Solving rough differential equations with the theory of regularity structures

Antoine Brault

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Abstract

The purpose of this article is to solve rough differential equations with the theory of the regularity structures. These new tools recently developed by Martin Hairer for solving semi-linear partial differential stochastic equations were inspired by the rough path theory. We take a pedagogical approach to facilitate the understanding of this new theory. We recover results of the rough path theory with the regularity structure framework. Hence, we show how to formulate a fixed point problem in the abstract space of modelled distributions to solve the rough differential equations. We also give a proof of the existence of a rough path lift with the regularity structure theory.

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1 Introduction

Suppose that we want to solve the following ordinary differential equation

$$\mathrm{d}y_t = F(y_t)\mathrm{d}W_t, \qquad y_0 = \xi \tag{1}$$

where $W : [0,T] \to \mathbb{R}^n$ and $F : \mathbb{R}^d \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^d)$ are regular functions. The equation (1) can be reformulated as

$$\forall t \in [0,T], \quad y_t = \xi + \int_0^t F(y_u)\mathrm{d}W_u. \tag{2}$$

As W is smooth, the equation (1) is well-defined as

$$y_t = \xi + \int_0^t F(y_u)\dot{W}_u\mathrm{d}u.$$

Therefore, it can be solved by a fixed-point argument.

Unfortunately, there are many natural situations in which we would like to consider the equation of type (2) for an irregular path W. This is notably the case when dealing with stochastic processes. For example the paths of the Brownian motion are almost surely nowhere differentiable [KS12]. It is then impossible to interpret (1) in a classical sense. Indeed, even if $\dot{W}$ is understood as a distribution, it is not possible in general to define a natural product between distributions, as y is itself to be thought as a distribution.

On the one hand, to overcome this issue, Itô's theory [KS12] was built to define properly an integral against a martingale M (for example the Brownian motion) : $\int_0^t Z_u \mathrm{d}M_u$, where Z must have some good properties. The

definition is not pathwise as it involves a limit in probability. Moreover, this theory is successful to develop a stochastic calculus with martingales but fails when this property vanishes. This is the case for the fractional Brownian motion, a natural process in modelling. Another bad property is that the map $W \mapsto y$ is not continuous in general with the associated uniform topology.

On the other hand, L.C. Young proved in [You36] that we can define the integral of f against g if f is α-Hölder, g is β-Hölder with $\alpha + \beta > 1$ as

$$\int_0^t f \mathrm{d}g = \lim_{|P|\to 0} \sum_{u,v\in P} f(u)(g(v) - g(u)),$$

where P is a subdivision of $[0, t]$ and $|P|$ denotes its mesh. This result is sharp, so that it is not possible to extend it to the case $\alpha + \beta \leq 1$ [You36]. If W is α-Hölder it would seem natural to think that y is α-Hölder, too. So assuming $\alpha < 1/2$ then $2\alpha < 1$, and Young's integral fails to give a meaning to (1). The fractional Brownian motion which depends on a parameter H giving its Hölder regularity cannot be dealt with Young's integral as soon as $H \leq 1/2$.

T. Lyons introduced in [Lyo98] the rough path theory which overcomes Young's limitation. The main idea is to construct for $0 \leq s \leq t \leq T$ an object $\mathbb{W}_{s,t}$ which "looks like" $\int_s^t (W_u - W_s)\mathrm{d}W_u$ and then define an integral against $(W, \mathbb{W})$. This theory enabled to solve (1) in most of the cases and to define a topology such that the Itô map $(W, \mathbb{W}) \mapsto y$ is continuous. Here, the rough path $(W, \mathbb{W})$ "encodes" the path W with algebraic operations. It is an extension of the Chen series developed in [Che57] and [Lyo94] to solve controlled differential equations. Since the original article of T. Lyons, other approaches of the rough paths theory were developed in [Dav07], [Gub04] and [Bai15]. The article [CL14] deals with the linear rough equations with a bounded operators. For monographs about the rough path theory, the reader can refer to [LQ02] or [FV10].

Recently, M. Hairer developed in [Hai14] the theory of the regularity structures which can be viewed as a generalisation of the rough path theory to deal with singular stochastic partial differential equation (SPDE). Another fruitful theory was introduced to solve SPDE in [GIP12] and also studied in [BBF15].

Here we propose to solve (1), in the case where the Hölder regularity of W is in $(1/3, 1/2]$, with the theory of the regularity structures. In particular, we build the rough integral (Theorem 4.16) and the integral of W againt itself (Theorem 4.6). Our proofs of these results are different from the classical approach. There are based on the reconstruction theorem. Moreover, we use a different way that what is done in [FH14] (chapter 13).

The main goal of this approach is to make this new theory understandable to people who are familiar with rough differential equations or ordinary

differential equations. In Section 3.1 we introduce the Hölder space which allows us to "measure" the regularity of a function. Then, we present the rough path theory in Section 4. In the Sections 5 and 6 we give the framework of the theory of the regularity structures and the modelled distributions for solving (1). We prove in Sections 7 and 8 the existence of the controlled rough path integral and the existence of a rough path lift. Finally, after having defined the composition of a function with a modelled distribution in Section 9, we solve the rough differential equation (1) in Section 10.

2 Notation

We denote by $\mathcal{L}(A, B)$, the set of linear continuous maps between two vector spaces A and B. Throughout the article, C denotes a positive constant whose value may change. For two functions f and g, we write $f \lesssim g$ if there is a constant C such that $f \leq Cg$. The symbol $:=$ means that the right hand side of the equality defines the left hand side. For a function Z from $[0,T]$ to a vector space, its increment is denoted by $Z_{s,t} := Z_t - Z_s$. If $X_1, ..., X_k$ are k vectors of a linear space, we denote by $\text{Vect}\langle X_1, ..., X_k\rangle$ the subspace generated by the linear combinations of these vectors. Let T be a non-negative real, we denote by $[0,T]$ a compact interval of $\mathbb{R}$. For a continuous function $f : [0,T] \to E$, where $(E, \|\cdot\|)$ is a Banach space, we denote by $\|f\|_{\infty,T}$ the supremum of $\|f(t)\|$ for $t \in [0,T]$. The tensor product is denoted by $\otimes$.

3 Hölder spaces

3.1 Classical Hölder spaces with a positive exponent

We introduce Hölder spaces which allow us to characterize the regularity of a non-differentiable function.

Definition 3.1. For $0 \leq \alpha < 1$ and $T > 0$, the function $f : [0,T] \to E$ is α-Hölder if

$$\sup_{s \neq t \in [0,T]} \frac{\|f(t) - f(s)\|}{|t-s|^\alpha} < +\infty.$$

We denote by $C^\alpha(E)$ the *space of α-Hölder functions* equipped with the semi-norm

$$\|f\|_{\alpha,T} := \sup_{s \neq t \in [0,T]} \frac{\|f(t) - f(s)\|}{|t-s|^\alpha}.$$

If $\alpha \geq 1$ such that $\alpha = q + \beta$ where $q \in \mathbb{N}$ and $\beta \in [0,1)$, we set $f \in C^\alpha(E)$ if f has q derivatives and $f^{(q)}$ is β-Hölder.

We denote by $C^\alpha = C^\alpha(\mathbb{R}^n)$. For $q \in \mathbb{N}$, we denote by C_b^q the set of all functions $f \in C^q$ such that

$$\|f\|_{C_b^q} := \sum_{k=0}^{q} \left\| f^{(k)} \right\|_\infty , \tag{3}$$

where $f^{(k)}$ is the derivative of order k ($f^{(0)} := f$).

Remark 3.2. The linear space of α-Hölder functions $C^\alpha(E)$ is a non separable Banach space endowed with one of the two equivalent norms $\|f(0)\| + \|f\|_{\alpha,T}$ or $\|f\|_\infty + \|f\|_{\alpha,T}$.

Remark 3.3. If f is α-Hölder on $[0,T]$, then f is β-Hölder for $\beta < \alpha$, *i.e.* $C^\alpha(E) \subset C^\beta(E)$.

3.2 Localised test functions and Hölder spaces with a negative exponent

In the next section, we define a model of a regularity structure. A model maps $\mathcal{T}$ with the space of Schwartz distributions $\mathcal{D}'(\mathbb{R}, \mathbb{R}^n)$, i.e., the space of linear continuous forms from $\mathcal{D}(\mathbb{R}, \mathbb{R}^n)$ (the space of smooth functions compactly supported from $\mathbb{R}$ to $\mathbb{R}^n$ with the topology of uniform convergence over all compact intervals). For $r > 0$, we denote by B_r the space of all functions in $\eta \in C_b^r$ compactly supported on $[-1,1]$, such that $\|\eta\|_{C_b^r} \leq 1$.

We define Hölder spaces with negative exponent which is a space of distributions.

Definition 3.4. For $\lambda > 0$, $s \in \mathbb{R}$ and a test function $\eta \in B_r$, we define the *test function localised* in s by

$$\eta_s^\lambda(t) := \frac{1}{\lambda} \eta \left(\frac{t-s}{\lambda} \right),$$

for all $t \in \mathbb{R}$.

Remark 3.5. The lower is λ, the more η_s^λ is localised around s, as can be seen in Figure 1.

Remark 3.6. We work here with $t, s \in \mathbb{R}$, because we want to solve stochastic ordinary differential equations. But in the case of stochastic partial differential equations, the parameters t and s belong to $\mathbb{R}^e$ where e is an integer, see [Hai14].

Definition 3.7. For $\alpha < 0$, we define the *Hölder space* C^α as the dual of C_0^r where r is an integer strictly greater than $-\alpha$ and such that for any $\xi \in C^\alpha$ the following estimate holds

$$|\xi(\eta_s^\lambda)| \leq C(T) \lambda^\alpha, \tag{4}$$

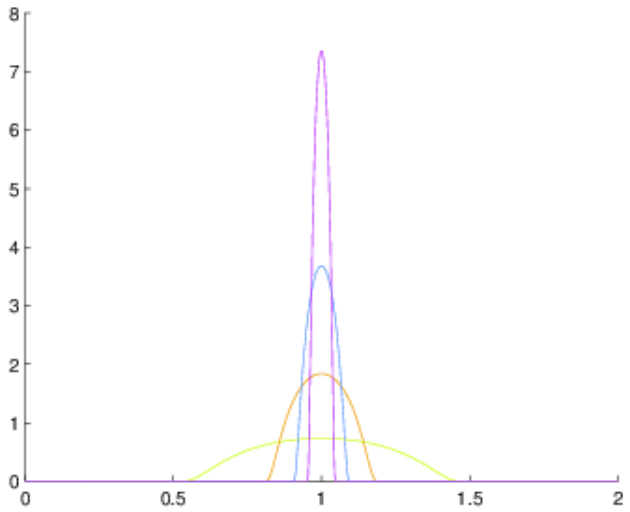

Figure 1: Representation of η_s^λ for $s = 1$, $\lambda \in \{0.5, 0.2, 0.1, 0.05\}$ and with $\eta(s) = \exp(-1/(1-s^2))\mathbb{1}_{(-1,1)}(s)$.

where $C(T) \geq 0$ is a constant uniform over all $s \in [0,T]$, $\lambda \in (0,1]$ and $\eta \in B_r$.

We define the semi-norm on C^α as the lowest constant $C(T)$ for a fixed compact $[0,T]$, i.e

$$\|\xi\|_{\alpha,T} := \sup_{s\in[0,T]} \sup_{\eta\in B_r} \sup_{\lambda\in(0,1]} \left| \frac{\xi(\eta_s^\lambda)}{\lambda^\alpha} \right|.$$

Remark 3.8. The space C^α does not depend on the choice of r, see for example [FH14] Exercise 13.31, p. 209.

Remark 3.9. With Definition 3.1, we can give a meaning of an α-Hölder function for $\alpha \in \mathbb{R}$. Moreover it is possible to show that if f is a function in C^α with $\alpha = q + \beta > 0$ where q is an integer and $\beta \in (0,1)$, then for every $x \in [0,T]$ and localised functions ϕ_x^λ,

$$|(f - P_x)(\phi_x^\lambda)| \leq C\lambda^\beta,$$

where C is uniform over $x \in [0,T]$, $\lambda \in (0,1]$ and $\phi \in B_r$ (r a positive integer), P_x is the Taylor expansion of f of the order q in x, and $f - P_x$ is view as the canonical function associated.

Now, when we say that $f \in C^\alpha$ we should distinguish two cases :

- if $\alpha \geq 0$, f is an α-Hölder *function* in the sense of Definition 3.1
- if $\alpha < 0$, f is an α-Hölder *distribution* in the sense of Definition 3.7.

We give here a characterization of the space C^α for $\alpha \in (-1, 0)$ which is useful to make a link between the rough path and the regularity structures theories.

Lemma 3.10. *For any $\beta \in (0, 1)$, the distribution $\xi \in C^{\beta-1}$ if and only if there exist a function $z \in C^\beta$ such that $z(0) = 0$ and*

$$\forall \eta \in B_1,\ \xi(\eta) = -\langle z, \eta' \rangle. \tag{5}$$

Which means if z is view as a distribution that $z' = \xi$.

The proof of Lemma 3.10 requires introducing elements of the wavelet theory which can be found in [Mey95].

Theorem 3.11. *There exist $\phi, \psi \in C_0^1(\mathbb{R})$ such that for all $n \in \mathbb{N}$*

$$\{\phi_k^i := 2^{i/2}\phi(2^i \cdot -k),\ k \in \mathbb{Z}\} \cup \{\psi_k^j := 2^{j/2}\psi(2^j \cdot -k),\ k \in \mathbb{Z},\ j \geq i\} \tag{6}$$

is an orthonormal basis of $L^2(\mathbb{R})$. This means that for all $f \in L^2(\mathbb{R})$, $i \in \mathbb{N}$ we can write

$$f(t) = \sum_{j \geq i}^{+\infty} \sum_{k \in \mathbb{Z}} \langle f, \psi_k^j \rangle \psi_k^j(t) + \sum_{k \in \mathbb{Z}} \langle f, \phi_k^i \rangle \phi_k^i(t), \tag{7}$$

where the convergence is in $L^2(\mathbb{R})$. Moreover, we have the very useful property,

$$\int \psi(t) t^k \mathrm{d}t = 0, \tag{8}$$

for $k \in \{0, 1\}$.

Remark 3.12. The notation in Definition 3.4 for η_s^λ and in Theorem 3.11 for ϕ_k^i, ψ_k^j are different.

We now procede to the proof of Lemma 3.10.

Proof of Lemma 3.10. The first implication is trivial and does not require the wavelet analysis. If there exists $z \in C^\alpha$ such that for any $\eta \in C_0^r$, $\xi(\eta) = -\langle z, \dot{\eta} \rangle$, then for $\lambda \in (0, 1)$ and $s \in \mathbb{R}$,

$$\begin{aligned} \xi(\eta_s^\lambda) &= -\frac{1}{\lambda^2} \int_{\mathbb{R}} z(u) \dot{\eta}\left(\frac{u-s}{\lambda}\right) \mathrm{d}u \\ &= -\frac{1}{\lambda^2} \int_{\mathbb{R}} (z(u) - z(s)) \dot{\eta}\left(\frac{u-s}{\lambda}\right) \mathrm{d}u, \end{aligned}$$

where the last equality holds because η is compactly supported. So finally, with $\eta \in B_r$ we obtain

$$|\xi(\eta_s^\lambda)| \leq \frac{1}{2} \|\eta\|_{C^1} \|z\|_\alpha \lambda^{\alpha-1}, \tag{9}$$

so $\xi \in C^{\alpha-1}$.

Now if $\xi \in C^{\alpha-1}$ for $\alpha \in (0,1)$ we define for $t \in [0,1]$,

$$z := \sum_{k\in\mathbb{Z}} \langle \xi, \phi_k^0 \rangle \int_0^t \phi_k^0 + \sum_{j\in\mathbb{N}} \sum_{k\in\mathbb{Z}} \langle \xi, \psi_k^j \rangle \int_0^t \psi_k^j, \tag{10}$$

where $\phi, \psi \in C_0^1$ are defined in Definition 3.11 and are compactly supported in $[0,1]$.

We can assume that ϕ, ψ are compactly supported in $[0,1]$, thus the functions ψ_k^i are compactly supported in $[2^{-j}k, 2^{-j}(k+1)]$. We rewrite (10) as

$$z := \langle \xi, \phi \rangle \int_0^t \phi + \sum_{j\in\mathbb{N}} \sum_{k=0}^{2^j-1} \langle \xi, \psi_k^j \rangle \int_0^t \psi_k^j, \tag{11}$$

because the terms $\langle \psi_k^j, \mathbb{1}_{[0,1]} \rangle$ vanish for $k \notin [0, 2^j - 1]$.

We need to justify that the series on the right hand side of (11) is convergent in C^α. We denote for any integer $N \in \mathbb{N}$,

$$S_N^z := \sum_{j=0}^{N} S_j, \tag{12}$$

where $S_j(t) := \sum_{k=0}^{2^j-1} \langle \xi, \psi_k^j \rangle \int_0^t \psi_k^j$. According to the continuity of ξ and (4),

$$|\langle \xi, \phi \rangle| \leq C, \text{ and } |\langle \xi, \psi_k^j \rangle| \leq C 2^{j/2 - j\alpha}. \tag{13}$$

For $|t-s| \leq 1$, let $j_0 \leq N$ be an integer such that $2^{-j_0} \leq |t-s| < 2^{-j_0+1}$. This is always possible for N large enough. On the one hand, for $j \leq j_0$,

$$\begin{aligned} |S_j(t) - S_j(s)| &\leq \left\| S_j' \right\|_\infty |t-s| \\ &\leq |t-s| \sup_{u\in[0,1]} \sum_{k=0}^{2^j-1} |\langle \xi, \psi_k^j \rangle| \cdot |\psi_k^j(u)| \\ &\leq C 2^{j(1-\alpha)} |t-s|, \end{aligned}$$

where we use the fact that $\sum_{k=0}^{2^j u - 1} |\psi(2^j - k)| \leq 1$ because ψ is compactly supported on $[0,1]$. On the other hand, for $j > j_0$,

$$|S_j(t) - S_j(s)| \leq 2 \left\| S_j \right\|_\infty \tag{14}$$

$$\leq 2 \sup_{u\in[0,1]} \sum_{k=0}^{2^j-1} |\langle \xi, \psi_k^j \rangle| \cdot |2^{-j} \hat{\psi}_k^j(u)|, \tag{15}$$

$$\leq 2C 2^{-j\alpha} \sup_{u\in[0,1]} \sum_{k=0}^{2^j-1} |\hat{\psi}(2^j u - k)| \tag{16}$$

where $\hat{\psi} := \int_0^t \psi$. But $\int_{\mathbb{R}} \psi = 0$, which implies that $\hat{\psi}$ is compactly supported, and consequently $\sum_{k=0}^{2^j-1} |\hat{\psi}(2^j u - k)| < C'$. So finally, for $j > j_0$,

$$|S_j(t) - S_j(s)| \leq C 2^{-j\alpha}, . \tag{17}$$

Thus, combining (16), (17),

$$\begin{aligned}\sum_{j=0}^{N} |S_j(t) - S_j(s)| &\leq C|t-s| \sum_{j=0}^{j_0} 2^{j(1-\alpha)} + C \sum_{j=j_0+1}^{\infty} 2^{-j\alpha} \\ &\leq C'|t-s|^{\alpha},\end{aligned}$$

where C' is a new constant. This proves that S_N^z is converges in C^α and justifies the definition of z in (11).

Now, we want to check that $\xi = \dot{z}$ in the distribution framework. For any $\eta \in C_0^2$,

$$\begin{aligned}\langle z, \dot{\eta}\rangle &= \sum_{k\in\mathbb{Z}} \langle \xi, \phi_k^0\rangle \langle \int_0^t \phi_k^0, \dot{\eta}\rangle + \sum_{j\in\mathbb{N}} \sum_{k\in\mathbb{Z}} \langle \xi, \psi_k^j\rangle \langle \int_0^t \psi_k^j, \dot{\eta}\rangle \\ &= -\sum_{k\in\mathbb{Z}} \langle \xi, \phi_k^0\rangle \langle \phi_k^0, \eta\rangle - \sum_{j\in\mathbb{N}} \sum_{k\in\mathbb{Z}} \langle \xi, \psi_k^j\rangle \langle \psi_k^j, \eta\rangle \\ &= -\langle \xi, \sum_{k\in\mathbb{Z}} \phi_k^0 \langle \phi_k^0, \eta\rangle - \sum_{j\in\mathbb{N}} \sum_{k\in\mathbb{Z}} \psi_k^j \langle \psi_k^j, \eta\rangle\rangle \\ &= -\langle \xi, \eta\rangle,\end{aligned}$$

where the commuting of the serie and ξ is justified by the continuity of ξ in C_0^1 and the convergence of the following serie in C_0^1,

$$S_N^\eta := \sum_{j=0}^{N} \sum_{k=0}^{2^j-1} \psi_k^j \langle \psi_k^j, \eta\rangle. \tag{18}$$

Indeed, we have

$$\begin{aligned}|\langle \psi_k^j, \eta\rangle| &\leq 2^{-j/2} \left| \int \psi(x) \eta(2^{-j}(x+k)) \mathrm{d}x \right| \\ &\leq 2^{-j/2} \int |\psi(x)| |\eta(2^{-j}(x+k)) - \eta(2^{-j}k) - \eta'(2^{-j}k)(2^j x)| \mathrm{d}x \\ &\leq \|\psi\|_\infty \left\|\eta'\right\|_\infty 2^{-j/2} 2^{-2j},\end{aligned}$$

where we use the fact that $\int \psi(t) t^k = 0$ for an integer $k \leq 1$. This implies

that

$$\sum_{j=0}^{N}\left\|\sum_{k=0}^{2^j-1}\psi_k^j\langle\psi_k^j,\eta\rangle\right\|_{C^1}\leq\sum_{j=0}^{N}\left\|\sum_{k=0}^{2^j-1}\psi_k^j\langle\psi_k^j,\eta\rangle\right\|_{\infty}+\sum_{j=0}^{N}\left\|\sum_{k=0}^{2^j-1}2^j\psi_k'^j\langle\psi_k^j,\eta\rangle\right\|_{\infty}$$
$$\leq\left(\sum_{j=0}^{N}2^{-2j}+\sum_{j=0}^{N}2^{-j}\right)\|\psi\|_{\infty}\left\|\eta'\right\|_{\infty},$$

which proves that S_N^η is absolutely convergent in C_0^1.

Now by density of C_0^2 in C_0^1 and the continuity of ξ on C_0^1 we conclude that $\langle z,\dot\eta\rangle=-\langle\xi,\eta\rangle$ holds for $\eta\in C_0^1$. ■

Remark 3.13. According to Theorem 3.11 and (11), we have for all integer $i\in\mathbb{N}$,

$$z(t)=\sum_{k=0}^{2^i-1}\langle\xi,\phi_k^i\rangle\int_0^t\phi_k^i+\sum_{j\geq i}\sum_{k=0}^{2^{j-1}}\langle\xi,\psi_k^j\rangle\int_0^t\psi_k^j. \tag{19}$$

4 Elements of rough path theory

We introduce here the elements of the rough path theory for solving equation (2). The notions discussed are reformulated in the regularity structure framework in the following sections. For an extensive introduction the reader can refer to [FH14], and for complete monographs to [LQ02, FV10].

4.1 The space of rough paths

Let W be a continuous function from $[0,T]$ to $\mathbb{R}^n$.

We set $\alpha\in(1/3,1/2]$. We define a rough path associated to $W\in C^\alpha$. Because W is not smooth enough, we should improve it by adding a new structure. The main idea of the rough path theory is to say that we only need to define an object which has the same algebraic and analytical properties as $\int_s^t W_{s,u}\otimes\mathrm{d}W_u$, the integral of the increment of the path against itself.

The importance of the iterated integrals can be understood with the classical linear differential equations where the solutions are provided with the exponential function. Indeed, if W is a differential function from $[0,T]$ to $\mathbb{R}$, the solutions of

$$\mathrm{d}y_t=y_t\mathrm{d}W_t, \tag{20}$$

are

$$y_{s,t}=\exp(W_{s,t})=1+\int_s^t\mathrm{d}W_{t_1,s}+\int_s^t\int_s^{t_1}\mathrm{d}W_{t_2,s}\mathrm{d}W_{t_1,s}+\cdots. \tag{21}$$

Definition 4.1. An *α-Hölder rough path* with $\alpha\in(1/3,1/2]$ is an ordered pair $\mathbf{W}:=(W,\mathbb{W})$ of functions, where $W:[0,T]\to\mathbb{R}^n$ and $\mathbb{W}:[0,T]^2\to\mathbb{R}^n\otimes\mathbb{R}^n$ such that

1. For $s,u,t \in [0,T]$, $\mathbb{W}_{s,t} - \mathbb{W}_{s,u} - \mathbb{W}_{u,t} = W_{s,u} \otimes W_{u,t}$ (Chen's relation), i.e., for every $1 \leq i,j \leq n$, $\mathbb{W}^{i,j}_{s,t} - \mathbb{W}^{i,j}_{s,u} - \mathbb{W}^{i,j}_{u,t} = W^i_{s,u} W^j_{u,t}$.
2. The function W is α-Hölder and $\mathbb{W}$ is 2α-Hölder in the sense

$$\|\mathbb{W}\|_{2\alpha,T} := \sup_{s \neq t \in [0,T]} \frac{\|\mathbb{W}_{s,t}\|}{|t-s|^{2\alpha}} < +\infty.$$

One calls $\mathbb{W}$ the *second order process*. We denote by $\mathscr{C}^\alpha$ the *space of α-Hölder rough paths* endowed with the semi-norm

$$\|\mathbf{W}\|_{\alpha,T} = \|W\|_{\alpha,T} + \|\mathbb{W}\|_{2\alpha,T}\,.$$

Remark 4.2. The second order process $\mathbb{W}_{s,t}$ can be thought of as $\int_s^t W_{s,u} \otimes \mathrm{d}W_u$.

Remark 4.3. The first condition which is called Chen's relation represents the algebraic property of $\int_s^t W_{s,u} \otimes \mathrm{d}W_u$. Indeed, if W is smooth,

$$\int_s^t W^i_{s,v} \dot{W}^j_v \mathrm{d}v - \int_s^u W^i_{s,v} \dot{W}^j_v \mathrm{d}v - \int_u^t W^i_{u,v} \dot{W}^j_v \mathrm{d}v = W^i_{s,u} W^j_{u,t}$$

for all $1 \leq i,j \leq n$ and $0 \leq s \leq u \leq t$.

Remark 4.4. The second condition is a natural analytic property. Indeed, if $W \in C^\alpha$, then typically for $\lambda \in \mathbb{R}$, $W_{\lambda s,\lambda t}$ looks like $\lambda^\alpha W_{s,t}$. Hence, according to the first property $\mathbb{W}_{\lambda s,\lambda t}$ looks like $\lambda^{2\alpha}\mathbb{W}_{s,t}$. So we should think of $\mathbb{W}$ as a 2α-Hölder function.

Remark 4.5. If $\mathbb{W}$ is a second order process of W, for any 2α-Hölder function F taking values in $\mathbb{R}^n \otimes \mathbb{R}^n$, $\mathbb{W}_{s,t} + F_t - F_s$ satisfies also the two properties of Definition 4.1. So if $\mathbb{W}$ exists, it is not unique at all.

It is always possible to build $\mathbb{W}$ from W (see [LV07]), but it is not easy in practice.

Theorem 4.6. *For any $W \in C^\alpha$ with $\alpha \in (1/3, 1/2]$ there exists a rough path lift $\mathbb{W}$,* i.e. *$\boldsymbol{W} = (W, \mathbb{W}) \in \mathscr{C}^\alpha$ in a way that the map $W \to \boldsymbol{W}$ is continuous for the topology defined in Definition 4.1.*

Proof. We will prove of this theorem in the case $\alpha \in (1/3, 1/2]$ in Section 8 as an application of the theory of regularity structures. ∎

4.2 Controlled rough paths

The aim of this section is to define an integrand against W which we are going to call a controlled rough path by W. This approach was developed by Gubinelli in [Gub04]. We introduce a function with the same regularity of W which is not differentiable with respect to time but with respect to W itself. This is the concept of the Gubinelli's derivative.

Definition 4.7. Let W be in C^α, we call a *controlled rough path by* W the pair $(y, y') \in C^\alpha(\mathbb{R}^d) \times C^\alpha(\mathbb{R}^{d\times n})$ such that

$$y_{s,t} = y'_s W_{s,t} + R^y_{s,t} \tag{22}$$

with $\|R^y\|_{2\alpha,T} < +\infty$. The function y' is the *Gubinelli's derivative* of y with respect to W.

We denote $\mathscr{D}^{2\alpha}_W$ the *space of the controlled rough paths* (y, y') driven by W endowed with the semi-norm

$$\|(y, y')\|^W_{2\alpha,T} := \|y'\|_{\alpha,T} + \|R^y\|_{2\alpha,T}\,. \tag{23}$$

Remark 4.8. The identity (22) looks like a Taylor expansion of first order

$$f_t = f_s + f'_s(t-s) + O(|t-s|^2),$$

but $(W_t - W_s)$ substitutes the usual polynomial expression $(t-s)$, y'_s the normal derivative and the remainder term is of order 2α whereas order 2. The theory of regularity structures is a deep generalization of this analogy.

Remark 4.9. The Gubinelli's derivative y' is matrix-valued which depends on y and W.

Remark 4.10. Unlike $\mathscr{C}^\alpha$ in the Definition 4.1, which is not a linear space, $\mathscr{D}^{2\alpha}_W$ is a Banach space with the norm $\|y_0\| + \|y'_0\| + \|(y, y')\|^W_{2\alpha,T}$ or the norm $\|y\|_{\infty,T} + \|y'\|_{\infty,T} + \|(y, y')\|^W_{2\alpha,T}$. These two norms are equivalent.

Remark 4.11. The uniqueness of y' depends on the regularity of W. If W is too smooth, for example in $C^{2\alpha}$, then y is in $C^{2\alpha}$, and every continuous function y' matches with the definition of the Gubinelli's derivative, particularly $y' = 0$. But we can prove that y' is uniquely determined by y when W is irregular enough. The reader can refer to the Chapter 4 of [FH14] for detailed explanations.

Remark 4.12. It is not trivial to exhibit a non trivial controlled rough path. In fact, most of functions of the space match with solutions of rough differential equations.

4.3 Integration against rough paths

In the linear case, the differential equation (1) can be restated on an integral form as

$$y_t = \xi + \int_0^t y_u \mathrm{d}W_u. \tag{24}$$

To give a meaning to this equation we must define an integral driven by an irregular noise W for a set of suitable integrands y that we make precise now.

First we define Young's integral for the case where $W \in C^\alpha$, $y \in C^\beta$ with $\alpha + \beta > 1$. The solution y should inherit of the regularity of W. So Young's theory allows us to solve (24) when $\alpha > 1/2$ ($2\alpha > 1$).

Then, we show that if $\alpha + \beta \leq 1$, Young's integration fails and that we have to "improve" the path of W. We build the rough integral of a controlled rough path $(y, y') \in \mathscr{D}_W^{2\alpha}$ against a rough path $(W, \mathbb{W}) \in \mathscr{C}^\alpha$.

4.4 Young's integration

Young's integral was developed by Young in [You36] and then used by T. Lyons in [Lyo94] to deal with differential equations driving by a stochastic process in the nineties.

It is the same construction as for the Riemann sum. Let $\mathcal{P}$ be a subdivision of $[s,t]$, we denote by $|\mathcal{P}|$ the mesh of $\mathcal{P}$. We want to define the integral as follows:

$$\int_s^t y_u \mathrm{d}W_u = \lim_{|\mathcal{P}|\to 0} \sum_{u,v\in\mathcal{P}} y_u W_{u,v},$$

where $u, v \in \mathcal{P}$ denotes successive points of the subdivision.

Theorem 4.13. *If $W \in C^\alpha$ and $y \in C^\beta$ with $\alpha + \beta > 1$, $\sum_{u,v\in\mathcal{P}} y_u W_{u,v}$ converges when $|\mathcal{P}| \to 0$. The limit is independent of the choice of $\mathcal{P}$, and it is denoted as $\int_s^t y_u \mathrm{d}W_u$. Moreover the bilinear map $(W, y) \to \int_s^t y_u \mathrm{d}W_u$ is continuous from $C^\alpha \times C^\beta$ to C^α.*

Proof. For the original proof cf. [You36]. ∎

Some important properties of the classical Riemann integration holds.

Proposition 4.14. *1. Chasles' relation holds.*

2. *When $t \to s$ we have the following approximation*

$$\int_s^t y_u \mathrm{d}W_u = y_s W_{s,t} + O(|t-s|^{\alpha+\beta}). \tag{25}$$

3. *The map $t \mapsto \int_s^t y_u \mathrm{d}W_u$ is α-Hölder continuous.*

4. *If F is C^1, $F(y)$ is C^β-Hölder and the Young integral $\int_s^t F(y_u)\mathrm{d}W_u$ is well-defined as above.*

Remark 4.15. Unfortunately with Young's construction, when $\alpha \leq 1/2$, we can find two sequences of smooth functions $W^{1,n}$ and $W^{2,n}$ converging to W in C^α but such that $\int_s^t F(W^{1,n})\mathrm{d}W^{1,n}$ and $\int_s^t F(W^{2,n})_n \mathrm{d}W^{2,n}$ converge to two different limits for a smooth function F . See for an example the Lejay's area bubbles in [Lej12].

4.5 Controlled rough path integration

The rough integral relies on the controlled rough paths introduced previously. Remark 4.15 shows that if $y, W \in C^\alpha$, we cannot define a continuous integral such as $\int_s^t y_u \mathrm{d}W_u$ looks like $y_s W_{s,t}$ when $t \to s$. We must use the structure of controlled rough paths to define a "good" integral of y against W. Then, given a rough path $\mathbf{W} \in \mathscr{C}^\alpha$ and considering a controlled rough path $(y, y') \in \mathscr{D}_W^{2\alpha}$ we would like to build an integral $\int_s^t y_u \mathrm{d}W_u$ as a good approximation of $y_s W_{s,t} + y'_s \mathbb{W}_{s,t}$ when $t \to s$.

Theorem 4.16. *For $\alpha \in (1/3, 1/2]$, let $\boldsymbol{W} = (W, \mathbb{W}) \in \mathscr{C}^\alpha$ be an α-Hölder rough path. Given a controlled rough path driven by W : $(y, y') \in \mathscr{D}_W^{2\alpha}$ we consider the sum $\sum_{u,v\in\mathcal{P}} y_u W_{u,v} + y'_u \mathbb{W}_{u,v}$ where $\mathcal{P}$ is a subdivision of $[s,t]$ ($s \le t \in [0,T]$). This sum converges when the mesh of $\mathcal{P}$ goes to 0. We define the integral of y against $\boldsymbol{W}$ as*

$$\int_s^t y_u \mathrm{d}\,\boldsymbol{W}_u := \lim_{|\mathcal{P}|\to 0} \sum_{u,v\in\mathcal{P}} y_u W_{u,v} + y'_u \mathbb{W}_{u,v}.$$

The limit exists and does not depend on the choice of the subdivision. Moreover, the map $(y, y') \to (t \in [0,T] \mapsto \int_0^t y_u \mathrm{d}\,\boldsymbol{W}_u, y)$ from $\mathscr{D}_W^{2\alpha}$ into itself is continuous.

Proof. The classical proof uses the Sewing Lemma. We can find all the details in [FH14]. Here we are going to give a proof with the reconstruction map which is one of the main theorems of the theory of the regularity structures. ■

For solving (1), we need to show that $F(y_t)$ remains a controlled rough path if F is a smooth function. The following Proposition shows that $(F(y), (F(y))')$ which is defined by :

$$F(y)_t = F(y_t), \qquad F(y)'_t = F'(y_t) y_t, \tag{26}$$

is a controlled rough path.

Proposition 4.17. *Let $F \in C_b^2$ be a function from $\mathbb{R}^n$ to $\mathbb{R}^d \times \mathbb{R}^n$ twice continuously differentiable such that F and its derivative are bounded. Given $(y, y') \in \mathscr{D}_W^{2\alpha}$ let $(F(y), F(y)') \in \mathscr{D}_W^{2\alpha}$ defined as above (26). Then, there is a constant $C_{\alpha,T}$ depending only on α and T such as*

$$\left\|F(y), F(y)'\right\|_{2\alpha,T}^W \le C_{\alpha,T} \left\|F\right\|_{C_b^2} (1 + \|W\|_\alpha)^2 (\left\|y'_0\right\| + \left\|y, y'\right\|_{2\alpha,T}^W)^2,$$

where $\|F\|_{C_b^2} = \|F\|_\infty + \|F'\|_\infty + \|F''\|_\infty$.

Proof. We can find the proof in [FH14]. This Theorem is equivalent to Theorem 9.1, which is formulated in the regularity structure framework. ■

5 Regularity structures

5.1 Definition of a regularity structure

The theory of the regularity structures was introduced by Martin Hairer in [Hai14]. The tools developed in this theory allow us to solve a very wide range of semi-linear partial differential equations driving by an irregular noise.

This theory can be viewed as a generalisation of the Taylor expansion theory to irregular functions. The main idea is to describe the regularity of a function in each point and then to reconstruct this function with an operator.

First we give the definition of a regularity structure.

Definition 5.1. A *regularity structure* is a 3-tuple $\mathscr{T} = (\mathcal{A}, \mathcal{T}, \mathcal{G})$ where

- The set of index $\mathcal{A} \subset \mathbb{R}$ is bounded from below, locally finite and such that $0 \in \mathcal{A}$.
- The *model space* $\mathcal{T}$ is a graded linear space indexed by $\mathcal{A}$: $\mathcal{T} = \bigoplus_{\alpha \in \mathcal{A}} \mathcal{T}_\alpha$, where each $\mathcal{T}_\alpha$ is a non empty Banach space. The elements of $\mathcal{T}_\alpha$ are said of *homogeneity* α. For $\underline{\tau} \in \mathcal{T}$, we denote $\|\underline{\tau}\|_\alpha$ the norm of the component of $\underline{\tau}$ in $\mathcal{T}_\alpha$. Furthermore, $\mathcal{T}_0 = \mathrm{Vect}\langle \underline{1} \rangle$ is isomorphic to $\mathbb{R}$.
- The set $\mathcal{G}$ is a set of continuous linear operators acting on $\mathcal{T}$ such as for $\Gamma \in \mathcal{G}$, $\Gamma(\underline{1}) = \underline{1}$ and $\underline{\tau} \in \mathcal{T}_\alpha$, $\Gamma\underline{\tau} - \underline{\tau} \in \bigoplus_{\beta < \alpha} \mathcal{T}_\beta$. The set $\mathcal{G}$ is called *structure group*.

Remark 5.2. We underline the elements of the model space for the sake of clarity.

Remark 5.3. For $m := \min \mathcal{A}$ so $\Gamma\underline{\tau} = \underline{\tau}$ for every $\underline{\tau} \in \mathcal{T}_m$.

Let us explain the motivations of this definition. The classic polynomial Taylor expansion of order $m \in \mathbb{N}$ is given, between $t \in \mathbb{R}$ and 0, where t converges to 0 by

$$f(t) = P(t) + o(t^m), \quad \text{where } P(t) = \sum_{k=0}^{m} \frac{f^{(k)}(0)}{k!} t^k.$$

In this case the approximation P of f is indexed by integers and the space $\mathcal{T}$ is the polynomial space. For all $h \in \mathbb{R}$, the operator Γ_h associates a Taylor expansion at point t, a Taylor expansion at a point $t + h$. The polynomial $\Gamma_h(P(t)) - P(t)$ is of order less than $m - 1$:

$$\Gamma_h(P(t)) - P(t) := P(t+h) - P(t) = \sum_{k=0}^{m} \frac{f^{(k)}(0)}{k!} ((t+h)^k - t^k).$$

Moreover we have the structure of group on $(\Gamma_h, h \in \mathbb{R})$:

$$\begin{aligned}\Gamma_{h'} \circ \Gamma_h P(t) = \Gamma_{h'} \left(\sum_{k=0}^{d-1} \frac{f^k(0)}{k!}(t+h)^k \right) &= \sum_{k=0}^{d-1} \frac{f^k(0)}{k!} \Gamma_{h'}((t+h)^k) \\ &= \sum_{k=0}^{d-1} \frac{f^k(0)}{k!}(t+h+h')^k \\ &= \Gamma_{h+h'} P(t).\end{aligned}$$

Hence, we can define the polynomial regularity structure as following.

Definition 5.4. We define $\mathscr{T}^1 = (\mathcal{A}^1, \mathcal{T}^1, \mathcal{G}^1)$ the *canonical polynomial regularity structure* as

- $\mathcal{A}^1 = \mathbb{N}$ is the index set.
- For $k \in \mathcal{A}^1$ we define $\mathcal{T}^1_k = \text{Vect}\langle \underline{X}^k \rangle$. The subspace $\mathcal{T}^1_k$ contains the monomial of order k. The polynomial model space is $\mathcal{T}^1 = \bigoplus_{k \in \mathcal{A}} \mathcal{T}^1_k$.
- For $h \in \mathbb{R}$, $\Gamma^1_h \in \mathcal{G}^1$ is given by

$$\Gamma^1_h(\underline{X}^k) = (\underline{X} + h\underline{1})^k.$$

  For $P_k \in \mathcal{T}^1_k$, there is $a_k \in \mathbb{R}$ such that $P_k = a_k \underline{X}^k$. We define the norm on $\mathcal{T}^1_k$ by $\|P_k\|_k = |a_k|$.

With the same arguments we define the polynomial regularity structure and its model associated in $\mathbb{R}^n$.

Definition 5.5. We define $\mathscr{T}^p = (\mathcal{A}^p, \mathcal{T}^p, \mathcal{G}^p)$ the *canonical polynomial regularity structure* on $\mathbb{R}^n$ as

- $\mathcal{A}^p = \mathbb{N}$ is the index set.
- For $\delta \in \mathcal{A}^p$, and k a multi-index of $\mathbb{N}^n$ such that $|k| := k_1 + \cdots + k_n = \delta$, we define $T^p_\delta = \text{Vect}\langle \underline{X}^k := \prod_{i=1}^n \underline{X}_i^{k_i}, |k| = \delta \rangle$. This space T^p_k is a linear space of homogeneous polynomial with n variables and of order δ. For $P_\delta \in T^p_\delta$, there are real coefficients $(a_k)_{|k|=\delta}$ such that $P_\delta = \sum_{|k|=\delta} a_k \underline{X}^k$. We chose the norm on T^p_δ such that $\|P_\delta\|_\delta := \sum_{|k|=\delta} |a_k|$. We define $\mathcal{T}^p = \bigoplus_{\delta \in A} T^p_\delta$ as the polynomial model space.
- For $h \in \mathbb{R}^n$, $\Gamma^p_h \in G^p$ is given by

$$\Gamma^p_h(\underline{X}^k) = \prod_{i=1}^n (\underline{X}_i + h_i \underline{1})^{k_i}.$$

Remark 5.6. The polynomial regularity structure is a trivial example of regularity structure which we introduce for a better understanding. But the strength of this theory is to deal with negative degree of homogeneity.

5.2 Definition of a model

Definition 5.7. Given a regularity structure $\mathscr{T} = (\mathcal{A}, \mathcal{T}, \mathcal{G})$, a *model* $M = (\Pi, \Gamma)$ is two sets of functions such that for $s, t, u \in \mathbb{R}$

- The operator Π_s is continuous and linear from $\mathcal{T}$ to $\mathcal{D}'(\mathbb{R}, \mathbb{R}^n)$.
- $\Gamma_{t,s}$ belongs to $\mathcal{G}$, so it is a linear operator acting on $\mathcal{T}$.
- The following algebraic relations hold: $\Pi_s \Gamma_{s,t} = \Pi_t$ and $\Gamma_{s,t}\Gamma_{t,u} = \Gamma_{s,u}$.
- The following analytic relations hold : for every $\gamma > 0$, $\beta < \alpha \leq \gamma$ with $\alpha, \beta \in \mathcal{A}$ and $\underline{\tau} \in \mathcal{T}_\alpha$, there is a constant $C(T, \gamma)$ uniform over $s, t \in [0, T]$, $\lambda \in (0, 1]$, $\phi \in B_r$ such that

$$|\Pi_s(\underline{\tau})(\phi_s^\lambda)| \leq C(T,\gamma)\lambda^\alpha \left\|\underline{\tau}\right\|_\alpha \text{ and } \left\|\Gamma_{s,t}(\underline{\tau})\right\|_\beta \leq C(T,\gamma)|t-s|^{\alpha-\beta} \left\|\underline{\tau}\right\|_\alpha . \tag{27}$$

We denote respectively by $\left\|\Pi\right\|_{\gamma,T}$ and $\left\|\Gamma\right\|_{\gamma,T}$ the smallest constants such that the bounds (27) hold. Namely,

$$\left\|\Pi\right\|_{\gamma,T} := \sup_{s\in[0,T]} \sup_{\phi\in B_r} \sup_{\lambda\in(0,1]} \sup_{\alpha<\gamma} \sup_{\underline{\tau}\in T_\alpha} \frac{|\Pi_s(\underline{\tau})(\phi_s^\lambda)|_\alpha}{\lambda^\alpha \left\|\underline{\tau}\right\|_\alpha}$$

$$\text{and} \quad \left\|\Gamma\right\|_{\gamma,T} := \sup_{s\neq t\in[0,T]} \sup_{\beta<\alpha<\gamma} \sup_{\underline{\tau}\in T_\alpha} \frac{\left\|\Gamma_{s,t}(\underline{\tau})\right\|_\beta}{|t-s|^{\alpha-\beta} \left\|\underline{\tau}\right\|_\alpha}.$$

The two operators $\left\|\cdot\right\|_{\gamma,T}$ define semi-norms.

The easiest regularity structure which we can describe is the polynomial one (see Definition 5.5). We can now define the model associated to this regularity structure.

Definition 5.8. Given that $\mathscr{T}^p = (\mathcal{A}^p, \mathcal{T}^p, \mathcal{G}^p)$ the canonical polynomial regularity structure on $\mathbb{R}^n$ defined in the Definition 5.5, we define the *model of the polynomial regularity structure* $M^p = (\Pi^p, \Gamma^p)$ such that for all $x, y \in \mathbb{R}^n$ and k a multi-index of order n,

$$\Pi_x^p(\underline{X}^k)(y) := ((y_1 - x_1)^{k_1}, \dots, (y_n - x_n)^{k_n}),$$

$$\Gamma_{x,y}^p(\underline{X}^k) := \Gamma_{x-y}(\underline{X}^k).$$

Proof. It is straightforward to check that this definition is in accordance with the definition of a model (Definition 6.1 below). ∎

Remark 5.9. The operator Π_s which associates to an element of the abstract space a distribution which approximates this element in s. Typically for polynomial regularity structure on $\mathbb{R}$,

$$\Pi_s^p(\underline{X}^k) = (t \to (t - s)^k).$$

Remark 5.10. In the model space, the operator $\Gamma_{s,t}$ gives an expansion in a point s, given an expansion in a point t. For example

$$\Gamma^p_{s,t}(\underline{X}^k) = \Gamma^p_{s-t}(\underline{X}^k) = (\underline{X} + (s-t)\underline{1})^k. \tag{28}$$

Remark 5.11. The first algebraic relation means that if a distribution looks like $\underline{\tau}$ near t, the same distribution looks like $\Gamma_{s,t}(\underline{\tau})$ near s. In practice, we use this relation to find the suitable operator $\Gamma_{t,s}$. The second algebraic relation is natural. It says that moving an expansion from u to s is the same as moving an expansion from u to t and then from t to s.

Remark 5.12. The first analytic relation has to be understood as Π_s approximating $\underline{\tau} \in T_\alpha$ in s with the precision λ^α. The relation (28) shows that the second analytic relation is natural. Indeed,

$$(\underline{X} + (t-s)\underline{1})^k = \sum_{i=0}^{k} \binom{k}{i} (t-s)^{k-i} \underline{X}^i,$$

so for $\ell \leq k$, $\left\|\Gamma^p_{s,t}(\underline{X}^k)\right\|_\ell = \binom{k}{i}|t-s|^{k-\ell}$, where $\binom{k}{i} = \frac{k!}{i!(k-i)!}$ are the binomial coefficients.

5.3 The rough path regularity structure

We now reformulate the results of Subsections 4.1 and 4.2 on rough paths to build up a regularity structure.

For finding the regularity structure, we make some computations for $n = 1$, then we give the proof in the general case after the Definition 5.13.

We fix $\alpha \in (1/3, 1/2]$ and a rough path $\mathbf{W} = (W, \mathbb{W}) \in \mathscr{C}^\alpha$. We want to show that $\mathscr{D}^{2\alpha}_W$ can be viewed as a regularity structure.

Let $(y, y') \in \mathscr{D}^{2\alpha}_W$ be a controlled rough path. According to Definition 4.7, $y_t = y_s + y'_s W_{s,t} + O(|t-s|^{2\alpha})$. So to describe the expansion of y with the regularity structure framework, we need the symbol $\underline{1}$ constant of homogeneity 0 and the symbol $\underline{W}$ of homogeneity α. This leads us to define the elements of the regularity structure of the controlled rough path (y, y') evaluated at time t by

$$\underline{Y}(t) = y_t \underline{1} + y'_t \underline{W}.$$

Moreover, we would like to build the rough path integral $\int y \mathrm{d}\mathbf{W}$ in the regularity structure context. So we introduce abstract elements $\underline{\dot{W}}$ and $\underline{\dot{\mathbb{W}}}$ which "represent" $\mathrm{d}\mathbf{W} = \mathrm{d}(W, \mathbb{W})$. The function W is α-Hölder, so we define the homogeneity of $\underline{\dot{W}}$ as $\alpha - 1$. The second order process $\mathbb{W}$ is 2α-Hölder, which leads us to define the homogeneity of $\underline{\dot{\mathbb{W}}}$ as $2\alpha - 1$.

Finally, with the notation of Definition 5.1, $A = \{\alpha-1, 2\alpha-1, 0, \alpha\}$, $\mathcal{T} = \mathrm{Vect}\langle \underline{\dot{W}}, \underline{\dot{\mathbb{W}}}, \underline{1}, \underline{W} \rangle$. Besides, we order the elements in $\mathrm{Vect}\langle\cdot\rangle$ by homogeneity.

It remains to define $\mathcal{G}$ and an associated model. We start by building the model (Π, Γ). For $s \in [0, T]$, Π_s should transform the elements of $\mathcal{T}$

to distributions (or functions when it is possible) which approximate this elements at the point s. On the one hand we define

$$\Pi_s(\underline{\dot{W}})(\phi) := \int \phi(t)\mathrm{d}W_t, \qquad \Pi_s(\underline{\dot{\mathbb{W}}})(\phi) := \int \phi(t)\mathrm{d}\mathbb{W}_{s,t},$$

where ϕ is a test function. Both integrals are well-defined because ϕ is smooth. The homogeneities of $\underline{\dot{W}}$ and $\underline{\dot{\mathbb{W}}}$ are negative, so they are mapped with distributions. On the other hand, $\underline{1}$ and $\underline{W}$ have positive homogeneities, so we can approximate them in s with functions as

$$\Pi_s(\underline{1})(t) := 1, \qquad \Pi_s(\underline{W})(t) := W_{s,t}.$$

Now, we define $\Gamma_{s,t}(\underline{\tau})$ for every $\beta \in A$ and $s,t \in [0,T]$ and $\underline{\tau} \in \mathcal{T}_\beta$. According to Definition 5.7 : $\Pi_s\Gamma_{s,t}(\underline{\tau})(\phi) = \Pi_t(\underline{\tau})(\phi)$. Moreover, following Definition 5.1, $\Gamma_{s,t}$ should be a linear combination of elements of homogeneity lower than $\underline{\tau}$ and with the coefficient 1 in front of $\underline{\tau}$. First, it seems obvious to set $\Gamma_{s,t}(\underline{1}) = \underline{1}$, because $\underline{1}$ represents a constant. Then we can look for $\Gamma_{s,t}(\underline{W}) = \underline{W} + a_{s,t}\underline{1}$ as a function where $a_{s,t}$ has to be determined. If it is not enough, we would look for $\Gamma_{s,t}(\underline{W})$ with more elements of our structure T. By linearity

$$\Pi_s(\underline{W} + a_{s,t}\underline{1})(u) = W_{s,u} + a_{s,t},$$

so we want that $W_{s,u} + a_{s,t} = \Pi_t(\underline{W})(u) = W_{t,u}$. Finally, we have to choose $a_{s,t} = W_{t,s}$. Given that $\underline{\dot{W}}$ has the lowest homogeneity of our structure, we set $\Gamma_{s,t}(\underline{\dot{W}}) = \underline{\dot{W}}$ in order to respect the last item of Definition 5.1. With the same reason as for $\underline{W}$ and using the Chen's relation of Defintion 4.1, we find that $\Gamma_{s,t}(\underline{\dot{\mathbb{W}}}) = \underline{\dot{\mathbb{W}}} + W_{t,s}\underline{\dot{W}}$ (see the proof of Definition 5.13).

All we did here is in one dimension. With the same arguments we can find the regularity structure of a rough path in $\mathbb{R}^n$.

Definition 5.13. For $\alpha \in (1/3, 1/2]$, given a rough path $\mathbf{W} = (W, \mathbb{W}) \in \mathscr{C}^\alpha$ which take value in $\mathbb{R}^n \bigoplus (\mathbb{R}^n \otimes \mathbb{R}^n)$. We define the *regularity structure of rough paths* $\mathscr{T}^r = (\mathcal{A}^r, \mathcal{T}^r, \mathcal{G}^r)$ and the model associated $M^r = (\Pi^r, \Gamma^r)$ as

i) Index set $\mathcal{A}^r := \{\alpha - 1, 2\alpha - 1, 0, \alpha\}$.

ii) Model space $\mathcal{T}^r := \mathcal{T}^r_{\alpha-1} \bigoplus \mathcal{T}^r_{2\alpha-1} \bigoplus \mathcal{T}^r_0 \bigoplus \mathcal{T}^r_\alpha$, with

$$\mathcal{T}^r_{\alpha-1} := \mathrm{Vect}\langle \underline{\dot{W}}^i, i = 1, \cdots, n\rangle, \qquad \mathcal{T}^r_{2\alpha-1} := \mathrm{Vect}\langle \underline{\dot{\mathbb{W}}}^{i,j}, i,j = 1, \cdots, n\rangle$$

$$\mathcal{T}^r_0 := \mathrm{Vect}\langle \underline{1}\rangle, \qquad \mathcal{T}^r_\alpha := \mathrm{Vect}\langle \underline{W}^i, i = 1, \cdots, n\rangle.$$

iii) For i,j integers between 1 and n, $h \in \mathbb{R}^n$ and Γ^r_h in the structure group $\mathcal{G}^r$, the following relations hold

$$\Gamma^r_h(\underline{\dot{W}}^i) := \underline{\dot{W}}^i, \qquad \Gamma^r_h(\underline{\dot{\mathbb{W}}}^{i,j}) := \underline{\dot{\mathbb{W}}}^{i,j} + h^i\underline{\dot{W}}^j$$

$$\Gamma^r_h(\underline{1}) := \underline{1}, \text{ and } \quad \Gamma^r_h(\underline{W}^i) := \underline{W}^i + h^i\underline{1}.$$

iv) For i, j two integers between 1 and n, for $s, t \in [0, T]$,

$$\Pi^r_s(\underline{\dot{W}^i})(\phi) := \int \phi(t) \mathrm{d}W^i_t, \qquad \Pi^r_s(\underline{\dot{\mathbb{W}}^{i,j}})(\phi) := \int \phi(t) \mathrm{d}\mathbb{W}^{i,j}_{s,t}$$

$$\Pi^r_s(\underline{1})(t) := 1, \qquad \Pi^r_s(\underline{W^i})(t) := W^i_{s,t},$$

where ϕ is a test function.

v) For $s, t \in \mathbb{R}$, $\Gamma^r_{s,t} := \Gamma^r_{|h=W_{t,s}}$.

Proof. Checking that this definition respects the definitions of a regularity structure (Definition 5.1) and of a model (Definition 5.7) is straightforward.

Here we only show where the Chen's relation of Definition 4.1 is fundamental to show that the algebraic condition of Definition 5.7 : $\Pi^r_s \Gamma^r_{s,t} \underline{\dot{\mathbb{W}}} = \Pi^r_t \underline{\dot{\mathbb{W}}}$ holds.

According to the definition above $\Gamma^r_{s,t} \underline{\dot{\mathbb{W}}^{i,j}} = \underline{\dot{\mathbb{W}}^{i,j}} + h^i \underline{\dot{W}^j}$. So we have

$$\Pi^r_s(\Gamma^r_{s,t} \underline{\dot{\mathbb{W}}^{i,j}})(\phi) = \int \phi(u) \mathrm{d}\mathbb{W}^{i,j}_{s,u} + W^i_{t,s} \int \phi(u) \mathrm{d}W^j_{s,u}. \tag{29}$$

In differentiating the Chen's relation $\mathbb{W}^{i,j}_{s,u} = \mathbb{W}^{i,j}_{s,t} + \mathbb{W}^{i,j}_{t,u} + W^i_{s,t} W^j_{t,u}$ with respect to u we get $\mathrm{d}\mathbb{W}^{i,j}_{s,u} = \mathrm{d}\mathbb{W}^{i,j}_{t,u} + W^i_{s,t} \mathrm{d}W^j_{t,u}$. It follows that

$$\Pi^r_s(\Gamma^r_{s,t} \underline{\dot{\mathbb{W}}^{i,j}})(\phi) = \int \phi(u) \mathrm{d}\mathbb{W}^{i,j}_{t,u} + W^i_{s,t} \int \phi(u) \mathrm{d}W^j_{s,u} + W^i_{t,s} \int \phi(u) \mathrm{d}W^j_{s,u}. \tag{30}$$

Finally $\Pi^r_s(\Gamma^r_{s,t} \underline{\dot{\mathbb{W}}^{i,j}})(\phi) = \int \phi(u) \mathrm{d}\mathbb{W}^{i,j}_{t,u} = \Pi^r_t \underline{\dot{\mathbb{W}}}$, which is the algebraic condition required. ■

6 Modelled distributions

6.1 Definition and the reconstruction operator

We have defined a regularity structure. We now introduce the space of functions from $[0, T]$ to $\mathcal{T}$, the *model space* of a regularity structure. These abstract functions should represent at each point of $[0, T]$, a "Taylor expansion" of a real function.

We showed in Section 5.3 how to build an abstract function $\underline{Y}(t) = y_t \underline{1} + y'_t \underline{W}$ which represents the expansion of a real controlled rough path (y, y') at a point t. The most important result of the theory of regularity structures is to show how to build a real function or distribution from an abstract function. Namely, given an approximation of a function at each time, how to reconstruct "continuously" the function. This is given by the *reconstruction map* theorem.

Definition 6.1. Given a regularity structure $(\mathcal{A}, \mathcal{T}, \mathcal{G})$ and a model $M = (\Pi, \Gamma)$, for $\gamma \in \mathbb{R}$ we define the space D^γ_M of *modelled distributions* as functions $\underline{f} : [0,T] \to \mathcal{T}_{<\gamma} := \bigoplus_{\beta<\gamma} \mathcal{T}_\beta$ such that for all $s,t \in [0,T]$ and for all $\beta < \gamma$

$$\left\| \underline{f}(s) - \Gamma_{s,t}(\underline{f}(t)) \right\|_\beta \leq C(T)|t-s|^{\gamma-\beta},$$

where $C(T)$ is a constant which depends only on T.

Recalling that $\|\cdot\|_\beta$ is the norm of the component in $\mathcal{T}_\beta$, we define by

$$\left\| \underline{f} \right\|_{\gamma,T} := \sup_{s\neq t\in[0,T]} \sup_{\beta<\gamma} \frac{\left\| \underline{f}(t) - \Gamma_{t,s}(\underline{f}(s)) \right\|_\beta}{|t-s|^{\gamma-\beta}}$$

a semi-norm on the space D^γ_M. It is also possible to consider the norm

$$\left\| \underline{f} \right\|^*_{\gamma,T} := \sup_{t\in[0,T]} \sup_{\beta<\gamma} \left\| \underline{f}(t) \right\|_\beta + \left\| \underline{f} \right\|_{\gamma,T}.$$

Moreover $\|\cdot\|^*_{\gamma,T}$ is equivalent to

$$\sup_{\beta<\gamma} \left\| \underline{f}(0) \right\|_\beta + \left\| \underline{f} \right\|_{\gamma,T},$$

so from now we use these two norms without distinction.

Remark 6.2. For a fixed model M, the modelled distributions space D^γ_M is a Banach space with the norm $\| \ \|^*_{\gamma,T}$.

Remark 6.3. We choose the same notation for the semi-norm on $\mathscr{D}^\gamma_W$ the space of modelled distributions and on C^α the space of Hölder functions or distributions.

So when $\underline{f} \in D^\gamma_M$, we have to understand $\left\| \underline{f} \right\|_{\gamma,T}$ with Definition 6.1 but when $f \in C^\alpha$, $\|f\|_{\gamma,T}$ is the Hölder norm of Definition 3.1 (for functions $\alpha > 0$) or 3.7 (for distributions $\alpha < 0$).

Remark 6.4. The modelled distribution space D^γ_M can be thought of as abstract γ-Hölder functions. Indeed, for an integer p and $\delta \in [0,1)$ such that $\gamma = p + \delta$, if f is a smooth function

$$\left| f(x) - \sum_{k=0}^{p} \frac{f^{(k)}(y)}{k!}(y-x)^k \right| \leq C|t-s|^\delta,$$

according to the Taylor's inequality. So Definition 6.1 of modelled distributions has to be seen as an extension of the Taylor inequality in a no classical way.

Now we are able to outline the main theorem of the theory of the regularity structures which given a modelled distribution allows us to build a "real" distribution approximated at each point by the modelled distribution.

Theorem 6.5 (Reconstruction map). *Given a regularity structure $\mathscr{T} = (\mathcal{A}, \mathcal{T}, \mathcal{G})$ and a model $M = (\Pi, \Gamma)$, for a real $\gamma > \alpha_* = \min \mathcal{A}$ and an integer $r > |\alpha_*|$ there is a linear continuous map $\mathcal{R} : D^\gamma_M \to C^{\alpha_*}$ such that for all $\underline{f} \in D^\gamma_M$,*

$$|\left[\mathcal{R}(\underline{f}) - \Pi_s(\underline{f}(s))\right](\phi^\lambda_s)| \leq C \|\Pi\|_{\gamma,T} \left\|\underline{f}\right\|^*_{\gamma,T} \lambda^\gamma, \tag{31}$$

where C depends uniformly over $\phi \in B_r$, $\lambda \in (0,1]$, $s \in [0,T]$.

Moreover if $\gamma > 0$, the bound (31) defined $\mathcal{R}(\underline{f})$ uniquely.

If $(\tilde{\Pi}, \Gamma)$ is an other model for $\mathscr{T}$ and $\tilde{\mathcal{R}}$ the reconstruction map associated to the model, we have the bound

$$\begin{aligned}&|\mathcal{R}(\underline{f}) - \tilde{\mathcal{R}}(\tilde{\underline{f}}) - \Pi_s(\underline{f}(s)) + \tilde{\Pi}_s(\tilde{\underline{f}}(s))](\eta^\lambda_s)| \\ &\leq C \left(\left\|\tilde{\Pi}\right\|_{\gamma,T} \left\|f - \tilde{\underline{f}}\right\|^*_{\gamma,T} + \left\|\Pi - \tilde{\Pi}\right\|_{\gamma,T} \left\|\underline{f}\right\|^*_{\gamma,T} \right) \lambda^\gamma,\end{aligned} \tag{32}$$

where C depends uniformly over $\phi \in B_r$, $\lambda \in (0,1]$, $s \in [0,T]$, as above.

Proof. The proof uses the wavelet analysis in decomposing the function $\underline{f}$ in a smooth wavelet basis. The proof requires many computation. A complete on can be found in [Hai14] and a less exhaustive one is in [FH14]. The construction of $\mathcal{R}(\underline{f})$ is the following. We define a sequence $(\mathcal{R}^j(\underline{f}))_{j\in\mathbb{N}}$ such that

$$\mathcal{R}^j(\underline{f}) := \sum_{k\in\mathbb{Z}} \Pi_{k/2^j}(\underline{f}(k/2^j))(\phi^j_k)\phi^j_k, \tag{33}$$

where ϕ^j_k is defined in Definition 3.11 with a regularity at almost r. Then, we show that $\mathcal{R}^j(\underline{f})$ converges weakly to a distribution $\mathcal{R}(\underline{f})$ which means that $\mathcal{R}^j(\underline{f})(\eta)$ converges to $\mathcal{R}(\underline{f})(\eta)$ for all $\eta \in C^r_0$. And we show that the bound (31) holds. ∎

Remark 6.6. It can be proved that if for all $s \in [0,T]$ and $\underline{\tau} \in \mathcal{T}$, $\Pi_s\underline{\tau}$ is a continuous function then $\mathcal{R}(\underline{f})$ is also a continuous function such that

$$\mathcal{R}(\underline{f})(s) = \Pi_s(\underline{f}(s))(s). \tag{34}$$

Corollary 6.7. *With the same notation as in Theorem 6.5, for every $\gamma > 0$, there is a constant C such as*

$$\left\|\mathcal{R}(\underline{f})\right\|_{\alpha,T} \leq C \|\Pi\|_{\gamma,T} \left\|\underline{f}\right\|^*_{\gamma,T}.$$

Proof. According to Theorem 6.5, for $\phi \in B_r$,

$$\frac{|\mathcal{R}(\underline{f})(\phi^\lambda_s)|}{\lambda^\alpha} \leq \frac{|\Pi_s(\underline{f}(s))(\phi^\lambda_s)|}{\lambda^\alpha} + C \|\Pi\|_{\gamma,T} \left\|\underline{f}\right\|^*_{\gamma,T} \lambda^{\gamma-\alpha},$$

and according to the Definition 5.7,

$$\frac{|\Pi_s(\underline{f}(s))(\phi_x^\lambda)|}{\lambda^\alpha} \leq \|\Pi\|_{\gamma,T} \left\|\underline{f}\right\|_{\gamma,T}.$$

So finally

$$\begin{aligned}\left\|\mathcal{R}(\underline{f})\right\|_{\alpha,T} &\leq \|\Pi\|_{\gamma,T} \left\|\underline{f}\right\|_{\gamma,T} + C\,\|\Pi\|_{\gamma,T} \left\|\underline{f}\right\|^*_{\gamma,T} \lambda^{\gamma-\alpha} \\ &\leq C\,\|\Pi\|_{\gamma,T} \left\|\underline{f}\right\|^*_{\gamma,T},\end{aligned}$$

which, by letting λ going to 0 proves the inequality. ■

6.2 Modelled distribution of controlled rough paths

We reformulate the definition of a controlled rough path in the regularity structures framework.

Definition 6.8. Given $(W, \mathbb{W}) \in \mathscr{C}^\alpha$, $(y, y') \in \mathscr{D}^{2\alpha}_W$, the rough path regularity structure $(\mathcal{A}^r, \mathcal{T}^r, \mathcal{G}^r)$ and $M^r = (\Pi^r, \Gamma^r)$ the model associated (cf. Definition 5.13), we define a modelled distribution $\underline{Y} \in D^{2\alpha}_{M^r}$ such that

$$\underline{Y}(t) = y_t \underline{1} + y'_t \underline{W}, \quad \forall t \in [0, T].$$

The space $D^{2\alpha}_{M^r}$ is the space of the *modelled distributions of the controlled rough paths.*

Remark 6.9. This definition is a particular case of modelled distributions of Definition 6.1.

Proof. Let check that $\underline{Y}$ is in $D^{2\alpha}_{M^r}$. For every $s, t \in [0, T]$,

$$\begin{aligned}\underline{Y}(t) - \Gamma^r_{t,s}(\underline{Y}(s)) &= \underline{Y}(t) - \Gamma^r_{t,s}(y_s \underline{1} + y'_s \underline{W}) \\ &= \underline{Y}(t) - (y_s \underline{1} + y'_s \underline{W} + y'_s W_{s,t} \underline{1}),\end{aligned}$$

using the Definition 5.13. Then, we have

$$\left\|\underline{Y}(t) - \Gamma^r_{t,s}(\underline{Y}(s))\right\|_0 = \left\|y(t) - y(s) - y'(s) W_{s,t}\right\| \leq C|t-s|^{2\alpha},$$

according to the definition 4.7 of controlled rough paths. Besides,

$$\left\|\underline{Y}(t) - \Gamma^r_{t,s}(\underline{Y}(s))\right\|_\alpha = \left\|y'(t) - y'(s)\right\| \leq C|t-s|^\alpha,$$

which proves that $\underline{Y} \in D^{2\alpha}_{M^r}$. ■

Proposition 6.10. *With the notations of Definition 6.8, the application* $(y, y') \in \mathscr{D}^{2\alpha}_W \mapsto \underline{Y} \in D^{2\alpha}_{M^r}$ *is an isomorphism and the norms* $\|y\|_{\infty,T} + \|y'\|_{\infty,T} + \|(y, y')\|^W_{2\alpha,T}$ *and* $\|\underline{Y}\|^*_{2\alpha,T}$ *are equivalent.*

Proof. We prove the only equivalence between the two norms.

With the notation of Definition 4.7, we recall that

$$y_{s,t} = y'_s W_{s,t} + R^y_{s,t}, \tag{35}$$

and that $\|(y,y')\|^W_{2\alpha,T} = \|y'\|_{\alpha,T} + \|R^y\|_{2\alpha,T}$. Then according to the previous proof and Definition 6.1,

$$\|\underline{Y}\|_{2\alpha,T} = \sup\left\{\|y'\|_{\alpha,T}, \|R^y\|_{2\alpha,T}\right\}. \tag{36}$$

So we have $\|\underline{Y}\|_{2\alpha,T} \leq \|(y,y')\|^W_{2\alpha,T}$ and $\|(y,y')\|^W_{2\alpha,T} \leq 2\|\underline{Y}\|_{2\alpha,T}$. In adding the terms $\|y\|_{\infty,T} + \|y'\|_{\infty,T}$ to each semi-norms, we obtain the result. ■

7 Rough path integral with the reconstruction map

The power of the theory of the regularity structures is to give a sense in some cases of a product of distributions. Indeed, it is not possible in general to extend the natural product between functions to the distribution space.

To build the controlled rough path integral of Theorem 4.16, with the theory of regularity structures we need to give a meaning to the product between y and $\dot{W}$, where $\dot{W}$ is a distribution. We start by giving a meaning to the abstract product between $\underline{Y}$ and $\underline{\dot{W}}$. If this product has good properties, it will be possible the use the reconstruction map (Theorem 6.5) to define a "real" multiplication.

Definition 7.1 (Multiplication in the model space)**.** Given a regularity structure $(\mathcal{A},\mathcal{T},\mathcal{G})$, we say that the continuous bilinear map $\star : \mathcal{T}^2 \to \mathcal{T}$ defines a *multiplication* (*product*) on the model space $\mathcal{T}$ if

- for all $\underline{\tau} \in \mathcal{T}$, on has $\underline{1} \star \underline{\tau} = \underline{1}$,
- for every $\underline{\tau} \in T_\alpha$ and $\underline{\sigma} \in T_\beta$, on has $\underline{\tau} \star \underline{\sigma} \in T_{\alpha+\beta}$, with the condition $\alpha + \beta \in \mathcal{A}$.

We denote by $|\underline{\tau}|$ the homogeneity α of the symbol $\underline{\tau}$. The last item of the definition can be rephrased as $|\underline{\tau} \star \underline{\sigma}| = |\underline{\tau}| + |\underline{\sigma}|$.

Remark 7.2. For example in the following Theorem 7.3, we define in the regularity structure of rough paths the multiplication described in the table below:

$\star$	$\underline{\dot{W}}$	$\underline{\dot{\mathbb{W}}}$	$\underline{1}$	$\underline{W}$
$\underline{\dot{W}}$			$\underline{W}$	$\underline{\dot{\mathbb{W}}}$
$\underline{\dot{\mathbb{W}}}$			$\underline{\dot{\mathbb{W}}}$	
$\underline{1}$	$\underline{\dot{W}}$	$\underline{\dot{\mathbb{W}}}$	$\underline{1}$	$\underline{W}$
$\underline{W}$	$\underline{\dot{\mathbb{W}}}$		$\underline{W}$	

Theorem 7.3. *We set $\alpha \in (1/3, 1/2]$. There is a linear map $I : D^{2\alpha}_{M^r} \to C^\alpha$ such that for all $\underline{Y} \in D^{2\alpha}_{M^r}$, $I(\underline{Y})(0) = 0$ and such that the map L defined by*

$$\forall t \in [0,T],\ L(\underline{Y})(t) := I(\underline{Y})(t)\underline{1} + \langle \underline{Y}(t), \underline{1}\rangle \underline{W}$$

is linear and continuous from $D^{2\alpha}_{M^r}$ into itself. The symbol $\langle \cdot, \underline{1}\rangle$ denotes the coordinate along $\underline{1}$.

Remark 7.4. Recalling that if $\underline{Y} \in D^{2\alpha}_{M^r}$, according to the Definition 6.8 there is $(y, y') \in \mathscr{D}^{2\alpha}_W$ such that

$$\underline{Y}(t) = y_t \underline{1} + y'_t \underline{W}, \tag{37}$$

we show in the proof of the Theorem 7.3 that

$$I(\underline{Y})(t) = \int_0^t Y_s \mathrm{d}\mathbf{W}_s, \tag{38}$$

where $\int_0^t Y_s \mathrm{d}\mathbf{W}_s$ is defined in Theorem 4.16. Thus L is the equivalent in the modeled distribution space of the map

$$(y, y') \in \mathscr{D}^{2\alpha}_W \mapsto \left(\int_0^\cdot y_s \mathrm{d}\mathbf{W}_s, y_\cdot \right) \in \mathscr{D}^{2\alpha}_W. \tag{39}$$

Remark 7.5. The proof of the existence of I is the same as in Theorem 4.16 (classical sewing lemma). But we show how Theorem 6.5 (reconstruction map) can be adapted to recover the result.

Proof. For $\underline{Y}$ in $D^{2\alpha}_{M^r}$, we define the point-wise product between $\underline{Y}$ and $\dot{\underline{W}}$ as in Remark 7.2, i.e $\underline{Y}(t) \star \dot{\underline{W}} := y_t \dot{\underline{W}} + y'_t \underline{W}\dot{\underline{W}}$, where $\underline{W}\dot{\underline{W}} := \underline{W} \star \dot{\underline{W}} := \dot{\mathbb{W}}$. We denote this product $\underline{Y}\dot{\underline{W}}(t)$, to simplify the notation. Using the fact that $|\underline{W}| + |\dot{\underline{W}}| = 2\alpha - 1 = |\dot{\mathbb{W}}|$ it is straightforward to check that the product is consistent with the Definition 7.1.

We check now that $\underline{Y}\dot{\underline{W}}$ is in $D^{3\alpha-1}_{M^r}$. According to Definition 5.13 item v, we compute

$$\Gamma^r_{t,s}\left(\underline{Y}\dot{\underline{W}}(s)\right) = (y_s + y'_s W_{s,t})\dot{\underline{W}} + y'_s \dot{\mathbb{W}},$$

since $\underline{Y} \in D^{2\alpha}_{M^r}$ with Definition 6.8,

$$\left\| \underline{Y}\dot{\underline{W}}(t) - \Gamma^r_{t,s}\left(\underline{Y}\dot{\underline{W}}(s)\right) \right\|_{\alpha-1} = \left\| y_{s,t} - y'_s W_{s,t} \right\| \lesssim |t-s|^{2\alpha}, \tag{40}$$

$$\left\| \underline{Y}\dot{\underline{W}}(t) - \Gamma^r_{t,s}\left(\underline{Y}\dot{\underline{W}}(s)\right) \right\|_{2\alpha-1} = \left\| y'_{s,t} \right\| \lesssim |t-s|^{\alpha}. \tag{41}$$

Thus, by Definition 6.1, we get that $\underline{Y}\dot{\underline{W}} \in D^{3\alpha-1}_{M^r}$.

Thus, given that $3\alpha - 1 > 0$, we can apply the reconstruction theorem in the positive case. So there is a unique distribution $\mathcal{R}(\underline{Y}\dot{\underline{W}})$ in $C^{\alpha-1}$

such that for every $s \in [0,T]$, $\lambda > 0$ and every localized test function η_s^λ of Definition 3.4,

$$\left|\mathcal{R}(\underline{Y}\dot{\underline{W}})(\eta_s^\lambda) - y_s \int \eta_s^\lambda(u)\mathrm{d}W_u - y'_s \int \eta_s^\lambda(u)\mathrm{d}\mathbb{W}_{s,u}\right| \leq C\,\|\eta\|_{C^1}\,\lambda^{3\alpha-1}, \tag{42}$$

where we use relations of the item iv of Definition 5.13.

We define with Lemma 3.10 the operator $I : D^{2\alpha}_{M^r} \to C^\alpha$ such that $I(\underline{Y}) \in C^\alpha$ is associated to $\mathcal{R}(\underline{Y}\dot{\underline{W}})$. It means that $I(\underline{Y})(0) := 0$ and $\langle I(\underline{Y}), \eta'\rangle := -\langle \mathcal{R}(\underline{Y}\dot{\underline{W}}), \eta\rangle$. Moreover according to Remark 3.13, we have for $|t-s| \leq 1$ and all integer $j_0 \in \mathbb{N}$,

$$I(\underline{Y})_{s,t} = \sum_{k=0}^{2^{j_0}-1} \langle \mathcal{R}(\underline{Y}\dot{\underline{W}}), \phi_k^{j_0}\rangle \int_s^t \phi_k^{j_0} + \sum_{j\geq j_0}\sum_{k=0}^{2^j-1} \langle \mathcal{R}(\underline{Y}\dot{\underline{W}}), \psi_k^j\rangle \int_s^t \psi_k^j. \tag{43}$$

We choose the integer j_0 such that $2^{-j_0} \leq |t-s| < 2^{-j_0+1}$, then we have

$$I(\underline{Y})_{s,t} - y_s W_{s,t} - y'_s \mathbb{W}_{s,t} = \sum_{k=0}^{2^{j_0}-1} \langle \mathcal{R}(\underline{Y}\dot{\underline{W}}) - \Pi_s(\underline{Y}\dot{\underline{W}}(s)), \phi_k^{j_0}\rangle \int_s^t \phi_k^{j_0} \tag{44}$$

$$+ \sum_{j\geq j_0}\sum_{k=0}^{2^j-1} \langle \mathcal{R}(\underline{Y}\dot{\underline{W}}) - \Pi_s(\underline{Y}\dot{\underline{W}}(s)), \psi_k^j\rangle \int_s^t \psi_k^j. \tag{45}$$

We have

$$\langle \mathcal{R}(\underline{Y}\dot{\underline{W}}) - \Pi_s(\underline{Y}\dot{\underline{W}}(s)), \psi_k^j\rangle = \langle \mathcal{R}(\underline{Y}\dot{\underline{W}}) - \Pi_{k/2^j}(\underline{Y}\dot{\underline{W}}(k/2^j)), \psi_k^j\rangle + \langle \Pi_{k/2^j}(\underline{Y}\dot{\underline{W}}(k/2^j)) - \Pi_s(\underline{Y}\dot{\underline{W}}(s)), \psi_k^j\rangle. \tag{46}$$

The first term of the right side of (46) is bounded by (31),

$$|\langle \mathcal{R}(\underline{Y}\dot{\underline{W}}) - \Pi_{k/2^j}(\underline{Y}\dot{\underline{W}}(k/2^j)), \psi_k^j\rangle| \leq C 2^{-j/2} 2^{j(1-3\alpha)}, . \tag{47}$$

For bounding the second term of the right side of (46) we use the algebraic relations between Π and Γ as well as the relations (27),

$$\langle \Pi_{k/2^j}(\underline{Y}\dot{\underline{W}}(k/2^j)) - \Pi_s(\underline{Y}\dot{\underline{W}}(s)), \psi_k^j\rangle = \langle \Pi_{k/2^j}\left(\underline{Y}\dot{\underline{W}}(k/2^j) - \Gamma_{k/2^j,s}\underline{Y}\dot{\underline{W}}(s)\right), \psi_k^j\rangle.$$

Yet $\underline{Y}\dot{\underline{W}} \in D^{3\alpha-1}_{M^r}$, so with (40) and (41), we have

$$\left\|\underline{Y}\dot{\underline{W}}(k/2^j) - \Gamma_{k/2^j,s}\underline{Y}\dot{\underline{W}}(s)\right\|_\beta \leq C|k/2^j - s|^{3\alpha-1-\beta},$$

for $\beta \in \{2\alpha-1, \alpha-1\}$. Finally, we obtain with the bounds (27),

$$\left|\langle \Pi_{k/2^j}(\underline{Y}\dot{\underline{W}}(k/2^j)) - \Pi_s(\underline{Y}\dot{\underline{W}}(s)), \psi_k^j\rangle\right| \leq \sum_{\beta\in\{2\alpha-1,\alpha-1\}} 2^{-j\beta-j/2}|k/2^j - s|^{3\alpha-1-\beta}.$$

Moreover ψ has a compact support in $[0,1]$, so ψ_k^j has a compact support in $[k/2^j,(k+1)/2^j]$. The only non-zero term in (45) satisfies $|k/2^j - s| \le 2^{-j}$. Finally,

$$\left|\langle \Pi_{k/2^j}(\underline{Y}\dot{\underline{W}}(k/2^j)) - \Pi_s(\underline{Y}\dot{\underline{W}}(s)), \psi_k^j\rangle\right| \le C2^{+j/2}2^{-3j\alpha}. \tag{48}$$

Summing (47) and (48) we have

$$|\langle \mathcal{R}(\underline{Y}\dot{\underline{W}}) - \Pi_s(\underline{Y}\dot{\underline{W}}(s)), \psi_k^j\rangle| \le C2^{-j/2}2^{j(1-3\alpha)}. \tag{49}$$

With (49) and (42), we bound the first sum of the right side of (45)

$$\left|\sum_{k=0}^{2^{j_0}-1} \langle \mathcal{R}(\underline{Y}\dot{\underline{W}}) - \Pi_s(\underline{Y}\dot{\underline{W}}(s)), \psi_k^{j_0}\rangle \int_s^t \psi_k^{j_0}\right| \le C|t-s|2^{-j_0(3\alpha-1)} \tag{50}$$

$$\le C|t-s|^{3\alpha}, \tag{51}$$

where we use the fact that ψ has a compact support. In the same way,

$$\left|\sum_{k=0}^{2^{j_0}-1} \langle \mathcal{R}(\underline{Y}\dot{\underline{W}}) - \Pi_s(\underline{Y}\dot{\underline{W}}(s)), \phi_k^{j_0}\rangle \int_s^t \phi_k^{j_0}\right| \le C|t-s|^{3\alpha}. \tag{52}$$

It remains to obtain a bound of (45). We observe that

$$\left|\sum_{k=0}^{2^{j}-1} \int_s^t \psi_k^{j}\right| \le C2^{-j/2}, \tag{53}$$

because a primitive of ψ has a compact support on $[0,1]$ thanks to the fact that $\int \psi = 0$. Then, combining (49) and (53) we obtain from the choice of j_0,

$$\left|\sum_{j>j_0}\sum_{k=0}^{2^j-1} \langle \mathcal{R}(\underline{Y}\dot{\underline{W}}) - \Pi_s(\underline{Y}\dot{\underline{W}}(s)), \psi_k^{j}\rangle \int_s^t \psi_k^{j}\right| \le \sum_{j>j_0} 2^{-3j\alpha}$$

$$\le C2^{-3j_0\alpha}$$

$$\le C|t-s|^{3\alpha}. \tag{54}$$

With (51), (52) and (54) we obtain the bound of (45),

$$|I(\underline{Y})_{s,t} - y_s W_{s,t} - y'_s \mathbb{W}_{s,t}| \le C|t-s|^{3\alpha}. \tag{55}$$

For showing that $L(\underline{Y})$ is in $D^{2\alpha}_{M^r}$, we compute $\Gamma^r_{t,s}(L(\underline{Y})(s)) = (I(\underline{Y})(s) + y_s W_{s,t})\underline{1} + y_s\underline{W}$ and we use the estimation (55). Thus, we have

$$\left\|L(\underline{Y})(t) - \Gamma^r_{t,s}(L(\underline{Y})(s))\right\|_0 = \|I(\underline{Y})(t) - I(\underline{Y})(s) - y_s W_{s,t}\| \tag{56}$$

$$\le \left\|y'\right\|_{\infty,T} \|\mathbb{W}\|_{2\alpha,T} |t-s|^{2\alpha} + C|t-s|^{3\alpha}, \tag{57}$$

$$\text{and } \|L(\underline{Y})(t) - \Gamma_{t,s}(L(\underline{Y})(s))\|_\alpha = \|y_{s,t}\| \leq \|y\|_\alpha |t-s|^\alpha, \tag{58}$$

which proves that $L(\underline{Y})$ is in $D^{2\alpha}_{M^r}$.

It remains to prove the continuity of L. According to (31), the constant C in (55) is proportional to $\|\underline{Y}\|^*_{\gamma,T}$. So we have,

$$|I(\underline{Y})_{s,t} - y_s W_{s,t}| \leq \left\|y'\right\|_\infty \|\mathbb{W}\|_{2\alpha,T} |t-s|^{2\alpha} + C \|Y\|^*_{2\alpha,T} |t-s|^{3\alpha},$$

which allows with the previous computation (57) and (58) to bound

$$\|L(\underline{Y})\|^*_{2\alpha,T} \leq C \|\underline{Y}\|^*_{2\alpha,T}. \tag{59}$$

This concludes the proof. ■

8 Existence of a rough path lift

As an application of the reconstruction operator in the case $\gamma \leq 0$, we prove Theorem 4.6 which states that for any $W \in C^\alpha$ ($\alpha \in (1/3, 1/2]$) with values in $\mathbb{R}^n$, it exists a rough path lift $\mathbb{W}$ and that the map $W \mapsto \mathbf{W}$ is continuous from C^α to $\mathscr{C}^\alpha$.

Proof (Theorem 4.6). We consider the regularity structure $(\mathcal{A}^e, \mathcal{T}^e, \mathcal{G}^e)$ such that $\mathcal{A}^e = \{\alpha-1, 0\}$, $\mathcal{T}^e = \text{Vect}\langle \underline{\dot{W}}^i, i = 1, \dots, n\rangle \bigoplus \text{Vect}\langle \underline{1}\rangle$ and for $\Gamma^e_h \in \mathcal{G}$, $\Gamma^e_h(\underline{\dot{W}}) = \underline{\dot{W}}$, $\Gamma^e_h(\underline{1}) = \underline{1}$. We associate the model $M^e = (\Pi^e, \Gamma^e)$ such that for every $s, t \in [0, T]$, $\eta \in B_1$ we have

$$\Pi^e_s(\underline{\dot{W}})(\eta) := \int \eta(t) \mathrm{d}W_t, \qquad \Pi^e_s(\underline{1})(t) := 1,$$

and $\Gamma^e_{s,t} := \Gamma^e_{W_{t,s}}$.

For $0 \leq s \leq t \leq 1$, and integers $0 \leq i, j \leq n$, the modelled distribution $\underline{\dot{\mathbb{W}}}$ given by $\underline{\dot{\mathbb{W}}^{i,j}}(s) := W^i_s \underline{\dot{W}}^j$ is in $D^{2\alpha-1}_M$. Indeed $\underline{\dot{\mathbb{W}}^{i,j}}(t) - \Gamma^e_{t,s}\left(\underline{\dot{\mathbb{W}}^{i,j}}(s)\right) = W^i_t \underline{\dot{W}}^j - W^i_s \underline{\dot{W}}^j = W^i_{s,t} \underline{\dot{W}}^j$, then

$$\left\|\underline{\dot{\mathbb{W}}^{i,j}}(t) - \Gamma^e_{t,s}\left(\underline{\dot{\mathbb{W}}^{i,j}}(s)\right)\right\|_{\alpha-1} \leq |t-s|^\alpha.$$

So, $\gamma - (\alpha - 1) = \alpha$, we have $\gamma = 2\alpha - 1$. We conclude using the Definition 6.1.

Given that $\alpha \in (1/3, 1/2]$, we have $2\alpha - 1 \leq 0$ so the uniqueness of the reconstruction map does not hold. But, according to Theorem 6.5, there exists $\mathcal{R}(\underline{\dot{\mathbb{W}}}) \in C^{\alpha-1}$ such that

$$|[\mathcal{R}(\underline{\dot{\mathbb{W}}}) - \Pi^e_s(\underline{\dot{\mathbb{W}}})](\eta^\lambda_s)| \leq C\lambda^{2\alpha-1}, \tag{60}$$

where $\eta \in B_1$. With Lemma 3.10, we define $z \in C^\alpha$ as the primitive of $\mathcal{R}(\underline{\dot{\mathbb{W}}})$ such that $z(0) = 0$. Moreover, according to Remark 3.13, we have for all integer $j_0 \in \mathbb{N}$ and $s, t \in [0, 1]$,

$$z_{s,t} = \sum_{k=0}^{2^{j_0}-1} \langle \mathcal{R}(\underline{\dot{\mathbb{W}}}), \phi^{j_0}_k \rangle \int_s^t \phi^{j_0}_k + \sum_{j \geq j_0} \sum_{k=0}^{2^j-1} \langle \mathcal{R}(\underline{\dot{\mathbb{W}}}), \psi^j_k \rangle \int_s^t \psi^j_k. \tag{61}$$

and,

$$W_{s,t} = \sum_{k=0}^{2^{j_0}-1} \langle \Pi_s^e(\dot{\underline{W}}), \phi_k^{j_0} \rangle \int_s^t \phi_k^{j_0} + \sum_{j \geq j_0} \sum_{k=0}^{2^j-1} \langle \Pi_s^e(\dot{\underline{W}}), \psi_k^j \rangle \int_s^t \psi_k^j, \tag{62}$$

which yields to

$$W_s \otimes W_{s,t} = \sum_{k=0}^{2^{j_0}-1} \langle \Pi_s^e(\dot{\underline{\mathbb{W}}}(s)), \phi_k^{j_0} \rangle \int_s^t \phi_k^{j_0} + \sum_{j \geq j_0} \sum_{k=0}^{2^j-1} \langle \Pi_s^e(\dot{\underline{\mathbb{W}}}(s)), \psi_k^j \rangle \int_s^t \psi_k^j. \tag{63}$$

If there is a constant $C \geq 0$ such that,

$$|z_{s,t} - W_s \otimes W_{s,t}| \leq C|t-s|^{2\alpha}, \tag{64}$$

then setting $\mathbb{W}_{s,t} := z_{s,t} - W_s \otimes W_{s,t}$, the pair $(W, \mathbb{W})$ belongs to $\mathscr{C}^\alpha$ according to the Definition 4.1. Let us prove (64). We choose the integer j_0 such that $2^{-j_0} \leq |t-s| < 2^{-j_0+1}$. Thus, we have

$$\begin{aligned} z_{s,t} - W_s \otimes W_{s,t} = & \sum_{k=0}^{2^{j_0}-1} \langle \mathcal{R}(\dot{\underline{\mathbb{W}}}) - \Pi_s^e(\dot{\underline{\mathbb{W}}}(s)), \phi_k^{j_0} \rangle \int_s^t \phi_k^{j_0} \\ & + \sum_{j \geq j_0} \sum_{k=0}^{2^j-1} \langle \mathcal{R}(\dot{\underline{\mathbb{W}}}) - \Pi_s^e(\dot{\underline{\mathbb{W}}}(s)), \psi_k^j \rangle \int_s^t \psi_k^j. \end{aligned}$$

We proceed exactly as in the proof of the Theorem 7.3 equation (49) to show that

$$|\langle \mathcal{R}(\dot{\underline{\mathbb{W}}}) - \Pi_s^e(\dot{\underline{\mathbb{W}}}(s)), \phi_k^j \rangle| \leq C 2^{-j/2 - j(2\alpha-1)}, \tag{65}$$

and

$$|\langle \mathcal{R}(\dot{\underline{\mathbb{W}}}) - \Pi_s^e(\dot{\underline{\mathbb{W}}}(s)), \psi_k^j \rangle| \leq C 2^{-j/2 - j(2\alpha-1)}. \tag{66}$$

Then, using the Taylor's inequality,

$$\begin{aligned} \sum_{k=0}^{2^{j_0}-1} \langle \mathcal{R}(\dot{\underline{\mathbb{W}}}) - \Pi_s^e(\dot{\underline{\mathbb{W}}}(s)), \phi_k^{j_0} \rangle \int_s^t \phi_k^{j_0}| &\leq C|t-s| 2^{-j_0(2\alpha-1)} \\ &\leq C|t-s|^\alpha, \end{aligned} \tag{67}$$

and we have the same estimation if we replace ϕ by ψ. It remains to control

$$\begin{aligned} \left| \sum_{j > j_0} \sum_{k=0}^{2^j-1} \langle \mathcal{R}(\dot{\underline{\mathbb{W}}}) - \Pi_s^e(\dot{\underline{\mathbb{W}}}(s)), \psi_k^j \rangle \int_s^t \psi_k^j \right| &\leq C 2^{-j/2} 2^{-j(2\alpha-1)} \sup_{s,t \in [0,1]} \sum_{k=0}^{2^j-1} \int_s^t |\psi_k^j| \\ &\leq C 2^{-j2\alpha} 2 \sup_{t \in [0,1]} \sum_{k=0}^{2^j-1} |\hat{\psi}(2^j t - k)| \\ &\leq C 2^{-j2\alpha} \\ &\leq C|t-s|^{2\alpha} \end{aligned} \tag{68}$$

where $\hat{\psi}$ is a primitive of ψ with a compact support in $[0,1]$ because $\int \psi = 0$. Combining the bounds (67),(68), we obtain the bound (64), which concludes the proof of the existence of $\mathbb{W}$.

It remains to show the continuity. If there is another path $\tilde{W} \in C^\alpha$, we define as for W, a model $(\tilde{\Pi}, \tilde{\Gamma})$, a modelled distribution $\underline{\tilde{\mathbb{W}}}$, a reconstruction map $\tilde{\mathcal{R}}$ and then $\tilde{\mathbb{W}}$. By denoting

$$\Delta\Pi_{s,k/2^j} := [\Pi(\underline{\dot{\mathbb{W}}}(k/2^j)) - \Pi(\underline{\dot{\mathbb{W}}}(s)) - \tilde{\Pi}(\underline{\tilde{\dot{\mathbb{W}}}}(k/2^j)) + \tilde{\Pi}(\underline{\tilde{\dot{\mathbb{W}}}}(s))](\psi_k^j), \quad (69)$$

we have

$$\begin{aligned} &|\Delta\Pi_{s,k/2^j}| \\ &\leq \left\|W - \tilde{W}\right\|_{\alpha,T} \left(\|W\|_{\alpha,T} + \left\|\tilde{W}\right\|_{\alpha,T} \right) |s - k/2^j| 2^{j/2(1-\alpha)} 2^{-j/2}. \end{aligned} \quad (70)$$

According to the bounds (32), (67),(68), (70) and in writing

$$\begin{aligned} &[\mathcal{R}(\underline{\dot{\mathbb{W}}}) - \Pi_s(\underline{\dot{\mathbb{W}}}(s)) - \tilde{\mathcal{R}}(\underline{\tilde{\dot{\mathbb{W}}}}) + \Pi_s(\underline{\tilde{\dot{\mathbb{W}}}}(s))](\psi_k^j) \\ &= \mathcal{R}(\underline{\dot{\mathbb{W}}}) - \Pi_{k/2^j}(\underline{\dot{\mathbb{W}}}(k/2^j)) - \tilde{\mathcal{R}}(\underline{\tilde{\dot{\mathbb{W}}}}) + \tilde{\Pi}_{k/2^j}(\underline{\tilde{\dot{\mathbb{W}}}}(k/2^j)) + \Delta\Pi_{s,k/2^j} \end{aligned}$$

we get

$$\begin{aligned} |\mathbb{W}_{s,t} - \tilde{\mathbb{W}}_{s,t}| \leq C\Big(&\left\|\tilde{\Pi}\right\|_{2\alpha-1,T} \left\|\underline{\dot{\mathbb{W}}} - \underline{\tilde{\dot{\mathbb{W}}}}\right\|^*_{2\alpha-1,T} + \left\|\Pi - \tilde{\Pi}\right\|_{2\alpha-1,T} \left\|\underline{\dot{\mathbb{W}}}\right\|^*_{2\alpha-1,T} \\ &+ \left\|W - \tilde{W}\right\|_{\alpha,T} \left(\|W\|_{\alpha,T} + \left\|\tilde{W}\right\|_{\alpha,T} \right) \Big) |t-s|^{2\alpha}. \end{aligned}$$

Yet we have, $\left\|\underline{\dot{\mathbb{W}}} - \underline{\tilde{\dot{\mathbb{W}}}}\right\|^*_{2\alpha-1,T} = \left\|W - \tilde{W}\right\|_{\infty,T} + \left\|W - \tilde{W}\right\|_{\alpha,T}$, and

$$\left\|\Pi - \tilde{\Pi}\right\|_{2\alpha-1,T} \leq C \left\|W - \tilde{W}\right\|_{\alpha,T}. \quad (71)$$

So finally,

$$\left\|\mathbf{W} - \tilde{\mathbf{W}}\right\|_{\alpha,T} \leq C \left\|W - \tilde{W}\right\|_{\alpha,T}, \quad (72)$$

which proves the continuity.

■

Remark 8.1. Given that $2\alpha - 1$ is negative, the uniqueness of $\mathbb{W}$ does not hold, which is in accordance with Remark 4.5.

9 Composition with a smooth function

Before solving the general rough differential equation (1) with the theory of regularity structures, we should give a sense of the composition of a modelled

distribution with a function. Then we will be able to consider (1) in the space of the modelled distributions.

The composition of a modelled distribution $\underline{f} \in D^{\gamma}_{M}$ with a smooth function F is developed in [Hai14]. The author gives a general theorem which allows the composition with an arbitrary smooth function F when $\underline{f}$ takes its values in a model space $\mathcal{T}$ such that the smallest index of homogeneity is equal to 0, *i.e.* $\forall t \in \mathbb{R}$, $\underline{f}(t) \in \mathrm{Vect}\langle \underline{1}, ... \rangle$. Thus, it is possible to define the composition as a Taylor expansion

$$\hat{F} \circ \underline{f}(t) = \sum_{k} \frac{F^{(k)}(\bar{\underline{f}}(t))}{k!} (\underline{f}(t) - \bar{\underline{f}}(t)\underline{1})^{k}, \tag{73}$$

where $\bar{\underline{f}}$ is the coordinate of $\underline{f}$ onto $\underline{1}$. The definition above makes no sense if the product between elements of the regularity structure is not defined. We can also find the general definition in [Hai14] but it is not useful here. The idea of the decomposition (73) is to compute a Taylor expansion of F in $\bar{\underline{f}}$ the part of $\underline{f}$ which is the first approximation of $\mathcal{R}\underline{f}$.

Here we just prove (what is needed for solving (1)) that $\hat{F} \circ \underline{f}$ lives in the same space as $\underline{f}$ and that $\hat{F}$ is Lipschitz in the particular case of modelled distribution of controlled rough paths.

Theorem 9.1. *Let be $F \in C^{2}_{b}(\mathbb{R}^{d}, \mathcal{L}(\mathbb{R}^{n}, \mathbb{R}^{d}))$. For $\alpha \in (1/3, 1/2]$, given a rough path $\boldsymbol{W} = (W, \mathbb{W}) \in \mathscr{C}^{\alpha}$, the controlled rough path $(y, y') \in \mathscr{D}^{2\alpha}_{W}$, for all $\underline{Y} \in D^{2\alpha}_{M^{r}}$ defined by $\underline{Y}(t) = y_t\underline{1} + y'_t\underline{W}$, the map $\hat{F}$ such that*

$$\hat{F} \circ \underline{Y}(t) := F(y_t)\underline{1} + F'(y_t)y'_t\underline{W}, \tag{74}$$

is in $D^{2\alpha}_{M^{r}}$. Moreover if $F \in C^{3}_{b}$ the function associated $\hat{F}$ is Lipschitz, i.e. *for all $\underline{Y}, \widetilde{\underline{Y}} \in D^{2\alpha}_{M^{r}}$*

$$\left\| \hat{F}(\underline{Y}) - \hat{F}(\widetilde{\underline{Y}}) \right\|^{*}_{2\alpha,T} \leq C \left\| \underline{Y} - \widetilde{\underline{Y}} \right\|^{*}_{2\alpha,T}, \tag{75}$$

where C is a constant.

Proof. Firstly, let us show that $\hat{F}$ is a map from $D^{2\alpha}_{M^{r}}$ to $D^{2\alpha}_{M^{r}}$. A straightforward computation leads us to the two following expressions

$$\left\| \hat{F}(\underline{Y})(t) - \Gamma^{r}_{t,s}\left(\hat{F}(\underline{Y})(s)\right) \right\|_{0} = \left\| F'(y_t)y'_t - F'(y_s)y'_s \right\|,$$
$$\left\| \hat{F}(\underline{Y})(t) - \Gamma^{r}_{t,s}\left(\hat{F}(\underline{Y})(s)\right) \right\|_{\alpha} = \left\| F(y_t) - F(y_s) - F'(y_s)y'_s W_{s,t} \right\|.$$

Let us denote the left-hand of the first equality $\Delta^{0}_{s,t}$ and of the second one $\Delta^{\alpha}_{s,t}$. We obtain

$$\begin{aligned} \Delta^{0}_{s,t} &\leq \left\| F'(y_t) \right\| \left\| y'_t - y'_s \right\| + \left\| y'_s \right\| \left\| F'(y_t) - F'(y_s) \right\| \\ &\leq \left\| F' \right\|_{\infty,T} \left\| y' \right\|_{\alpha} |t-s|^{\alpha} + \left\| y' \right\|_{\infty,T} \left\| F'' \right\|_{\infty,T} \left\| Y \right\|_{\alpha,T} |t-s|^{\alpha} \end{aligned}$$

and

$$\begin{aligned}\Delta^{\alpha}_{s,t} &= \left\|F(y_t) - F(y_s) - F'(y_s)(y_{s,t} - R^y_{s,t})\right\| \\ &\leq \left\|F(y_t) - F(y_s) - F'(y_s)y_{s,t}\right\| + \left\|F'(y_s)R^y_{s,t}\right\| \\ &\leq \frac{1}{2}\left\|F''\right\|_{\infty,T}\|y_{s,t}\|^2 + \left\|F'\right\|_{\infty,T}\|R^y\|_{2\alpha,T}\,|t-s|^{2\alpha} \\ &\leq \frac{1}{2}\left\|F''\right\|_{\infty,T}\|y\|^2_{\alpha,T}\,|t-s|^{2\alpha} + \left\|F'\right\|_{\infty,T}\|R^y\|_{2\alpha,T}\,|t-s|^{2\alpha}.\end{aligned}$$

This proves that $\hat{F}(\underline{Y}) \in D^{2\alpha}_{M^r}$.

We now prove the inequality (75). A more general proof can be found in [Hai14]. We define $\widetilde{\underline{Z}} = \underline{Y} - \widetilde{\underline{Y}}$, which is in $D^{2\alpha}_{M^r}$ by linearity. We denote by $Q_{<2\alpha}$ the projection onto $\mathcal{T}_{<2\alpha}$. Using the integration by parts formula, one can check that

$$\hat{F}(\underline{Y}(s)) - \hat{F}(\widetilde{\underline{Y}}(s)) = \sum_{k=0}^{1}\int_0^1 F^{(k)}(\widetilde{y}_s + uz_s)Q_{<2\alpha}\left[\left[(\widetilde{y}'_s + uz'_s)\underline{W}\right]^k\widetilde{\underline{Z}}(s)\right]\mathrm{d}u.$$

Then, we compute the expansion between s and t of $\underline{\Delta}(s) := \hat{F}(\underline{Y}(s)) - \hat{F}(\widetilde{\underline{Y}}(s))$. We denote $\underline{A}_u(s) := \widetilde{\underline{Y}}(s) + u\widetilde{\underline{Z}}(s)$. When u is fixed, $\underline{A}_u$ is in $D^{2\alpha}_{M^r}$. We have

$$\begin{aligned}\Gamma_{t,s}\underline{\Delta}(s) &= \sum_{k=0}^{1}\int_0^1 F^{(k)}(\underline{A}_u(s))\Gamma_{t,s}Q_{<2\alpha}\left([A'_u(s)\underline{W}]^k\widetilde{\underline{Z}}(s)\right)\mathrm{d}u \\ &= \sum_{k=0}^{1}\int_0^1 F^{(k)}(\underline{A}_u(s))[\Gamma_{t,s}(A'_u(s)\underline{W})]^k\Gamma_{t,s}\widetilde{\underline{Z}}(s)\mathrm{d}u + \underline{R}(s,t),\end{aligned}$$

where $\underline{R}$ is a remainder such that $\|\underline{R}(s,t)\|_\beta \lesssim |t-s|^{2\alpha-\beta}$ for $\beta \in \{0,\alpha\}$. From now, we denote by $\underline{R}$ all the remainder terms which satisfy this property.

We now shift the last expression from s to t. On the one hand

$$\Gamma^r_{t,s}(A'_u(s)\underline{W}) = \Gamma^r_{t,s}\underline{A}_u(s) - A_u(s)\underline{1} = \underline{A}_u(t) - A_u(s)\underline{1} + \underline{R}(s,t).$$

On the other hand

$$\Gamma^r_{t,s}\widetilde{\underline{Z}}(s) = \widetilde{\underline{Z}}(t) + \underline{R}(s,t).$$

This yields

$$\Gamma^r_{t,s}\underline{\Delta}(s) = \sum_{k=0}^{1}\int_0^1 F^{(k)}(\underline{A}_u(s))[A'_u(t)\underline{W} + (A_u(t) - A_u(s))\underline{1}]^k\widetilde{\underline{Z}}(s)\mathrm{d}u + \underline{R}(s,t).$$

It remains to shift $F^{(k)}$ from s to t. With the classical Taylor expansion formula,

$$F^{(k)}(A_u(s)) = \sum_{0\le l+k\le 1} F^{(k+l)}(A_u(t))(A_u(s)-A_u(t))^l + O(|t-s|^{2\alpha-k\alpha}),$$

because $\|A_u(t)-A_u(s)\| \le |t-s|^\alpha$. And the bound

$$\left\| [A'_u(t)\underline{W} + (A_u(t)-A_u(s))\underline{1}]^k \right\|_\beta \lesssim |t-s|^{k\alpha-\beta}$$

holds. Finally, with the two previous expressions,

$$\begin{aligned}
\Gamma^r_{t,s}\underline{\Delta}(s) &= \sum_{0\le l+k\le 1} F^{(k+l)}(A_u(t))(A_u(s)-A_u(t))^l \\
&\times [A'_u(t)\underline{W} + (A_u(t)-A_u(s))\underline{1})]^k \widetilde{\underline{Z}}(t) + \left\|\widetilde{\underline{Z}}\right\|^*_{2\alpha,T} O(|t-s|^{2\alpha-\beta}) \\
&= \sum_{0\le k\le 1} F^{(k)}(A_u(t))[A'_u(t)\underline{W}]^k \widetilde{\underline{Z}}(t) + \left\|\widetilde{\underline{Z}}\right\|^*_{2\alpha,T} O(|t-s|^{2\alpha-\beta}) \\
&= \underline{\Delta}(t) + \left\|\widetilde{\underline{Z}}\right\|^*_{2\alpha,T} O|t-s|^{2\alpha-\beta},
\end{aligned}$$

which proves the inequality. ∎

10 Solving the rough differential equations

Theorem 7.3 combined with Theorem 9.1 allow us to solve the rough differential equations in the modelled distribution space $D^{2\alpha}_{M^r}$.

Theorem 10.1. *Given $\xi \in \mathbb{R}^d$, $F \in C^3(\mathbb{R}^d, \mathcal{L}(\mathbb{R}^n, \mathbb{R}^d))$, a rough path $\boldsymbol{W} = (W, \mathbb{W}) \in \mathscr{C}^\beta$ with $\beta \in (1/3, 1/2)$, there is a unique modelled distribution $\underline{Y} \in D^{2\beta}_{M^r}$ such that for all $t < \delta$,*

$$\underline{Y}(t) = \xi\underline{1} + L(\hat{F}(\underline{Y}))(t), \tag{76}$$

where $\delta > 0$ and L is defined in Theorem 7.3. If $F \in C^3_b$ the solution is global in time.

Proof. We prove that the operator $N(\underline{Y}) := \xi\underline{1} + L(\hat{F}(\underline{Y}))$ where L is defined in Theorem 7.3, has a unique fixed point. For this we show that the unit ball of $D^{2\alpha}_{M^r}$ is invariant under the action of N, and then that N is a strict contraction.

These two properties can be obtained by choosing a wise time interval $[0, T]$. We take a rough path $\mathbf{W} = (W, \mathbb{W}) \in \mathscr{C}^\beta \subset \mathscr{C}^\alpha$ with $1/3 < \alpha < \beta < 1/2$ and $\underline{Y} \in D^{2\alpha}_{M^r}$. This trick allows us to have a $T^{\beta-\alpha}$ in our estimates. Thus, with a T small enough we prove the fixed point property. We start by choosing $T \le 1$.

According to Theorem 9.1 $\hat{F}(\underline{Y}) \in D^{2\alpha}_{M^r}$, thus Theorem 7.3 shows that $N(\underline{Y}) \in D^{2\alpha}_{M^r}$. If $\underline{Y}$ is a fixed point of N then $\underline{Y} \in D^{2\beta}_{M^r}$, thanks to the fact that $\mathbf{W} \in \mathscr{C}^\beta$. Indeed,

$$\|\underline{Y}(t) - \Gamma_{t,s}\underline{Y}(s)\|_\beta = \|y_{s,t}\| \leq \left\|y'\right\|_{\infty,T} \|W\|_{2\beta,T} |t-s|^{2\beta} + \|R^y\|_{2\alpha,T} |t-s|^{2\alpha},$$

and

$$\|\underline{Y}(t) - \Gamma_{t,s}\underline{Y}(s)\|_0 = \left\|y_{s,t} - y'_s W_{s,t}\right\| \leq \left\|y'\right\|_{\infty,T} \|\mathbb{W}_{s,t}\| + O(|t-s|^{3\alpha}).$$

As a result of the fixed point property $y' = F(y)$. This proves that $\underline{Y} \in D^{2\beta}_{M^r}$.

We recall that $\|\underline{Y}\|^*_{2\alpha,T} = \sup_{\epsilon\in\{0,\alpha\}} \|\underline{Y}(0)\|_\epsilon + \|\underline{Y}\|_{2\alpha,T}$, where

$$\|\underline{Y}\|_{2\alpha,T} = \sup_{t,s\in[0,T],\epsilon\in\{0,\alpha\}} \frac{\|\underline{Y}(t) - \Gamma_{t,s}\underline{Y}(s)\|_\epsilon}{|t-s|^\epsilon}.$$

It is more convenient to work with the semi-norm $\|\cdot\|_{2\alpha,T}$, so we define the affine ball unit on $[0,T]$

$$B_T = \{\underline{Y} \in D^{2\alpha}_{M^r},\ \underline{Y}(0) = \xi\underline{1} + f(\xi)\underline{W},\ \|\underline{Y}\|_{2\alpha,T} \leq 1\}.$$

Invariance: For $\underline{Y} \in B_T$, on has $\left\|\hat{F}(\underline{Y})\right\|_{2\alpha,T} \leq \|\underline{Y}\|_{2\alpha,T}$ and $\|N(\underline{Y})\|_{2\alpha,T} = \left\|L(\hat{F}(\underline{Y}))\right\|_{2\alpha,T}$. On the on hand, according to the reconstruction map,

$$\begin{aligned}
&\left\|(I\hat{F}(\underline{Y}))_{s,t} - F(y_s)W_{s,t}\right\| \\
\leq\ & \left\|F'(y)y'\right\|_{\infty,T} \|\mathbb{W}\|_{2\alpha,T} |t-s|^{2\alpha} + C\left\|\hat{F}(\underline{Y})\right\|^*_{2\alpha,T} |t-s|^{3\alpha} \\
\leq\ & \left\|\hat{F}(\underline{Y})\right\|^*_{2\alpha,T} \|\mathbb{W}\|_{2\alpha} |t-s|^{2\alpha} + C\left\|\hat{F}(\underline{Y})\right\|^*_{2\alpha,T} |t-s|^{3\alpha} \\
\leq\ & \left\|F'\right\|_\infty \left[\|(\underline{Y})\|^*_{2\alpha,T} \|\mathbb{W}\|_{2\beta,T} T^{\beta-\alpha}|t-s|^{2\alpha} + C\|\underline{Y}\|^*_{2\alpha,T} |t-s|^{2\alpha}T^\alpha\right],
\end{aligned}$$

because $\|\cdot\|_\beta \leq \|\cdot\|_\alpha T^{\beta-\alpha}$. Using the fact that $T^\alpha \leq T^{\beta-\alpha}$ and that $\underline{Y} \in B_T$ we obtain

$$\|N(\underline{Y})\|_0 \leq CT^{\beta-\alpha},$$

where C is independent of $\underline{Y}$. On the other hand,

$$\begin{aligned}
\|y_{s,t}\| &\leq \left\|y'\right\|_{\infty,T} \|W\|_{\alpha,T} |t-s|^\alpha + \|R^y\|_{2\alpha,T} |t-s|^{2\alpha} \\
&\leq \|\underline{Y}\|^*_{2\alpha,T} \|W\|_{\beta,T} T^{\beta-\alpha}|t-s|^\alpha + \|R^y\|_{2\alpha,T} T^\alpha|t-s|^\alpha \\
&\leq \|\underline{Y}\|^*_{2\alpha,T} \|W\|_{\beta,T} T^{\beta-\alpha}|t-s|^\alpha + \|\underline{Y}\|^*_{2\alpha,T} T^\alpha|t-s|^\alpha.
\end{aligned}$$

Using the last inequality

$$\begin{aligned}
\|F(y)\|_{\alpha,T} &\leq \left\|F'\right\|_\infty \|y\|_{\alpha,T} \\
&\leq \|\underline{Y}\|^*_{2\alpha,T} \|W\|_{\beta,T} T^{\beta-\alpha}|t-s|^\alpha + \|\underline{Y}\|^*_{2\alpha,T} T^\alpha|t-s|^\alpha,
\end{aligned}$$

which leads to $\|N(\underline{Y})\|_\alpha \leq CT^{\beta-\alpha}$. Finally, we obtain the following estimate $\|N(\underline{Y})\|_{2\alpha,T} \leq CT^{\beta-\alpha}$, where C does not depend on $\underline{Y}$. By choosing $T = T_0$ small enough, we show that $N(B_{T_0}) \subset B_{T_0}$.

Contraction: For $\underline{Y}, \widetilde{\underline{Y}} \in D^{2\alpha}_{M^r}$,

$$\begin{aligned}\left\|N(\underline{Y}) - N(\widetilde{\underline{Y}})\right\|_{2\alpha,T} &\leq \left\|N(\underline{Y}) - N(\widetilde{\underline{Y}})\right\|_0 + \left\|N(\underline{Y}) - N(\widetilde{\underline{Y}})\right\|_\alpha \\ &\leq C\left\|\hat{F}(Y) - \hat{F}(\widetilde{\underline{Y}})\right\|^*_{2\alpha,T} T^{\beta-\alpha} + \|F(y) - F(\widetilde{y})\|_\alpha \\ &\leq C\left\|\underline{Y} - \widetilde{\underline{Y}}\right\|^*_{2\alpha,T} T^{\beta-\alpha} + \left\|F'\right\|_\infty \|y - \widetilde{y}\|_\alpha ,\end{aligned}$$

according to (75). Then it is easy to show that

$$\|y - \widetilde{y}\|_\alpha \leq CT^{\beta-\alpha}\left\|\underline{Y} - \widetilde{\underline{Y}}\right\|_{2\alpha,T}.$$

Finally, $\left\|N(\underline{Y}) - N(\widetilde{\underline{Y}})\right\|_{2\alpha,T} \leq CT^{\beta-\alpha}\left\|\underline{Y} - \widetilde{\underline{Y}}\right\|_{2\alpha,T}$ where C does not depend on neither $\underline{Y}$ nor $\widetilde{\underline{Y}}$. So with T small enough, $N(B_T) \subset B_T$ and N is a strict contraction. So, there is a unique solution $\underline{Y} \in D^{2\alpha}_{M^r}$ to (76) on $[0, T]$. As mentioned at the beginning of the proof, $\underline{Y}$ is in$D^{2\beta}_{M^r}$. ■

Corollary 10.2. *Given $\xi \in \mathbb{R}^d$, $F \in C^3(\mathbb{R}^d, \mathcal{L}(\mathbb{R}^n, \mathbb{R}^d))$, a rough path $\boldsymbol{W} = (W, \mathbb{W}) \in \mathscr{C}^\beta$ with $\beta \in (1/3, 1/2)$, there is a unique controlled rough path $(y, y') \in \mathscr{D}^{2\beta}_W$ such that for all $t < \delta$,*

$$y(t) = \xi + \int_0^t F(y_u)\mathrm{d}\,\boldsymbol{W}_u, \tag{77}$$

where $\delta > 0$ and the integral has to be understood as the controlled rough path integral (Theorem 4.16). Furthermore, we have $y' = F(y)$. If $F \in C^3_b$ the solution is global in time.

Proof. It suffices to project Equation (76) onto $\underline{1}$ and onto $\underline{W}$. ■

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List of symbols

C^α α-Hölder continuous functions, page 4

$\mathscr{D}_W^{2\alpha}$ The controlled rough path space, page 6

$:=$ Symbol to define an object, page 4

$\lesssim$ Estimation, page 4

$\mathcal{G}$ Structure group of a regularity structure, page 10

$\mathcal{A}$ Set index of a regularity structure, page 10

$\langle \cdot, \underset{\sim}{1} \rangle$ The projection onto $\underset{\sim}{1}$, page 21

$\mathcal{L}(A, B)$ Linear continuous applications between two vector spaces A and B., page 4

D_M^γ Modelled distributions of regularity γ, page 17

$\mathcal{T}$ Model space of a regularity structure, page 10

$\mathcal{T}_{<\gamma}$ Elements of $\mathcal{T}$ of homogeneity strictly less than γ, page 17

$\star$ Abstract product, page 20

$\left\| \underset{\sim}{f} \right\|_{\gamma,T}^*$ Norm of the modelled distribution space, page 17

$\left\| \underset{\sim}{\tau} \right\|_\alpha$ Norm of the component of $\underset{\sim}{\tau}$ in $\mathcal{T}_\alpha$, page 10

$\|\cdot\|_{2\alpha,T}^W$ Semi-norm of the controlled rough path space, page 6

$\|f\|_{\infty,T}$ Supremum of $\|f(t)\|$ for $t \in [0,T]$, page 4

$\otimes$ Tensor product, page 4

$Q_{<2\alpha}$ The projection onto $\mathcal{T}_{<2\alpha}$, page 25

$\mathcal{R}$ The reconstruction operator, page 18

$\mathbf{W}$ Rough path associated to W, page 5

$\mathscr{T}$ A regularity structure, page 10

$\mathscr{T}^p$ Model associated to the regularity structure of the rough paths, page 16

$\mathscr{T}^p$ Polynomial regularity structure, page 14

$\mathscr{T}^p$ Regularity structure of the rough paths, page 16

$\mathrm{Vect}\langle \cdot \rangle$ Subspace generated by the linear combinations of $\cdot$., page 4

B_r The space of all functions in $\phi \in C_b^r$ compactly supported on $[-1, 1]$, such that $\|\phi\|_{C_b^r} \leq 1$., page 11

C_b^q Bounded functions with all derivative up to order q bounded. , page 4

$D_{M^r}^{2\alpha}$ Modelled distributions of the controlled rough paths, page 19

M Model associated to a regularity structure, page 13

M^p Model of the polynomial regularity structure, page 14

$Z_{s,t}$ Means $Z_t - Z_s$, page 4

$\mathcal{P}$ A subdivision, page 8

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