{N+2s}}\qquad \forall\,(t,x)\in Q_\infty. \tag{5.1}$$

for some $c_{10}>0$.

Proof. By Theorem C, for any $x_0\in\mathcal{S}$,

$$u(t,x)\ge u_\infty(t,x-x_0)=t^{-\frac{1+\beta}{p-1}}v(t^{-\frac{1}{2s}}(x-x_0))\qquad \forall\,(t,x)\in Q_\infty,$$

which implies that

$$u(t,x)\ge t^{-\frac{1+\beta}{p-1}}\sup_{x_0\in\mathcal{S}} v(t^{-\frac{1}{2s}}(x-x_0))\qquad \forall\,(t,x)\in Q_\infty. \tag{5.2}$$

The maximum of V is achieved at 0, hence, for any $x\in\mathcal{S}$,

$$u(t,x)\ge t^{-\frac{1+\beta}{p-1}}V(0)=c_{11}t^{-\frac{1+\beta}{p-1}}. \tag{5.3}$$

If $x \in \mathcal{S}^c$, there exists $\bar{x} \in \mathcal{S}$ such that $\text{dist}\,(x, \mathcal{S}) = |x - \bar{x}|$. It follows from [28, Theorem 1.2] that,

$$u(t,x) \geq \frac{c_{10} t^{-\frac{1+\beta}{p-1}}}{1 + (t^{-\frac{1}{2s}} \text{dist}\,(x, \mathcal{S}))^{N+2s}}. \tag{5.4}$$

Then (5.1) holds true. □

The next result shows that any closed set can be the singular set of the initial trace of a positive solution of (1.26).

Proposition 5.1 *Assume that* $1 + \frac{2s(1+\beta)}{1+2s} < p < 1 + \frac{2s(1+\beta)}{N}$ *and* $\mathcal{S} \subset \mathbb{R}^N$ *is a nonempty closed set. Then there exists a minimal solution* $u := u_{\mathcal{S},\infty}$ *with initial trace* $(\mathcal{S}, 0)$. *Furthermore it satisfies (4.35).*

Proof. We first notice that the condition $1 + \frac{2s(1+\beta)}{1+2s} < p < 1 + \frac{2s(1+\beta)}{N}$ is equivalent to the conditions stated in Theorem J, i.e.

$$\begin{aligned} &(i) \qquad \text{either} \quad N = 1 \text{ and } 1 + \tfrac{2s(1+\beta)}{1+2s} < p < 1 + 2s(1+\beta), \\ &(ii) \qquad \text{or} \quad N = 2,\ \tfrac{1}{2} \leq s < 1 \text{ and } 1 + \tfrac{2s(1+\beta)}{1+2s} < p < 1 + s(1+\beta). \end{aligned} \tag{5.5}$$

Let $A := \{z_n\}_{n\in\mathbb{N}}\}$ be a countable dense subset of $\mathcal{S}$. For $k \in \mathbb{N}_*$, set

$$\mu_k = k \sum_{j=1}^{k} \delta_{z_j}, \tag{5.6}$$

and let $u = u_{\mu_k}$ be the solution of

$$\begin{aligned} \partial_t u + (-\Delta)^s u + t^\beta u^p = 0 \qquad &\text{in} \quad Q_\infty \\ u(0,.) = \mu_k \qquad &\text{in} \quad \mathbb{R}^N. \end{aligned} \tag{5.7}$$

The sequence $\{u_{\mu_k}\}$ is increasing. If $a \in \mathcal{S}^c$ and $d_a = \text{dist}\,(a, \mathcal{S})$. By construction there holds

$$u_{\mu_k} \leq u_{B^c_{d_a}(a),\infty}. \tag{5.8}$$

Hence u_{μ_k} converges to some solution $\tilde{u}$ of (1.26) in Q_∞ which has zero initial trace on $B_{d_a}(a)$, for any $a \in \mathcal{S}^c$ since (5.8) still holds with $\tilde{u}$ instead of u_{μ_k}, and satisfies $\tilde{u} \geq u_{z_j,\infty}$ for any $z_j \in A$. Hence $Tr(\tilde{u}) = (\mathcal{S}, 0)$. Estimate (4.35) is independent of the geometry of $\mathcal{S}$. □

Proof of Theorem J. It is similar to the one of Theorem I . We consider an increasing sequence of compact sets $\{K_k\}_{k\in\mathbb{N}^*}$ included in $\mathcal{R}$ such that $\bigcup_k K_k = \mathcal{R}$, set $\nu_k = \chi_{K_k}\nu$ and $\tilde{\nu}_k = \nu_k + \mu_k$, where μ_k is defined by (5.6). Then the solution of (1.26) with initial data $\tilde{\nu}_k$ satisfies

$$\sup\{u_{\nu_k}, u_{\mu_k}\} \leq u_{\tilde{\nu}_k} \leq u_{\nu_k} + u_{\mu_k}. \tag{5.9}$$

By the same argument as in the proof of Theorem I , the sequence $\{u_{\nu_k}\}$ is increasing and converges to a solution u_ν (1.26) with initial trace $(\{\emptyset\}, \nu)$. Hence the sequence $\{u_{\tilde{\nu}_k}\}$ which is also increasing. converges to some solution $\tilde{u}$ of (1.26) which satisfies

$$\sup\{u_\nu, u_{\infty,\mathcal{S}}\} \leq \tilde{u} \leq u_\nu + u_{\infty,\mathcal{S}}. \tag{5.10}$$

Then $\tilde{u}$ has initial trace $(\mathcal{S}, \nu)$. □

The proof of Corollary K, part (b) is straightforward.

Remark. We conjecture that the following more general version of Theorem J holds: *For any integer* $\kappa \in [1, N]$ *any* $p > 1$ *such that* $1 + \frac{2s(1+\beta)}{\kappa+2s} < p < 1 + \frac{2s(1+\beta)}{N}$, *any closed set* $\mathcal{S}$ *contained in an affine plane of codimension* κ *and any bounded measure in* $\mathcal{S}^c$, *there exists a solution* u *of problem (1.32).* We notice that the condition on p can be fulfilled for some p if and only if $N - \kappa < 2s$, hence either $\kappa = N$ i.e. $\mathcal{S}$ is a single point and no condition on s, or $\kappa = N - 1$ hence $\mathcal{S}$ is contained in a straight line and $\frac{1}{2} < s < 1$.

5.2 Proof of Theorem L

The proof uses the method developed in [48]. The function

$$\tilde{\phi}(x) = \inf\{\phi(y) : |y| \geq |x|\}, \tag{5.11}$$

is radial, nondecreasing, smaller that ϕ and we write it as $\tilde{\phi}(|x|)$. Furthermore it satisfies

$$\lim_{|x|\to\infty} |x|^{-\frac{2s}{p-1}} \tilde{\phi}(x) > 0. \tag{5.12}$$

We set

$$\tilde{\phi}_n(x) = \begin{cases} \tilde{\phi}(x) & \text{if } |x| \leq n \\ \tilde{\phi}(n) & \text{if } |x| > n. \end{cases}$$

The existence of a solution $u_{\tilde{\phi}_n}$ of (1.26) with initial trace $(\{\emptyset, \tilde{\phi}_n\}$ follows from the fact that $\mathbb{H}_s[\tilde{\phi}_n]$ exists by [15] and that $\mathbb{H}_s[\tilde{\phi}_n] \geq u_{\tilde{\phi}_n \chi_{B_k}}$ for any $k \in \mathbb{N}^*$. Hence $u_{\tilde{\phi}_n}$ is the increasing limit of $u_{\tilde{\phi}_n \chi_{B_k}}$ when $k \to \infty$. it is obtained by replacing $\tilde{\phi}_n$ by $\tilde{\phi}_n \chi_{B_k}$ and by letting $k \to \infty$. The solution Y_n of the differential equation $Y'(t) + t^\beta Y^p(t) = 0$ with initial data $Y(0) = \tilde{\phi}(n)$ is expressed by

$$Y_n(t) = \frac{\tilde{\phi}(n)}{\left(1 + \frac{p-1}{\beta+1} t^{\beta+1} (\tilde{\phi}(n))^{p-1}\right)^{\frac{1}{p-1}}}. \tag{5.13}$$

It larger than $u_{\tilde{\phi}_n}$. Let us denote by w_n the solution of

$$\begin{aligned} \partial_t w_n + (-\Delta)^s w_n + t^\beta Y_n^{p-1}(t) w_n = 0 \qquad & \text{in} \quad (0,\infty) \times \mathbb{R}^N \\ w_n(0,x) = \tilde{\phi}_n(x) \qquad & \text{in} \quad \mathbb{R}^N. \end{aligned} \tag{5.14}$$

Since $Y_n^{p-1} \geq u_{\tilde{\phi}_n}^{p-1}$, w_n is smaller that $u_{\tilde{\phi}_n}$, moreover w_n can be explicitely computed

$$w_n(t,x) = e^{-\int_0^t s^\beta Y_n^{p-1}(s) ds} \int_{\mathbb{R}^N} H_s(t, x-y) \tilde{\phi}_n(y) dy. \tag{5.15}$$

In particular,

$$\begin{aligned} w_n(t,0) &= e^{-\int_0^t s^\beta Y_n^{p-1}(s)ds}\left(\int_{B_n} H_s(t,y)\tilde{\phi}_n(y)dy + \int_{B_n^c} H_s(t,y)\tilde{\phi}_n(y)dy\right) \\ &\geq cte^{-\int_0^t s^\beta Y_n^{p-1}(s)ds}\tilde{\phi}(n)\int_{B_n^c}\frac{dy}{t^{\frac{N+2s}{2s}}+|y|^{N+2s}} \\ &\geq cte^{-\int_0^t s^\beta Y_n^{p-1}(s)ds}\tilde{\phi}(n)\int_n^\infty\frac{r^{N-1}dr}{t^{\frac{N+2s}{2s}}+r^{N+2s}} \\ &\geq ctn^{-2s}e^{-\int_0^t s^\beta Y_n^{p-1}(s)ds}\tilde{\phi}(n). \end{aligned}$$

Next

$$-\int_0^t s^\beta Y_n^{p-1}(s)ds = -\int_0^t \frac{s^\beta(\tilde{\phi}(n))^{p-1}ds}{1+\frac{p-1}{\beta+1}t^{\beta+1}(\tilde{\phi}(n))^{p-1}} = -\frac{1}{p-1}\ln\left(1+\tfrac{p-1}{\beta+1}t^{\beta+1}(\tilde{\phi}(n))^{p-1}\right).$$

We write

$$\begin{aligned} tn^{-2s}e^{-\int_0^t s^\beta Y_n^{p-1}(s)ds}\tilde{\phi}(n) &= e^{\ln\left(t\tilde{\phi}(n)\right)-2s\ln n-\frac{1}{p-1}\ln\left(1+\frac{p-1}{\beta+1}t^{\beta+1}(\tilde{\phi}(n))^{p-1}\right)} \\ &= e^{\frac{1}{p-1}\ln\left(\frac{t^{p-1}(\tilde{\phi}(n))^{p-1}}{n^{2s(p-1)}\left(1+\frac{p-1}{\beta+1}t^{\beta+1}(\tilde{\phi}(n))^{p-1}\right)}\right)}. \end{aligned}$$

If we take $t=t_n$ such that

$$\lim_{n\to\infty} t_n^{\frac{\beta+1}{p-1}}\tilde{\phi}(n) = +\infty, \tag{5.16}$$

the expansion of the term in the logarithm gives

$$\frac{t^{p-1}(\tilde{\phi}(n))^{p-1}}{n^{2s(p-1)}\left(1+\frac{p-1}{\beta+1}t^{\beta+1}(\tilde{\phi}(n))^{p-1}\right)} = \frac{\beta+1}{p-1}n^{-2s(p-1)}t_n^{p-2-\beta}(1+o(1)) \quad \text{as } n\to\infty. \tag{5.17}$$

If besides (5.16) it is assumed that

$$\lim_{n\to\infty} n^{-2s(p-1)}t_n^{p-2-\beta} = +\infty, \tag{5.18}$$

we infer that $w_n(0,t_n)\to\infty$ as $n\to\infty$. Clearly the origin can be replaced by any $z\in\mathbb{R}^N$ and the previous calculation shows that this limit is uniform for z belonging to compact sets on $\mathbb{R}^N$. Since $u_{\tilde{\phi}_n}\geq w_n$, we infer that

$$\lim_{n\to\infty} u_{\tilde{\phi}_n}(t_n,z) = +\infty \Longrightarrow \lim_{n\to\infty} u_{\tilde{\phi}_n} = u_\infty = U_{p,\beta}, \tag{5.19}$$

by using (1.29). □

6 Appendix: symmetry and monotonicity results

The following is a variant of the maximum principle which will be used in the sequel.

Lemma 6.1 *Let $R, T > 0$, $\delta \in [0, T)$ and Q be a domain of Q_∞ such that $\overline{Q} \subset (\delta, T) \times B_R$. Assume that $h \geq 0$ in Q and $\psi \in C(\bar{Q})$ satisfies*

$$\begin{aligned} \partial_t \psi + (-\Delta)^s \psi + h(t,x)\psi \geq 0 \quad &\text{in} \quad Q \\ \psi \geq 0 \quad &\text{in} \quad ([\delta, T) \times B_R) \setminus Q. \end{aligned} \tag{6.1}$$

Then ψ is nonnegative in $[\delta, T) \times B_R$.

Proof. Let $\epsilon \in (0, T - \delta]$. We first claim that ψ is nonnegative in $[\delta, T - \epsilon] \times B_R$. If it does not hold, and since $\psi \geq 0$ in $([\delta, T) \times B_R) \setminus Q$, then there exists $(t_0, x_0) \in Q \cap ([\delta, T - \epsilon] \times B_R)$ such that

$$\psi(t_0, x_0) = \min_{(t,x) \in [\delta, T-\epsilon] \times B_R} \psi(t, x) < 0.$$

Then $\partial_t \psi(t_0, x_0) \leq 0$ and $(-\Delta)^s \psi(t_0, x_0) < 0$. Since $h \geq 0$ in Q and $(t_0, x_0) \in Q$, there holds

$$\partial_t \psi(t_0, x_0) + (-\Delta)^s \psi(t_0, x_0) + h(t_0, x_0)\psi(t_0, x_0) < 0,$$

which is a contradiction. Thus, ψ is nonnegative in $[\delta, T - \epsilon] \times B_R$. Since ϵ is arbitrary, the result follows. Notice that we can take $R = \infty$ in the above proof provided Q is a bounded domain. □

Next we prove the following result.

Proposition 6.1 *Let $N \geq 1$, $\beta > -1$, $p > 1$ and $g \in C(\mathbb{R}^N)$ be a nonnegative continuous radially symmetric and nonincreasing function which tends to 0 when $|x| \to \infty$. If $u \in L^1_{loc}(0, \infty; \mathcal{L}^s(\mathbb{R}^N) \cap C(\overline{Q}_\infty)$ is a nonnegative solution of (1.26) in Q_∞ which converges to g uniformly when $t \to 0$, then u is radially symmetric and nonincreasing.*

Proof. Since $u \in L^1_{loc}\left(0, \infty; \mathcal{L}^s(\mathbb{R}^N)\right) \cap C(\overline{Q}_\infty)$, it is bounded from above by $\mathbb{H}_s[g]$ and uniqueness holds as for the linear equation [15]. Since the initial data is radially symmetric and the equation is invariant by rotations in $\mathbb{R}^N$, $u(t, .)$ is also radially symmetric. Because of uniqueness and stability, it is sufficient to prove the result for a function u which initial data is obtained from the previous one by multiplying it by a smooth, even, nonincreasing and nonnegative function with compact support. The corresponding solution of (1.26) in Q_∞, still denoted by u, is smooth in Q_∞ and bounded from above by $\mathbb{H}_s[g]$. Hence it satisfies

$$\begin{aligned} &(i) && \lim_{t \to \infty} u(t,x) = 0 && \text{uniformly in } \ x \in \mathbb{R}^N, \\ &(ii) && \lim_{|x| \to \infty} u(t,x) = 0 && \text{uniformly in } \ t \in \mathbb{R}_+, \\ &(iii) && \lim_{t \to 0} u(t,x) = g(x) && \text{uniformly in } \ x \in \mathbb{R}^N. \end{aligned} \tag{6.2}$$

Next we use a moving plane method (see [44] for other applications). For $\lambda \in \mathbb{R}$, we set

$x_\lambda = (2\lambda - x_1, x')$ if $x = (x_1, x') \in \mathbb{R}^N$,

$$\Sigma_\lambda = \{x = (x_1, x') \in \mathbb{R}^N \mid x_1 < \lambda\} \tag{6.3}$$

and

$$T_\lambda = \{x = (x_1, x') \in \mathbb{R}^N \mid x_1 = \lambda\}.$$

We observe that if $\lambda > 0$, then $\{x_\lambda \mid x \in \Sigma_\lambda\} = \{x \in \mathbb{R}^N | \ x_1 > \lambda\}$ and

$$|x_\lambda| > |x| \quad \text{ for } x \in \Sigma_\lambda. \tag{6.4}$$

We claim that for any $\lambda > 0$,

$$u(t,x) \geq u(t, x_\lambda) \qquad \forall\, (t,x) \in \mathbb{R}_+^* \times \Sigma_\lambda. \tag{6.5}$$

Set $\varphi(t,x) = u(t,x) - u(t,x_\lambda)$ and suppose that (6.5) does not hold. Because of (6.2) there holds $\lim\limits_{|x|\to\infty} \varphi(t,x) = 0$ uniformly with respect to $t \geq 0$, $\lim\limits_{t\to\infty} \varphi(t,x) = 0$ uniformly with respect to $x \in \mathbb{R}^N$ and $\lim\limits_{t\to 0} \varphi(t,x) = g(x) - g(x_\lambda) \geq 0$ uniformly with respect to $x \in \mathbb{R}^N$. It follows that there exists $\varepsilon_0 > 0$ and $(t_0, x_0) \in \mathbb{R}_+^* \times \Sigma_\lambda$ such that

$$\varphi(t_0, x_0) = \min_{(t,x)\in\overline{\Sigma}_\lambda} \varphi(t,x) = -\varepsilon_0 < 0. \tag{6.6}$$

The function ϕ satisfies

$$\partial_t \phi + (-\Delta)^s \phi + h(t,x)\phi = 0 \qquad \text{in } \ Q_\infty, \tag{6.7}$$

for some $h(t,x) \geq 0$, and it has initial data $\phi(0,x) = g(x) - g(x_\lambda)$ in $\mathbb{R}^N$. Take $\epsilon \in (0, \epsilon_0)$ and set $\phi_\epsilon = \phi + \epsilon$. Using (6.2) we see that there exists $T_0 > t_0 > 0$ and $R_0 > |x_0| > 0$ such that $\phi_\epsilon(t,x) \geq 0$ for $(t,x) \in \big([T,\infty) \times \mathbb{R}^N\big) \bigcup ([0,\infty) \times B_R^c)$, for all $T \geq T_0$ and $R \geq R_0$. Furthermore there exists $\delta_0 \in (0, t_0)$ such that for any $\delta \in (0, \delta_0)$ such that $\phi_\epsilon(t,x) \geq 0$ for $(t,x) \in [0,\delta) \times \big(\mathbb{R}^N \cap \Sigma_\lambda\big)$. We set

$$Q = \Sigma_\lambda \cap (\delta, T_0) \times B_{R_0}.$$

We apply Lemma 6.1 in $[\frac{\delta}{2}, T) \times B_R$ and conclude that $\phi_\epsilon \geq 0$ in $[\frac{\delta}{2}, T) \times B_R$, which contradicts the fact that $\phi_\epsilon(t_0, x_0) = \epsilon - \epsilon_0 < 0$. Hence (6.5) holds. Since $\lambda > 0$ is arbitrary, this implies in particular by continuity that

$$\frac{\partial u}{\partial x_1}(t, x_1, x') \leq 0 \qquad \forall\, (t, x_1, x') \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^{N-1}. \tag{6.8}$$

Similarly, we can get that

$$\frac{\partial u}{\partial x_1}(t, x_1, x') \geq 0 \qquad \forall\, (t, x_1, x') \in \mathbb{R}_+ \times \mathbb{R}_- \times \mathbb{R}^{N-1}. \tag{6.9}$$

Since $u(t,x)$ is radially symmetric with respect to x, it implies that $u(t,x) \geq u(t,x')$ if $|x| \leq |x'|$, which ends the proof. □

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