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HÖLDER STABILITY FOR AN INVERSE MEDIUM PROBLEM WITH INTERNAL DATA

MOURAD CHOULLI† AND FAOUZI TRIKI‡

ABSTRACT. We are interested in an inverse medium problem with internal data. This problem is originated from multi-waves imaging. We aim in the present work to study the well-posedness of the inversion in terms of the boundary conditions. We precisely show that we have actually a stability estimate of Hölder type. For sake of simplicity, we limited our study to the class of Helmholtz equations $\Delta + V$ with bounded potential V .

Mathematics subject classification : 35R30.

Key words : Helmholtz equation, inverse medium problem, internal data, Hölder stability, unique continuation.

1. INTRODUCTION

Let Ω be a C^2 -smooth bounded domain of \mathbb{R}^n , $n = 2, 3$, with boundary Γ . We denote the geodesic distance on Ω associated to the round metric by d_g . That is

$$d_g(x, y) = \inf \left\{ \int_0^1 |\dot{\psi}(t)| dt; \psi : [0, 1] \rightarrow \Omega \text{ is a piecewise } C^1 \text{ path joining } x \text{ to } y \right\}.$$

Denote by \mathbf{d}_g the diameter of Ω with respect to the distance d_g :

$$\mathbf{d}_g = \sup \{d_g(x, y); x, y \in \Omega\}.$$

Set

$$\mathcal{D} = \{V \in L^\infty(\Omega); 0 \text{ is not an eigenvalue of } A_V\},$$

where $A_V : L^2(\Omega) \rightarrow L^2(\Omega)$ is the unbounded operator defined by

$$A_V = -\Delta - V \text{ with } D(A_V) = H^2(\Omega) \cap H_0^1(\Omega).$$

Pick $0 < k < 1$ and $\bar{V} \in \mathcal{D}$ so that $2v_0 \leq \bar{V}$, where $v_0 > 0$ is given, and let $\mathcal{D}_0(k, v_0, \bar{V})$ be the subset of \mathcal{D} of those functions $V \in L^\infty(\Omega)$ satisfying

$$\|V - \bar{V}\|_{L^\infty(\Omega)} \leq \min \left(\frac{k}{\|A_{\bar{V}}^{-1}\|_{\mathcal{B}(L^2(\Omega))}}, v_0 \right).$$

Fix, for some $p > n$, $h \in W^{2-\frac{1}{p}, p}(\Gamma)$ non identically equal to zero and denote by u_V the solution of the following BVP for the Helmholtz equation:

$$\Delta u + Vu = 0 \text{ in } \Omega \text{ and } u = h \text{ on } \Gamma.$$

According to [14, Theorem 3.1, page 1782], $u_V \in W^{2,p}(\Omega)$ and the following estimate holds

$$(1.1) \quad \|u\|_{W^{2,p}(\Omega)} \leq M = M(\Omega, p, v_0, k, \bar{V}, h), \quad V \in \mathcal{D}_0(k, v_0, \bar{V}).$$

We are mainly interested in determining the absorption coefficient V from the internal data

$$I_V = Vu_V^2.$$

This inverse problem is originated from multi-waves imaging. The term multi-waves refers to the fact that two types of physical waves are used to probe the medium under study. Usually, the first wave is sensitive

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to the contrast of the desired parameter, the other types can carry the information revealed by the first type of waves to the boundary of the medium where measurements can be taken. In this paper we assume that the first inversion has been performed, that is the internal data I_V is retrieved, and we focus on the second step. We refer to [4, 5, 7, 8, 9, 10, 14] and reference therein for further details.

Choose $K > 0$ sufficiently large in such a way that

$$\mathcal{D}_1(k, v_0, \bar{V}, K) = \mathcal{D}_0(k, v_0, \bar{V}) \cap \{V \in C^{0,1}(\bar{\Omega}); \|V\|_{C^{0,1}(\bar{\Omega})} \leq K\} \neq \emptyset.$$

Note that, according to Rademacher's theorem, $C^{0,1}(\bar{\Omega})$ is continuously embedded in $W^{1,\infty}(\Omega)$.

For the inverse problem under consideration we are going to prove various Hölder stability estimates.

Theorem 1.1. (*interior stability*) *Assume that $\mathbf{d}_g < \infty$ and let $\omega \Subset \Omega$. Then, there exist two constants $C = C(\Omega, p, v_0, k, \bar{V}, h, K, \omega) > 0$ and $\mu = \mu(\Omega, p, v_0, k, \bar{V}, h, K, \omega)$ so that, for any $V, \tilde{V} \in \mathcal{D}_1(k, v_0, \bar{V}, K)$ satisfying $V = \tilde{V}$ on Γ , we have*

$$\|V - \tilde{V}\|_{L^\infty(\omega)} \leq C \left\| \sqrt{I_V} - \sqrt{I_{\tilde{V}}} \right\|_{H^1(\Omega)}^\mu.$$

A similar result was already proved by G. Alessandrini in [1] under different assumptions.

When h does not vanish on a part of Γ , we obtain improved results which we state in the following theorems.

Theorem 1.2. *Under the assumptions $\mathbf{d}_g < \infty$ and $|h| > \kappa > 0$ on Γ , there exist two constants $C = C(\Omega, p, v_0, k, \bar{V}, h, K) > 0$ and $\mu = \mu(\Omega, p, v_0, k, \bar{V}, h, K)$ so that, for any $V, \tilde{V} \in \mathcal{D}_1(k, v_0, \bar{V}, K)$ satisfying $V = \tilde{V}$ on Γ , we have*

$$\|V - \tilde{V}\|_{L^\infty(\Omega)} \leq C \left\| \sqrt{I_V} - \sqrt{I_{\tilde{V}}} \right\|_{H^1(\Omega)}^\mu.$$

Theorem 1.3. *Let $\tilde{\Omega} \subset \Omega$ be a C^2 -smooth domain with boundary $\tilde{\Gamma}$ such that $\tilde{\gamma} := \tilde{\Gamma} \cap \Gamma$ satisfies $\tilde{\gamma} \neq \emptyset$, and let $\omega \Subset \tilde{\Omega}$. Under the assumptions $\mathbf{d}_g < \infty$ and $|h| > \kappa > 0$ on Γ , there exist two constants $C = C(\Omega, p, v_0, k, \bar{V}, h, K, \omega, \tilde{\Omega}) > 0$ and $\mu = \mu(\Omega, p, v_0, k, \bar{V}, h, K, \omega, \tilde{\Omega})$ so that, for any $V, \tilde{V} \in \mathcal{D}_1(k, v_0, \bar{V}, K)$ satisfying $V = \tilde{V}$ on Γ and $\nabla V = \nabla \tilde{V}$ on $\tilde{\gamma}$, we have*

$$\|V - \tilde{V}\|_{L^\infty(\omega)} \leq C \left\| \sqrt{I_V} - \sqrt{I_{\tilde{V}}} \right\|_{H^1(\tilde{\Omega})}^\mu.$$

In the following result we allow the Dirichlet boundary condition h to vanish on Γ . In the sequel denote

$$\Gamma_+ = \{x \in \Gamma; |h(x)| > 0\} \text{ and } \Gamma_0 = \{x \in \Gamma; h(x) = 0\}.$$

Theorem 1.4. *Assume that $\mathbf{d}_g < \infty$. Let γ be a compact subset of $\Gamma_+ \cup \mathring{\Gamma}_0$ and $\omega \Subset \Omega \cup \gamma$. Then, there exist two constants $C = C(\Omega, p, v_0, k, \bar{V}, h, K, \omega) > 0$ and $\mu = \mu(\Omega, p, v_0, k, \bar{V}, h, K, \omega)$ so that, for any $V, \tilde{V} \in \mathcal{D}_1(k, v_0, \bar{V}, K)$ satisfying $V = \tilde{V}$ on Γ , we have*

$$\|V - \tilde{V}\|_{L^\infty(\omega)} \leq C \left\| \sqrt{I_V} - \sqrt{I_{\tilde{V}}} \right\|_{H^1(\Omega)}^\mu.$$

We deduce from Theorem 1.3 that it is possible to recover the potential V on a small set ω of Ω if the medium is probed starting from $\tilde{\gamma}$ the part of the boundary where h is intense, and by covering a neighboring region $\tilde{\Omega}$ that contains ω . Unfortunately h may in general settings be zero on some parts of the boundary Γ , and Theorem 1.4 is an attempt to improve the results of Theorems 1.2 and 1.3 in this direction.

In [2], the authors were able to prove a lower bound for the gradient of solutions near the boundary when the boundary data is ‘‘qualitatively unimodal’’ (see [2] for the definition). Roughly speaking, the key in their proof is that even if the tangential gradient of solutions vanishes, there is, according to Hopf's maximum principle, a non zero contribution of the derivative of solutions in the normal direction. Unfortunately, there is no similar arguments that can be used in order to get lower bound for solutions on the boundary.

2. PROOF OF THE MAIN RESULT

We need several intermediate results before proving Theorem 1.1.

If \mathbf{d} is the diameter of Ω with respect to the euclidean metric, define, where $0 < \delta < \mathbf{d}$,

$$\Omega^\delta = \{x \in \Omega; \text{dist}(x, \Gamma) \geq \delta\} \text{ and } \Omega_\delta = \{x \in \Omega; \text{dist}(x, \Gamma) \leq \delta\}.$$

For $0 < v_0 \leq V_0$ and $M > 0$, let $\mathcal{V}(v_0, V_0) = \{V \in L^\infty(\Omega); v_0 \leq V \leq V_0\}$,

$$\begin{aligned} \mathcal{S}_w(v_0, V_0, \kappa, M) = \{u \in H^2(\Omega); \Delta u + Vu = 0, \text{ for some } V \in \mathcal{V}(v_0, V_0) \\ \text{and } \|u\|_{H^2(\Omega)} \leq M, \|u\|_{L^\infty(\Gamma)} \geq \kappa\}. \end{aligned}$$

and

$$\begin{aligned} \mathcal{S}_s(v_0, V_0, \kappa, M) = \{u \in H^2(\Omega); \Delta u + Vu = 0, \text{ for some } V \in \mathcal{V}(v_0, V_0) \\ \text{and } \|u\|_{H^2(\Omega)} \leq M, |u| \geq \kappa \text{ on } \Gamma\}. \end{aligned}$$

The main ingredient in our approach is the following theorem. Its proof, which is quite technical, is given in Appendix A.

Theorem 2.1. *There exists a constants $c = c(\Omega, v_0, V_0, \kappa, M) > 0$ so that, for any $x_0 \in \Omega^\delta$ and $u \in \mathcal{S}_w(v_0, V_0, \kappa, M)$, we have*

$$(2.1) \quad e^{-e^{\frac{c}{\delta}}} \leq \|u\|_{L^2(B(x_0, \delta))}.$$

Let $V \in \mathcal{V}(v_0, V_0)$, $0 < \delta < \mathbf{d}$ and $u \in H^2(\Omega)$ so that $\Delta u + Vu = 0$ in Ω . We define, for $x_0 \in \Omega^\delta$ and $0 < r < \delta$,

$$\begin{aligned} H(x_0, r) &= H_u(x_0, r) = \int_{|x-x_0|=r} u^2(x) dS(x), \\ D(x_0, r) &= D_u(x_0, r) = \int_{|x-x_0|<r} \{|\nabla u(x)|^2 + V(x)u^2(x)\} dx, \\ K(x_0, r) &= K_u(x_0, r) = \int_{|x-x_0|<r} u^2(x) dS(x). \end{aligned}$$

Since A_V satisfies the unique continuation property, we have $H(x_0, r) > 0$ for all $x_0 \in \Omega^\delta$ and $0 < r < \delta$. Define the frequency function N by

$$N(x_0, r) = N_u(x_0, r) = \frac{rD(x_0, r)}{H(x_0, r)}.$$

The following two lemmas can be deduced from the calculations developed in [16, 17].

Lemma 2.1. *We have*

$$K(x_0, r) \leq rH(x_0, r), \quad x_0 \in \Omega^\delta, \quad 0 < r < \delta_0 = \min(r_0, \delta),$$

where $r_0 = [(n-1)V_0^{-1}]^{\frac{1}{2}}$.

Lemma 2.2. *For $x_0 \in \Omega^\delta$ and $0 < r < \delta_0$,*

$$N(x_0, r) \leq C \max(N(x_0, \delta_0), 1).$$

Here $C = C(\Omega, V_0) > 0$ is a constant.

Fix $0 < \alpha \leq 1$. We say that $W \subset L^1_+(\Omega) = \{w \in L^1(\Omega); w \geq 0\}$ is a uniform set of weights for the weighted interpolation inequality

$$(2.2) \quad \|f\|_{L^\infty(\Omega)} \leq C \|f\|_{C^{0,\alpha}(\overline{\Omega})}^{1-\mu} \|fw\|_{L^1(\Omega)}^\mu,$$

if the constants $C > 0$ and $0 < \mu < 1$ in (2.2) can be chosen independently in $w \in W$ and $f \in C^{0,\alpha}(\overline{\Omega})$.

Similarly, we will say that $W \subset L^1_+(\Omega)$ is a uniform set of interior weights for the interior weighted interpolation inequality, where $\omega \Subset \Omega$ is arbitrary,

$$(2.3) \quad \|f\|_{L^\infty(\omega)} \leq C \|f\|_{C^{0,\alpha}(\overline{\Omega})}^{1-\mu} \|fw\|_{L^1(\Omega)}^\mu$$

if the constants $C > 0$ and $0 < \mu < 1$ in (2.3), depending on ω , can be chosen independently in $w \in W$ and $f \in C^{0,\alpha}(\overline{\Omega})$.

Theorem 2.2. 1) *The set*

$$\mathcal{W}_w(v_0, V_0, \kappa, M) = \{w = u^2; u \in \mathcal{S}_w(v_0, V_0, \kappa, M)\}$$

is a uniform set of interior weights for the interior weighted interpolation inequality (2.3).

2) *The set*

$$\mathcal{W}_s(v_0, V_0, \kappa, M) = \{w = u^2; u \in \mathcal{S}_s(v_0, V_0, \kappa, M)\}$$

is a uniform set of weights for the weighted interpolation inequality (2.2).

Before proving this theorem, we establish some preliminaries.

Lemma 2.3. *Let $0 < \delta < \mathbf{d}$. There exists a constant $C = C(\Omega, v_0, V_0, \kappa, M, \delta) > 0$ so that, for any $u \in \mathcal{S}_w(v_0, V_0, \kappa, M)$, we have*

$$\|N\|_{L^\infty(\Omega^\delta \times (0, \delta_0))} \leq C.$$

Here $N = N_u$ is the frequency function associated to u and δ_0 is as in Lemma 2.1.

Proof. Let $x_0 \in \Omega^\delta$. From Theorem 2.1

$$(2.4) \quad K(x_0, \delta_0) = \|u\|_{L^2(B(x_0, \delta_0))^n}^2 \geq C.$$

Combined with Lemma 2.1, this estimates yields

$$(2.5) \quad H(x_0, \delta_0) \geq C.$$

In light of Lemma 2.2 we end up getting

$$N(x_0, r) \leq C, \quad 0 < r < \delta_0,$$

which leads immediately to the expected inequality. \square

Proposition 2.1. *Let $0 < \delta < \mathbf{d}$. There exist two constants $C = C(\Omega, v_0, V_0, \kappa, M, \delta) > 0$ and $c = c(\Omega, v_0, V_0, \kappa, M, \delta) > 0$ so that, for any $u \in \mathcal{S}_w(v_0, V_0, \kappa, M)$, we have*

$$Cr^c \leq \|u\|_{L^2(B(x_0, r))}, \quad x_0 \in \Omega^\delta, \quad 0 < r < \delta_0,$$

where δ_0 is as in Lemma 2.1.

Proof. We set for simplicity $H = H_u$. From the calculations carried out in [16, 17], we have

$$\partial_r H(x_0, r) = \frac{n-1}{r} H(x_0, r) + 2D(x_0, r).$$

Whence

$$\partial_r \left(\ln \frac{H(x_0, r)}{r^{n-1}} \right) = \frac{\partial_r H(x_0, r)}{H(x_0, r)} - \frac{n-1}{r} = \frac{2N(x_0, r)}{r}.$$

This and Lemma 2.3 entail

$$\partial_r \left(\ln \frac{H(x_0, r)}{r^{n-1}} \right) \leq \frac{C}{r}, \quad 0 < r < \delta_0.$$

Thus

$$\int_{sr}^{s\delta_0} \partial_t \left(\ln \frac{H(x_0, t)}{t^{n-1}} \right) dt = \ln \frac{H(x_0, s\delta_0)r^{n-1}}{H(x_0, sr)\delta_0^{n-1}} \leq \ln \frac{\delta_0^C}{r^C}, \quad 0 < s < 1, \quad 0 < r < \delta_0.$$

Hence

$$H(x_0, s\delta_0) \leq \frac{C}{r^c} H(x_0, sr)$$

and then

$$\|u\|_{L^2(B(x_0, \delta_0))}^2 = \delta_0^{n-1} \int_0^1 H(x_0, s\delta_0) s^{n-1} ds \leq \frac{C}{r^c} r^{n-1} \int_0^1 H(x_0, sr) s^{n-1} ds = \frac{C}{r^c} \|u\|_{L^2(B(x_0, r))}^2.$$

Combined with (2.4), this estimate yields

$$Cr^c \leq \|u\|_{L^2(B(x_0, r))}.$$

The proof is then complete \square

Proof of Theorem 2.2. 1) Let $w \in \mathcal{W}_w(v_0, V_0, \kappa, M)$ and $u \in \mathcal{S}_w(v_0, V_0, \kappa, M)$ so that $w = u^2$. Fix $\omega \Subset \Omega$. We need to prove that (2.3) holds with constants C and μ that are independent of w and $f \in C^{0,\alpha}(\bar{\Omega})$. By homogeneity it is enough to establish (2.3) when $\|f\|_{C^{0,\alpha}(\bar{\Omega})} = 1$. To this end, pick $f \in C^{0,\alpha}(\bar{\Omega})$ satisfying $\|f\|_{C^{0,\alpha}(\bar{\Omega})} = 1$.

Pick $x_0 \in \bar{\omega}$ so that $|f(x_0)| = \|f\|_{L^\infty(\omega)}$. According to Proposition 2.1, there exist $\delta > 0$ and two constants $C = C(\Omega, v_0, V_0, \kappa, M, \delta) > 0$ and $c = c(\Omega, v_0, V_0, \kappa, M, \delta) > 0$ so that, for any $u \in \mathcal{S}_w(v_0, V_0, \kappa, M)$, we have

$$(2.6) \quad Cr^c \leq \|u\|_{L^2(B(x_0, r))}, \quad 0 < r < \delta_0 = \min(\delta, r_0),$$

where r_0 is as in Lemma 2.1.

But

$$|f(x_0)| = \|f\|_{L^\infty(\omega)} \leq |f(x)| + r^\alpha, \quad \text{for any } x \in B(x_0, r).$$

Therefore

$$\begin{aligned} \|f\|_{L^\infty(\omega)} \int_{B(x_0, r)} u(x)^2 dx &\leq 2 \int_{B(x_0, r)} |f(x)| u(x)^2 dx + 2r^\alpha \int_{B(x_0, r)} u(x)^2 dx \\ &\leq 2 \int_{\omega} |f(x)| u(x)^2 dx + 2r^\alpha \int_{B(x_0, r)} u(x)^2 dx. \end{aligned}$$

Note that, according to the unique continuation property,

$$\int_{B(x_0, r)} u(x)^2 dx \neq 0.$$

Hence

$$\|f\|_{L^\infty(\omega)} \leq 2 \frac{\|fu^2\|_{L^1(\Omega)}}{\|u\|_{L^2(B(x_0, r))}^2} + 2r^\alpha.$$

Combined with (2.6), this estimate yields

$$(2.7) \quad \|f\|_{L^\infty(\omega)} \leq C(\|fu^2\|_{L^1(\Omega)} r^{-c} + r^\alpha), \quad 0 < r < \delta_0.$$

When $\|fu^2\|_{L^1(\Omega)} < \delta_0^{c+\alpha}$, we can take $r = \|fu^2\|_{L^1(\Omega)}^{1/(c+\alpha)}$ in (2.7) in order to get

$$(2.8) \quad \|f\|_{L^\infty(\omega)} \leq C \|fu^2\|_{L^1(\Omega)}^\mu,$$

with $\mu = \frac{\alpha}{c+\alpha}$.

If $\|fu^2\|_{L^1(\Omega)} \geq \delta_0^{c+\alpha}$, we have

$$(2.9) \quad \|f\|_{L^\infty(\omega)} \leq \delta_0^{-(c+\alpha)} \|fu^2\|_{L^1(\Omega)} \leq \delta_0^{-(c+\alpha)} M^{2-2\mu} \|fu^2\|_{L^1(\Omega)}^\mu.$$

The expected inequality follows then from (2.8) and (2.9).

2) Let $w \in \mathcal{W}_s(v_0, V_0, \kappa, M)$ and $u \in \mathcal{S}_s(v_0, V_0, \kappa, M)$ so that $w = u^2$. As in 1) we have to prove that (2.2) holds with constants C and μ independent on w and $f \in C^{0,\alpha}(\bar{\Omega})$. As we have seen before, by homogeneity it is enough to establish (2.2) when $\|f\|_{C^{0,\alpha}(\bar{\Omega})} = 1$. Let then $f \in C^{0,\alpha}(\bar{\Omega})$ satisfying $\|f\|_{C^{0,\alpha}(\bar{\Omega})} = 1$.

Since $H^2(\Omega)$ is continuously embedded in $C^{0,1/2}(\bar{\Omega})$, there exists a constant $c = c(\Omega) > 0$ so that

$$[u]_{1/2} = \sup \left\{ \frac{|u(x) - u(y)|}{|x - y|^{1/2}}; x, y \in \bar{\Omega}, x \neq y \right\} \leq c \|u\|_{H^2(\Omega)} \leq cM.$$

Fix $\delta_1 \leq \left(\frac{\kappa}{2cM}\right)^2$. Then a straightforward computation leads

$$|u| \geq \kappa/2 \text{ in } \Omega_{\delta_1}.$$

From the Hölder continuity of f , we get, where $\eta = \delta_1/4$,

$$\|f\|_{L^\infty(\Omega_\eta)} = |f(x_0)| \leq |f(x)| + r^\alpha, \quad x \in B(x_0, r), \quad 0 < r < \delta_0 = \delta_0(\eta).$$

Whence, proceeding as in 1), we get

$$\|f\|_{L^\infty(\Omega_\eta)} \leq \frac{\|fu^2\|_{L^1(\Omega)}}{\|u\|_{L^2(B(x_0, r))}^2} + r^\alpha.$$

This and Proposition 2.1 yield

$$(2.10) \quad \|f\|_{L^\infty(\Omega_\eta)} \leq C(\|fu^2\|_{L^1(\Omega)}r^{-c} + r^\alpha), \quad 0 < r < \delta_0.$$

On the other hand, noting that $B(y, r) \subset \Omega_{\delta_1}$ when $y \in \Omega_{2\eta}$ and $0 < r < \eta$, we have

$$\begin{aligned} \|f\|_{L^\infty(\Omega_{2\eta})} \int_{B(y_0, r)} u(x)^2 dx &= |f(y_0)| \int_{B(y_0, r)} u(x)^2 dx \\ &\leq \int_{B(y_0, r)} |f(x)|u(x)^2 dx + r^\alpha \int_{B(y_0, r)} u(x)^2 dx. \end{aligned}$$

But, as

$$\int_{B(y_0, r)} |f(x)|u(x)^2 dx \geq (\kappa^2 r^n |B(0, 1)|)/2,$$

we see that an estimate of the form (2.10) holds with $\|f\|_{L^\infty(\Omega_\eta)}$ substituted by $\|f\|_{L^\infty(\Omega_{2\eta})}$. Consequently,

$$\|f\|_{L^\infty(\Omega)} \leq C(\|fu^2\|_{L^1(\Omega)}r^{-c} + r^\alpha), \quad 0 < r < \bar{\delta} = \min(\eta, \delta_0).$$

We can then mimic the end of the proof of 1) in order to obtain the expected inequality. \square

Proof of Theorem 1.1. Pick $\omega \Subset \Omega$. We firstly observe that, according to (1.1),

$$\{u_V; V \in \mathcal{D}_0(k, v_0, \bar{V})\} \subset \mathcal{S}_\omega(v_0, V_0, \kappa, M),$$

with $\kappa = \|h\|_{L^\infty(\Gamma)}$, $V_0 = v_0 + \|\bar{V}\|_{L^\infty(\Omega)}$ and $M = M(\Omega, p, v_0, k, \bar{V}, h)$ is as in (1.1). We can then apply 1) of Theorem 2.2 in order to obtain

$$(2.11) \quad \|V - \tilde{V}\|_{L^\infty(\omega)} \leq C\|(V - \tilde{V})u_V^2\|_{L^1(\Omega)}^{2\mu},$$

with $C = C(\Omega, p, v_0, k, \bar{V}, h, \omega)$ and $\mu = \mu(\Omega, p, v_0, k, \bar{V}, h, \omega)$.

On the other hand, we have from [14, Theorem 2.2, page 1781]

$$(2.12) \quad \|\sqrt{I_V}(V - \tilde{V})\|_{H^1(\Omega)} \leq C\|\sqrt{I_V} - \sqrt{I_{\tilde{V}}}\|_{H^1(\Omega)}^{1/2}.$$

Set for simplicity $u = u_V$ (resp. $I = I_V$) and $\tilde{u} = u_{\tilde{V}}$ (resp. $\tilde{I} = I_{\tilde{V}}$). Then

$$\begin{aligned} (V - \tilde{V})u^2 &= Vu^2 - \tilde{V}\tilde{u}^2 + \tilde{V}(u^2 - \tilde{u}^2) \\ &= I - \tilde{I} + \tilde{V}(|u| + |\tilde{u}|)(|u| - |\tilde{u}|). \end{aligned}$$

Hence

$$(2.13) \quad \|(V - \tilde{V})u^2\|_{L^1(\Omega)} \leq C\left(\|\sqrt{I} - \sqrt{\tilde{I}}\|_{L^1(\Omega)} + \| |u| - |\tilde{u}| \|_{L^1(\Omega)}\right).$$

But

$$|\tilde{u}| - |u| = \frac{1}{\sqrt{\tilde{V}}} (I - \tilde{I}) + \frac{\sqrt{I}}{V\sqrt{\tilde{V}}} \left[(V - \tilde{V})\sqrt{I} \right]$$

implying

$$(2.14) \quad \|(V - \tilde{V})u^2\|_{L^1(\Omega)} \leq C\left(\|\sqrt{I} - \sqrt{\tilde{I}}\|_{L^1(\Omega)} + \|(V - \tilde{V})\sqrt{I}\|_{L^1(\Omega)}\right).$$

Now a combination of (2.12), (2.13) and (2.14) yields

$$\|V - \tilde{V}\|_{L^\infty(\omega)} \leq C \|\sqrt{I_V} - \sqrt{I_{\tilde{V}}}\|_{H^1(\Omega)}^\mu,$$

which is the expected inequality. \square

Proof of Theorem 1.2. The proof is quite similar to that of Theorem 1.1. We have only to apply 2) of Theorem 2.2 instead of 1) of Theorem 2.2. \square

Proof of Theorem 1.3. We next improve a weighted stability estimate derived in [14, Theorem 2.2, page 1781].

Lemma 2.4. *Under the same assumptions of Theorem 1.4, there exists $C(v_0, \bar{V}, h, K, \omega, \tilde{\Omega}) > 0$ and $1 > \mu'(v_0, \bar{V}, h, K, \omega, \tilde{\Omega}) > 0$ such that*

$$(2.15) \quad \|\sqrt{I_V}(V - \tilde{V})\|_{L^2(\omega)} \leq C \|\sqrt{I_V} - \sqrt{I_{\tilde{V}}}\|_{H^1(\tilde{\Omega})}^{\mu'}.$$

Proof. We set $\theta = \frac{1}{\sqrt{V}}$ and $J = \sqrt{I_V}$. We deduce from the proof of [14, Theorem 2.2, page 1781], that θ satisfies

$$(2.16) \quad J\Delta(J\theta) = -\frac{J^2}{\theta} \quad \text{in } \Omega.$$

Theorem 2.2 implies that θ indeed verifies

$$(2.17) \quad \Delta(J\theta) = -\frac{J}{\theta} \quad \text{in } \Omega.$$

Let $\tilde{\theta} = \frac{1}{\sqrt{\tilde{V}}}$ and $\tilde{J} = \sqrt{I_{\tilde{V}}}$. Identity (2.17) with V substituted by \tilde{V} yields

$$(2.18) \quad \Delta(\tilde{J}\tilde{\theta}) = -\frac{\tilde{J}}{\tilde{\theta}} \quad \text{in } \Omega.$$

Taking the difference of each side of equations (2.17) and (2.18), we obtain

$$(2.19) \quad \Delta(J(\theta - \tilde{\theta})) - \frac{1}{\tilde{\theta}}J(\theta - \tilde{\theta}) = \frac{\tilde{J} - J}{\tilde{\theta}} + \Delta(\tilde{\theta}(\tilde{J} - J)).$$

As $J(\theta - \tilde{\theta})$ has zero Cauchy data on $\tilde{\gamma}$, we deduce from [3, Theorem 1.7] that there exists $C(v_0, \bar{V}, h, K, \omega, \tilde{\Omega}) > 0$ and $1 > \mu'(v_0, \bar{V}, h, K, \omega, \tilde{\Omega}) > 0$ so that

$$(2.20) \quad \|J(\theta - \tilde{\theta})\|_{L^2(\omega)} \leq C \left(\left\| \frac{\tilde{J} - J}{\tilde{\theta}} \right\|_{L^2(\tilde{\Omega})} + \left\| \tilde{\theta}(\tilde{J} - J) \right\|_{H^1(\tilde{\Omega})} \right)^{\mu'}.$$

Whence the expected inequality follows. \square

The rest of the proof is quite similar to that of Theorem 1.1. We apply again 1) of Theorem 2.2 to inequality (2.15). \square

Proof of Theorem 1.4. We split γ into two components $\gamma_+ = \gamma \cap \Gamma_+$ and $\gamma_0 = \gamma \cap \Gamma_0$.

Let V and \tilde{V} be as in the statement of Theorem 1.4. As $p > n$, $W^{2,p}(\Omega)$ is continuously embedded in $C^1(\bar{\Omega})$. Whence $u_V^2 \in W^{2,p}(\Omega)$. Inspecting the proof of [6, Proposition 3.1] we get that there exists a constant $\delta = \delta(\Omega, p, v_0, k, \bar{V}, h, \gamma_0)$ and a neighborhood \mathcal{U}_0 of γ_0 in $\omega \cup \Gamma_0$ so that $|u_V|^{-\delta} \in L^1(\mathcal{U}_0)$. We get from the proof of [6, Lemma 1.3] that there exists $C_0 = C_0(\Omega, p, v_0, k, \bar{V}, h, \gamma_0)$ such that

$$\|V - \tilde{V}\|_{L^2(\mathcal{U}_0)} \leq C_0 \|(V - \tilde{V})u_V^2\|_{L^1(\mathcal{U}_0)}^{\frac{\delta}{2+\delta}}.$$

In light of Lemma [13, Lemma B.1], this inequality entails

$$\|V - \tilde{V}\|_{L^\infty(\mathcal{U}_0)} \leq C_0 \|(V - \tilde{V})u_V^2\|_{L^1(\mathcal{U}_0)}^{2\mu_0},$$

for some $\mu_0 = \mu_0(\Omega, p, v_0, k, \bar{V}, h, \gamma_0)$.

As in Theorem 1.1, this inequality leads to the following one

$$(2.21) \quad \|V - \tilde{V}\|_{L^\infty(\mathcal{U}_0)} \leq C_0 \|\sqrt{I_V} - \sqrt{I_{\tilde{V}}}\|_{H^1(\Omega)}^{\mu_0}.$$

On the other hand, we easily check that

$$|u_V| \geq \frac{1}{2} \min_{\gamma_+} |u_V| = \frac{1}{2} \min_{\gamma_+} |h| := t > 0$$

in a neighborhood \mathcal{U}_+ of γ_+ in $\omega \cup \Gamma_+$, depending only $\Omega, p, v_0, k, \bar{V}, h$ and γ_+ . We apply once again Lemma [13, Lemma B.1] in order to get

$$\|V - \tilde{V}\|_{L^\infty(\mathcal{U}_+)} \leq C_+ \|(V - \tilde{V})u_V^2\|_{L^1(\mathcal{U}_+)}^{\mu_+},$$

for some $\mu_+ = \mu_+(\Omega, p, v_0, k, \bar{V}, h, \gamma_+)$ and $C_+ = C_+(\Omega, p, v_0, k, \bar{V}, h, \gamma_+)$.

From this inequality we derive, again similarly as in the proof of Theorem 1.1,

$$(2.22) \quad \|V - \tilde{V}\|_{L^\infty(\mathcal{U}_+)} \leq C_+ \|\sqrt{I_V} - \sqrt{I_{\tilde{V}}}\|_{H^1(\Omega)}^{\mu_+},$$

Let $\tilde{\omega} \Subset \Omega$ so that $\omega \subset \tilde{\omega} \cup \mathcal{U}_0 \cup \mathcal{U}_+$. By the interior stability estimate in Theorem 1.1, there exists $\tilde{\mu} = \tilde{\mu}(\Omega, p, v_0, k, \bar{V}, h, \omega)$ and $C = C(\Omega, p, v_0, k, \bar{V}, h, \omega)$ so that

$$(2.23) \quad \|V - \tilde{V}\|_{L^\infty(\tilde{\omega})} \leq C \|\sqrt{I_V} - \sqrt{I_{\tilde{V}}}\|_{H^1(\Omega)}^{\tilde{\mu}}$$

We end up getting the expected inequality by combining (2.21), (2.22) and (2.23), with $\mu = \min(\mu_0, \mu_+, \tilde{\mu})$. \square

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APPENDIX A

In this appendix, Ω is a bounded domain of \mathbb{R}^n , $n \geq 2$, with Lipschitz boundary Γ . Let

$$L = \operatorname{div}(A\nabla \cdot) + V,$$

where $V \in L^\infty(\Omega)$, $A = (a^{ij})$ is a symmetric matrix with coefficients in $W^{1,\infty}(\Omega)$ and there exist $\kappa > 0$ and $\Lambda > 0$ so that

$$(A.1) \quad A(x)\xi \cdot \xi \geq \kappa|\xi|^2, \quad x \in \Omega, \quad \xi \in \mathbb{R}^n,$$

and

$$(A.2) \quad \|V\|_{L^\infty(\Omega)} + \|a^{ij}\|_{W^{1,\infty}(\Omega)} \leq \Lambda, \quad 1 \leq i, j \leq n.$$

Recall the following three-ball interpolation inequality, proved in [11] when $V = 0$ but still holds for any bounded V .

Theorem A.1. *Let $0 < k < \ell < m$. There exist $C > 0$ and $0 < s < 1$, depending only on $\Omega, k, \ell, m, \kappa$ and Λ , such that*

$$(A.3) \quad \|v\|_{L^2(B(y,\ell r))} \leq C \|v\|_{L^2(B(y,kr))}^s \|v\|_{L^2(B(y,mr))}^{1-s},$$

for all $v \in H^1(\Omega)$ satisfying $Lv = 0$ in Ω , $y \in \Omega$ and $0 < r < \frac{1}{m} \operatorname{dist}(y, \Gamma)$.

Recall that Ω has the uniform interior cone condition, abbreviated to **UICP** in the sequel, if there exist $R > 0$ and $\theta \in]0, \frac{\pi}{2}[$ so that, to any $\tilde{x} \in \Gamma$ corresponds $\xi = \xi(\tilde{x}) \in \mathbb{S}^{n-1}$ for which

$$\mathcal{C}(\tilde{x}) = \{x \in \mathbb{R}^n; |x - \tilde{x}| < R, (x - \tilde{x}) \cdot \xi > |x - \tilde{x}| \cos \theta\} \subset \Omega.$$

A classical result says that any Lipschitz domain possesses the **UICP** (see for instance [18]).

Proof of Theorem 2.1. In this proof C denote a generic constant that can depend only on Ω , v_0 , V_0 , κ and M .

Step 1. Let $y, y_0 \in \Omega^{3\delta}$ and $\psi : [0, 1] \rightarrow \Omega$ be a C^1 -piecewise smooth path joining y_0 to y so that $\ell(\psi) \leq d_g(y_0, y) + 1$. Let $t_0 = 0$ and $t_{k+1} = \inf\{t \in [t_k, 1]; \psi(t) \notin B(\psi(t_k), \delta)\}$, $k \geq 0$. We claim that there exists an integer $N \geq 1$ so that $\psi(1) \in B(\psi(t_N), \delta)$. If not, we would have $\psi(1) \notin B(\psi(t_k), \delta)$ for any $k \geq 0$. As the sequence (t_k) is non decreasing and bounded from above by 1, it converges to $\hat{t} \leq 1$. In particular, there exists an integer $k_0 \geq 1$ so that $\psi(t_k) \in B(\psi(\hat{t}), \frac{\delta}{2})$, $k \geq k_0$. But this contradicts the fact that $|\psi(t_{k+1}) - \psi(t_k)| = \delta$, $k \geq 0$.

Let us check that $N \leq N_0$ where N_0 depends only on \mathbf{d}_g and δ . Pick $1 \leq j \leq n$ so that

$$\max_{1 \leq i \leq n} |\psi_i(t_{k+1}) - \psi_i(t_k)| = |\psi_j(t_{k+1}) - \psi_j(t_k)|.$$

Then

$$\delta \leq n |\psi_j(t_{k+1}) - \psi_j(t_k)| = n \left| \int_{t_k}^{t_{k+1}} \dot{\psi}_j(t) dt \right| \leq n \int_{t_k}^{t_{k+1}} |\dot{\psi}(t)| dt.$$

Consequently, where $t_{N+1} = 1$,

$$(N+1)\delta \leq n \sum_{k=0}^N \int_{t_k}^{t_{k+1}} |\dot{\psi}(t)| dt = n\ell(\psi) \leq n(\mathbf{d}_g + 1).$$

Therefore

$$N \leq N_0 = \left\lfloor \frac{n(\mathbf{d}_g + 1)}{\delta} \right\rfloor.$$

Let $y_k = \psi(t_k)$, $0 \leq k \leq N$. If $|z - y_{k+1}| < \delta$, then $|z - y_k| \leq |z - y_{k+1}| + |y_{k+1} - y_k| < 2\delta$. In other words $B(y_{k+1}, \delta) \subset B(y_k, 2\delta)$.

We get from Theorem A.1

$$(A.4) \quad \|u\|_{L^2(B(y_j, 2\delta))}^n \leq C \|u\|_{L^2(B(y_j, 3\delta))}^{1-s} \|u\|_{L^2(B(y_j, \delta))}^s, \quad 0 \leq j \leq N.$$

Set $I_j = \|u\|_{L^2(B(y_j, \delta))}^n$, $0 \leq j \leq N$ and $I_{N+1} = \|u\|_{L^2(B(y, \delta))}^n$. Since $B(y_{j+1}, \delta) \subset B(y_j, 2\delta)$, $1 \leq j \leq N-1$, estimate (A.4) implies

$$(A.5) \quad I_{j+1} \leq C M_0^{1-s} I_j^s, \quad 0 \leq j \leq N,$$

where we set $M_0 = \|u\|_{L^2(\Omega)}$.

Let $C_1 = C^{1+s+\dots+s^{N+1}}$ and $\beta = s^{N+1}$. Then by a simple induction argument estimate (A.5) yields

$$(A.6) \quad I_{N+1} \leq C_1 M_0^{1-\beta} I_0^\beta.$$

Without loss of generality, we assume in the sequel that $C \geq 1$ in (A.5). Using that $N \leq N_0$, we have

$$\beta \geq \beta_0 = s^{N_0+1} \geq s e^{-\frac{\kappa}{s}} = \psi(\delta), \quad \text{where } \kappa = n(\mathbf{d}_g + 1) |\ln s|,$$

$$C_1 \leq C^{\frac{1}{1-s}},$$

$$\left(\frac{I_0}{M_0} \right)^\beta \leq \left(\frac{I_0}{M_0} \right)^{\beta_0}.$$

These estimates in (A.6) gives

$$\frac{I_{N+1}}{M_0} \leq C \left(\frac{I_0}{M_0} \right)^{\psi(\delta)}.$$

In the other words,

$$\frac{\|u\|_{L^2(B(x,\delta))}}{\|u\|_{L^2(\Omega)}} \leq C \left(\frac{\|u\|_{L^2(B(x_0,\delta))}}{\|u\|_{L^2(\Omega)}} \right)^{\psi(\delta)}.$$

Applying Young's inequality, we get from this inequality

$$(A.7) \quad \|u\|_{L^2(B(y,\delta))} \leq C \left(\epsilon^{\frac{1}{1-\psi(\delta)}} \|u\|_{L^2(\Omega)} + \epsilon^{-\frac{1}{\psi(\delta)}} \|u\|_{L^2(B(y_0,\delta))} \right), \quad \epsilon > 0, \quad y, y_0 \in \Omega^{3\delta}.$$

Step 2. Fix $\tilde{x} \in \Gamma$ so that $|u(\tilde{x})| = \|u\|_{L^\infty(\Gamma)}$. Let $\xi = \xi(\tilde{x})$ be as in the definition of the **UICP**. Let $x_0 = \tilde{x} + \delta\xi$, $\delta \leq \frac{R}{2}$, $d_0 = |x_0 - \tilde{x}| = \delta$ and $\rho_0 = \left(\frac{d_0}{3}\right) \sin \theta$. Note that $B(x_0, 3\rho_0) \subset \mathcal{C}(\tilde{x})$.

By induction in k , we construct a sequence of balls $(B(x_k, 3\rho_k))$, contained in $\mathcal{C}(\tilde{x})$, as follows

$$\begin{cases} x_{k+1} = x_k - \alpha_k \xi, \\ \rho_{k+1} = \mu \rho_k, \\ d_{k+1} = \mu d_k, \end{cases}$$

where

$$d_k = |x_k - \tilde{x}|, \quad \rho_k = c d_k, \quad \alpha_k = (1 - \mu) d_k,$$

with

$$c = \frac{\sin \theta}{3}, \quad \mu = \frac{3 - 2 \sin \theta}{3 - \sin \theta}.$$

Note that this construction guaranties that, for each k , $B(x_k, 3\rho_k) \subset \mathcal{C}(\tilde{x})$ and

$$(A.8) \quad B(x_{k+1}, \rho_{k+1}) \subset B(x_k, 2\rho_k).$$

We get by applying Theorem A.1 that there exist $C > 0$ and $0 < s < 1$, depending only on Ω , v_0 and V_0 , so that

$$(A.9) \quad \begin{aligned} \|u\|_{L^2(B(x_k, 2\rho_k))} &\leq C \|u\|_{L^2(B(x_k, 3\rho_k))}^{1-s} \|u\|_{L^2(B(x_k, \rho_k))}^s \\ &\leq C M^{1-s} \|u\|_{L^2(B(x_k, \rho_k))}^s. \end{aligned}$$

In light of (A.8), (A.9) gives

$$(A.10) \quad \|u\|_{L^2(B(x_{k+1}, \rho_{k+1}))} \leq C M^{1-s} \|u\|_{L^2(B(x_k, \rho_k))}^s.$$

Let $J_k = \|u\|_{L^2(B(x_k, \rho_k))}$. Then (A.10) is rewritten as follows

$$J_{k+1} \leq C M^{1-s} J_k^s.$$

An induction argument in k yields

$$J_k \leq C^{1+s+\dots+s^{k-1}} M^{(1-s)(1+s+\dots+s^{k-1})} J_0^{s^k}.$$

That is

$$(A.11) \quad J_k \leq \left[C^{\frac{1}{1-s}} M \right]^{1-s^k} J_0^{s^k}.$$

Applying Young's inequality we obtain, for any $\epsilon > 0$,

$$(A.12) \quad \begin{aligned} J_k &\leq (1 - s^k) \epsilon^{\frac{1}{1-s^k}} C^{\frac{1}{1-s}} M + s^k \epsilon^{-\frac{1}{s^k}} J_0 \\ &\leq \epsilon^{\frac{1}{1-s^k}} C^{\frac{1}{1-s}} M + \epsilon^{-\frac{1}{s^k}} J_0 \\ &\leq C \epsilon^{\frac{1}{1-s^k}} M + \epsilon^{-\frac{1}{s^k}} J_0. \end{aligned}$$

Now, since $u \in C^{0, \frac{1}{2}}(\overline{\Omega})$,

$$|u(\tilde{x})| \leq [u]_{\frac{1}{2}} |\tilde{x} - x|^{\frac{1}{2}} + |u(x)|, \quad x \in B(x_k, \rho_k).$$

Hence

$$|\mathbb{S}^{n-1} |\rho_k^n |u(\tilde{x})|^2 \leq 2[u]_{\frac{1}{2}}^2 \int_{B(x_k, \rho_k)} |\tilde{x} - x| dx + 2 \int_{B(x_k, \rho_k)} |u(x)|^2 dx.$$

Or equivalently

$$|u(\tilde{x})|^2 \leq 2|\mathbb{S}^{n-1}|^{-1}\rho_k^{-n} \left([u]_{\frac{1}{2}}^2 \int_{B(x_k, \rho_k)} |\tilde{x} - x| dx + \int_{B(x_k, \rho_k)} |u(x)|^2 dx \right).$$

A simple computation shows that $d_k = \mu^k d_0$. Then

$$|\tilde{x} - x| \leq |\tilde{x} - x_k| + |x_k - x| \leq d_k + \rho_k = (1+c)d_k = (1+c)\mu^k d_0.$$

Therefore,

$$|u(\tilde{x})|^2 \leq 2 \left(M^2(1+c)^{\frac{1}{2}} d_0^{\frac{1}{2}} \mu^k + |\mathbb{S}^{n-1}|^{-1} (cd_0)^{-n} \mu^{-nk} \|u\|_{L^2(B(x_k, \rho_k))}^2 \right)$$

implying, when $d_0(= \delta) \leq 1$,

$$(A.13) \quad |u(\tilde{x})| \leq C \left(M\mu^{\frac{k}{2}} + \mu^{-\frac{nk}{2}} \delta^{-\frac{n}{2}} J_k \right).$$

Inequalities (A.12) and (A.13) gives

$$(A.14) \quad |u(\tilde{x})| \leq C \left(\mu^{\frac{k}{2}} M + \mu^{-\frac{nk}{2}} \epsilon^{\frac{1}{1-s^k}} \delta^{-\frac{n}{2}} M + \mu^{-\frac{nk}{2}} \epsilon^{-\frac{1}{s^k}} \delta^{-\frac{n}{2}} J_0 \right).$$

We get, by choosing $\epsilon = \mu^{(1-s^k)(n+1)\frac{k}{2}}$ in (A.14),

$$|u(\tilde{x})| \leq C \left(\mu^{\frac{k}{2}} M + \mu^{\frac{k}{2}} \delta^{-\frac{n}{2}} M + \mu^{-\frac{(n+1)k}{2s^k} + \frac{k}{2}} \delta^{-\frac{n}{2}} J_0 \right).$$

Hence

$$(A.15) \quad |u(\tilde{x})| \leq C\delta^{-\frac{n}{2}} \left(\mu^{\frac{k}{2}} M + \mu^{-\frac{(n+1)k}{2s^k}} J_0 \right),$$

by using $\delta \leq \text{diam}(\Omega)$.

Let $t > 0$ and k be the integer so that $k \leq t < k+1$. It follows from (A.15)

$$(A.16) \quad |u(\tilde{x})| \leq C\delta^{-\frac{n}{2}} \left(\mu^{\frac{t}{2}} M + \mu^{-\frac{t(n+1)}{2s^t}} J_0 \right).$$

Let $\kappa = (n+1)/2 + |\ln s|$. Then (A.16) yields

$$(A.17) \quad |u(\tilde{x})| \leq C\delta^{-\frac{n}{2}} \left(\mu^{\frac{t}{2}} M + \mu^{-e^{\kappa t}} J_0 \right).$$

letting $e^{\kappa t} = \frac{1}{\epsilon}$, $0 < \epsilon < 1$, we get from (A.17)

$$(A.18) \quad |u(\tilde{x})| \leq C\delta^{-\frac{n}{2}} \left(\epsilon^\beta M + e^{\frac{|\ln \mu|}{\epsilon}} J_0 \right),$$

where $\beta = \frac{|\ln \mu|}{2\kappa}$.

Step 3. A combination of (A.7) and (A.18) entails, with $0 < \epsilon < 1$ and $\epsilon_1 > 0$,

$$|u(\tilde{x})| \leq C\delta^{-\frac{n}{2}} \left(\epsilon^\beta M + e^{\frac{|\ln \mu|}{\epsilon}} \left(\epsilon_1^{\frac{1}{1-\psi(\delta)}} M + \epsilon_1^{-\frac{1}{\psi(\delta)}} \|u\|_{L^2(B(y_0, \delta))^n} \right) \right).$$

Hence, where $\ell = \frac{n}{2}$ and $\rho = |\ln \mu|$,

$$\|u\|_{L^\infty(\Gamma)} \leq C\delta^{-\ell} \left(\epsilon^\beta M + e^{\frac{\rho}{\epsilon}} \left(\epsilon_1^{\frac{1}{1-\psi(\delta)}} M + \epsilon_1^{-\frac{1}{\psi(\delta)}} \|u\|_{L^2(B(y_0, \delta))^n} \right) \right).$$

In this inequality we take $\epsilon_1 = \epsilon^{\beta(1-\psi(\delta))} e^{-\frac{\rho}{\epsilon}(1-\psi(\delta))}$. Using that

$$\epsilon_1^{-\frac{1}{\psi(\delta)}} \leq e^{\frac{\rho+\beta}{\epsilon} \left(\frac{1-\psi(\delta)}{\psi(\delta)} \right)},$$

we obtain in a straightforward manner

$$\|u\|_{L^\infty(\Gamma)} \leq C\delta^{-\ell} \left(\epsilon^\beta M + e^{\frac{\rho+\beta}{\epsilon} \left(\frac{1-\psi(\delta)}{\psi(\delta)} \right)} \|u\|_{L^2(B(y_0, \delta))^n} \right).$$

By setting $\phi(\delta) = (\rho + \beta) \frac{1-\psi(\delta)}{\psi(\delta)}$, we can rewrite the previous estimate as follows

$$\|u\|_{L^\infty(\Gamma)} \leq C\delta^{-\ell} \left(\epsilon^\beta M + e^{\frac{\phi(\delta)}{\epsilon}} \|u\|_{L^2(B(y_0, \delta))^n} \right),$$

or equivalently

$$(A.19) \quad \|u\|_{L^\infty(\Gamma)} \leq C\delta^{-\ell} \left(t^{-\beta} M + e^{t\phi(\delta)} \|u\|_{L^2(B(y_0, \delta))^n} \right), \quad t > 1.$$

If $\frac{M}{\|u\|_{L^2(B(y_0, \delta))^n}} > e^{\phi(\delta)}$, we find $t > 1$ so that $\frac{M}{\|u\|_{L^2(B(y_0, \delta))^n}} = t^\beta e^{t\phi(\delta)}$. The estimate (A.19) with that t yields

$$\|u\|_{L^\infty(\Gamma)} \leq C\delta^{-\ell} M \left(\frac{1}{\beta + \phi(\delta)} \ln \left(\frac{M}{\|u\|_{L^2(B(y_0, \delta))^n}} \right) \right)^{-\beta}.$$

In light of the inequality $\|u\|_{L^\infty(\Gamma)} \geq \eta$, this estimate implies

$$\eta \leq C\delta^{-\ell} M \left(\frac{1}{\beta + \phi(\delta)} \ln \left(\frac{M}{\|u\|_{L^2(B(y_0, \delta))^n}} \right) \right)^{-\beta}.$$

This inequality is equivalent to the following one

$$(A.20) \quad M e^{-C(\beta + \phi(\delta))\delta^{-\frac{\ell}{\beta}} \left(\frac{M}{\eta}\right)^{\frac{1}{\beta}}} \leq \|u\|_{L^2(B(y_0, \delta))^n}.$$

Otherwise,

$$(A.21) \quad M e^{-\phi(\delta)} \leq \|u\|_{L^2(B(y_0, \delta))^n}.$$

We derive from (A.20) and (A.21) that, there exist $C > 0$ and δ^* so that

$$e^{-e^{c/\delta}} \leq \|u\|_{L^2(B(y_0, \delta))^n}, \quad 0 < \delta \leq \delta^*.$$

Obviously, a similar estimate holds for $\delta \geq \delta^*$. □

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