Initial trace of positive solutions to fractional diffusion equation with absorption

Laurent Veron, Huyuan Chen

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Huyuan Chen[1] **Laurent Véron**[2]

Abstract

In this paper, we prove the existence of an initial trace $\mathcal{T}_u$ of any positive solution u of the semilinear fractional diffusion equation (H)

$$\partial_t u + (-\Delta)^\alpha u + f(t,x,u) = 0 \quad \text{in} \quad \mathbb{R}^*_+ \times \mathbb{R}^N,$$

where $N \geq 1$ where the operator $(-\Delta)^\alpha$ with $\alpha \in (0,1)$ is the fractional Laplacian and $f : \mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{R}_+ \mapsto \mathbb{R}$ is a Caratheodory function satisfying $f(t,x,u)u \geq 0$ for all $(t,x,u) \in \mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{R}_+$. We define the regular set of the trace $\mathcal{T}_u$ as an open subset of $\mathcal{R}_u \subset \mathbb{R}^N$ carrying a nonnegative Radon measive ν_u such that

$$\lim_{t\to 0} \int_{\mathcal{R}_u} u(t,x)\zeta(x)dx = \int_{\mathcal{R}_u} \zeta d\nu \qquad \forall \zeta \in C^2_0(\mathcal{R}_u),$$

and the singular set $\mathcal{S}_u = \mathbb{R}^N \setminus \mathcal{R}_u$ as the set points a such that

$$\limsup_{t\to 0} \int_{B_\rho(a)} u(t,x)dx = \infty \qquad \forall \rho > 0.$$

We study the reverse problem of constructing a positive solution to (H) with a given initial trace $(\mathcal{S}, \nu)$ where $\mathcal{S} \subset \mathbb{R}^N$ is a closed set and ν is a positive Radon measure on $\mathcal{R} = \mathbb{R}^N \setminus \mathcal{S}$ and develop the case $f(t,x,u) = t^\beta u^p$ where $\beta > -1$ and $p > 1$.

Key words: Fractional heat equation, Initial trace, Singularities

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Contents

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1 Introduction

The first aim of this paper is to study the existence of an *initial trace* of positive solutions to the semilinear fractional diffusion equation

$$\partial_t u + (-\Delta)^\alpha u + f(t,x,u) = 0 \quad \text{in} \quad Q_\infty := \mathbb{R}^*_+ \times \mathbb{R}^N, \tag{1.1}$$

where $f : \mathbb{R}^*_+ \times \mathbb{R}^N \times \mathbb{R} \mapsto \mathbb{R}$ is a Caratheodory function satisfying

$$f(t,x,u)u \geq 0 \qquad (t,x,u) \in \mathbb{R}^*_+ \times \mathbb{R}^N \times \mathbb{R}, \tag{1.2}$$

and $(-\Delta)^\alpha$ is the fractional Laplacian with $\alpha \in (0,1)$ defined by

$$(-\Delta)^\alpha u(x) = \lim_{\varepsilon \to 0^+} (-\Delta)^\alpha_\varepsilon u(x),$$

where

$$(-\Delta)^\alpha_\varepsilon u(x) := -a_{N,\alpha} \int_{\mathbb{R}^N} \frac{u(z)-u(x)}{|z-x|^{N+2\alpha}} \chi_\varepsilon(|x-z|)dz \ , \ \ a_{N,\alpha} = \frac{\Gamma(\frac{N}{2}+\alpha)}{\pi^{\frac{N}{2}}\Gamma(2-\alpha)}\alpha(1-\alpha), \tag{1.3}$$

for $\varepsilon > 0$ and

$$\chi_\varepsilon(r) = \begin{cases} 0 & \text{if} \quad r \in [0,\varepsilon], \\ 1 & \text{if} \quad r > \varepsilon. \end{cases}$$

The solutions of (1.1) are intended in the classical sense and, in order $(-\Delta)^\alpha u(t,x)$ be defined, we always assume that $u(t,.) \in \mathcal{L}^\alpha(\mathbb{R}^N)$ for any $t > 0$, where

$$\mathcal{L}^\alpha(\mathbb{R}^N) = \left\{ \phi \in L^1_{loc}(\mathbb{R}^N) \ \text{ s.t. } \ \|\phi\|_{\mathcal{L}^\alpha} := \int_{\mathbb{R}^N} \frac{|\ \phi(x)\ |\, dx}{1+|\ x\ |^{N+2\alpha}} < \infty \right\}. \tag{1.4}$$

Notice that the constant functions belong to $\mathcal{L}^\alpha(\mathbb{R}^N)$. If $\omega \subset \mathbb{R}^N$ and $0 < T \leq \infty$ we set $Q^\omega_T = (0,T) \times \omega$, $Q^{\mathbb{R}^N}_T = Q_T$, $Q_\infty = \mathbb{R}^*_+ \times \mathbb{R}^N$ and denote by $B_\rho(z)$ (resp. $K_\rho(z)$) the open ball

(resp. open cube with sides parallel to the axis) with center $z \in \mathbb{R}^N$ and radius (side length) $\rho > 0$. We define the *regular set* of the initial trace of a positive solution u of (1.1) by

$$\mathcal{R}_u = \left\{ z \in \mathbb{R}^N : \exists\, \rho > 0 \text{ s.t.} \iint_{\mathrm{Q}_1^{\mathrm{B}_\rho(z)}} \mathrm{f(t,x,u)dxdt} < \infty \right\}. \tag{1.5}$$

Clearly $\mathcal{R}_u$ is open. The *conditional singular set* $\tilde{\mathcal{S}}_u$ is $\mathbb{R}^N \setminus \mathcal{R}_u$ and *the conditional initial trace* is the couple $Tr_c(u) := (\tilde{\mathcal{S}}_u, \nu)$. Our first result is the following statement which is the starting point of our work.

Theorem A *There exists a nonnegative Radon measure ν_u on $\mathcal{R}_u$ such that*

$$\lim_{t\to 0} \int_{\mathcal{R}_u} u(t,x)\zeta(x)dx = \int_{\mathcal{R}_u} \zeta d\nu_u \qquad \forall \zeta \in C_0^2(\mathcal{R}_u). \tag{1.6}$$

The problem of the initial trace of nonnegative solutions of semilinear heat equations was initiated by Marcus and Véron in [30] with equation

$$\partial_t u - \Delta u + u^p = 0 \qquad \text{in } Q_\infty, \tag{1.7}$$

with $p > 1$. They shew the existence of an initial trace $Tr(u)$ represented by a closed subset $\mathcal{S}_u$ of $\mathbb{R}^N$ and a nonnegative Radon measure ν_u on $\mathcal{R}_u = \mathbb{R}^N \setminus \mathcal{S}_u$. On $\mathcal{R}_u$ the initial trace is achieved as in (1.6). On $\mathcal{S}_u$ they proved that

$$\lim_{t\to 0} \int_{B_\rho(z)} u(t,x)dx = \infty \qquad \forall z \in \mathcal{S}_u,\, \forall \rho > 0. \tag{1.8}$$

They also highlighted the existence of a critical exponent $p_c = 1 + \frac{2}{N}$ which plays a crucial role in the fine analysis of the initial trace. For example they obtained that if p is *subcritical*, i.e. $1 < p < p_c$, (1.6) can be sharpened under the form

$$c_2(p,N) \le \liminf_{t\to 0} t^{\frac{1}{p-1}} u(z,t) \le \limsup_{t\to 0} t^{\frac{1}{p-1}} u(z,t) \le c_1(p). \tag{1.9}$$

for some positive constants $c_1(p) > c_2(p,N)$. Furthermore they proved that for any couple $(\mathcal{S},\nu)$ where $\mathcal{S}$ is a closed subset of $\mathbb{R}^N$ and ν a nonnegative Radon measure on $\mathcal{R} = \mathbb{R}^N \setminus \mathcal{S}$ there exists a unique nonnegative solution u of (1.7) with initial trace $Tr(u) = (\mathcal{S},\nu)$. The *supercritical* case $p \ge p_c$ turned out to be much more delicate and was finally elucidated in a series of works by Marcus and Véron [34] and Gkikas and Véron [25] following some deep ideas introduced by Marcus and Véron in [33] and Marcus [29] for solving similar questions dealing with semilinear elliptic equations. Al Sayed and Véron extended in [3] the subcritical analysis performed in [30] to the non-autonomous equation

$$\partial_t u - \Delta u + t^\beta u^p = 0 \quad \text{in } Q_\infty, \tag{1.10}$$

with $\beta > -1$ and $p > 1$.

The main difficulty to extend some of the previous results dealing with (1.7) and (1.10) comes from the fact that the fractional Laplacian is a non-local operator. A more precise characterization of the conditional singular set needs additional assumptions on u or on f. We define the *singular set* $\mathcal{S}_u$ of u by

$$\mathcal{S}_u = \left\{ z \in \mathbb{R}^N : \limsup_{t\to 0} \int_{B_\rho(z)} u(t,x)dx = \infty \quad \forall \rho > 0 \right\}. \tag{1.11}$$

This set is closed and it follows from Theorem A that $\mathcal{S}_u \subset \tilde{\mathcal{S}}_u$. The *initial trace* is the couple $Tr(u) := (\mathcal{S}_u, \nu)$. Notice that when $0 < \alpha < 1$, $Tr(u)$ could be different from $Tr_c(u)$, in sharp contrast with the case $\alpha = 1$, as a consequence of the non-local aspect of $(\Delta)^\alpha$.

Theorem B *Assume u is a nonnegative solution of (1.1). If $u \in L^1(0,T;\mathcal{L}^\alpha(\mathbb{R}^N))$, then $\mathcal{S}_u = \tilde{\mathcal{S}}_u$ and more precisely for any $z \in \mathcal{S}_u$ and $\rho > 0$,*

$$\lim_{t\to 0} \int_{B_\rho(z)} u(t,x)dx = \infty \qquad \forall \rho > 0. \tag{1.12}$$

The above assumption on u can be verified when the absorption is strong and the singular set is compact. Another type of characterization of the singular set needs the following assumptions on f: $f(t,x,u)$ satisfies $f(t,x,0) = 0$ and

$$0 \le f(t,x,u) \le t^\beta g(u) \qquad \forall (t,x,u) \in \mathbb{R}_+ \times \mathbb{R}^N \times \mathbb{R}_+, \tag{1.13}$$

where $\beta > -1$, g is nondecreasing, continuous and verifies the subcritical growth assumption,

$$\int_1^\infty g(s) s^{-1-p^*_\beta} ds < \infty, \tag{1.14}$$

with

$$p^*_\beta = 1 + \frac{2\alpha(1+\beta)}{N}. \tag{1.15}$$

Theorem C *Assume (1.13) and (1.14) hold and u is a nonnegative solution of (1.1) with initial trace $(\mathcal{S}_u, \nu_u)$. If $\mathcal{S}_u \neq \emptyset$ and $z \in \mathcal{S}_u$, then (1.12) holds. More precisely $u \ge u_{z,\infty}$ where $u_{z,\infty} = \lim_{k\to\infty} u_{k\delta_z}$ and $u_{k\delta_z}$ is the solution of*

$$\begin{aligned} \partial_t u + (-\Delta)^\alpha u + t^\beta g(u) = 0 \quad \text{in} \quad Q_\infty \\ u(0,.) = k\delta_z. \end{aligned} \tag{1.16}$$

The existence and uniqueness of a solution to (1.16) follows from [21, Th 1.1]. If $g : \mathbb{R} \mapsto \mathbb{R}_+$ is nondecreasing and satisfies

$$\int_1^\infty \frac{ds}{g(s)} < \infty, \tag{1.17}$$

and if $\beta > -1$, then the expression

$$\int_{U(t)}^{\infty} \frac{ds}{g(s)} = \frac{t^{\beta+1}}{\beta+1}, \tag{1.18}$$

defines the function U as the maximal solution of the ODE

$$\partial_t U + t^\beta g(U) = 0 \quad \text{on } \mathbb{R}_+^* \quad \text{s.t. } U(0) = \infty. \tag{1.19}$$

Theorem D *Assume $f(t,x,r) \geq t^\beta g(r)$ where $\beta > -1$ and g satisfies (1.17). If u is a nonnegative solution of (1.1) belonging to $L^1_{loc}(0,T;\mathcal{L}^\alpha(\mathbb{R}^N))$, then*

$$u(x,t) \leq U(t) \qquad \forall (t,x) \in Q_\infty. \tag{1.20}$$

Furthermore, if g satisfies

$$\int_1^\infty \frac{sds}{g(s)\left(\int_s^\infty \frac{d\tau}{g(\tau)}\right)^{\frac{\beta}{\beta+1}}} < \infty, \tag{1.21}$$

then $\mathcal{S}_u = \tilde{\mathcal{S}}_u$ and (1.12) holds for any $z \in \mathcal{S}_u$.

Theorem E *Assume $f(t,x,r) = t^\beta g(r)$ where $\beta > -1$ and g satisfies (1.17), is nondecreasing and is locally Lipschitz continuous. If u is a nonnegative solution of (1.1) belonging to $L^1_{loc}(0,T;\mathcal{L}^\alpha(\mathbb{R}^N))$, either*

$$u(x,t) < U(t) \qquad \forall (t,x) \in Q_\infty, \tag{1.22}$$

or

$$u(x,t) = U(t) \qquad \forall (t,x) \in Q_\infty. \tag{1.23}$$

In the second part of this paper we study in detail the initial trace problem for the equation

$$\partial_t u + (-\Delta)^\alpha u + t^\beta u^p = 0 \qquad \text{in} \quad Q_\infty, \tag{1.24}$$

when $\alpha \in (0,1)$, $\beta > -1$ and $p \in (1, p^*_\beta)$. A second critical value of p appears

$$p^{**}_\beta = 1 + \frac{2\alpha(1+\beta)}{N+2\alpha}. \tag{1.25}$$

Actually, if $u_k := u_{k\delta_0}$ is unique solution to

$$\begin{aligned} \partial_t u + (-\Delta)^\alpha u + t^\beta u^p = 0 \qquad & \text{in} \quad Q_\infty, \\ u(0,\cdot) = k\delta_0 \qquad & \text{in} \quad \mathbb{R}^N, \end{aligned} \tag{1.26}$$

it is proved in [21] that $u_\infty = \lim_{k\to\infty} u_k$ is very different according $1 < p < p^{**}_\beta$ or $p^{**}_\beta < p < p^*_\beta$. Notice that the case $p = p^{**}_\beta$ remained unsolved in [21].

(i) If $1 < p < p^{**}_\beta$, then

$$u_\infty(t,x) = U_{p,\beta}(t) := \left(\frac{1+\beta}{p-1}\right)^{\frac{1}{p-1}} t^{-\frac{1+\beta}{p-1}}. \tag{1.27}$$

The absorption is dominent, as if $\alpha = 0$.
(ii) If $p^{**}_\beta < p < p^*_\beta$, then

$$u_\infty(t,x) = V(t,x) := t^{-\frac{1+\beta}{p-1}} v\left(\frac{x}{t^{\frac{1}{2\alpha}}}\right), \tag{1.28}$$

where v is the minimal positive solution of

$$\begin{aligned} (-\Delta)^\alpha v - \frac{1}{2\alpha}\nabla v.\eta - \frac{1+\beta}{p-1} v + v^p = 0 \qquad \text{in} \quad \mathbb{R}^N \\ \lim_{|\eta|\to\infty} |\eta|^{\frac{2\alpha(1+\beta)}{p-1}} v(\eta) = 0. \end{aligned} \tag{1.29}$$

The function V is called the very singular solution of (1.24). In this case *the diffusion is dominent*, as when $\alpha = 1$.

We observe that $\tilde{\mathcal{S}}_{u_\infty} = \mathcal{S}_{u_\infty} = \{0\}$ when $p^{**}_\beta < p \le p^*_\beta$ and $\tilde{\mathcal{S}}_{u_\infty} = \mathcal{S}_{u_\infty} = \mathbb{R}^N$ when $1 < p < p^{**}_\beta$. In this paper, we first prove that $\mathcal{S}_{u_\infty} = \mathbb{R}^N$ for $p = p^{**}_\beta$. Our main result concerning (1.24) is the following.

Theorem F *Let u be a positive solution of (1.24).*
(i) If $p \in (1, p^{**}_\beta]$ and $\mathcal{S}_u \neq \emptyset$. Then $\mathcal{S}_u = \mathbb{R}^N$ and $u \ge U_{p,\beta}$. If we assume moreover that $u \in L^1_{loc}(0,\infty;\mathcal{L}^\alpha(\mathbb{R}^N))$, then $u = U_{p,\beta}$.
(ii) If there exists $\kappa \in [1,N] \cap \mathbb{N}$ such that $p \in (1,p^*_\beta) \cap \left(1, 1 + \frac{2\alpha(1+\beta)}{\kappa+2\alpha}\right]$ and $\mathcal{S}_u$ contains an affine plane $\mathcal{L}$ of codimension κ. Then the conclusions of (i) hold.

If $\kappa = N$, (ii) is just (i). Note that if $0 < \alpha \le \frac{1}{2}$ or if $\kappa \ge N-2$, then $(p^{**}_\beta, p^*_\beta) \cap \left(p^{**}_\beta, 1 + \frac{2\alpha(1+\beta)}{\kappa+2\alpha}\right] = (p^{**}_\beta, p^*_\beta)$, while, if $\frac{1}{2} < \alpha < 1$ and $\kappa = N-1$, then $(p^{**}_\beta, p^*_\beta) \cap \left(p^{**}_\beta, 1 + \frac{2\alpha(1+\beta)}{\kappa+2\alpha}\right] = \left(p^{**}_\beta, 1 + \frac{2\alpha(1+\beta)}{N-1+2\alpha}\right]$.

Conversely, given a closed set of $\mathcal{S} \subset \mathbb{R}^N$ and a nonnegative Radon measure on ν on $\mathcal{R} = \mathbb{R}^N \setminus \mathcal{S}$, we study the existence of solution of (1.24) with a given initial trace $Tr_c(u) = Tr(u) = (\mathcal{S}, \nu)$, that is a solution of the following problem

$$\begin{aligned} \partial_t u + (-\Delta)^\alpha u + t^\beta u^p = 0 \qquad \text{in} \quad Q_\infty, \\ Tr(u) = (\mathcal{S}, \nu). \end{aligned} \tag{1.30}$$

This means that u is a classical solution of the equation in Q_∞ and that (1.6) and (1.20) hold. By Theorem F any closed set cannot be the singular part of the initial trace of a positive solution of (1.24) if p is too small (diffusion effect) or if p is too large. In the same sense any positive bounded Radon measure ν cannot be the regular part of the initial trace of a positive solution

of (1.24) since condition (1.14) is equivalent to $p < p^*_\beta$. However this condition is restrictive and there exist several sufficient conditions linking ν, α, β and p. Hence we say that a nonnegative bounded measure ν is an *admissible measure* if the initial value problem

$$\begin{aligned} \partial_t u + (-\Delta)^\alpha u + t^\beta u^p = 0 \quad \text{in} \quad Q_\infty, \\ u(0,.) = \nu, \end{aligned} \tag{1.31}$$

admits a solution u_ν, always unique, and it is a *good measure* if it is stable in the sense that if ν is replaced by $\nu * \rho_n$ for some sequence of mollifiers, then $u_{\nu*\rho_n}$ and $t^\beta u^p_{\nu*\rho_n}$ converges to u_ν and $t^\beta u^p_\nu$ respectively in $L^1(Q_T)$. We denote by H_α is the kernel in $\mathbb{R}^*_+ \times \mathbb{R}^N$ associated to $(-\Delta)^\alpha$. It is expressed by

$$H_\alpha(t,x) = \frac{1}{t^{\frac{N}{2\alpha}}} \tilde{H}_\alpha \left(\frac{x}{t^{\frac{1}{2\alpha}}} \right) \quad \text{where} \quad \tilde{H}_\alpha(x) = \int_{\mathbb{R}^N} e^{ix.\xi - |\xi|^{2\alpha}} d\xi, \tag{1.32}$$

and let $\mathbb{H}_\alpha[\nu]$ be the associated potential of $\nu \in \mathfrak{M}^b(\mathbb{R}^N)$, defined by

$$\mathbb{H}_\alpha[\nu](t,x) = \int_{\mathbb{R}^N} H_\alpha(t, x-y) d\nu(y).$$

We first prove that a nonnegative bounded measure with Lebesgue decomposition $\nu = \nu_0 + \nu_s$ where $\nu_0 \in L^1(\mathbb{R}^N)$ and ν_s is singular with respect to the N-dim Lebesgue measure is a good measures if $t^\beta (\mathbb{H}_\alpha[\nu])^p \in L^1(Q_1)$. Our main existence result for solution of (1.31) is the following:

Theorem G *Let $N \geq 1$, $p > 1$ and $-1 < \beta < p-1$. A nonnegative bounded measure ν in $\mathbb{R}^N$ is an admissible measure if and only if ν vanishes on Borel subsets of $\mathbb{R}^N$ with zero $\mathrm{cap}^{\mathbb{R}^N}_{\frac{2\alpha(1+\beta)}{p},p'}$-Bessel capacity.*

Concerning problem (1.30) we have the following general result,

Theorem H *Assume $N \geq 1$ and $p > 1 + \frac{2\alpha(1+\beta)}{1+2\alpha}$. If $\mathcal{S}$ is a closed subset of $\mathbb{R}^N$ such that $\mathcal{S} = \overline{\mathrm{int}\, \mathcal{S}}$ and ν is a nonnegative good bounded Radon measure on $\mathcal{R} = \mathcal{S}^c$, then problem (1.30) admits a solution.*

To state the next result, Now we state the result as follows.

Theorem I *Assume $\beta > -1$, $p > 1$ and one of the following assumptions is fulfilled:*
(i) either $N = 1$ and $1 + \frac{2\alpha(1+\beta)}{1+2\alpha} < p < 1 + 2\alpha(1+\beta)$,
(i) or $N = 2$, $\frac{1}{2} < \alpha < 1$ and $1 + \frac{2\alpha(1+\beta)}{1+2\alpha} < p < 1 + \alpha(1+\beta)$.
Then for any closed set $\mathcal{S} \subset \mathbb{R}^N$ and any nonnegative bounded measure ν in $\mathcal{R} = \mathcal{S}^c$ there exists a nonnegative solution u to (1.30).

2 Initial trace with general nonlinearity

2.1 Existence of an initial trace

Proof of Theorem A. For any bounded domain $\omega \subset \mathbb{R}^N$, we denote by $C^2_0(\overline{\omega})$ the space of functions $\xi : \mathbb{R}^N \mapsto \mathbb{R}$ which are C^2 and have compact support in $\overline{\omega}$. We always assume that

$N \geq 1$ and $0 < \alpha < 1$. Let ϕ_ω be the first eigenfunction of $(-\Delta)^\alpha$ in $H_0^\alpha(\omega)$, with corresponding eigenvalue $\lambda_\omega > 0$, i.e. the solution of

$$\begin{array}{rll} (-\Delta)^\alpha \phi_\omega = \lambda_\omega \phi_\omega & \text{in} & \omega \\ \phi_\omega = 0 & \text{in} & \omega^c. \end{array} \tag{2.1}$$

Existence and basic properties of the eigenfunctions can be found in [4], [10]. We normalize ϕ_ω by $\sup \phi_\omega = 1$. We say that ω is of class **E. S. C.** if it satisfies the *exterior sphere condition*. It is known by [37, Prop 1.1] that $\phi_\omega(x) \leq c(\text{dist}\,(x, \partial\omega))^\alpha$ in ω, and there exists $q > 2$ such that $\phi_\omega^q \in C_0^2(\overline{\omega})$. We denote by $K_\rho(z)$ the open cube with sides parallel to the axis of center $z \in \mathbb{R}^N$ and length sides $\rho > 0$, and $K_1 := K_1(0)$. Then

$$\phi_{K_\rho(z)}(x) = \phi_{K_1}\left(\frac{x-z}{\rho}\right) \quad \text{and} \quad \lambda_{K_\rho(z)} = \frac{\lambda_{K_1}}{\rho^{2\alpha}}.$$

The next lemma is a precision of [19, Lemma 2.3].

Lemma 2.1 *Let $q \in \mathbb{N} \cap [2, \infty)$ and $\zeta \in C_0^2(\overline{\omega})$, $\zeta \geq 0$, then*

$$\begin{aligned} (-\Delta)^\alpha \zeta^q(x) &= q\zeta^{q-1}(x)(-\Delta)^\alpha \zeta(x) - \frac{a_{N,\alpha}}{q(q-1)} \int_{\mathbb{R}^N} \frac{\zeta^q(y) - \zeta^q(x) - q(\zeta(y) - \zeta(x))\zeta^{q-1}(y)}{|\, x-y\,|^{N+2\alpha}} dy \\ &\geq \zeta^{q-1}(x)(-\Delta)^\alpha \zeta(x) - \frac{a_{N,\alpha}}{q} \zeta^{q-2}(x) \int_{\mathbb{R}^N} \frac{(\zeta(y) - \zeta(x))^2}{|\, x-y\,|^{N+2\alpha}} dy. \end{aligned} \tag{2.2}$$

Proof. By [19, Lemma 2.3],

$$(-\Delta)^\alpha \zeta^q(x) = q\zeta^{q-1}(x)(-\Delta)^\alpha \zeta(x) - q(q-1)a_{N,\alpha} \int_{\mathbb{R}^N} \left(\int_{\zeta(x)}^{\zeta(y)} (\zeta(y) - t) t^{q-2} dt \right) \frac{dy}{|\, x-y\,|^{N+2\alpha}}.$$

By integration by parts

$$\begin{aligned} \int_{\zeta(x)}^{\zeta(y)} (\zeta(y) - t) t^{q-2} dt &= \frac{1}{q(q-1)} \left(\zeta^q(y) - \zeta^q(x) - q(\zeta(y) - \zeta(x))\zeta^{q-1}(x) \right) \\ &= \frac{\zeta(y) - \zeta(x)}{q(q-1)} \left[\zeta^{q-1}(y) + \zeta^{q-2}(y)\zeta(x) + ... + \zeta(y)\zeta^{q-2}(x) - (q-1)\zeta^{q-1}(x) \right]. \end{aligned}$$

Since for any $a, b \geq 0$

$$\begin{aligned} & b^{q-1} + ab^{q-2} + ...a^{q-2}b - (q-1)a^{q-1} \\ &\quad = b^{q-1} - a^{q-1} + a(b^{q-2} - a^{q-2}) + a^2(b^{q-3} - a^{q-3}) + ... + a^{q-2}(b-a) \\ &\quad = (b-a)\left[\left(b^{q-2} + ab^{q-3} + ... + a^{q-2} \right) + a\left(b^{q-3} + ab^{q-4} + ... + a^{q-3} \right) + ... + a^{q-2} \right] \\ &\quad \geq (q-1)(b-a)a^{q-2}, \end{aligned}$$

we derive (2.2). □

Remark. By the mean value theorem, we see that there exists $m_\zeta \in \{z = \zeta(w) : w \in \mathbb{R}^N\}$ such that

$$\begin{aligned} L(\zeta^q) &:= \frac{a_{N,\alpha}}{q(q-1)} \int_{\mathbb{R}^N} \frac{\zeta^q(y) - \zeta^q(x) - q(\zeta(y) - \zeta(x))\zeta^{q-1}(x)}{\mid x - y \mid^{N+2\alpha}} dy \\ &= \frac{a_{N,\alpha}}{2} m_\zeta^{q-2} \int_{\mathbb{R}^N} \frac{(\zeta(y) - \zeta(x))^2}{\mid x - y \mid^{N+2\alpha}} dy \end{aligned} \tag{2.3}$$

Proposition 2.2 *Assume f satisfies (1.2) and u is a nonnegative solution of (1.1) such that $u(t,.) \in \mathcal{L}^\alpha(\mathbb{R}^N)$ for all $t \in (0,T)$. If $f(.,.,u) \in L^1(Q_\omega^T)$ for some bounded domain $\omega \subset \mathbb{R}^N$ of class E. S. C. and $T > 0$. Then there exists $\ell_\omega \geq 0$ such that*

$$\lim_{t \to 0} \int_\omega u(t,x)\phi_\omega^q(x)dx = \ell_\omega. \tag{2.4}$$

Furthermore

$$\begin{aligned} \ell_\omega + \frac{a_{N,\alpha}}{q} \int_0^T \int_{\mathbb{R}^N} u(t,x)\phi_\omega^{q-2}(x) \left(\int_{\mathbb{R}^N} \frac{(\phi_\omega(y) - \phi_\omega(x))^2}{\mid x - y \mid^{N+2\alpha}} dy \right) dx \\ \leq e^{q\lambda_\omega T} X(T) + \int_0^T \int_\omega f(t,x,u)\phi_\omega^q(x) e^{q\lambda_\omega s} dxdt. \end{aligned} \tag{2.5}$$

Proof. Since $\phi_\omega^q \in C_0^2(\overline{\omega})$, there holds

$$\frac{d}{dt} \int_\omega u(t,x)\phi_\omega^q(x)dx + \int_{\mathbb{R}^N} u(t,x)(-\Delta)^\alpha \phi_\omega^q(x)dx + \int_\omega f(t,x,u)\phi_\omega^q(x)dx = 0. \tag{2.6}$$

Set

$$X(t) = \int_\omega u(t,x)\phi_\omega^q(x)dx,$$

then

$$\frac{d}{dt} \left(e^{q\lambda_\omega t} X(t) - \int_t^T \int_\omega f(t,x,u)\phi_\omega^q(x) e^{q\lambda_\omega s} dxds \right) = e^{q\lambda_\omega t} \int_{\mathbb{R}^N} L(\phi_\omega^q)(x)dx \geq 0. \tag{2.7}$$

This implies that $\lim_{t \to 0} X(t) = \ell_\omega$ exists and

$$\ell_\omega + \int_0^T \int_{\mathbb{R}^N} L(\phi_\omega^q)(x) e^{q\lambda_\omega s} u(s,x) dxds = e^{q\lambda_\omega T} X(T) + \int_0^T \int_\omega f(t,x,u)\phi_\omega^q(x) e^{q\lambda_\omega s} dxds,$$

which implies (2.5) by Lemma 2.1. □

The proof of Theorem A is completed by the following statement:

Proposition 2.3 *There exists a nonnegative Radon measure μ_u on $\mathcal{R}_u$ such that for any $\zeta \in C_0^2(\mathcal{R}_u)$ there holds*

$$\lim_{t \to 0} \int_\omega u(t,x)\zeta(x)dx = \int_{\mathcal{R}_u} \zeta d\mu_u. \tag{2.8}$$

Proof. Let $\zeta \in C_0^2(\mathcal{R}_u)$ with support K and let G be an open subset containing K such that ∂G is smooth and $\overline{G}$ is a compact subset of $\mathcal{R}_u$ and assume $0 \leq \zeta \leq 1$. We put

$$Y(t) = \int_{\mathbb{R}^N} u(t,x)\zeta(x)dx = \int_G u(t,x)\zeta(x)dx,$$

and

$$\int_{\mathbb{R}^N} f(t,x,u)\zeta(x)dx = \int_G f(t,x,u)\zeta(x)dx.$$

Then

$$Y'(t) + \int_{\mathbb{R}^N} u(t,x)(-\Delta)^\alpha \zeta(x)dx + \int_G f(t,x,u)\zeta(x)dx = 0.$$

Since $\zeta \geq 0$ we have

$$\begin{aligned}
&\frac{1}{a_{N,\alpha}} \int_{\mathbb{R}^N} u(t,x)(-\Delta)^\alpha \zeta(x)dx \\
&\qquad = \int_{\mathbb{R}^N} u(t,x) \int_G \frac{\zeta(x)-\zeta(y)}{|\ x-y\ |^{N+2\alpha}} dydx + \int_G u(t,x)\zeta(x) \int_{G^c} \frac{dy}{|\ x-y\ |^{N+2\alpha}} dx \\
&\qquad = \int_G u(t,x) \int_G \frac{\zeta(x)-\zeta(y)}{|\ x-y\ |^{N+2\alpha}} dydx - \int_{G^c} u(t,x) \int_G \frac{\zeta(y)}{|\ x-y\ |^{N+2\alpha}} dydx \qquad (2.9) \\
&\qquad\qquad + \int_G u(t,x)\zeta(x) \int_{G^c} \frac{dy}{|\ x-y\ |^{N+2\alpha}} dx \\
&\qquad \leq \int_G u(t,x) \int_G \frac{\zeta(x)-\zeta(y)}{|\ x-y\ |^{N+2\alpha}} dydx + \int_G u(t,x)\zeta(x) \int_{G^c} \frac{dy}{|\ x-y\ |^{N+2\alpha}} dx.
\end{aligned}$$

Hence

$$\frac{d}{dt}\left(Y(t) - \int_t^T \left(\int_{\mathbb{R}^N} u(t,x)(-\Delta)^\alpha \zeta(x)dx + \int_G f(t,x,u)\zeta(x)dx \right) ds \right) \geq 0. \qquad (2.10)$$

Notice that

$$a_{N,\alpha} \int_G \frac{\zeta(x)-\zeta(y)}{|\ x-y\ |^{N+2\alpha}} dy := (-\Delta)_G^\alpha \zeta(x),$$

is the regional fractional Laplacian of order α. Since ζ is C^2 with support in $K \subset G \subset \overline{G} \Subset \mathcal{R}_u$, there exists M such that

$$\zeta(x) + |(-\Delta)_G^\alpha \zeta(x)\chi_G(x)| + \left| \zeta(x) \int_{G^c} \frac{dy}{|\ x-y\ |^{N+2\alpha}} \right| \leq c_3 \phi_G^q(x) \quad \forall x \in \mathbb{R}^N.$$

This implies that

$$|u\chi_G(-\Delta)^\alpha \zeta| + |f(.,.,u)\zeta| \in L^1(Q_G^T). \qquad (2.11)$$

Combining (2.10) and (2.11) we infer that the following limit exists

$$\lim_{t\to 0} Y(t) = \lim_{t\to 0} \int_G u(t,x)\zeta(x)dx := \tilde{\mu}_u(\zeta). \qquad (2.12)$$

If we drop the restriction $\zeta \leq 1$ we get

$$0 \leq \lim_{t\to 0} \int_G u(t,x)\zeta(x)dx = \tilde{\mu}_u(\zeta) \leq c_3 \ell_G \sup_G \zeta. \tag{2.13}$$

Next we assume that $\zeta \in C_0(\mathcal{R}_u)$ is nonnegative, with support $K \subset G \subset \overline{G} \Subset \mathcal{R}_u$. Then there exists an increasing sequences $\{\zeta_n\} \subset C_0^2(\mathcal{R}_u)$ of nonnegative functions smaller than ζ which converges to ζ uniformly (take for example $\zeta_n = (\zeta - n^{-1})_+ * \rho_n$ for some sequence of mollifiers $\{\rho_n\}$ with $\mathrm{supp}(\rho_n) \subset B_{n^{-2}}$). The sequence $\{\tilde{\mu}_u(\zeta_n)\}$ is increasing and bounded from above by $M\ell_G \sup_G \zeta$. Hence it is convergent and its limit, still denoted by $\tilde{\mu}_u(\zeta)$ is independent of the sequence $\{\zeta_n\}$. We can also consider a uniform approximation of ζ from above in considering $\zeta_n' = (\sigma_n + \zeta) * \rho_n$ where $\sigma_n = n^{-1}\chi_{K_n}$ and $K_n = \{x \in \mathbb{R}^N : \mathrm{dist}\,(x,K) \leq n^{-1}\}$. Actually,

$$\tilde{\mu}_u(\zeta) = \sup\{\tilde{\mu}_u(\eta) : \eta \in C_0^2(\mathcal{R}_u), 0 \leq \eta \leq \zeta\} = \inf\{\tilde{\mu}_u(\eta') : \eta' \in C_0^2(\mathcal{R}_u), \zeta \leq \eta'\}. \tag{2.14}$$

This implies that for all η and η' belonging to $C_0^2(\mathcal{R}_u)$ such that $\eta \leq \zeta \leq \eta'$ we have

$$\tilde{\mu}_u(\eta) \leq \liminf_{t\to 0} \int_{\mathcal{R}_u} u(t,x)\zeta(x)dx \leq \limsup_{t\to 0} \int_{\mathcal{R}_u} u(t,x)\zeta(x)dx \leq \tilde{\mu}_u(\eta'). \tag{2.15}$$

Combined with (2.14) we derive the existence of the limit and

$$\lim_{t\to 0} \int_{\mathcal{R}_u} u(t,x)\zeta(x)dx = \tilde{\mu}_u(\zeta). \tag{2.16}$$

Finally, if $\zeta \in C_0(\mathcal{R}_u)$ is a signed function we write $\zeta = \zeta_+ - \zeta_-$ and $\mu_u(\zeta) = \tilde{\mu}_u(\zeta_+) - \tilde{\mu}_u(\zeta_-)$. Hence μ_u is a positive Radon measure on $\mathcal{R}_u$, and (2.8) follows from (2.16) with ζ replaced by ζ_+ and ζ_-. □

Lemma 2.4 *Assume that $G \subset \mathbb{R}^N$ is a bounded smooth domain and $\eta \in C_0^2(G)$. Then there exists $c_4 > 0$ such that*

$$|(-\Delta)^\alpha \eta(x)| \leq \frac{c_4 \|\eta\|_{C^2}}{1 + |x|^{N+2\alpha}} \qquad \forall\, x \in \mathbb{R}^N. \tag{2.17}$$

Moreover, assume that $\eta \geq 0$ in G, then $(-\Delta)^\alpha \eta \leq 0$ in G^c and for any $\delta > 0$ there exists $c_\delta > 1$ independent of η such that

$$\frac{\|\eta\|_{L^1}}{c_\delta(1 + |x|^{N+2\alpha})} \leq -(-\Delta)^\alpha \eta(x) \leq \frac{c_\delta \|\eta\|_{L^1}}{1 + |x|^{N+2\alpha}}, \tag{2.18}$$

for $x \in \{z \in \mathbb{R}^N : \mathrm{dist}\,(z,G) \geq \delta\}$.

Proof. Let $x \in G^c$ and $y \in \mathbb{R}^N$, then $\eta(x) - \eta(y) \leq 0$ and hence $(-\Delta)^\alpha \eta \leq 0$ in G^c. For $y \in G$ and $x \in G^c$ satisfying $\mathrm{dist}\,(x,G) > \delta$, there exists $c_6 > 1$ such that

$$c_6^{-1}(1 + |x|^{N+2\alpha}) \leq |x - y|^{N+2\alpha} \leq c_6(1 + |x|^{N+2\alpha}).$$

Together with

$$(-\Delta)^\alpha \eta(x) = -a_{N,\alpha} \int_G \frac{\eta(y)}{|y-x|^{N+2\alpha}} dy \qquad \forall\, x \in G^c,$$

one obtains the claim. □

Proof of Theorem B. Let $\rho > \rho' > 0$ and $\zeta \in C_0^2(B_\rho(z))$ such that $0 \le \zeta \le 1$ and $\zeta = 1$ on $B_{\rho'}(z)$). Then there holds

$$\int_{B_\rho(z)} u(t,x)\zeta(x)dx = \int_{B_\rho(z)} u(T,x)\zeta(x)dx + \int_t^T \int_{B_\rho(z)} f(s,x,u)\zeta(x)dxds + \int_t^T \int_{\mathbb{R}^N} u(s,x)(-\Delta)^\alpha \zeta(x)dxds.$$

The function ζ satisfies

$$|\ (-\Delta)^\alpha \zeta(x)\ | \le \frac{c_4 \|\zeta\|_{C^2}}{1+|\ x\ |^{N+2\alpha}} \qquad \forall x \in \mathbb{R}^N.$$

Since $(t,x) \mapsto (1+|\ x\ |^{N+2\alpha})^{-1} u((t,x) \in L^1(Q_T)$ we derive

$$\lim_{t\to 0} \int_{B_\rho(z)} u(t,x)\zeta(x)dx = \infty, \tag{2.19}$$

which implies the claim. □

2.2 Pointwise estimates

Proof of Theorem C. In what follows we characterize the singular set of the initial trace when the absorption reaction is subcritical, that is it satisfies (1.13), and (1.14) and (1.15) hold. Under these two last assumptions for any bounded Radon measure in $\mathbb{R}^N$, it is proved in [21, Th 1.1] that there exists a unique weak solution $u := u_\mu$ to

$$\begin{aligned} \partial_t u + (-\Delta)^\alpha u + t^\beta g(u) = 0 \qquad & \text{in } Q_\infty \\ u(0,.) = \mu \qquad & \text{in } \mathbb{R}^N. \end{aligned} \tag{2.20}$$

We recall by a weak solution, we mean a function $u \in L^1(Q_T)$ such that $t^\beta g(u) \in L^1(Q_T)$ for all $T > 0$ satisfying

$$\int_0^T \int_{\mathbb{R}^N} \left[(-\partial_t \xi + (-\Delta)^\alpha \xi)\, u + t^\beta g(u)\xi \right] dxdt = \int_{\mathbb{R}^N} \xi(0,x) d\mu(x), \tag{2.21}$$

for all ξ in the space $\mathbb{Y}_{\alpha,T}$ of functions defined in Q_∞ satisfying

(i) $\|\xi\|_{L^1(Q_T)} + \|\xi\|_{L^\infty(Q_T)} + \|\partial_t \xi\|_{L^\infty(Q_T)} + \|(-\Delta)^\alpha \xi\|_{L^\infty(Q_T)} < +\infty$

(ii) $\xi(T) = 0$ and for $0 < t < T$, there exist $M > 0$ and $\epsilon_0 > 0$such that for $0 < \epsilon \le \epsilon_0$, $\|(-\Delta)_\epsilon^\alpha \xi(t,\cdot)\|_{L^\infty(\mathbb{R}^N)} \le M$.

Furthermore, if μ_j converges to μ weakly in the sense of measures, then u_{μ_j} converges to u_μ locally uniformly in Q_∞. Up to translation we can assume that $z = 0$. Since (1.20) holds, for any $k > 0$ there exist two sequences $\{t_n\}$ and $\{\rho_n\}$ converging to 0 such that

$$\int_{B_{\rho_n}} u(t_n, x)dx = k. \tag{2.22}$$

For $R > 0$, let u_n^R be the solution of

$$\begin{array}{rll} \partial_t u + (-\Delta)^\alpha u + t^\beta g(u) = 0 & \text{in } & (t_n, \infty) \times B_R \\ u(x,t) = 0 & \text{in } & (t_n, \infty) \times B_R^c \\ u(t_n, .) = u(t_n, x)\chi_{B_{\rho_n}} & \text{in } & B_R. \end{array} \tag{2.23}$$

By comparison $u \geq u_n^R$ in $[t_n, \infty) \times B_R$. Letting $R \to \infty$ we infer that u_n^R increases and converges to the solution u_n^∞ of

$$\begin{array}{rll} \partial_t u + (-\Delta)^\alpha u + t^\beta g(u) = 0 & \text{in } & (t_n, \infty) \times \mathbb{R}^N \\ u(t_n, .) = u(t_n, x)\chi_{B_{\rho_n}} & \text{in } & \mathbb{R}^N. \end{array} \tag{2.24}$$

and there still holds $u \geq u_n^\infty$. Letting $n \to \infty$ and using the above mentioned stability result, we derive that u_n^∞ converges to $u_{k\delta_0}$ and $u \geq u_{k\delta_0}$. Since it holds for any k, the claim follows.

□

Proof of Theorem D. (i) *Proof of (1.20).* Let $\gamma \in C^2(\mathbb{R})$ be a convex nondecreasing function vanishing on $(-\infty, 0]$ such that $\gamma(r) \leq r_+$. For $\epsilon > 0$ let U_ϵ be the solution of

$$\begin{array}{rl} \partial_t U + t^\beta g(U) = 0 & \text{in } (\epsilon, \infty) \\ U(\epsilon) = \infty. & \end{array} \tag{2.25}$$

Hence

$$\int_{U_\epsilon(t)}^\infty \frac{ds}{g(s)} = \frac{1}{\beta + 1}\left(t^{\beta+1} - \epsilon^{\beta+1}\right). \tag{2.26}$$

Then there holds

$$\begin{aligned} (-\Delta)^\alpha \gamma(u(t,.) - U_\epsilon(t))(x) = \gamma'((u(t,x) - U_\epsilon(t))(-\Delta)^\alpha u(x) \\ - a_{N,\alpha}\frac{\gamma''(u(t,z_x) - U_\epsilon(t))}{2}\int_{\mathbb{R}^N} \frac{(u(t,y) - u(t,x))^2}{|\ x - y\ |^{N+2\alpha}}dy. \end{aligned}$$

Notice that the integral is convergent if $t > \epsilon$ since $\gamma(u(t,.) - U(t-\epsilon)) = \gamma(u^*(t,.) - U_\epsilon(t))$ where $0 \leq u^*(t,.) \leq U_\epsilon(t)$ and u satisfies

$$\int_{\mathbb{R}^N} \frac{u(.,x)dx}{1 + |x|^{N+\alpha}} < \infty \qquad \text{a.e. in } (0,T).$$

Then

$$\begin{aligned}\partial_t\gamma(u(t,x)-U_\epsilon(t))+(-\Delta)^\alpha\gamma(u(t,.)-U_\epsilon(t))(x)&\\ \leq \gamma'((u(t,x)-U_\epsilon(t))\left(\partial_t u(t,x)-\partial_t U_\epsilon(t)+(-\Delta)^\alpha u(x)\right)&\\ \leq \gamma'((u(t,x)-U_\epsilon(t))\left(t^\beta g(U_\epsilon(t))-f(t,x,u(t,x))\right))&\\ \leq 0.&\end{aligned}$$

Therefore $\gamma(u(.,.)-U_\epsilon(.))$ is a subsolution. Let $\eta\in C_0^\infty(\mathbb{R}^N)$, $\eta\geq 0$. Using Lemma 2.4 we have

$$\left|\int_{\mathbb{R}^N}\gamma(u(t,x)-U_\epsilon(t))(-\Delta)^\alpha\eta dx\right|\leq c_4\|\eta\|_{C^2}\int_{\mathbb{R}^N}\frac{u(t,x)dx}{1+\mid x\mid^{N+2\alpha}}.$$

Since $u\in L^1_{loc}(0,T;\mathcal{L}^\alpha(\mathbb{R}^N))$, for almost all s,t such that $\epsilon<s<t$ there holds

$$\begin{aligned}\int_{\mathbb{R}^N}\gamma(u(t,x)-U_\epsilon(t))\eta(x)dx+\int_s^t\int_{\mathbb{R}^N}\gamma(u(\tau,x)-U_\epsilon(t))(-\Delta)^\alpha\eta(x)dxd\tau&\\ \leq\int_{\mathbb{R}^N}\gamma(u(s,x)-U_\epsilon(s))\eta(x)dx.&\end{aligned}$$

Since $\gamma(u(s,x)-U_\epsilon(s))\eta(x)\leq u(s,x)\eta(x)$ and $u(s,.)\eta\in L^1(\mathbb{R}^N)$ we derive from the dominated convergence theorem that

$$\lim_{s\downarrow\epsilon}\int_{\mathbb{R}^N}\gamma(u(s,x)-U_\epsilon(s))\eta(x)dx=0.$$

Hence, letting $s\to\epsilon$ and $\gamma(r)\uparrow r_+$, we get

$$\begin{aligned}\int_{\mathbb{R}^N}(u(t,x)-U_\epsilon(t))_+\eta(x)dx&\leq\left|\int_\epsilon^t\int_{\mathbb{R}^N}(u(\tau,x)-U_\epsilon(t))_+(-\Delta)^\alpha\eta(x)dxd\tau\right|\\ &\leq c_4\|\eta\|_{C^2}\int_\epsilon^t\int_{\mathbb{R}^N}\frac{(u(\tau,x)-U_\epsilon(t))_+}{1+\mid x\mid^{N+2\alpha}}dxd\tau.\end{aligned}\tag{2.27}$$

Next, for $n\geq 1$, we replace η by $\eta_n(x)=\eta(n^{-1}x)$, where $0\leq\eta\leq 1$, $\eta(x)=1$ on B_1 and $\mathrm{supp}(\eta)\subset B_2$. We can also assume that η is radially decreasing and $\eta(0)=1$. Since $\|\eta_n\|_{C^2}\leq\|\eta\|_{C^2}$, we derive from (2.27) and the monotone convergence theorem that the following holds for almost all $t\in(\epsilon,T)$

$$\int_{\mathbb{R}^N}(u(t,x)-U_\epsilon(t))_+dx\leq c_4\|\eta\|_{C^2}\int_\epsilon^t\int_{\mathbb{R}^N}\frac{(u(\tau,x)-U_\epsilon(\tau))_+}{1+\mid x\mid^{N+2\alpha}}dxd\tau\tag{2.28}$$

This inequality implies that $(u(t,.)-U_\epsilon(t))_+\in L^1(\mathbb{R}^N)$ for almost all $t\in(\epsilon,T)$. We set

$$\Psi_\epsilon(t)=\int_\epsilon^t\int_{\mathbb{R}^N}\frac{(u(\tau,x)-U_\epsilon(\tau))_+}{1+\mid x\mid^{N+2\alpha}}dxd\tau.$$

Then

$$\Psi'_\epsilon(t) = \int_{\mathbb{R}^N} \frac{(u(t,x) - U_\epsilon(t))_+}{1+ \mid x \mid^{N+2\alpha}} dx \leq \int_{\mathbb{R}^N} (u(t,x) - U_\epsilon(t))_+ dx \leq c_4 \|\eta\|_{C^2} \Psi_\epsilon(t).$$

Since $\Psi_\epsilon(\epsilon) = 0$ we derive $\Psi_\epsilon(t) = 0$ on $(0,T)$, hence $u(t,x) \leq U_\epsilon(t)$ a.e. on $(\epsilon, T) \times \mathbb{R}^N$. Letting $\epsilon \to 0$, we derive the claim.

(ii) *End of the proof.* Because of Theorem B it is sufficient to prove that if (1.21) holds, then $U \in L^1(0,1)$. Indeed we denote by Φ the function

$$\Phi(\phi) = \int_\phi^\infty \frac{ds}{g(s)}.$$

Clearly Φ is an decreasing diffeomorphism from $\mathbb{R}^*_+$ onto $(0, \Phi(0)$ and $U(t) = \Phi^{-1}\left(\frac{t^{\beta+1}}{\beta+1}\right)$. In the next integral we set $U(t) = s$, then $t = ((\beta+1)\Phi(s))^{\frac{1}{\beta+1}}$ and we get

$$\begin{aligned}\int_0^1 U(t)dt &= \int_\infty^{U(1)} s\Phi'(s) \left((\beta+1)\Phi(s)\right)^{-\frac{\beta}{\beta+1}} ds \\ &= (\beta+1)^{-\frac{\beta}{\beta+1}} \int_{U(1)}^\infty \frac{sds}{g(s)\left(\int_s^\infty \frac{d\tau}{g(\tau)}\right)^{\frac{\beta}{\beta+1}}} < \infty.\end{aligned}$$

□

The following weight function plays an important role in the description of the initial trace problem for positive solutions of the fractional heat equation

$$\Phi(x) = \frac{1}{\left(1 + (\mid x \mid^2 -1)_+^4\right)^{\frac{N+2\alpha}{8}}} \qquad \forall x \in \mathbb{R}^N. \tag{2.29}$$

It has the remarkable property that

$$-c\Phi(x) \leq (-\Delta)^\alpha \Phi(x) \leq c\Phi(x) \qquad \forall x \in \mathbb{R}^N, \tag{2.30}$$

for some constant $c > 0$ (see [9], [8]). Furthermore

$$\frac{1}{c_1(1+ \mid x \mid^{N+2\alpha})} \leq \Phi(x) \leq \frac{c_1}{1+ \mid x \mid^{N+2\alpha}} \qquad \forall x \in \mathbb{R}^N. \tag{2.31}$$

Lemma 2.5 *Let $f : \mathbb{R}^*_+ \times \mathbb{R}^N \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ be a Caratheodory function which satisfies (1.2) and is nonnecreasing with respect to the variable u. If $u_0 \in L^1(\mathbb{R}^N)$ is nonnegative and $u \in C(\mathbb{R}^+; L^1(\mathbb{R}^N))$ is the unique the weak solution of*

$$\begin{aligned}\partial_t u + (-\Delta)^\alpha u + f(t,x,u) = 0 \qquad &\text{in } Q_\infty \\ u(0,.) = u_0 \qquad &\text{in } \mathbb{R}^N,\end{aligned} \tag{2.32}$$

satisfies, for some constant $c_{N,\alpha} > 0$

$$\int_{\mathbb{R}^N} u(t,x)\Phi(x)dx + \int_0^t \int_{\mathbb{R}^N} \left(u(s,x)(-\Delta)^\alpha \Phi(x) + f(s,x,u)\Phi(x)\right) dxds = \int_{\mathbb{R}^N} u_0(x)\Phi(x)dx. \tag{2.33}$$

Proof. Since u is a weak solution of (2.32) and the function Φ satisfies the assumptions (i)-(ii) in [21, Def. 1.1], we get (2.33). □

Corollary 2.6 *Assume f satisfies the assumptions of Lemma 2.5 and that inequalities (1.13)-(1.14)-(1.15) hold. Then for any nonnegative measure μ in $\mathbb{R}^N$ verifying*

$$\int_{\mathbb{R}^N} \Phi(x) d\mu(x) < \infty, \tag{2.34}$$

there exists a weak solution $u \in C_b(\mathbb{R}^+; \mathcal{L}^\alpha(\mathbb{R}^N)) \cap L^1(\mathbb{R}^+; \mathcal{L}^\alpha(\mathbb{R}^N))$ of (2.32) in the sense that

$$\int_0^t \int_{\mathbb{R}^N} \left(-(\partial_t \xi + (-\Delta)^\alpha \xi) u + \xi f(s,x,u)\right) dxds + \int_{\mathbb{R}^N} u(t,x)\xi(t,x)dx = \int_{\mathbb{R}^N} \xi(0,x) d\mu(x), \tag{2.35}$$

for all $\xi \in C_0^2(\overline{Q}_T)$ satisfying the assumptions (i)-(ii) in [21, Def. 1.1]. Furthermore

$$\int_{\mathbb{R}^N} u(t,x)\Phi(x)dx + \int_0^t \int_{\mathbb{R}^N} \left(u(s,x)(-\Delta)^\alpha \Phi(x) + f(s,x,u)\Phi(x)\right) dxds = \int_{\mathbb{R}^N} \Phi(x) d\mu(x). \tag{2.36}$$

Proof. By the assumptions on f, for any $n > 0$ there exists a unique $u_n \in L^1(Q_T)$ verifying $f(.,.,u_n) \in L^1(Q_T)$ and is a weak solution of

$$\begin{array}{rll} \partial_t u + (-\Delta)^\alpha u + f(t,x,u) = 0 & & \text{in } Q_\infty \\ u(0,.) = \mu_n := \chi_{B_n} u_0 & & \text{in } \mathbb{R}^N, \end{array} \tag{2.37}$$

from [21, Th. 1.1]. If ρ_k is a sequence of mollifiers with compact support and $\mu_{n,k} = (\chi_{B_n} u_0) * \rho_k$, the sequence $\{u_{n,k}\}$ of weak solutions of

$$\begin{array}{rll} \partial_t u + (-\Delta)^\alpha u + f(t,x,u) = 0 & & \text{in } Q_\infty \\ u(0,.) = \mu_{n,k} & & \text{in } \mathbb{R}^N, \end{array} \tag{2.38}$$

then $u_{n,k}$ satisfies

$$\begin{aligned} \int_{\mathbb{R}^N} u_{n,k}(t,x)\Phi(x)dx + \int_0^t \int_{\mathbb{R}^N} \left(u_{n,k}(s,x)(-\Delta)^\alpha \Phi(x) + f(s,x,u_{n,k})\Phi(x)\right) dxds \\ = \int_{\mathbb{R}^N} \mu_{n,k}(x)\Phi(x)dx. \end{aligned} \tag{2.39}$$

When $k \to \infty$, we know from the proof of [21, Th. 1.1] that, up to a subsequence, $\{u_{n,k}\}_k$ converges a.e. in Q_T to some function u_n, $\{f(.,.,u_{n,k})\}_k$ converges a.e. to $\{f(.,.,u_n)\}$ and that $\{u_{n,k}\}_k$ and $\{f(.,.,u_{n,k})\}_k$ are uniformly integrable in $L^1(Q_T)$. Furthermore $u_n \in C([0,T]; L^1(\mathbb{R}^N))$ and for any $t \in (0,T]$, $\{u_{n,k}(t,.)\}_k$ converges to $u_n(t,.)$ in $L^1(\mathbb{R}^N)$. This implies that

$$\int_{\mathbb{R}^N} u_n(t,x)\Phi(x)dx + \int_0^t \int_{\mathbb{R}^N} \left(u_n(s,x)(-\Delta)^\alpha \Phi(x) + f(s,x,u_n)\Phi(x)\right) dxds = \int_{\mathbb{R}^N} \Phi(x) d\mu_n(x). \tag{2.40}$$

Furthermore

$$\int_0^t \int_{\mathbb{R}^N} \left(-(\partial_t \xi + (-\Delta)^\alpha \xi) u_n + \xi f(s,x,u_n)\right) dxds + \int_{\mathbb{R}^N} u_n(t,x)\xi(t,x)dx = \int_{\mathbb{R}^N} \xi(0,x) d\mu_n(x), \tag{2.41}$$

for all $\xi \in C_0^2(\overline{Q}_T)$ satisfying the assumptions (i)-(ii) in [21, Def. 1.1]. When $n \to \infty$, $u_n \uparrow u$ and $f(s,x,u_n) \uparrow f(s,x,u)$. Using the monotone convergent theorem we see that u satisfies (2.36),and that the sequences $\{u_n\}_n$ and $\{f(.,.,u_n)\}_n$ converges to u and $f(.,.,u)$ in $L^1(0,T;\mathcal{L}^\alpha(\mathbb{R}^N))$ respectively. Using estimate (2.17) we can let n to infinity in (2.41) and derive (2.35). □

As it is pointed out in [9], the weight function Φ plays a role similar to an eigenfunction of $(-\Delta)^\alpha$ we prove a backward-forward uniqueness result for solutions of (1.1) inspired from [9, Lemma 4.2].

Theorem 2.7 *Assume $u \mapsto f(t,x,u)$ is locally Lipschitz continuous on $\mathbb{R}$, uniformly with respect to $x \in \mathbb{R}^N$ and locally uniformly with respect to $t \in \mathbb{R}_+^*$. If u_1 and u_2 belong to $L^1_{loc}(\mathbb{R}_+^*;\mathcal{L}^\alpha(\mathbb{R}^N)) \cap L^\infty_{loc}\mathbb{R}_+^*;L^\infty(\mathbb{R}^N))$ and are weak solutions of (1.1) in Q_T which coincide for $t = t_0 > 0$, then $u_1 = u_2$ in Q_T.*

Proof. For any $0 < \epsilon < t_0 < T < \infty$, u_1 and u_2 are uniformly bounded in $[\epsilon, T] \times \mathbb{R}^N$. Hence the function D defined by

$$D(t,x) = \begin{cases} \dfrac{f(t,x,u_1(t,x)) - f(t,x,u_2(t,x))}{u_1(t,x) - u_2(t,x)} & \text{if } \mathrm{u_1(t,x) \neq u_2(t,x)} \\ 0 & \text{if } \mathrm{u_1(t,x) = u_2(t,x)}, \end{cases}$$

is bounded in $[\epsilon, T] \times \mathbb{R}^N$ by some constant $M = M(\epsilon, T) > 0$. Set $w = u_1 - u_2$, it satisfies

$$\partial_t w + (-\Delta)^\alpha w + Dw = 0 \quad \text{in } \mathrm{Q_T}$$

and is uniformly bounded in $[\epsilon, T] \times \mathbb{R}^N$. Hence

$$\frac{d}{dt}\int_{\mathbb{R}^N} w(t,x)\Phi(x)dx + \int_{\mathbb{R}^N} w(t,x)(-\Delta)^\alpha \Phi(x)dx + \int_{\mathbb{R}^N} D(t,x)w(t,x)dx = 0.$$

Using (2.30) we derive

$$-(c+M)\int_{\mathbb{R}^N} w(t,x)\Phi(x)dx \leq \frac{d}{dt}\int_{\mathbb{R}^N} w(t,x)\Phi(x)dx \leq (c+M)\int_{\mathbb{R}^N} w(t,x)\Phi(x)dx. \tag{2.42}$$

This implies

$$-(c+M)\int_{\mathbb{R}^N} w(t,x)\Phi(x)dx \leq \frac{d}{dt}\int_{\mathbb{R}^N} w(t,x)\Phi(x)dx \leq (c+M)\int_{\mathbb{R}^N} w(t,x)\Phi(x)dx, \tag{2.43}$$

and

$$\begin{aligned} &(i) \qquad \int_{\mathbb{R}^N} w(t,x)\Phi(x)dx \leq e^{(c+M)(t-s)}\int_{\mathbb{R}^N} w(s,x)\Phi(x)dx \\ &(ii) \qquad e^{(c+M)(s-t)}\int_{\mathbb{R}^N} w(s,x)\Phi(x)dx \leq \int_{\mathbb{R}^N} w(t,x)\Phi(x)dx \end{aligned} \tag{2.44}$$

for all $\epsilon \le s \le t \le T$. Taking $s = t_0$ in (i) and $t = t_0$ in (ii) yields $w \equiv 0$ in $[\epsilon, T] \times \mathbb{R}^N$. □

Proof of Theorem E. By Theorem D we know that $u \le U$. If there exists some $(t_0, x_0) \in Q_T$ such that $u((t_0, x_0)) = U(t_0)$, then either $u((t_0, x)) = U(t_0)$ for all $x \in \mathbb{R}^N$, or

$$(-\Delta)^\alpha (u - U)(t_0, x_0) < 0 \qquad \forall x \in \mathbb{R}^N.$$

Since $f(t, x, u) - t^\beta g(U) \ge 0$ and $\partial_t (u - U)(t_0, x_0) = 0$ we derive that $u((t_0, .)) \equiv U(t_0)$. Since g is nondecreasing this situation is impossible, hence $u((t_0, .)) = U(t_0)$. Since g is locally Lipschitz continuous, this implies $u = U$ in Q_T by Theorem 2.7. □

A straightforward consequence of Theorems B1-B-4 is the next statement.

Corollary 2.8 *Assume $f(t, x, r) = t^\beta g(r)$ where $\beta > -1$ and $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is continuous and nondecreasing and satisfies (1.14), (1.17) and (1.21). If u is a nonnegative of (1.1) in Q_T belonging to $L^1_{loc}(0, T; \mathcal{L}^\alpha(\mathbb{R}^N))$ such that $\mathcal{S}_u \ne \emptyset$, there holds*

$$u(x, t) \ge u_{\infty,z}(x, t) = u_{\infty,0}(x - z, t) \qquad \forall (x, t) \in Q_T. \tag{2.45}$$

3 The case $f(t, x, u) = t^\beta u^p$

We denote by $(-\Delta)^\alpha_\kappa$ the fractional Laplacian in $\mathbb{R}^\kappa$ and $(-\Delta)^\alpha_N = (-\Delta)^\alpha$. The following standard lemma will be usefull in the sequel.

Lemma 3.1 *Let $1 \le \kappa \le N - 1$ be an integer. If $u \in C^2(\mathbb{R}^\kappa) \cap \mathcal{L}^\alpha(\mathbb{R}^\kappa)$ and $\tilde{u}(x_1, x') = u(x_1)$ if $(x_1, x') \in \mathbb{R}^\kappa \times \mathbb{R}^{N-\kappa}$, then*

$$(-\Delta)^\alpha \tilde{u}(x_1, x') = (-\Delta)^\alpha_\kappa u(x_1). \tag{3.46}$$

Proof. This more or less well known lemma is based upon the explicit value of the constant $a_{N,\alpha}$ in the definition of $(-\Delta)^\alpha$. For the sake of completeness we give here the proof.

$$\begin{aligned}
(-\Delta)^\alpha \tilde{u}(x_1, x') &= a_{N,\alpha} \int_{\mathbb{R}^\kappa} \int_{\mathbb{R}^{N-\kappa}} \frac{u(x_1) - u(y_1)}{((x_1 - y_1)^2 + |\, x' - y' \,|^2)^{\frac{N}{2}+\alpha}} dy' dy_1 \\
&= a_{N,\alpha} \int_{\mathbb{R}^\kappa} \left(\int_{\mathbb{R}^{N-\kappa}} \frac{dy'}{((x_1 - y_1)^2 + |\, y' \,|^2)^{\frac{N}{2}+\alpha}} \right) (u(x_1) - u(y_1))\, dy_1 \\
&= a_{N,\alpha} \left(\int_{\mathbb{R}^{N-\kappa}} \frac{dz'}{(1 + |\, z' \,|^2)^{\frac{N}{2}+\alpha}} \right) \int_{\mathbb{R}^\kappa} \frac{u(x_1) - u(y_1)}{|\, x_1 - y_1 \,|^{\kappa+2\alpha}} dy_1 \\
&= \frac{a_{N,\alpha}}{a_{\kappa,\alpha}} \left(\left| S^{N-1-\kappa} \right| \int_0^\infty \frac{r^{N-\kappa-1} dr}{(1 + r^2)^{\frac{N}{2}+\alpha}} \right) (-\Delta)^\alpha_\kappa u(x_1).
\end{aligned}$$

Since

$$\left| S^{N-1-\kappa} \right| = \frac{2\pi^{\frac{N-1-\kappa}{2}}}{\Gamma(\frac{N-1-\kappa}{2})},$$

and (see e.g. [38, p. 103])

$$\int_0^\infty \frac{r^{N-\kappa-1}dr}{(1+r^2)^{\frac{N}{2}+\alpha}} = \frac{1}{2}B\left(\frac{N-\kappa}{2}, \frac{\kappa}{2}+\alpha\right) = \frac{1}{2}\frac{\Gamma(\frac{\kappa}{2}+\alpha)\Gamma(\frac{N-\kappa}{2})}{\Gamma(\frac{N}{2}+\alpha)},$$

by Euler's formula, where B denotes beta function, we deduce that

$$\frac{a_{\kappa,\alpha}}{a_{N,\alpha}} = \left|S^{N-1-\kappa}\right| \int_0^\infty \frac{r^{N-\kappa-1}dr}{(1+r^2)^{\frac{N}{2}+\alpha}},$$

which yields (3.46). □

The next statement is a straightforward consequence.

Corollary 3.2 *Assume* $u(x) = u(x_1, ...x_N) = \sum_{j=1}^N u_j(x_j)$, *then*

$$(-\Delta)^\alpha \tilde{u}(x) = \sum_{j=1}^N \left(-\frac{d^2}{dx_j^2}\right)^\alpha u_j(x_j). \tag{3.47}$$

3.1 Proof of Theorem F (i)

When $f(t,x,u) = t^\beta g(u) := t^\beta u^p$, conditions (1.17) and (1.21) are fulfilled when $p > 1$ and $p > \beta + 2$ respectively. Condition $1 < p \le p_\beta^{**}$ is not compatible with $p > \beta + 2$, and condition $p_\beta^{**} < p < p_\beta^*$ necessitates $\beta + 2 < 1 + \frac{2\alpha(1+\beta)}{N}$, equivalently $\beta + 1 < \frac{2\alpha}{N}$.

Step 1. The case $1 < p < 1 + \frac{2\alpha(1+\beta)}{N+2\alpha}$). Let $z \in \mathcal{S}_u$. Since $r \mapsto r^p$ satisfies (1.14) there holds $u \ge u_{z,\infty}$ by Theorem C. Since $u_{z,\infty} = U_{p,\beta}$ by (1.27), we derived that $u \ge U_{p,\beta}$. If we assume that $u \in L^1_{loc}(0,T;\mathcal{L}^\alpha(\mathbb{R}^N))$ then $u = U_{p,\beta}$ by Theorem D.

Step 2. The case $p = 1 + \frac{2\alpha(1+\beta)}{N+2\alpha}$. We set $u_\infty = u_{0,\infty}$. From [21, Theorem 1.3 (ii)] ,

$$u_\infty(t,x) \ge \frac{c_7 t^{-\frac{N+2\alpha}{2\alpha}}}{1 + |t^{-\frac{1}{2\alpha}}x|^{N+2\alpha}} \quad \forall\, (t,x) \in (0,1) \times \mathbb{R}^N, \tag{3.48}$$

where $c_7 > 0$. Then

$$\begin{aligned}
\int_t^1 \int_{B_1(0)} u_\infty(s,x)dxds &\ge c_7 \int_t^1 \int_{B_1(0)} \frac{s^{-\frac{N+2\alpha}{2\alpha}}}{1 + |s^{-\frac{1}{2\alpha}}x|^{N+2\alpha}} dxds \\
&\ge c_7 \int_t^1 \int_{B_1(0)} \frac{s^{-1}}{1 + |x|^{N+2\alpha}} dxds \\
&\to +\infty \quad \text{as} \quad t \to 0^+,
\end{aligned}$$

thus,

$$\int_0^1 \int_{B_1(0)} u_\infty(t,x)dxdt = +\infty. \tag{3.49}$$

By Proposition 6.1 in Appendix, $x \mapsto u_k(t,x)$ is radially symmetric and decreasing, so is u_∞. Therefore, if we prove that there exists $x \in \mathbb{R}^N \setminus \{0\}$ such that $\lim_{t\to 0} u_\infty(t,x) = \infty$, it will imply

$$\lim_{t\to 0} u_\infty(t,z) = \infty \qquad \text{uniformly with respect to z in } \overline{B}_{|x|}.$$

Hence $\overline{B}_{|x|} \subset \mathcal{S}_{u_\infty}$ and by Theorem C,

$$u_\infty(t,x) \geq u_{z,\infty}(t,x) = u_\infty(t,x-z) \qquad \forall z \in \overline{B}_{|x|} \tag{3.50}$$

Because u_∞ is radially symmetric and decreasing, it implies that

$$u_\infty(t,x) = u_\infty(t,x-z) \qquad \forall z \in \overline{B}_{|x|}. \tag{3.51}$$

By iterating this process we infer that $u_\infty(t,x)$ is indeed independent of x and tends to ∞ when $t \to 0$. It coincides therefore to the maximal solution $U_{p,\beta}$ of (1.19) with $g(u) = u^p$.

Next we assume that $\mathcal{S}^c_{u_\infty} \neq \emptyset$ and let $x_0 \in \mathcal{S}^c_{u_\infty}$. Hence $\limsup_{t\to 0} u_\infty(t,x_0) < \infty$ and

$$\sup_{0<t\leq 1} u_\infty(t,x) \leq \sup_{0<t\leq 1} u_\infty(t,x_0) := M < \infty \quad \text{uniformly with respect to x in } \overline{B}^c_{|x_0|}. \tag{3.52}$$

By rescaling we can assume that $\mid x_0 \mid = 1$. Let $\bar{x} \in \overline{B}^c_3$ and $\eta \in C^2_0(B_1(\bar{x}))$ such that $\eta \geq 0$ and $\eta = 1$ on $\overline{B}_{\frac{1}{2}}(\bar{x})$. We denote

$$X_1(t) = \int_{\mathbb{R}^N} u_\infty(t,x)\eta(x)dx, \;\; Y_1(t) = t^\beta \int_{\mathbb{R}^N} u^p_\infty(t,x)\eta(x)dx, \;\; Z_1(t) = \int_{\mathbb{R}^N} u_\infty(t,x)(-\Delta)^\alpha \eta(x)dx,$$

and there holds

$$X_1'(t) + Z_1(t) + Y_1(t) = 0 \quad \text{on } (0,1]. \tag{3.53}$$

Since u_∞ is bounded in $(0,1] \times B_2(\bar{x})$ by (3.52), $X_1(t)$ and $Y_1(t)$ remains bounded on $(0,1]$.

$$Z_1(t) = \int_{B_1} u_\infty(t,x)(-\Delta)^\alpha \eta(x)dx + \int_{B^c_1} u_\infty(t,x)(-\Delta)^\alpha \eta(x)dx := Z_{1,1}(t) + Z_{1,2}(t)$$

Since η has it support in $B_1(\bar{x})$, there exists $c_8 > 0$ such that

$$(-\Delta)^\alpha \eta(x) \leq -c_8 \qquad \forall x \in B_1.$$

Using (3.49) we derive that

$$\int_0^1 Z_{1,1}(s)ds = -\infty.$$

Using (2.17) in Lemma 2.4, we have

$$\mid Z_{1,2}(t) \mid \leq c_4 \|\eta\|_{C^2} \int_{B^c_1} \frac{u_\infty(t,x)dx}{1+\mid x \mid^{N+2\alpha}} \leq c_5 \int_1^\infty \frac{r^{N-1}dr}{1+r^{N+2\alpha}}.$$

Hence

$$\int_0^1 Z_1(s)ds = -\infty.$$

Integrating (3.53) it contradicts the boundedness of X_1 and Y_1. Hence, for any $z \in \mathbb{R}^N$,

$$\limsup_{t\to 0} u_\infty(t,z) = \infty. \tag{3.54}$$

Using again the fact that $x \mapsto u_\infty(t,x)$ is radial and decreasing with respect to $\mid x \mid$, we derive

$$\limsup_{t\to 0} \int_{B_\rho(z)} u_\infty(t,x)dx = \infty \qquad \forall \rho > 0. \tag{3.55}$$

By Theorem C, we infer that $u_\infty(t,x) \geq u_{z,\infty}(t,x) = u_\infty(t,x-z)$. Interverting 0 and z we conclude again that $u_\infty(t,x)$ depends only on t, hence it coincides with $U_{\beta,p}(t)$, and clearly $\mathcal{S}_{u_\infty} = \mathbb{R}^N$. □

3.2 Proof of Theorem F (ii)

We assume that $\kappa \geq 1$ and $\mathcal{L} = \{0_{\mathbb{R}^\kappa}\} \times \mathbb{R}^{N-\kappa}$. We set $x = (x_1, x') \in \mathbb{R}^\kappa \times \mathbb{R}^{N-\kappa}$. We use Theorem F (i) with N replace by $N-\kappa$ to prove the part (ii). If $x = (x_1, x')$, then $\bar{x} = (0, x') \in \mathcal{L}$, hence by [21, Th 1.3 (ii)]

$$u_\infty(t, x-\bar{x}) \geq \frac{c_7 t^{-\frac{N+2\alpha}{2\alpha}}}{1 + (t^{-\frac{1}{2\alpha}} \mid x-\bar{x} \mid)^{N+2\alpha}} = \frac{c_7 t^{-\frac{N+2\alpha}{2\alpha}}}{1 + (t^{-\frac{1}{2\alpha}} \mid x_1 \mid)^{N+2\alpha}}.$$

By Theorem C, we obtain

$$u(t,x) \geq \frac{c_7 t^{-\frac{N+2\alpha}{2\alpha}}}{1 + (t^{-\frac{1}{2\alpha}} \mid x_1 \mid)^{N+2\alpha}} \qquad \forall (t,x) := (t, x_1, x') \in \mathbb{R}^*_+ \times \mathbb{R}^\kappa \times \mathbb{R}^{N-\kappa}. \tag{3.56}$$

For $n \in \mathbb{N}^*$, set $t_n = n^{-2\alpha}$, $\rho_0 = \int_{\mathbb{R}^{N-\kappa}} \frac{dy'}{(1+|y'|^{N+2\alpha}}$, $\lambda_0 = \rho_0^{\frac{1}{p-1}}$ and

$$f_n(x_1) = \frac{c_7 \lambda_0 (\rho_0 t_n)^{-\frac{1+\beta}{p-1}}}{1 + \left((\rho_0 t_n)^{-\frac{1}{2\alpha}} \mid x_1 \mid\right)^{N+2\alpha}}.$$

Then

$$\int_{\mathbb{R}^\kappa} f_n(x_1)dx_1 = c t_n^{\frac{\kappa}{2\alpha} - \frac{1+\beta}{p-1}} \int_0^\infty \frac{r^{\kappa-1}dr}{1+r^{N+2\alpha}},$$

for somme $c > 0$. Since $p < 1 + \frac{2\alpha(1+\beta)}{\kappa}$ the above integral is finite for any n but tends to ∞ with n. Hence, for any $n, k \in \mathbb{N}^*$ there exists $\epsilon_{n,k} > 0$ such that

$$\int_{|x_1| \leq \epsilon_{n,k}} f_n(x_1)dx_1 = k = c t_n^{\frac{\kappa}{2\alpha} - \frac{1+\beta}{p-1}} \int_0^{\epsilon_{n,k} t_n^{-\frac{N+2\alpha}{2\alpha}}} \frac{r^{\kappa-1}dr}{1+r^{N+2\alpha}}.$$

Hence $\epsilon_{n,k} \to 0$ when $n \to \infty$. This implies that for any $\zeta \in C_0^\infty(\mathbb{R}^\kappa)$,

$$\lim_{n\to\infty} \int_{|x_1|\le\epsilon_{n,k}} f_n(x_1)dx_1 = k\zeta(0).$$

Equivalently $f_{n,k} := f_n \chi_{B_{\epsilon_{n,k}}} \to k\delta_0$ in the sense of measures in $\mathbb{R}^\kappa$. Let $w_{n,k}$ be the solution of

$$\begin{aligned} \partial_t u + (-\Delta)^\alpha_\kappa u + t^\beta u^p = 0 \qquad & \text{in } \mathbb{R}^*_+ \times \mathbb{R}^\kappa \\ u(0,.) = f_{n,k} \quad & \text{in } \mathbb{R}^\kappa. \end{aligned} \tag{3.57}$$

Then $\tilde{w}_{n,k}(t, x_1, x') = w_{n,k}(t, x_1)$ is a solution of

$$\begin{aligned} \partial_t u + (-\Delta)^\alpha u + t^\beta u^p = 0 \qquad & \text{in } \mathbb{R}^*_+ \times \mathbb{R}^N \\ u(0,.) = \tilde{f}_{n,k} \quad & \text{in } \mathbb{R}^N, \end{aligned} \tag{3.58}$$

with $\tilde{f}_{n,k}(x_1, x') = f_{n,k}(x_1)$. Since $u(t_n, x) \ge \tilde{f}_n(x)$ in $\mathbb{R}^N$, we derive by the comparison principle that $u(t + t_n, x) \ge \tilde{w}_{n,k}(t, x)$ in $\mathbb{R}^*_+ \times \mathbb{R}^N$. Hence, by letting successively $n \to \infty$ and $k \to \infty$,

$$u(t + t_n, x_1, x') \ge w_{n,k}(t, x_1) \Longrightarrow u(t, x_1, x') \ge u^\kappa_{k\delta_0}(t, x_1) \Longrightarrow u(t, x_1, x') \ge u^\kappa_\infty(t, x_1), \tag{3.59}$$

where we have denoted by $u^\kappa_{k\delta_0}$ and u^κ_∞ respectively the solution of the equation in (3.57) with $k\delta_0$ as initial data and the limit of this solution when $k \to \infty$. Since $1 < p \le 1 + \frac{2\alpha}{\kappa+2\alpha}$, $u^\kappa_\infty = U_{p,\beta}$ by (i), which ends the proof. □

Remark. It appears interesting to investigate whether the fact that the singular set $\mathcal{S}_u$ contains a (N-κ)-dimensional plane can be replaced $\mathcal{S}_u$ contains a (N-κ)-dimensional submanifold.

4 Solution with a given initial trace: the general case

4.1 Problem with initial data measure

If ν is a bounded Radon measure on an open set $\mathcal{R} \subset \mathbb{R}^N$, that we note $\nu \in \mathfrak{M}^b(\mathcal{R})$), we extend it by 0 in $\mathcal{S} = \mathbb{R}^N \setminus \mathcal{R}$ and the new measure still denoted by ν, belongs to the space $\mathfrak{M}^b(\mathbb{R}^N)$ of bounded Radon measures on $\mathbb{R}^N$. Conversely, if $\nu \in \mathfrak{M}^b(\mathbb{R}^N)$ vanishes on $\mathcal{S} = \mathbb{R}^N \setminus \mathcal{R}$ its restriction to $\mathcal{R}$ belongs to $\mathfrak{M}^b(\mathcal{R})$.

Definition 4.1 *A nonnegative bounded Radon measure ν in $\mathbb{R}^N$ is an admissible measure if there exists a function $u = u_\nu \in L^1(Q_T)$ with $t^\beta u^p \in L^1(Q_T)$ to*

$$\begin{aligned} \partial_t u + (-\Delta)^\alpha u + t^\beta u^p = 0 \qquad & \text{in} \quad Q_\infty, \\ u(0,.) = \nu. \end{aligned} \tag{4.1}$$

It is a good measure if the sequence u_{ν_n} of solutions of (4.1) with initial data $\nu_n = \nu * \rho_n$ where $\{\rho_n\}$ is a sequence of mollifiers, converges to u_ν in $L^1(Q_T)$ and if $t^\beta u^p_{\nu_n}$ converges to $t^\beta u^p \in L^1(Q_T)$.

Uniqueness of solutions is proved in [21]. The following result will be useful in the sequel

Proposition 4.1 *Let $p > 1$ and $\beta > -1$. If $\nu, \mu \in \mathfrak{M}^b_+(\mathbb{R}^N)$ are good measures (resp. admissible measures), then $\nu + \mu$ is a good measure (resp. admissible measure).*

Proof. We set $\nu_n = \nu * \rho_n$ and $\mu_n = \mu * \rho_n$ and denote by u_{ν_n}, u_{μ_n} and $u_{\nu_n+\mu_n}$ the solutions of the initial value problem (4.1) with ν replaced by ν_n, μ_n and $\nu_n + \mu_n$ respectively. Since $p > 1$, $u_{\nu_n} + u_{\mu_n}$ is a supersolution of (1.24). Hence $u_{\nu_n+\mu_n} \le u_{\nu_n} + u_{\mu_n}$. When $n \to \infty$, $u_{\nu_n+\mu_n}$ converges a.e. to some function u (see [21]). Since u_{ν_n} and u_{μ_n} converges in $L^1(Q_T)$, the sequence $u_{\nu_n+\mu_n}$ is uniformly integrable in Q_T, it converges to some w (up to extraction of a subsequence). Furthermore

$$(u_{\nu_n+\mu_n})^p \le (u_{\nu_n} + u_{\mu_n})^p \le 2^{p-1}\left(u^p_{\nu_n} + u^p_{\mu_n}\right).$$

Since $t^\beta u^p_{\nu_n}$ and $t^\beta u^p_{\mu_n}$ converges in $L^1(Q_T)$ to $t^\beta u^p_\nu$ and $t^\beta u^p_\mu$ respectively, they are uniformly integrable. Hence the sequence $\{t^\beta (u_{\nu_n+\mu_n})^p\}$ is uniformly integrable in Q_T and thus, up to extraction of a second subsequence, $t^\beta (u_{\nu_n+\mu_n})^p$ converges to $t^\beta w^p$ in $L^1(Q_T)$. Going to the limit in the integral expression of the fact that $u_{\nu_n+\mu_n}$ satisfies (4.1) with initial data $\nu_n + \mu_n$, it follows that w satisfies the same equation with initial data $\mu + \nu$. Hence $w = u_{\nu+\mu}$ and by uniqueness, the whole sequence $\{u_{\nu_n+\mu_n}\}$ converges to $u_{\nu+\mu}$. The proof in the case of admissible measures is similar. □

Proposition 4.2 *Let $p > 1$ and $\beta > -1$. If $\{\nu_k\} \subset \mathfrak{M}^b_+(\mathbb{R}^N)$ is a nondecreasing sequence of admissible measures converging to $\nu \in \mathfrak{M}^b_+(\mathbb{R}^N)$, then ν is an admissible measure.*

Proof. The sequence $\{u_{\nu_k}\}$ is increasing. Furthermore.

$$u_{\nu_k} \le \mathbb{H}_\alpha[\nu_k] \le \mathbb{H}_\alpha[\nu].$$

Hence there exists some $u \in L^1(Q_T)$ for any $T > 0$, $u \ge 0$, such that $u_{\nu_n} \to u$ in $L^1(Q_T)$ and a.e. in Q_∞. By identity (3.25) in the proof of [21, Th. 1.1], we have for $\tau \ge T$,

$$\int_0^1 \int_{\mathbb{R}^N} \left(u_{\nu_k} + (\tau - t)t^\beta u^q_{\nu_k}\right) dxdt + (\tau - T)\int_{\mathbb{R}^N} u_{\nu_k}(T, x)dx = \tau \int_{\mathbb{R}^N} d\nu_k \le \tau \|\nu\|_{\mathfrak{M}^b_+}. \tag{4.2}$$

Hence $t^\beta u^q \in L^1(Q_T)$ and $t^\beta u^q_{\nu_k} \to t^\beta u^q$ in $L^1(Q_T)$. By (2.21) there holds

$$\int_0^1 \int_{\mathbb{R}^N} \left[u_{\nu_k}\left(-\partial_t \xi + (-\Delta)^\alpha \xi\right) + t^\beta u^q_{\nu_k}\xi\right] dxdt = \int_{\mathbb{R}^N} \xi(0, x)d\nu_k(x),$$

for all $\xi \in \mathbb{Y}_{\alpha,T}$, it follows that $u = u_\nu$. Hence ν is an admissible measure. □

The whole description of the set of admissible measures necessitates the introduction of Bessel capacities as in the case $\alpha = 1$, see [30], MV5). We have a first partial answer.

Lemma 4.3 *Let $p > 1$ and $\beta > -1$. If $\nu \in \mathfrak{M}^b_+(\mathbb{R}^N)$ satisfies $t^\beta(\mathbb{H}_\alpha[\nu])^p \in L^1(Q_T)$, then ν is a good measure.*

Proof. Let $\nu_n = \nu * \rho_n$. By the maximum principle

$$u_{\nu_n} \le \mathbb{H}_\alpha[\nu_n] = \mathbb{H}_\alpha[\nu * \rho_n] = \mathbb{H}_\alpha[\nu] * \rho_n.$$

Since $\mathbb{H}_\alpha[\nu] \in L^1(Q_T)$, $\mathbb{H}_\alpha[\nu] * \rho_n \to \mathbb{H}_\alpha[\nu]$ in $L^1(Q_T)$. Similarly $t^\beta(\mathbb{H}_\alpha[\nu] * \rho_n)^p \to t^\beta(\mathbb{H}_\alpha[\nu])^p$ in $L^1(Q_T)$. Since $u_{\nu_n} \le \mathbb{H}_\alpha[\nu] * \rho_n$, we conclude that the sequences $\{u_{\nu_n}\}$ and $\{t^\beta u^p_{\nu_n}\}$ are uniformly integrable in $L^1(Q_T)$, hence they are precompact by Vitali's convergence theorem and subsequences are Cauchy sequences in $L^1(Q_T)$. We end the proof as above, using uniqueness. □

Proposition 4.4 *Let $p > 1$ and $\beta > -1$. Let $\nu \in \mathfrak{M}^b_+(\mathbb{R}^N)$ with Lebesgue decomposition $\nu = \nu_0 + \nu_s$ where ν_0 and ν_s belong to $\mathfrak{M}^b_+(\mathbb{R}^N)$, $\nu_0 \in L^1(\mathbb{R}^N)$ and ν_s is singular with respect to the N-dim Lebesgue measure. If $t^\beta(\mathbb{H}_\alpha[\nu_s])^p \in L^1(Q_T)$, then ν is a good measure.*

Proof. By [21, Lemma 3.2] there exists a unique solution u_{ν_0} (resp. ν_s) to problem (4.1) with ν replaced by ν_0. (resp. ν_s. By [21, Lemma 3.2] the sequences $\{u_{\nu_0 * \rho_n}\}$ and $\{t^\beta u^p_{\nu_0 * \rho_n}\}$ are Cauchy sequences in $L^1(Q_T)$. By Lemma 4.3, the sequences $\{u_{\nu_s * \rho_n}\}$ and $\{t^\beta u^p_{\nu_s * \rho_n}\}$ share the same property. Hence ν_0. and ν_s are good measures and we conclude with Proposition 4.1. □

We recall some classical results about Bessel potentials, capacities and interpolation. For $0 < \gamma < N$, the Bessel kernel J_γ is defined in $\mathbb{R}^N \setminus \{0\}$ by $J_\gamma(x) = \mathcal{F}^{-1}\left((1 + |\xi|^2)^{-\frac{\gamma}{2}}\right)$ where $\mathcal{F}$ is the Fouriier transform in $\mathbb{R}^N$, and the Bessel potential of a positive measure is

$$\mathbf{J}_\gamma[\mu](x) = J_\gamma * \mu(x) = \int_{\mathbb{R}^N} J_\gamma(x - y) d\mu(y). \tag{4.3}$$

For $1 \le r < \infty$, the Bessel capacity $\mathrm{cap}^{\mathbb{R}^N}_{\gamma,r}$ of a compact set is

$$\mathrm{cap}^{\mathbb{R}^N}_{\gamma,r}(K) = \inf\{\|\mathbf{J}_\gamma[\zeta]\|^r_{L^r(\mathbb{R}^N)} : \zeta \in \omega_K\} \tag{4.4}$$

where ω_K is the subset of nonnegative function belonging to the Schwartz space $\mathcal{S}(\mathbb{R}^N)$, with value larger or equal to 1 on K. Furthermore

$$(-\Delta)^{\frac{\gamma}{2}}\phi = \zeta \Longleftrightarrow \mathbf{J}_\gamma[\zeta] = \phi. \tag{4.5}$$

If a linear m-accretive operator A in $L^r(\mathbb{R}^N)$ with domain $D(A)$ is the infinitesimal generator of an analytic semigroup of bounded linear operators $S^A(t)$, i.e.

$$u(t) = S^{-A}_t \upsilon \quad \forall t \ge 0 \Longleftrightarrow \frac{du}{dt} + A\upsilon = 0 \quad \text{on } \mathbb{R}^*_+\,,\ u(0) = \upsilon, \tag{4.6}$$

the real intermolation classes between $D(A)$ and $L^r(\mathbb{R}^N)$ can be obtained (see [39, p. 96]) by

$$\left[D(A), L^r(\mathbb{R}^N)\right]_{\theta,r} = \left\{\upsilon \in L^r(\mathbb{R}^N) : \int_0^1 \int_{\mathbb{R}^N} t^{(1-\theta)r} |AS^A_t \upsilon|^r dx \frac{dt}{t} < \infty\right\}, \tag{4.7}$$

and

$$\|v\|_{[D(A),L^r]_{\theta,r}} \equiv \|v\|_{L^r} + \left(\int_0^1 \|t^{1-\theta} A S_t^A v\|_{L^r}^r \frac{dt}{t}\right)^{\frac{1}{r}}. \tag{4.8}$$

If $A = (-\Delta)^\alpha + \delta I$ for some $\delta > 0$, its domain $D(A)$ in $L^r(\mathbb{R}^N)$ is the Bessel potential space $(I-\Delta)^{-\alpha}(L^r(\mathbb{R}^N)) = L^{2\alpha,r}(\mathbb{R}^N)$ (see [27, Th 1]). By classical interpolation properties of Bessel potential spaces (see e.g. [39]),

$$\left[D(A), L^r(\mathbb{R}^N)\right]_{\theta,r} = L^{2\theta\alpha,r}(\mathbb{R}^N) = (I-\Delta)^{-\theta\alpha}(L^r(\mathbb{R}^N)). \tag{4.9}$$

Furthermore (4.8) can be replaced by

$$\|v\|_{[D(A),L^r]_{\theta,r}} \equiv \left(\int_0^1 \|t^{1-\theta} A S_t^A v\|_{L^r}^r \frac{dt}{t}\right)^{\frac{1}{r}}. \tag{4.10}$$

Proposition 4.5 *Let $N \geq 1$, $p > 1$ and $-1 < \beta < p-1$. If problem (4.1) admits a positive solution u_ν for some $\nu \in \mathfrak{M}_b^+(\mathbb{R}^N)$, then ν vanishes on Borel subsets of $\mathbb{R}^N$ with zero Bessel capacity $\mathrm{cap}_{\frac{2\alpha(1+\beta)}{p},p'}^{\mathbb{R}^N}$, i.e.*

$$\forall K \subset \mathbb{R}^N, K \text{ Borel}, \mathrm{cap}_{\frac{2\alpha(1+\beta)}{p},p'}^{\mathbb{R}^N}(K) = 0 \Longrightarrow \nu(K) = 0. \tag{4.11}$$

Proof. Assume $u := u_\nu$ is the solution of (4.1). Since $\mathrm{cap}_{\frac{2\alpha(1+\beta)}{p},p'}^{\mathbb{R}^N}$ is a Choquet capacity we can assume that $K \subset \mathbb{R}^N$ is compact and let $\zeta \in \mathcal{S}(\mathbb{R}^N)$ be such that $0 \leq \zeta$ in $\mathbb{R}^N$ and $\zeta \geq 1$ on K. We set $\Phi = e^{-\delta t}\mathbb{H}^\alpha[\zeta]$ for some $\delta \in (0,1)$ and take $\Phi^{p'}$ as a test function. Then

$$\int_{\mathbb{R}^N} u\Phi^{p'}(1,.)dx + \int_0^1 \int_{\mathbb{R}^N} \left[\left(-\partial_t \Phi^{p'} + (-\Delta)^\alpha \Phi^{p'}\right) u + t^\beta u^p \Phi^{p'}\right] dxdt = \int_{\mathbb{R}^N} \Phi^{p'} d\nu. \tag{4.12}$$

Note that $(-\Delta)^\alpha \Phi^{p'} \geq p'\Phi^{p'-1}(-\Delta)^\alpha \Phi$ and $\partial_t \Phi + (-\Delta)^\alpha \Phi + \delta\Phi = 0$, hence

$$-\partial_t \Phi^{p'} + (-\Delta)^\alpha \Phi^{p'} \geq 2p'\Phi^{p'-1}(-\Delta)^\alpha \Phi,$$

and we derive from Hölder's inequality

$$\int_{\mathbb{R}^N} u\Phi^{p'}(1,x)dx + 2p' \left(\int_0^1 \int_{\mathbb{R}^N} t^{-\frac{p'\beta}{p}} |(-\Delta)^\alpha \Phi + \delta\Phi|^{p'} dxdt\right)^{\frac{1}{p'}} \left(\int_0^1 \int_{\mathbb{R}^N} t^\beta u^p \Phi^{p'} dxdt\right)^{\frac{1}{p}} + \int_0^1 \int_{\mathbb{R}^N} t^\beta u^p \Phi^{p'} dxdt \geq \nu(K).$$

Applying (4.7), (4.9) with $r = p'$, $\theta = \frac{1+\beta}{p}$, we obtain directly for some $c > 1$,

$$\frac{1}{c}\|\zeta\|_{L^{\frac{2\alpha(1+\beta)}{p},p'}} \leq \left(\int_0^1 \int_{\mathbb{R}^N} t^{-\frac{p'\beta}{p}} |(-\Delta)^\alpha \Phi + \delta\Phi|^{p'} dxdt\right)^{\frac{1}{p'}} \leq c\|\zeta\|_{L^{\frac{2\alpha(1+\beta)}{p},p'}}. \tag{4.13}$$

If $\mathrm{cap}^{\mathbb{R}^N}_{\frac{2\alpha(1+\beta)}{p},p'}(K)=0$, there exists a sequence $\{\zeta_n\}\subset\omega_K$ such that $\|\zeta_n\|_{L^{\frac{2\alpha(1+\beta)}{p},p'}}\to 0$ as $n\to\infty$. Furthermore it is possible to assume $\zeta_n\le 1$ in $\mathbb{R}^N$ (see [2]). Hence, up to a subsequence, $\zeta_n\to 0$ a.e. in $\mathbb{R}^N$. This implies $\Phi_n\le 1$ and $\Phi_n\to 0$ a.e. in Q_∞. Therefore

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}u\Phi_n^{p'}(1,x)dx\ \text{ and }\quad \lim_{n\to\infty}\int_0^1\int_{\mathbb{R}^N}t^\beta u^p\Phi_n^{p'}dxdt=0.$$

Combining the previous inequalities we derive that $\nu(K)=0$. □

The following result provides a complete characterization of good measure.

Theorem 4.6 *Let $N\ge 1$, $p>1$ and $-1<\beta<p-1$. A nonnegative bounded measure ν in $\mathbb{R}^N$ is an admissible measure if and only if ν vanishes on Borel subsets of $\mathbb{R}^N$ with zero $\mathrm{cap}^{\mathbb{R}^N}_{\frac{2\alpha(1+\beta)}{p},p'}$-Bessel capacity.*

Proof. If ν vanishes Borel subsets with zero $\mathrm{cap}^{\mathbb{R}^N}_{\frac{2\alpha(1+\beta)}{p},p'}$, there exists an increasing sequence of nonnegative measures $\{\nu_n\}\subset\left(L^{\frac{2\alpha(1+\beta)}{p-1},p'}(\mathbb{R}^N)\right)'=L^{-\frac{2\alpha(1+\beta)}{p-1},p}(\mathbb{R}^N)$ such that $\nu_n\to\nu$ in the sense of measures. This results is classical in the integer case and a proof in the Bessel case (similar in fact) can be found in [40, Prop. 3.6]. Let $\zeta\in\mathcal{S}(\mathbb{R}^N)$ and $\Phi=e^{-\delta t}\mathbb{H}_\alpha[\zeta]$, then

$$\int_{\mathbb{R}^N}\Phi\mathbb{H}_\alpha[\nu_n](1,x)dx+\int_0^1\int_{\mathbb{R}^N}\mathbb{H}_\alpha[\nu_n]\left(2(-\Delta)^\alpha\Phi+\delta\Phi\right)dxdt=\int_{\mathbb{R}^N}\zeta d\nu_n.$$

Hence

$$\int_0^1\int_{\mathbb{R}^N}\mathbb{H}_\alpha[\nu_n]\left((-\Delta)^\alpha\Phi+\delta\Phi\right)dxdt\le\|\nu_n\|_{L^{-\frac{2\alpha(1+\beta)}{p-1},p}}\|\zeta\|_{L^{\frac{2\alpha(1+\beta)}{p-1},p'}}.$$

Consider the mapping

$$\zeta\mapsto L(\zeta)=\int_0^1\int_{\mathbb{R}^N}t^{\frac{\beta}{p}}\mathbb{H}_\alpha[\nu_n]t^{-\frac{\beta}{p}}\left((-\Delta)^\alpha\Phi+\delta\Phi\right)dxdt.$$

It satisfies

$$\begin{aligned}|\,L(\zeta)\,|&\le\|\nu_n\|_{L^{-\frac{2\alpha(1+\beta)}{p-1},p}}\|\zeta\|_{L^{\frac{2\alpha(1+\beta)}{p-1},p'}}\\&\le c\|\nu_n\|_{L^{-\frac{2\alpha(1+\beta)}{p-1},p}}\left(\int_0^1\int_{\mathbb{R}^N}t^{-\frac{p'\beta}{p}}\left|(-\Delta)^\alpha\Phi+\delta\Phi\right|^{p'}dxdt\right)^{\frac{1}{p'}}.\end{aligned}\tag{4.14}$$

by (4.13). Hence $t^{\frac{\beta}{p}}\mathbb{H}_\alpha[\nu_n]\in L^p(Q_1)$ and

$$\left(\int_0^1\int_{\mathbb{R}^N}t^\beta\left(\mathbb{H}_\alpha[\nu_n]\right)^p dxdt\right)^{\frac{1}{p}}\le c\|\nu_n\|_{L^{-\frac{2\alpha(1+\beta)}{p-1},p}}.\tag{4.15}$$

We end the proof with Proposition 4.2. □

4.2 Barrier function for $N = 1$

We set

$$W(z) = \begin{cases} \dfrac{\ln(e+z^2)}{1+z^{1+2\alpha}} & \text{if} \quad z \geq 0 \\ 1 & \text{if} \quad \text{z} < 0, \end{cases} \tag{4.16}$$

and

$$w(t,x) = t^{-\frac{1+\beta}{p-1}} W(t^{-\frac{1}{2\alpha}}x), \qquad (t,x) \in \mathbb{R}_+^* \times \mathbb{R}, \tag{4.17}$$

where e is Neper constant. When $t \to 0$, the function w satisfies

$$\begin{aligned} &(i) \qquad w(t,x) = \frac{2t^{\frac{1+2\alpha}{2\alpha}-\frac{1+\beta}{p-1}}\ln t}{|x|^{1+2\alpha}}(1+o(1)) \qquad \text{if} \quad x > 0 \\ &(ii) \qquad w(t,x) = t^{-\frac{1+\beta}{p-1}} \qquad \text{if} \quad x \leq 0. \end{aligned} \tag{4.18}$$

Lemma 4.1 *Assume that $p > 1 + \frac{2\alpha(1+\beta)}{1+2\alpha}$. Then there exists $\lambda_0 > 0$ such that for $\lambda \geq \lambda_0$, the function $w_\lambda := \lambda w$ satisfies*

$$\begin{aligned} \partial_t w_\lambda + (-\Delta)_1^\alpha w_\lambda + t^\beta w_\lambda^p \geq 0 \qquad &\text{in} \quad \mathbb{R}_+^* \times \mathbb{R} \\ \lim_{t\to 0} w(t,x) = 0 \qquad &\text{if} \quad x > 0 \\ \lim_{t\to 0} w(t,x) = \infty \qquad &\text{if} \quad x \leq 0 \end{aligned} \tag{4.19}$$

Proof. Clearly the assertions concerning the limit of $w(x,t)$ when $t \to 0$ are satisfied since $\frac{1+2\alpha}{2\alpha} - \frac{1+\beta}{p-1} > 0$ by assumption. Then

$$\partial_t w_\lambda(t,x) = -\frac{\lambda(1+\beta)}{p-1} t^{-\frac{1+\beta}{p-1}-1} w(z) - \frac{\lambda}{2\alpha} t^{-\frac{1+\beta}{p-1}-1} w'(z) z,$$

with $z = t^{-\frac{1}{2\alpha}}x$, and

$$(-\Delta)_1^\alpha w_\lambda(t,x) = \lambda t^{-\frac{1+\beta}{p-1}-1}(-\Delta)_1^\alpha w(z).$$

Hence

$$\begin{aligned} &\partial_t w_\lambda(t,x) + (-\Delta)_1^\alpha w_\lambda(t,x) + t^\beta w_\lambda^p(t,x) \\ &\qquad = \lambda t^{-\frac{1+\beta}{p-1}-1}\left[(-\Delta)_1^\alpha w(z) - \frac{1}{2\alpha} w'(z) z - \frac{1+\beta}{p-1} w(z) + \lambda^{p-1} w^p(z)\right]. \end{aligned} \tag{4.20}$$

If $z > 0$, we obtain that

$$-\frac{1}{2\alpha} w'(z) z - \frac{1+\beta}{p-1} w(z) = \left[\frac{1+2\alpha}{2\alpha}\frac{z^{1+2\alpha}}{1+z^{1+2\alpha}} - \frac{1+\beta}{p-1} - \frac{z^2(e+z^2)^{-1}}{\alpha \ln(e+z^2)}\right] w(z).$$

Since $\frac{1+2\alpha}{2\alpha} > \frac{1+\beta}{p-1}$, $\lim_{z\to\infty} \frac{z^{1+2\alpha}}{1+z^{1+2\alpha}} = 1$ and $\lim_{z\to\infty} \frac{1}{\ln(e+z^2)} = 0$, then there exist $R_0 > 0$ and $\sigma_0 > 0$ such that

$$-\frac{1}{2\alpha} w'(z) z - \frac{1+\beta}{p-1} w(z) \geq \sigma_0 w(z) \qquad \forall z \geq R_0. \tag{4.21}$$

Next we deal with $(-\Delta)_1^\alpha w(z)$ and put

$$\tilde{w}(z) = \frac{\ln(e+z^2)}{1+|z|^{1+2\alpha}} \quad \forall z \in \mathbb{R},$$

so that $(-\Delta)_1^\alpha w(z) = (-\Delta)_1^\alpha \tilde{w}(z) + (-\Delta)_1^\alpha (1-\tilde{w}\chi_{\mathbb{R}_-})(z)$.
For $z > 2$, using the alternative definition of fractional Laplacian, we have that

$$\begin{aligned} -(-\Delta)^\alpha \tilde{w}(z) &= \frac{a_{1,\alpha}}{2}\int_{-\infty}^{\infty} \frac{\frac{\ln(e+|z+\tilde{y}|^2)}{1+|z+\tilde{y}|^{1+2\alpha}} + \frac{\ln(e+|z-\tilde{y}|^2)}{1+|z-\tilde{y}|^{1+2\alpha}} - \frac{2\ln(e+z^2)}{1+z^{1+2\alpha}}}{|\tilde{y}|^{1+2\alpha}} d\tilde{y} \\ &= \frac{a_{1,\alpha} w(z)}{2z^{2\alpha}} \int_{-\infty}^{\infty} \frac{I_z(y)}{|y|^{1+2\alpha}} dy, \end{aligned} \tag{4.22}$$

where

$$I_z(y) = \frac{1+z^{N+2\alpha}}{1+z^{1+2\alpha}|1+y|^{1+2\alpha}} \frac{\ln(e+z^2|1+y|^2)}{\ln(e+z^2)} + \frac{1+z^{1+2\alpha}}{1+z^{1+2\alpha}|1-y|^{1+2\alpha}} \frac{\ln(e+z^2|1-y|^2)}{\ln(e+z^2)} - 2.$$

Step 1: There exists $c_{12} > 0$ such that

$$\int_{\frac{1}{2}\le|y|\le\frac{3}{2}} \frac{I_z(y)}{|y|^{1+2\alpha}} dy \le \frac{c_{12}}{w(z)z}. \tag{4.23}$$

Actually, for $-\frac{3}{2} < y < -\frac{1}{2}$, there exists $c_{13} > 0$ such that

$$\frac{1+z^{1+2\alpha}}{1+z^{1+2\alpha}|1-y|^{1+2\alpha}} \frac{\ln(e+z^2|1-y|^2)}{\ln(e+z^2)} \le c_{13},$$

and then

$$\begin{aligned} \int_{-\frac{3}{2}}^{-\frac{1}{2}} \frac{I_z(y)}{|y|^{1+2\alpha}} dy &\le 2\int_0^{\frac{1}{2}} \frac{1+z^{1+2\alpha}}{1+(zr)^{1+2\alpha}} \frac{\ln(e+z^2r^2)}{\ln(e+z^2)} dr + c_{14} \\ &\le \frac{2}{w(z)z}\int_0^{\infty} \frac{\ln(e+t^2)}{1+t^{1+2\alpha}} dt + c_{14} \\ &\le \frac{c_{15}}{w(z)z}, \end{aligned}$$

where $c_{14}, c_{15} > 0$, and the last inequality holds since $w(z)z \to 0$ as $z \to +\infty$. Similarly,

$$\int_{\frac{1}{2}}^{\frac{3}{2}} \frac{I_z(y)}{y^{N+2\alpha}} dy_1 \le \frac{c_{16}}{w(z)z}.$$

Step 2: There exists $c_{17} > 0$ such that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{I_z(y)}{|y|^{1+2\alpha}} dy \le c_{17}. \tag{4.24}$$

Indeed, since function I_z is C^2 in $[-\frac{1}{2},\frac{1}{2}]$ and satisfies

$$I_z(0)=0 \ \text{ and } \ I_z(y)=I_z(-y),$$

then $I_z'(0)=0$ and there exists $c_{18}>0$ such that

$$|I_z''(y)|\le c_{18} \quad y\in[-\tfrac{1}{2},\tfrac{1}{2}].$$

Then we have

$$|I_z(y)|\le \frac{c_{18}}{2}y^2 \quad \forall y\in[-\tfrac{1}{2},\tfrac{1}{2}],$$

which implies that

$$\left|\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{I_z(y)}{|y|^{1+2\alpha}}dy\right|\le c_{19}.$$

Step 3: There exists $c_{17}>0$ such that

$$\left|\int_A \frac{I_z(y)}{|y|^{1+2\alpha}}dy\right|\le c_{20}, \tag{4.25}$$

where $A=(-\infty,-\frac{3}{2})\cup(\frac{3}{2},+\infty)$. In fact, for $y\in A$, we observe that there exists $c_{21}>0$ such that $I_z(y)\le c_{21}$ and

$$\int_A \frac{I_z(y)}{|y|^{1+2\alpha}}dy\le 2\int_{\frac{3}{2}}^{+\infty}\frac{c_{21}}{|y|^{1+2\alpha}}dy\le c_{22},$$

for some $c_{22}>0$. Consequently, by (4.22)-(4.25), there exists $c_{23}>0$ such that

$$(-\Delta)^\alpha\tilde{w}(z)\ge -\frac{c_{23}}{1+z^{1+2\alpha}} \quad \forall z\ge 2.$$

Since $1-\tilde{w}\chi_{\mathbb{R}_-}=1$ in $\mathbb{R}_+$ and $1-\tilde{w}\chi_{\mathbb{R}_-}\le 1$ in $\mathbb{R}_-$, we have also

$$(-\Delta)^\alpha(1-\tilde{w}\chi_{\mathbb{R}_-})(z)\ge 0 \quad \forall z>0.$$

Therefore, we derive

$$(-\Delta)^\alpha w(z)\ge -\frac{c_{23}}{1+z^{1+2\alpha}} \quad \forall z\ge 2. \tag{4.26}$$

Combining (4.21) and (4.26), we infer that there exists $R_1\ge R_0+2$ such that for $z>R_1$,

$$\begin{aligned}(-\Delta)^\alpha w(z)-\frac{1}{2\alpha}w'(z)z-\frac{1+\beta}{p-1}w(z)&\ge \sigma_0 w(z)-\frac{c_{23}}{1+z^{1+2\alpha}}\\ &=w(z)\left(\sigma_0-\frac{c_{23}}{\ln(e+z^2)}\right)\\ &\ge 0.\end{aligned}$$

For $z\le R_1$, there exists $c_{24}>0$ such that

$$(-\Delta)^\alpha w(z)-\frac{1}{2\alpha}w'(z)z-\frac{1+\beta}{p-1}w(z)\ge -c_{24},$$

and there exists $c_{25} > 0$ dependent of R_1 such that

$$w(z) \geq c_{25}.$$

Therefore, one can find $\Lambda_0 > 0$ such that for $\lambda \geq \Lambda_0$,

$$(-\Delta)^\alpha w(z) - \frac{1}{2\alpha} w'(z)z - \frac{1+\beta}{p-1} w(z) + \lambda^{p-1} w^p(z) \geq 0 \quad \forall z \in \mathbb{R}, \tag{4.27}$$

which together with (4.20), implies that (4.19) holds, which ends the proof. □

4.3 Solutions with initial trace $(\mathcal{S}, 0)$

Lemma 4.7 *Assume $N \geq 1$ and $p > 1 + \frac{2\alpha(1+\beta)}{1+2\alpha}$. Then for any $R > 0$ there exists a positive function $u = u_{\infty,B_R}$ minimal among the solutions of (1.24) in Q_∞ which satisfy*

$$\begin{array}{lll} \lim_{t\to 0} u(t,x) = \infty & \text{uniformly in} & \overline{B}_R \\ \lim_{t\to 0} u(t,x) = 0 & \text{uniformly in} & B^c_{R'} \quad \forall R' > R. \end{array} \tag{4.28}$$

Furthermore $R \mapsto u_{\infty,B_R}$ is increasing.

Proof. By scaling we can assume that $R = 1$ and we fix $\lambda \geq \lambda_0$. We denote by $\mathbf{e}_1$ the point with coordinates $(1, 0, ..., 0)$ in $\mathbb{R}^N$. The function

$$(t,x) \mapsto w_{e_1}(t, x_1, x') = \lambda t^{-\frac{1+\beta}{p-1}} W(t^{-\frac{1}{2\alpha}}(x_1 - 1)), \tag{4.29}$$

is a super solution of (1.24) in Q_∞ which satisfies

$$\begin{array}{ll} \lim_{t\to 0} w_{e_1}(t, x_1, x') = \infty & \text{uniformly in} \quad (-\infty, 1] \times \mathbb{R}^{N-1} \\ \lim_{t\to 0} w_{e_1}(t, x) = 0 & \text{uniformly in} [1+\epsilon, \infty) \times \mathbb{R}^{N-1}. \end{array} \tag{4.30}$$

Since equation (1.24) is invariant under rotations and translations, for any $a \in \partial B_1$ there exists a rotation $\mathcal{R}_a$ with center 0 such that $\mathcal{R}_a(a) = e_1$. Therefore the function $(t,x) \mapsto w_a(t,x) := w_{e_1}(t, \mathcal{R}_a(x))$ is a solution of (1.24) in Q_∞ and it satisfies

$$\begin{array}{lll} \lim_{t\to 0} w_a(t,x) = \infty & \text{uniformly in} & \{x \in \mathbb{R}^N : \langle x, a\rangle \leq 1\} \\ \lim_{t\to 0} w_{e_1}(t,x) = 0 & \text{uniformly in} & \{x \in \mathbb{R}^N : \langle x, a\rangle \geq 1 + ge\}. \end{array} \tag{4.31}$$

For $k \in \mathbb{N}^*$ let $u_{k\chi_{B_1}}$ be the solution of

$$\begin{array}{rll} \partial_t u + (-\Delta)^\alpha u + t^\beta u^p = 0 & \text{in} & Q_\infty \\ u(0,.) = k\chi_{B_1} & \text{in} & \mathbb{R}^N. \end{array} \tag{4.32}$$

Then $\{u_{k\chi_{B_1}}\}_k$ is increasing. For any $a \in \partial B_1$, $u_{k\chi_{B_1}} \leq w_a$, the following limit exists,

$$u_{\infty,B_1} = \lim_{k\to\infty} u_{k\chi_{B_1}},$$

and there holds

$$u_{\infty,B_1} \le \inf\{w_a : a \in \partial B_1\}.$$

This solution u is clearly minimal by construction and the monotonicity of the mapping $R \mapsto u_{\infty,B_R}$ follows. □

Remark. In the previous result, the ball B_R can be replaced by any closed convex set with a non-empty interior. If $a \in \partial K$ let H_a be an affine separation hyperplane, with outer normal vector $\mathbf{n}_a$ and

$$H_a^+ = \{x \in \mathbb{R}^N : \langle x-a, \mathbf{n}_a\rangle > 0\} \quad \text{and} \quad H_a^- = \{x \in \mathbb{R}^N : \langle x-a, \mathbf{n}_a\rangle < 0\}.$$

The supersolutions w_a are expressed by

$$(t,x) \mapsto w_a(t,x) = \lambda t^{-\frac{1+\beta}{p-1}} W(t^{-\frac{1}{2\alpha}}\langle x-a, \mathbf{n}_a\rangle),$$

and have initial trace $(0, \overline{H_a^-})$. Then we construct the minimal solution $u = u_{\infty,K}$ of (1.24) with initial trace $(0,K)$ such that

$$\begin{array}{lll} \lim_{t\to 0} u(t,x) = \infty & \text{uniformly in} & K \\ \lim_{t\to 0} u(t,x) = 0 & \text{uniformly in} & \{x \in K^c : \operatorname{dist}(x,K) \ge \epsilon\} \quad \forall \epsilon > 0. \end{array} \tag{4.33}$$

Furthermore the mapping $K \mapsto u_{\infty,K}$ is nondecreasing.

Proposition 4.8 *Assume $N \ge 1$ and $p > 1 + \frac{2\alpha(1+\beta)}{1+2\alpha}$. Then for any closed set $\mathcal{S}$ such that $\overline{\operatorname{int}(\mathcal{S})} = \mathcal{S}$ there exists a positive function $u = u_{\infty,S}$ minimal among the solutions of (1.24) in Q_∞ which satisfy*

$$\begin{array}{lll} \lim_{t\to 0} u(t,x) = \infty & \text{locally uniformly in} & \mathcal{S} \\ \lim_{t\to 0} u(t,x) = 0 & \text{locally uniformly in} & \{x \in \mathcal{S}^c : \operatorname{dist}(x,\mathcal{S}) \ge \epsilon\} \quad \forall \epsilon > 0. \end{array} \tag{4.34}$$

In particular $Tr(u_{\infty,S}) = (\mathcal{S}, 0)$. Furthermore

$$u_{\mathcal{S},\infty}(t,x) \le c_9 t^{-\frac{1+\beta}{p-1}} \frac{\ln\left(e + t^{-\frac{1}{\alpha}}(\operatorname{dist}(x,\mathcal{S}))^2\right)}{1 + t^{-\frac{1+2\alpha}{2\alpha}}(\operatorname{dist}(x,\mathcal{S}))^{1+2\alpha}} \qquad \forall (t,x) \in Q_\infty. \tag{4.35}$$

Proof. We first assume that $\mathcal{S}$ is compact, hence precompact, and for any $\delta > 0$ there exists a finite number of points ξ_j, $1 \le j \le n_\delta$ such that

$$\mathcal{S} \subset \bigcup_{j=I}^{n_\delta} \overline{B}_\epsilon(\xi_j) := \mathcal{S}_\delta.$$

Clearly n_δ is nondecreasing, furthermore we can choose the points ξ_j such that $\epsilon \mapsto \mathcal{S}_\delta$ is nonincreasing. Since $p > 1$, the function

$$w_{\mathcal{S}_\delta} := \sum_{j=1}^{n_\delta} u_{\infty,\overline{B}_\delta(\xi_j)}, \tag{4.36}$$

is a supersolution of (1.24) in Q_∞ and it satisfies

$$\begin{array}{lll} \lim_{t\to 0} w_{\mathcal{S}_\delta}(t,x) = \infty & \text{uniformly in} & \mathcal{S}_\delta \\ \lim_{t\to 0} w_{\mathcal{S}_\delta}(t,x) = 0 & \text{uniformly in} & \{x \in \mathcal{S}_\delta^c : \text{dist}\,(x,\mathcal{S}_\delta) \geq \epsilon\} \quad \forall \epsilon > 0. \end{array} \tag{4.37}$$

For $k \in \mathbb{N}^*$ let $u_{k\chi_{\mathcal{S}}}$ be the solution of (1.24) in Q_∞ with initial data $k\chi_{\mathcal{S}}$. It exists since $\mathcal{S}$ has a non-empty interior, and it coincides with the solution of (1.24) in Q_∞ with initial data $k\chi_{\text{int}(\mathcal{S})}$. Clearly there holds $u_{k\chi_{\mathcal{S}}} \leq w_{\mathcal{S}_\delta}$ and the sequence $\{u_{k\chi_{\mathcal{S}}}\}_k$ is increasing. There exists

$$u_{\infty,\mathcal{S}} = \lim_{k\to\infty} u_{k\chi_{\mathcal{S}}}.$$

It is a positive solution of (1.24) in Q_∞ which tends to infinity on $\mathcal{S}$, by construction, and satisfies $u_{\infty,\mathcal{S}} \leq w_{\mathcal{S}_\delta}$. This implies in particular that for any $\epsilon > 0$,

$$\lim_{t\to 0} u_{\infty,\mathcal{S}}(t,x) = 0 \quad \text{uniformly in} \quad \{x \in \mathcal{S}_\delta^c : \text{dist}\,(x,\mathcal{S}_\delta) \geq \epsilon\}.$$

Since this holds for any $\delta, \epsilon > 0$, the second assertion in (4.34) follows.

If S is unbounded, for any $\rho > 0$ large enough, $\mathcal{S}^\rho := \mathcal{S} \cap \overline{B}_\rho$ is a nonempty compact set and $\mathcal{S}^\rho = \overline{\text{int}}(\mathcal{S}^\rho)$. Hence there exists a solution $u_{\infty,\mathcal{S}^\rho}$ of (1.24) in Q_∞ with initial trace $(0,\mathcal{S}^\rho)$. By construction $\rho \mapsto u_{\infty,\mathcal{S}^\rho}$ is nondecreasing and converges to a nonnegative solution $u_{\infty,\mathcal{S}}$ of (1.24) in Q_∞. Let $a = (a_1, ..., a_N) \in \mathcal{S}^c$ and $\tau > 0$ such that

$$Q_a^\tau = \{x = (x_1, ..., x_N) :| x_j - a_j |\leq \tau\} \subset \mathcal{S}^c.$$

We put

$$W_j(t,x_j) = \lambda t^{-\frac{1+\beta}{p-1}} \left(W(t^{-\frac{1}{2\alpha}}(x_j - a_j + \tau) + W(t^{-\frac{1}{2\alpha}}(a_j + \tau - x_j) \right),$$

with $\lambda \geq \lambda_0$, then W_j is a supersolution of (1.24) in $\mathbb{R}_+ \times \mathbb{R}$ which satisfies

$$\begin{array}{lll} (i) & \lim_{t\to 0} W_j(t,x) = 0 & \text{locally uniformly in} \quad (a_j - \tau, a_j + \tau) \\ (i) & \lim_{t\to 0} W_j(t,x) = \infty & \text{uniformly in} \quad (-\infty, a_j - \tau] \bigcup [a_j + \tau, \infty). \end{array}$$

Hence $W_{Q_a^\tau}(t,x) = \sum_j W_j(t,x)$ is a supersolution of (1.24) in Q_∞ which satisfies

$$\begin{array}{lll} (i) & \lim_{t\to 0} W_{Q_a^\tau}(t,x) = 0 & \text{locally uniformly in} \quad Q_a^\tau \\ (i) & \lim_{t\to 0} W_{Q_a^\tau}(t,x) = \infty & \text{uniformly in} \quad \mathbb{R}^N \setminus Q_a^\tau. \end{array}$$

The estimate from above can be made more precise (it does not depend on the fact that $\mathcal{S} = \overline{\text{int}}\,\mathcal{S}$) using (4.16).

By construction $u_{\infty,\mathcal{S}^\rho} \leq W_{Q_a^\tau}$ which implies $u_{\infty,\mathcal{S}} \leq W_{Q_a^\tau}$. Hence $u_{\infty,\mathcal{S}}$ satisfies (4.34). The estimate from above can be made more precise (it does not depend from the fact that $\mathcal{S} = \overline{\text{int}}\,\mathcal{S}$) using (4.16) since.

$$W_{Q_a^\tau}(a) \leq 2N\lambda t^{-\frac{1+\beta}{p-1}} \frac{\ln\left(e + t^{-\frac{1}{\alpha}}\tau^2\right)}{1 + t^{-\frac{1+2\alpha}{2\alpha}}\tau^{1+2\alpha}} \tag{4.38}$$

If we take $\tau = \frac{\text{dist}\,(a,\mathcal{S})}{\sqrt{N}}$ we derive (4.35). Furthermore $u_{\infty,\mathcal{S}}$ is clearly minimal as the limit of an increasing sequence of solutions with bounded initial data with compact support. □

4.4 Proof of Theorem G

If $\nu \in \mathfrak{M}^b(\mathcal{R})$, we extend it by zero and still denote by $\nu \in \mathfrak{M}^b(\mathbb{R}^N)$ its extension. For $\rho > 0$, $\mathcal{S}^\rho := \mathcal{S} \cap \overline{B}_\rho$ and for $k \in \mathbb{N}^*$, $k\chi_{\mathcal{S}_\rho} dx$ is a good measure. Since ν is a good measure, $\nu + k\chi_{\mathcal{S}_\rho} dx$ is a good measure by Proposition 4.1 and there exists a solution $u := u_{\nu + k\chi_{\mathcal{S}_\rho} dx}$ of (1.24) in Q_∞ with initial data $\nu + k\chi_{\mathcal{S}_\rho} dx$ and it satisfies

$$\sup\left\{u_\nu, u_{k\chi_{\mathcal{S}_\rho} dx}\right\} \le u_{\nu + k\chi_{\mathcal{S}_\rho} dx} \le u_\nu + u_{k\chi_{\mathcal{S}_\rho} dx} \le u_\nu + u_{\infty,\mathcal{S}}. \tag{4.39}$$

Since $(k, \rho) \mapsto u_{k\chi_{\mathcal{S}_\rho} dx}$ is increasing, we can let (k and ρ go to infinity succesively and obtain that $u_{\nu + k\chi_{\mathcal{S}_\rho} dx}$ converges to a positive solution $\tilde{u}$ of (1.24) in Q_∞ and

$$\sup\left\{u_\nu, u_{\infty,\mathcal{S}}\right\} \le \tilde{u} \le u_\nu + u_{\infty,\mathcal{S}}. \tag{4.40}$$

This estimate implies that $Tr(\tilde{u}) = (\mathcal{S}, \nu)$. □

5 The subcritical case

For equation (1.24), the subcritical case corresponds to the fact that

$$u_\infty(t,x) = V(t,x) = t^{-\frac{1+\beta}{p-1}} v(t^{-\frac{1}{2\alpha}} x) \qquad \forall (t,x) \in Q_\infty,$$

where v is the minimal positive solution of (1.29).

5.1 Proof of Theorem I

Proposition 5.1 *Assume that* $1 + \frac{2\alpha(1+\beta)}{N+2\alpha} < p < 1 + \frac{2\alpha(1+\beta)}{N}$ *and* u *is a nonnegative solution of (1.30) where* $\mathcal{S} \neq \emptyset$. *Then*

$$u(t,x) \ge \frac{c_{10} t^{-\frac{1+\beta}{p-1}}}{1 + (t^{-\frac{1}{2\alpha}} d(x,\mathcal{S}))^{N+2\alpha}} \qquad \forall (t,x) \in Q_\infty. \tag{5.1}$$

for some $c_{10} > 0$ *such that for*

Proof. By Theorem C, for any $x_0 \in \mathcal{S}$,

$$u(t,x) \ge u_\infty(t, x - x_0) = t^{-\frac{1+\beta}{p-1}} v(t^{-\frac{1}{2\alpha}}(x - x_0)) \qquad \forall (t,x) \in Q_\infty,$$

which implies that

$$u(t,x) \ge t^{-\frac{1+\beta}{p-1}} \sup_{x_0 \in \mathcal{S}} v(t^{-\frac{1}{2\alpha}}(x - x_0)) \quad \forall (t,x) \in Q_\infty. \tag{5.2}$$

The maximum of V is achieved at 0, hence, for any $x \in \mathcal{S}$,

$$u(t,x) \ge t^{-\frac{1+\beta}{p-1}} V(0) = c_{11} t^{-\frac{1+\beta}{p-1}}. \tag{5.3}$$

If $x \in \mathcal{S}^c$, there exists $\bar{x} \in \mathcal{S}$ such that $\operatorname{dist}(x, \mathcal{S}) = |x - \bar{x}|$. It follows from [21, Theorem 1.2] that,

$$u(t,x) \geq \frac{c_{10} t^{-\frac{1+\beta}{p-1}}}{1 + (t^{-\frac{1}{2\alpha}} \operatorname{dist}(x, \mathcal{S}))^{N+2\alpha}}. \tag{5.4}$$

Then (5.1) holds. □

The next result shows that any closed set can be the singular set of the initial trace of a positive solution of (1.24).

Proposition 5.1 *Assume $1 + \frac{2\alpha(1+\beta)}{1+2\alpha} < p < 1 + \frac{2\alpha(1+\beta)}{N}$ and $\mathcal{S} \subset \mathbb{R}^N$ is a nonempty closed set. Then there exists a minimal solution $u := u_{\mathcal{S},\infty}$ with initial trace $(\mathcal{S}, 0)$. Furthermore it satisfies (4.35).*

Proof. We first notice that condition $1 + \frac{2\alpha(1+\beta)}{1+2\alpha} < p < 1 + \frac{2\alpha(1+\beta)}{N}$ is equivalent to the conditions stated in Theorem I, i.e.

$$\begin{aligned} &(i) \qquad \text{either } N = 1 \text{ and } 1 + \tfrac{2\alpha(1+\beta)}{1+2\alpha} < p < 1 + 2\alpha(1+\beta), \\ &(ii) \qquad \text{or } N = 2, \tfrac{1}{2} \leq \alpha < 1 \text{ and } 1 + \tfrac{2\alpha(1+\beta)}{1+2\alpha} < p < 1 + \alpha(1+\beta). \end{aligned} \tag{5.5}$$

Let $A := \{z_n\}_{n \in \mathbb{N}}\}$ be a countable dense subset of $\mathcal{S}$. For $k \in \mathbb{N}_*$, set

$$\mu_k = k \sum_{j=1}^{k} \delta_{z_j}, \tag{5.6}$$

and let $u = u_{\mu_k}$ be the solution of

$$\begin{aligned} \partial_t u + (-\Delta)^\alpha u + t^\beta u^p = 0 \qquad &\text{in} \quad Q_\infty \\ u(0,.) = \mu_k \qquad &\text{in} \quad \mathbb{R}^N. \end{aligned} \tag{5.7}$$

The sequence $\{u_{\mu_k}\}$ is increasing. If $a \in \mathcal{S}^c$ and $d_a = \operatorname{dist}(a, \mathcal{S})$. By construction there holds

$$u_{\mu_k} \leq u_{B^c_{d_a}(a),\infty}. \tag{5.8}$$

Hence u_{μ_k} converges to some solution $\tilde{u}$ of (1.24) in Q_∞ which has zero initial trace on $B_{d_a}(a)$, for any $a \in \mathcal{S}^c$ since (5.8) still holds with $\tilde{u}$ instead of u_{μ_k}, and satisfies $\tilde{u} \geq u_{z_j,\infty}$ for any $z_j \in A$. Hence $Tr(\tilde{u}) = (\mathcal{S}, 0)$. Estimate (4.35) is independent of the geometry of $\mathcal{S}$. □

Proof of Theorem I. It is similar to the one of Theorem G. We set $\nu_k = \nu + \mu_k$ where μ_k is defined by (5.6). Then the solution of (1.24) with initial data ν_k satisfies

$$\sup\{u_\nu, u_{\mu_k}\} \leq u_{\nu_k} \leq u_\nu + u_{\mu_k}. \tag{5.9}$$

The sequence $\{u_{\nu_k}\}$ is increasing and converges to some solution $\tilde{u}$ of (1.24) which satisfies

$$\sup\{u_\nu, u_{\infty,\mathcal{S}}\} \leq \tilde{u} \leq u_\nu + u_{\infty,\mathcal{S}}. \tag{5.10}$$

Then $\tilde{u}$ has initial trace $(\mathcal{S}, \nu)$. □

Remark. We conjecture that the following more general version of Theorem I holds: *For any integer $\kappa \in [1, N]$ any $p > 1$ such that $1 + \frac{2\alpha(1+\beta)}{\kappa+2\alpha} < p < 1 + \frac{2\alpha(1+\beta)}{N}$, any closed set $\mathcal{S}$ contained in an affine plane of codimension κ and any bounded measure in $\mathcal{S}^c$, there exists a solution u of problem (1.30).* We notice that the condition on p can be fulfilled for some p if and only if $N - \kappa < 2\alpha$, hence either $\kappa = N$ i.e. $\mathcal{S}$ is a single point and no condition on α, or $\kappa = N - 1$ hence $\mathcal{S}$ is contained in a straight line and $\frac{1}{2} < \alpha < 1$.

6 Appendix: symmetry and monotonicity results

The following is a variant of the maximum principle which will be used in the sequel.

Lemma 6.1 *Let $R, T > 0$, $\delta \in [0, T)$ and Q be a domain of Q_∞ such that $\overline{Q} \subset (\delta, T) \times B_R$. Assume that $h \geq 0$ in Q and $\psi \in C(\bar{Q})$ satisfies*

$$\begin{aligned} \partial_t \psi + (-\Delta)^\alpha \psi + h(t,x)\psi \geq 0 \quad &\text{in} \quad Q, \\ \psi \geq 0 \quad &\text{in} \quad ([\delta, T) \times B_R) \setminus Q. \end{aligned} \tag{6.1}$$

Then ψ is nonnegative in $[\delta, T) \times B_R$.

Proof. Let $\epsilon \in (0, T - \delta]$. We first claim that ψ is nonnegative in $[\delta, T - \epsilon] \times B_R$. If it does not hold, and since $\psi \geq 0$ in $([\delta, T) \times B_R) \setminus Q$, there exists $(t_0, x_0) \in Q \cap ([\delta, T - \epsilon] \times B_R)$ such that

$$\psi(t_0, x_0) = \min_{(t,x) \in [\delta, T-\epsilon] \times B_R} \psi(t, x) < 0.$$

Then $\partial_t \psi(t_0, x_0) \leq 0$ and $(-\Delta)^\alpha \psi(t_0, x_0) < 0$. Since $h \geq 0$ in Q and $(t_0, x_0) \in Q$, there holds

$$\partial_t \psi(t_0, x_0) + (-\Delta)^\alpha \psi(t_0, x_0) + h(t_0, x_0)\psi(t_0, x_0) < 0,$$

contradiction. Thus, ψ is nonnegative in $[\delta, T - \epsilon] \times B_R$. Since ϵ is arbitrary, the result follows. Notice that we can take $R = \infty$ in the above proof provided Q is a bounded domain.

□

Next we prove the following result.

Proposition 6.1 *Let $N \geq 1$, $\beta > -1$, $p > 1$ and $g \in C(\mathbb{R}^N)$ is a nonnegative continuous radially symmetric and nonincreasing function which tends to 0 when $|x| \to \infty$. If $u \in L^1_{loc}(0, \infty; \mathcal{L}^\alpha(\mathbb{R}^N) \cap C(\overline{Q}_\infty)$ is a nonnegative solution of (1.24) in Q_∞ which converges to g uniformly when $t \to 0$, then u is radially symmetric and nonincreasing.*

Proof. Since $u \in L^1_{loc}\left(0, \infty; \mathcal{L}^\alpha(\mathbb{R}^N)\right) \cap C(\overline{Q}_\infty)$, it is dominated by $\mathbb{H}^\alpha[g]$ and uniqueness holds as for the linear equation [9]. Since the initial data is radially symmetric and the equation is invariant by rotations in $\mathbb{R}^N$, $u(t, .)$ is also radially symmetric. Because of uniqueness and stability, it is sufficient to prove the result for a function u which initial data is obtained from the previous one by multiplying it by a smooth, even, nonincreasing and nonnegative function

with compact support. The corresponding solution of (1.24) in Q_∞ still denoted by u, is smooth in Q_∞ and bounded from above by $\mathbb{H}^\alpha[g]$. Hence it satisfies

$$\begin{aligned} &(i) && \lim_{t\to\infty} u_(t,x) = 0 && \text{uniformly in } x \in \mathbb{R}^N \\ &(ii) && \lim_{|x|\to\infty} u_(t,x) = 0 && \text{uniformly in } t \in \mathbb{R}_+ \\ &(iii) && \lim_{t\to 0} u_(t,x) = g(x) && \text{uniformly in } x \in \mathbb{R}^N. \end{aligned} \tag{6.2}$$

For $\lambda \in \mathbb{R}$, we set $x_\lambda = (2\lambda - x_1, x')$ if $x = (x_1, x') \in \mathbb{R}^N$,

$$\Sigma_\lambda = \{x = (x_1, x') \in \mathbb{R}^N \mid x_1 < \lambda\} \tag{6.3}$$

and

$$T_\lambda = \{x = (x_1, x') \in \mathbb{R}^N \mid x_1 = \lambda\}.$$

We observe that if $\lambda > 0$, then $\{x_\lambda \mid x \in \Sigma_\lambda\} = \{x \in \mathbb{R}^N | \ x_1 > \lambda\}$ and

$$|x_\lambda| > |x| \quad \text{ for } x \in \Sigma_\lambda. \tag{6.4}$$

We claim that for any $\lambda > 0$

$$u(t,x) \geq u(t,x_\lambda) \quad \forall (t,x) \in \mathbb{R}_+^* \times \Sigma_\lambda. \tag{6.5}$$

We set $\varphi(t,x) = u(t,x) - u(t,x_\lambda)$ and suppose that (6.5) does not hold. Because of (6.2) there holds $\lim\limits_{|x|\to\infty} \varphi(t,x) = 0$ uniformly with respect to $t \geq 0$, $\lim\limits_{t\to\infty} \varphi(t,x) = 0$ uniformly with respect to $x \in \mathbb{R}^N$ and $\lim\limits_{t\to 0} \varphi(t,x) = g(x) - g(x_\lambda) \geq 0$ uniformly with respect to $x \in \mathbb{R}^N$. It follows that there exists $\varepsilon_0 > 0$ and $(t_0, x_0) \in \mathbb{R}_+^* \times \Sigma_\lambda$ such that

$$\varphi(t_0, x_0) = \min_{(x,t)\in\overline{\Sigma}_\lambda} \varphi(t,x) = -\varepsilon_0 < 0. \tag{6.6}$$

The function ϕ satisfies

$$\partial_t \phi + (-\Delta)^\alpha \phi + h(t,x)\phi = 0 \qquad \text{in } \ Q_\infty, \tag{6.7}$$

for some $h(t,x) \geq 0$, and it has initial data $\phi(0,x) = g(x) - g(x_\lambda)$ in $\mathbb{R}^N$. Take $\epsilon \in (0, \epsilon_0)$ and set $\phi_\epsilon = \phi + \epsilon$. Using (6.2) we see that there exists $T_0 > t_0 > 0$ and $R_0 > |x_0| > 0$ such that $\phi_\epsilon(x,t) \geq 0$ for $(t,x) \in \big([T,\infty) \times \mathbb{R}^N\big) \bigcup ([0,\infty) \times B_R^c)$, for all $T \geq T_0$ and $R \geq R_0$. Furthermore there exists $\delta_0 \in (0, t_0)$ such that for any $\delta \in (0, \delta_0)$ such that $\phi_\epsilon(x,t) \geq 0$ for $(t,x) \in [0,\delta) \times \big(\mathbb{R}^N \cap \Sigma_\lambda\big)$. We set

$$Q = \Sigma_\lambda \cap (\delta, T_0) \times B_{R_0}.$$

We apply Lemma 6.1 in $[\frac{\delta}{2}, T) \times B_R$ and conclude that $\phi_\epsilon \geq 0$ in $[\frac{\delta}{2}, T) \times B_R$, which contradicts the fact that $\phi_\epsilon(t_0, x_0) = \epsilon - \epsilon_0 < 0$. Hence (6.5) holds. Since $\lambda > 0$ is arbitrary, this implies in particular by continuity that

$$\frac{\partial u}{\partial x_1}(t, x_1, x') \leq 0 \qquad \forall (t, x_1, x') \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^{N-1}. \tag{6.8}$$

Similarly

$$\frac{\partial u}{\partial x_1}(t, x_1, x') \geq 0 \qquad \forall (t, x_1, x') \in \mathbb{R}_+ \times \mathbb{R}_- \times \mathbb{R}^{N-1}. \tag{6.9}$$

Since $u(t,x)$ is radially symmetric with respect to x, it implies that $u(t,x) \geq u(t,x')$ if $|x| \leq |x'|$, which ends the proof. □

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