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Stability analysis of linear impulsive delay dynamical systems via looped-functionals

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Abstract

This article investigates the asymptotic stability of impulsive delay dynamical systems (IDDS) by using the Lyapunov-Krasovskii method and looped-functionals. The proposed conditions reduce the conservatism of the results found in the literature by allowing the functionals to grow during both the continuous dynamics and the discrete dynamics. Sufficient conditions for asymptotic stability in the form of linear matrix inequalities (LMI) are provided for the case of impulsive delay dynamical systems with linear and time-invariant (LTI) base systems (non-impulsive actions). Several numerical examples illustrate the effectiveness of the method.

Key words: Impulsive systems; stability; time-delay systems;

1 Introduction

Impulsive dynamical systems naturally arise as a framework for mathematical modeling of many real world processes that undergo abrupt state changes. Moreover, impulsive control techniques [26] have attracted increasing interest in the last two decades, because of its wide variety of applications, such as ecosystems management [12], orbital satellite [24], reset control systems [3, 14]. Roughly speaking, an impulsive dynamical system consists of three elements; a continuous-time dynamical equation, which governs the evolution of the system between reset (impulsive) events; a difference equation, which describes the way the system states are instantaneously changed; and finally a criterion for determining when the states of the system are to be reset [1, 16]. In addition, it is well known that time-delays phenomena frequently appear in many practical problems, such as biological systems, mechanical transmissions, fluid transmissions, networked control systems. Therefore, it is not surprising that impulsive delay dy-

namical systems have become an attractive research area (see e.g. [25]). In particular, during the last years their stability properties have been intensively studied [10, 17, 19, 27, 29].

The two main approaches (Lyapunov-Razumikhin technique and the Lyapunov-Krasovskii method) for investigating time-delay systems are also applied for the stability analysis of impulsive delay dynamical systems. In general, the stability results involve two conditions: a first condition imposes constraints over the derivative of the function/functional along system trajectories between two consecutive reset instants (as it has been made for non-impulsive systems with time-delay), and a second condition deals with the instantaneously change of the function/functional along the system trajectory at a reset instant. In fact, it is this last condition that makes the Lyapunov-Krasovskii method usually more difficult than Lyapunov-Razumikhin. Under these two conditions, the recent works [20, 21] provide sufficient conditions for the stability analysis of hybrid systems with time-delays, based on an extension of the hybrid inclusion model exposed in [14]. In general, those conditions mean that if the continuous dynamics is unstable, then the impulses must be frequent and their amplitude must be suitably related to the growth rate of the function/functional. On the contrary, impulses are not required to be very frequent and stabilizing (decrement of the function/functional) when the continuous dy-

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namics is stable.

In this work, the stability properties of impulsive delay dynamical systems with dwell-time constraints is addressed, by relying on the method of Lyapunov-Krasovskii (LK) functionals. This work takes inspiration from the recent results developed in the field of sampled-data systems [22, 23], where a Lyapunov functional/functional is combined with an auxiliary functional (referred to as looped-functional) to relax the required conditions. The approach of looped-functionals has been already applied to impulsive systems in [8, 9]. Nevertheless, the results are restricted to impulsive systems without time-delay. As opposed to previous results, the proposed criteria will not directly result in conditions on the instantaneous changes of the functional; instead of that, they will result in conditions based on a combination of the continuous dynamics and the impulsive dynamics. As a result, very general cases may be analyzed including, as a major contribution of this work, the case of unstable continuous dynamics and destabilizing impulses (they may increase the functional).

The rest of the paper is structured as follows. Section 2 contains some necessary notation and the formulation of impulsive delay dynamical systems. Section 3 provides a Lyapunov-Krasovskii-like proposition for guaranteeing the asymptotic stability of impulsive delay dynamical systems, and sufficient conditions in the form of LMI. To illustrate the results several examples are presented in Section 4. The main contributions are highlighted in Section 5. The proofs of the technical results are given in Appendices A and B.

2 Impulsive delay dynamical systems

2.1 Notation and background

The sets \mathbb{S}^n and \mathbb{S}_+^n denote the set of $n \times n$ symmetric matrices and the set of definite positive matrices, respectively. For a matrix $P \in \mathbb{S}^n$, $P > 0$ ($P < 0$) means that P is positive definite (negative definite). For a matrix $A \in \mathbb{R}^{n \times n}$, $\text{He}(A) = A + A^\top$. Given two vectors \mathbf{x}_1 and \mathbf{x}_2 , we write $(\mathbf{x}_1, \mathbf{x}_2)$ to denote $[\mathbf{x}_1^\top, \mathbf{x}_2^\top]^\top$. For a symmetric matrix $A \in \mathbb{R}^{n \times n}$, $\lambda_m(A)$ and $\lambda_M(A)$ stand for the minimum and maximum eigenvalue, respectively. The notation $\|\mathbf{x}\|$ is the Euclidean norm for $\mathbf{x} \in \mathbb{R}^n$. $\mathcal{PC}([a, b], \mathbb{R}^n)$ is the set of piecewise left-continuous functions with right limits, from $[a, b]$ to \mathbb{R}^n . The left and right limit are denoted by $\psi(\theta^-) = \psi(\theta) = \lim_{\epsilon \rightarrow 0, \epsilon < 0} \psi(\theta + \epsilon)$ and $\psi(\theta^+) = \lim_{\epsilon \rightarrow 0, \epsilon > 0} \psi(\theta + \epsilon)$, respectively. Therefore, for a function $\psi \in \mathcal{PC}([a, b], \mathbb{R}^n)$, a norm is defined as $\|\psi\| = \max_{\theta \in [a, b]} \|\psi(\theta)\|$. Analogously, $\mathcal{PC}([a, b] \times [c, d], \mathbb{R}^n)$ stands for the set of functions that are piecewise continuous on both arguments except in a finite number of points. The set of piecewise absolutely left-continuous functions $\psi : [a, b] \rightarrow \mathbb{R}^n$, with $\dot{\psi}$ the upper right-hand derivative, defined as

$\dot{\psi}(\theta) = \lim_{\epsilon \rightarrow 0, \epsilon > 0} \frac{\psi(\theta + \epsilon) - \psi(\theta)}{\epsilon}$, belonging to the set of square integrable functions, is denoted by $\mathcal{PAC}([a, b], \mathbb{R}^n)$, and its norm is defined by

$$\|\psi\|_{\mathcal{A}} = \max_{\theta \in [a, b]} \|\psi(\theta)\| + \left(\int_a^b \|\dot{\psi}(s)\|^2 \right)^{\frac{1}{2}}. \quad (1)$$

A function $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is of class \mathcal{K} if it is continuous, strictly increasing, and $f(0) = 0$.

2.2 Impulsive delay dynamical systems

This work focuses on linear impulsive delay dynamical systems, which are described by the following impulsive functional differential equation

$$\begin{cases} \dot{\mathbf{x}}(t) = A\mathbf{x}(t) + A_d\mathbf{x}(t-h), & t \neq t_k, \\ \mathbf{x}(t^+) = A_R\mathbf{x}(t), & t = t_k, \\ \mathbf{x}(t) = \phi(t), & t \in [-h, 0], \end{cases} \quad (2)$$

where $A, A_d, A_R \in \mathbb{R}^{n \times n}$, $h > 0$ is a given constant time-delay, and for $t \geq -h$, $\mathbf{x}(t) \in \mathbb{R}^n$ is the system state, $\dot{\mathbf{x}}(t)$ is the state upper right-hand derivative, $\mathbf{x}(t^+)$ is the after-reset state, and $\phi \in \mathcal{PC}([-h, 0], \mathbb{R}^n)$ is the initial condition. The sequence of reset instants $\{t_k\}_{k \in \mathbb{N}}$, $t_k > 0$ satisfies the dwell-time constraints $\Delta_k = t_{k+1} - t_k \geq \Delta_m$ and $\Delta_k \leq \Delta_M$ for some real values $\Delta_M \geq \Delta_m > 0$ (Zeno solutions do not exist). Since the provided results are characterized in terms of the dwell-time constraints, the sequence of reset instants can be generated by any reset law as long as the dwell-time constraints hold.

Note that, for $t \geq 0$, it is important to distinguish between the instantaneous state $\mathbf{x}(t) \in \mathbb{R}^n$ and the true (distributed) state $\mathbf{x}_t \in \mathcal{PC}([-h, 0], \mathbb{R}^n)$, which is defined by $\mathbf{x}_t(\theta) = \mathbf{x}(t+\theta)$, $\theta \in [-h, 0]$. For the sake of simplicity, the shifted-distributed state χ_k , as defined in [22], will be used; for any initial condition ϕ and any reset instant t_k , the shifted-distributed state χ_k is defined as the function $\chi_k \in \mathcal{PC}([0, \Delta_k] \times [-h, 0], \mathbb{R}^n)$, such that $\chi_k(\tau, \theta) = \mathbf{x}(t_k + \tau + \theta)$ for any $(\tau, \theta) \in [0, \Delta_k] \times [-h, 0]$. For any $t > 0$, it is clear that the distributed state may be recovered by using the shifted-distributed state, simply by doing $\mathbf{x}_t(\theta) = \chi_k(t - t_k, \theta)$, where t_k is the greatest reset instant such that $t_k \leq t$.

It is clear that both the functional differential equation and the impulsive equation are Lipschitz continuous, and that there exists a solution to (2) for all initial condition $\phi \in \mathcal{PC}([-h, 0], \mathbb{R}^n)$. The reader is referred to [2] for details of the notion of solutions and conditions for the existence and uniqueness of the solutions.

3 Stability analysis

In this section, we analyze the global asymptotic stability of system (2) by developing sufficient conditions

based on LK functionals, where in addition an auxiliary looped-functional is used to reduce the conservativeness.

3.1 Stability definitions and LK functional

The stability criterion developed in this section is based on a LK functional that depends on both the solution $\mathbf{x}(t)$ and its derivative¹ $\dot{\mathbf{x}}(t)$. Therefore, the initial conditions are restricted to the space of piecewise absolutely left-continuous functions with square integrable derivative (see, e.g. [13] and references therein), that is $\phi \in \mathcal{PAC}([-h, 0], \mathbb{R}^n)$.

Definition 3.1 *The trivial solution to system (2) is:*

- stable if for any $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that $\|\phi\|_{\mathcal{A}} < \delta$ implies $\|\mathbf{x}(t)\| < \epsilon$ for any $t \geq 0$,
- asymptotically stable if it is stable and there exists $\delta_a > 0$ such that $\lim_{t \rightarrow \infty} \|x(t)\| = 0$ whenever $\|\phi\|_{\mathcal{A}} < \delta_a$.
- globally asymptotically stable if it is asymptotically stable and $\delta_a > 0$ can be chosen arbitrarily large.

3.2 An improved LK theorem for stability of impulsive delay dynamical systems

The following proposition extends the idea of looped-functionals from [8, 23] to IDDSs. Roughly speaking, the main stability result, to be developed in the next Proposition, will be based on two functionals with the following properties: i) a LK functional satisfies a boundedness condition, ii) the sum of the LK functional and an auxiliary functional satisfy a decrescent condition, iii) a set of inequality conditions between both functionals guarantees that the increments of the LK functional between reset instants are bounded and converge to zero.

Let $V : \mathcal{PAC}([-h, 0], \mathbb{R}^n) \rightarrow \mathbb{R}_+$ be a functional, continuously differentiable with respect to its argument. If $\mathbf{x}(t)$ is a solution to (2) for some $\phi \in \mathcal{PAC}([-h, 0], \mathbb{R}^n)$, then $V(\mathbf{x}_t)$ has jump discontinuities at those values of t_k in which $\mathbf{x}(t_k)$ is discontinuous. For any $t \in [0, \infty) \setminus \{t_k\}_{k \in \mathbb{N}}$, the upper right-hand derivative of V along the solution $\mathbf{x}(t)$ to the system (2), is defined by

$$\dot{V}(\mathbf{x}_t) = \limsup_{\epsilon \rightarrow 0, \epsilon > 0} \frac{V(\mathbf{x}_{t+\epsilon}) - V(\mathbf{x}_t)}{\epsilon}. \quad (3)$$

Let us define $\mathbf{I}(\psi)$ for a function $\psi \in \mathcal{PAC}([-h, 0], \mathbb{R}^n)$, as follows:

$$\mathbf{I}(\psi)(\theta) = \begin{cases} \psi(\theta), & \text{if } \theta \in [-h, 0), \\ A_R \psi(0), & \text{if } \theta = 0. \end{cases} \quad (4)$$

¹ For the sake of clarity, the dependence of the functionals on the state derivative is not explicitly shown as an argument of the functionals.

Proposition 3.1 *Suppose $u, v, w : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ are continuous nondecreasing functions and in addition $u, v \in \mathcal{K}$. If there exist a real number $\eta \geq 0$ a functional $V : \mathcal{PAC}([-h, 0], \mathbb{R}^n) \rightarrow \mathbb{R}_+$ that satisfy*

$$u(\|\psi(0)\|) \leq V(\psi) \leq v(\|\psi\|_{\mathcal{A}}) \quad (5)$$

and

$$V(\mathbf{I}(\psi)) \leq \eta V(\psi) \quad (6)$$

for all $\psi \in \mathcal{PAC}([-h, 0], \mathbb{R}^n)$, and for any solution $\mathbf{x}(t)$ to (2) with initial condition $\phi \in \mathcal{PAC}([-h, 0], \mathbb{R}^n)$, there exists a continuous functional $\mathcal{V}_k : [0, \Delta_k] \times \mathcal{PAC}([0, \Delta_k] \times [-h, 0], \mathbb{R}^n) \rightarrow \mathbb{R}$ for every $k > 0$, which satisfies

$$\mathcal{V}_k(\Delta_k, \chi_k) - \mathcal{V}_k(0, \chi_k) = V(\chi_k(0^+, \cdot)) - V(\chi_k(0, \cdot)), \quad (7)$$

$$\mathcal{V}_k(0, \chi_k) \leq \eta V(\chi_k(0, \cdot)), \quad (8)$$

$$-\eta V(\chi_k(0, \cdot)) \leq \mathcal{V}_k(\tau, \chi_k) \quad (9)$$

for all $\tau \in [0, \Delta_k]$ and $k > 0$, and in addition, the following inequality hold

$$\dot{W}(\sigma, \chi_k) \leq -w(\|\chi_k(\sigma^+, 0)\|), \quad \sigma \in [0, \Delta_k], \quad k > 0, \quad (10)$$

where \dot{W} is evaluated along the solution to the system, and

$$W(\sigma, \chi_k) = \begin{cases} V(\chi_k(\sigma, \cdot)) + \mathcal{V}_k(\sigma, \chi_k), & \text{if } \sigma \in (0, \Delta_k], \\ V(\chi_k(0^+, \cdot)) + \mathcal{V}_k(0, \chi_k), & \text{if } \sigma = 0, \end{cases} \quad (11)$$

then the trivial solution to system (2) is stable. In addition, if $\lim_{s \rightarrow \infty} u(s) = \infty$ and $w(s) > 0$ for $s > 0$ then the trivial solution to system (2) is globally asymptotically stable.

Remark 3.1 *Proposition 3.1 can be directly applied to IDDSs where both the functional differential equation and the impulsive equation are nonlinear with local Lipschitz continuity. In addition, it is important to remark that the time-delay affects the continuous dynamics instead of only the sampling instants. In this way, for general IDDSs, it is not enough to guarantee the pointwise decrease of the LK functional. Indeed, the increments of the LK functional between reset instants should be bounded and converge to zero.*

Remark 3.2 *Slightly different versions of condition (7) have been used in [8, 9]. This condition provides a link between the discontinuities in V and the functionals \mathcal{V}_k , which allows to compensate the increments of the functional V due to the jumps with enough decrement during the flow, and the other way around.*

Remark 3.3 *The negativity of \dot{W} is not required when $k = 0$, which recalls the behavior of some LK functional for time-delay systems on the first delay interval. An illustration of this phenomenon is pointed out in the examples section.*

3.3 Stability criterion in term of LMIs

In this section, Prop. 3.1 and the LK functional proposed in [15] are used to provide sufficient LMI conditions for the global asymptotic stability of the IDDS (2).

Proposition 3.2 *For a given time-delay $h > 0$, and constants Δ_m and Δ_M satisfying $\frac{h}{r} < \Delta_m \leq \Delta_M$ for a given integer $r \geq 1$, the trivial solution to the system defined by (2) is globally asymptotically stable if there exist matrices $P, R, U, S_1 \in \mathbb{S}_+^n$, a matrix $Q \in \mathbb{S}_+^{nr}$, a matrix $X \in \mathbb{S}^n$, a matrix $S_2 \in \mathbb{R}^n$, and a matrix $Y \in \mathbb{R}^{(r+2)n \times n}$ such that*

$$\Pi_4 \leq 0 \text{ if } \Delta_m < \Delta_M, \quad (12)$$

$$M_0^\top \Pi_0 M_0 + \Pi_1 + N_R + \alpha(\Pi_2 + \Pi_3) + \frac{1}{\Delta_M} \Pi_4 < 0, \quad (13)$$

$$\begin{bmatrix} \Upsilon(\alpha, \beta) & \frac{h}{r} Y \\ \frac{h}{r} Y^\top & -\frac{h}{r} U \end{bmatrix} < 0, \quad (14)$$

$$\begin{bmatrix} M_0^\top \Pi_0 M_0 + \Pi_1 - \alpha \Pi_3 + \frac{1}{\Delta_M} \Pi_4 & \alpha Y \\ \alpha Y^\top & -\alpha U \end{bmatrix} < 0 \quad (15)$$

hold for $\alpha \in \{\Delta_m, \Delta_M\}$ and $\beta \in \{0, 1\}$, where the different matrices are defined in (18).

A common term in the LK functional to obtain delay-dependent criteria for time-delay systems is $\int_{-\frac{h}{r}}^0 \int_{\beta}^0 \dot{\chi}_k^\top(\tau^+, \alpha) R \dot{\chi}_k(\tau^+, \alpha) d\alpha d\beta$ (or similar). As it is shown in [11], the effect of the reset actions appears in the derivative of the LK functional through this term, and it may lead to increments of the LK functional. The idea behind looped-functionals allows to directly deal with this effect, when only one reset action occurs in the interval $[t - \frac{h}{r}, t]$. This is guaranteed by imposing $\frac{h}{r} < \Delta_m$. The advantage of the LK functional proposed in [15] is that this condition can be satisfied by taking r large enough.

Regarding the computational complexity of the conditions in Proposition 3.2, a simple computation leads to $(5 + 0.5r^2 + r)n^2 + (0.5r + 2.5)n$ variables and the size of the LMIs are $n(r + 2)$ for conditions (12) and (13), and $n(r + 3)$ for conditions (14) and (15). It is worth mentioning that the approach based on clock-dependent Lyapunov functions, described in [6], has been shown to provide better results (mainly in computational complexity, see [7]) than looped-functionals. Therefore, this approach should be considered for a possible extension of this work.

4 Examples

In this section, Proposition 3.2 is applied to several numerical examples.

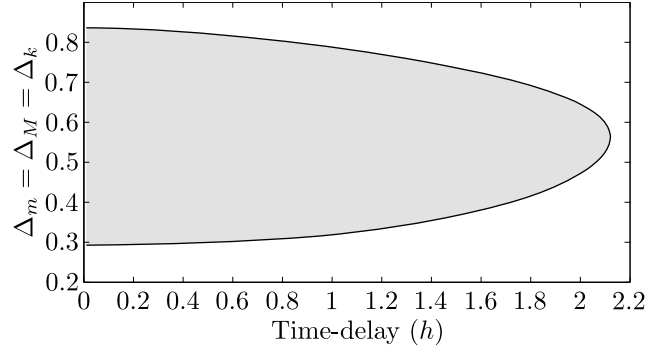


Fig. 1. Allowable reset period as a function of the time-delay for the IDDS of the Example 4.2 ($r = 10$).

4.1 An example with unstable continuous dynamics

Consider an IDDS (the example is taken from [29]) with matrices

$$A = \begin{bmatrix} 2 & \frac{5}{3} \\ \frac{1}{2} & 4 \end{bmatrix}, A_d = \begin{bmatrix} 3 & 5 \\ \frac{1}{3} & 1 \end{bmatrix}, A_R = \begin{bmatrix} \frac{3}{5} & 0 \\ 0 & \frac{3}{5} \end{bmatrix}. \quad (17)$$

The system without impulses is unstable for the time-delay $h = 0.1$. In addition, [29] shows that if $\Delta_M < 0.0095$ then the trivial solution to the system is globally exponentially stable for any fixed delay $h \in (0, \infty)$. Consider $h \in [0.001, 1]$, then applying Prop. 3.2 with $r = 40$, $\Delta_m > h/40$ and $\Delta_M = 0.0671$, it follows that the impulsive system is globally asymptotically stable. It is deduced that the bound $\Delta_M < 0.0095$ is very conservative for $h \in [0.001, 1]$. Finally, note that Δ_m strongly depends on r , hence if it is desired to reduce Δ_m or increase h then r should be increased, which may result in a longer computation time.

4.2 An example with both unstable continuous dynamics and unstable discrete dynamics

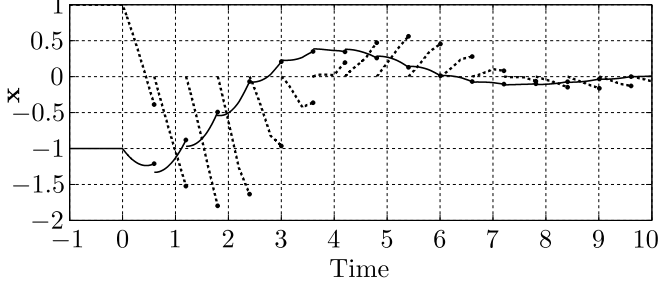
Consider the IDDS with matrices

$$A = \begin{bmatrix} 0.1 & 2 \\ -1 & 0.1 \end{bmatrix}, A_d = \begin{bmatrix} 1 & 1 \\ 0.1 & 0 \end{bmatrix}, A_R = \begin{bmatrix} 0 & 0 \\ 0 & 1.1 \end{bmatrix}. \quad (18)$$

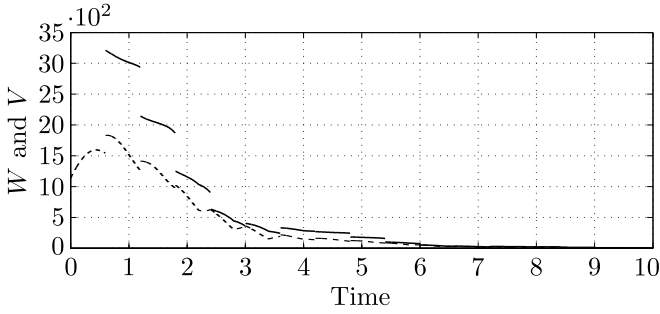
This system without impulses has been proved to be unstable for $h \in [0, 4]$ (using the results in [28]). Note that $\|A_R\| = 1.1$ which, roughly speaking, means that the discrete dynamics is also unstable (to the knowledge of the authors, there are not previous published results that can be applied to this type of IDDSs). Fig. 1 shows the minimum and the maximum period of reset which globally asymptotically stabilize the IDDS, as a function of h . Finally, the evolution of the system and the functionals V and W are shown in Fig. 2. Note that the functional V increases instantaneously due to both the reset actions.

Table 1
Maximum allowable time-delay in Example 4.3.

	No reset	Prop. 2, [11]	Prop. 4, [11]	Prop. 3.2 $\Delta_m = 0.081, \Delta_M = 1.78$	Prop. 3.2 $\Delta_m = 0.101, \Delta_M = 0.98$	Prop. 3.2 $\Delta_m = 0.121, \Delta_M = 0.5$
h	0.87	0.442	0.693	0.8	1	1.2



(a) Trajectory: x_1 (dotted) and x_2 (solid).



(b) LK functional V (dashed) and functional W (solid).

Fig. 2. Trajectory and value of V and W for the IDDS of the Example 4.2 with $h = 1, r = 10, \Delta_m = \Delta_M = 0.6$, and initial condition $\phi(\theta) = (1, -1), \theta \in [-1, 0]$.

4.3 An example of a time-delay reset control system

Time-delay reset control systems [3] are a particular class of the IDDSs, where the reset events are usually deter-

mined by the intersection of the trajectory with some surface (state-dependent). In the case of LTI plants and reset controllers with LTI base systems, the reset control system can be described by (2). The available results (see, e.g. [4, 11]) are too conservative to guarantee the asymptotic stability in many cases, mainly due to the state-dependent reset instants. In addition, let us remark that only a general Lyapunov-Krasovskii-like theorem is proposed in [20], but no computational conditions are given, for instance in term of LMIs. The LK theorem in [20] is similar to the one used in [11], where LMI conditions are developed. Therefore, we focus on [11] for comparison purpose. The proposed stability criterion can be applied to guarantee the stability of time-delay reset systems by imposing the dwell-time constraint $\Delta_m \leq t_{k+1} - t_k \leq \Delta_M$.

Consider a time-delay reset control system where the plant is $P = \frac{1}{s}$ and the reset controller is the parallel connection of a first order reset element (FORE) (see e.g. [3]) and a proportional term. The FORE is assumed to be endowed with a mechanism which forces and inhibits reset actions in order to guarantee the dwell-time constraint. The system is described by (2), following [11], with matrices

$$A = \begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix}, A_d = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, A_R = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \quad (19)$$

Table 1 shows a comparison of the maximum time-delay obtained by Prop. 3.2 (for some values of Δ_m and Δ_M) and the previous results. For the periodic reset case, the

$$\begin{aligned}
M_0 &= \begin{bmatrix} I_{(r+1)n} & 0_{(r+1)n \times n} \end{bmatrix}, \quad M_P = \begin{bmatrix} A & 0_{n, (r-1)n} & A_d \\ I_n & 0_{n, rn} \end{bmatrix}, \quad M_Q = \begin{bmatrix} I_{rn} & 0_{rn \times n} \\ 0_{rn \times n} & I_{rn} \end{bmatrix}, \quad M_R = \begin{bmatrix} A & 0_{n, (r-1)n} & A_d \\ -I_n & I_n & 0_{n, (r-1)n} \end{bmatrix}, \\
N_0 &= \begin{bmatrix} A & 0_{n, n(r-1)} & A_d & 0_n \end{bmatrix}, \quad N_1 = \begin{bmatrix} I_n & 0_{n, n(r+1)} \end{bmatrix}, \quad N_2 = \begin{bmatrix} 0_n & I_n & 0_{n, nr} \end{bmatrix}, \quad N_3 = \begin{bmatrix} 0_{n, n(r+1)} & I_n \end{bmatrix}, \\
N_4 &= (A_R - I)N_3, \quad N_{13} = N_1 - A_R N_3, \quad \Pi_1 = -N_{13}^\top S_1 N_{13} - \text{He}(N_{13}^\top S_2 N_3) + \text{He}(Y N_{13}), \\
\Pi_2 &= \text{He}(N_0^\top S_1 N_{13}) + \text{He}(N_0^\top S_2 N_3) + N_0^\top U N_0, \quad \Pi_3 = N_3^\top X N_3, \quad \Pi_4 = N_3^\top (A_R^\top P A_R - P) N_3, \\
\Upsilon(\alpha, \beta) &= M_0^\top \Pi_0 M_0 + \Pi_1 + \beta N_R + (\alpha - \frac{h}{r}) \Pi_2 + (\alpha - 2\frac{h}{r}) \Pi_3 + \frac{1}{\Delta_M} \Pi_4 \\
\Pi_0 &= M_P^\top \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} M_P + M_R^\top \begin{bmatrix} \frac{h}{r} R & 0 \\ 0 & -\frac{r}{h} R \end{bmatrix} M_R + M_Q^\top \begin{bmatrix} Q & 0 \\ 0 & -Q \end{bmatrix} M_Q, \quad N_R = \frac{r}{h} \begin{bmatrix} N_1^\top \\ N_2^\top \\ N_4^\top \end{bmatrix}^\top \begin{bmatrix} 0 & 0 & R \\ 0 & 0 & -R \\ R & -R & -R \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \\ N_4 \end{bmatrix}. \quad (18)
\end{aligned}$$

maximum time-delay is 1.433. Note that Prop. 4 in [11] requires $t_k - t_{k-1} > h$.

5 Conclusions

This article provides sufficient conditions for the asymptotic stability of linear IDDSs. The method allows relaxing the conditions on the Lyapunov-Krasovskii functionals that are traditionally employed in the stability analysis by Lyapunov-like methods. The proposed criterion is expressed in terms of LMIs. By solving these LMIs, it can be found positive constants that determine lower and upper bounds of the reset intervals for which the stability of the system is guaranteed. Several numerical examples show the effectiveness of the results and the reduction of the conservatism compared to previous results in the literature. The main advantage of the results is that the stability of IDDSs, with unstable continuous dynamics and unstable discrete dynamics, can be analyzed.

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A Proof of Prop. 3.1

The proof is divided in two parts: stability and asymptotic stability.

Stability: Assume $\eta \geq 1/3$ without loss of generality, and consider a solution $\mathbf{x}(t)$ with initial condition ϕ , and with reset and interval sequences (t_1, t_2, \dots) and $(\Delta_0, \Delta_1, \Delta_2, \dots)$, respectively. For any $k > 0$, and $\tau \in (0, \Delta_k]$, integrating (10) with respect to σ over $[0, \tau]$, results in the following inequality

$$V(\chi_k(\tau, \cdot)) - V(\chi_k(0^+, \cdot)) + \mathcal{V}_k(\tau, \chi_k) - \mathcal{V}_k(0, \chi_k) \leq 0 \quad (\text{A.1})$$

and now, by using (6), (8) and (9), it results in

$$V(\chi_k(\tau, \cdot)) \leq 3\eta V(\chi_k(0, \cdot)). \quad (\text{A.2})$$

In addition, by doing $\tau = \Delta_k$ in (A.1) and using condition (7), it directly follows

$$V(\chi_{k+1}(0, \cdot)) = V(\chi_k(\Delta_k, \cdot)) \leq V(\chi_k(0, \cdot)). \quad (\text{A.3})$$

for any $k > 0$. Considering all the previous reset instants, for $k = 2, 3, \dots$ and $\tau \in (0, \Delta_k]$, it results from (A.2) and (A.3) that $V(\chi_k(\tau, \cdot)) \leq 3\eta V(\chi_1(0, \cdot))$. Moreover, since $v \in \mathcal{K}$ and $\Delta_0 < \infty$ ($\Delta_0 = t_1$), there exists $\mu = \mu(\Delta_0)$ such that it is satisfied $V(\chi_0(\tau, \cdot)) \leq \mu v(\|\phi\|_{\mathcal{A}})$, where $\tau \in [0, \Delta_0]$. Finally, the above equations, (5), and $V(\chi_1(0, \cdot)) = V(\chi_0(\Delta_0, \cdot))$ imply

$$u(\|\chi_k(\tau, \cdot)\|) \leq V(\chi_k(\tau, \cdot)) \leq 3\eta V(\chi_1(0, \cdot)) \leq 3\eta \mu v(\|\phi\|_{\mathcal{A}}) \quad (\text{A.4})$$

For a given $\epsilon > 0$, choose $\delta = \delta(\epsilon)$, such as $0 < \delta < \min\{\epsilon, v^{-1}(\frac{u(\epsilon)}{3\eta\mu})\}$. Therefore, for all $\|\phi\|_{\mathcal{A}} < \delta$ it is satisfied $u(\|\chi_k(\tau, \cdot)\|) < 3\eta \mu v(\delta) < u(\epsilon)$ for $k > 0$ and $\tau \in (0, \Delta_k]$, and for $k = 0$ and $\tau \in [0, \Delta_0]$. The above condition directly implies $\|\mathbf{x}(t)\| < \epsilon$, $t \geq 0$, and thus the trivial solution to the system (2) is stable.

Asymptotic stability: This proof is based on the proof of asymptotic stability of retarded functional differential equations provided in [18]. The key idea is a contradiction argument based on a sequence of time intervals where the norm of the instantaneous state is greater than some value, which leads to a sequence of decrements of the functional V . The contradiction follows by considering a sufficiently large sequence, which makes the functional V negative, contradicting its definition.

The proof is as follows. For $\epsilon > 0$ choose δ_a as the constant for stability, that is $0 < \delta_a < \min\{\epsilon, v^{-1}(\frac{u(\epsilon)}{3\eta\mu})\}$,

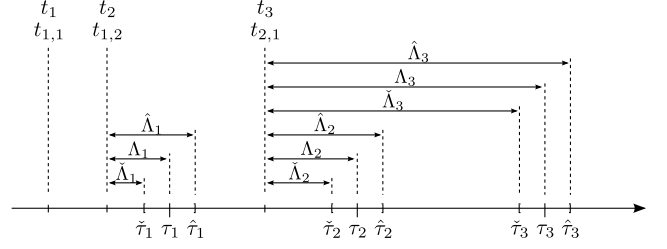


Fig. A.1. A notation example: $N_1 = 2, N_2 = 1, N_3 = 0, R(1) = 2, R(2) = 3, R(3) = 3, r(1) = 1, r(2) = 2, r(3) = 2. (t_1, t_2, t_3, \dots) = (t_{1,1}, t_{1,2}, t_{2,1}, \dots)$

then it is true that $\|\phi\|_{\mathcal{A}} < \delta_a$ implies $\|\mathbf{x}(t)\| < \epsilon$ for $t \geq 0$. Moreover, it has to be shown that for any $\varphi > 0$ there exists some $T = T(\delta_a, \varphi)$ such that $\|\mathbf{x}(t)\| < \varphi$ for $t \geq T$. Let $\delta_b = \delta(\varphi)$ be the constant for stability, then it is sufficient to prove that $\|\mathbf{x}_t\| < \delta_b$ for $t \geq T$. By contradiction, suppose that there does not exist such T , then the solution $\mathbf{x}(t)$ for an initial condition ϕ , with $\|\phi\|_{\mathcal{A}} < \delta_a$, satisfies $\|\mathbf{x}_t\|_{\mathcal{A}} \geq \delta_b$ for all $t \geq 0$. In addition, there exist $\delta_c > 0$ and a sequence² $(\tau_j)_{j=1}^{\infty}$ such that

$$(2j - 1)2h \leq \tau_j \leq 4jh, \quad (\text{A.5})$$

where τ_j is not a reset instant and $\|\mathbf{x}(\tau_j)\| \geq \delta_c$. Since $t_{k+1} - t_k \geq \Delta_m$ for $k = 1, 2, \dots$, and in addition, from the stability property there exists a constant $L > 0$ such that $\|\dot{\mathbf{x}}(t)\| < L$, for any $t \in [0, \infty) \setminus \{t_k\}_{k \in \mathbb{N}}$, then it is possible to build a set of intervals $I_j = [\tau_j - \frac{\delta_c}{2L}\hat{\alpha}_j, \tau_j + \frac{\delta_c}{2L}\hat{\alpha}_j]$ with $\hat{\alpha}_j, \hat{\alpha}_j \in \{0, 1\}$, $\hat{\alpha}_j + \hat{\alpha}_j > 0$, that do not contain reset instants and do not overlap (by using L large enough), that is $I_j \cap \{t_k\}_{k \in \mathbb{N}} = \emptyset, j = 1, 2, \dots$

For the sake of clarity, the reset instants t_k are rewritten as $(t_{j,l}), j = 1, 2, \dots, l = 1, 2, \dots, N_j$ (for some integers N_1, N_2, \dots), where the reset instant $t_{j,l}$ corresponds to the l^{th} -reset instant prior to $\tau_j \in I_j$, that is

$$0 < t_{1,1} < t_{1,2} < \dots < t_{1,N_1} < \tilde{\tau}_1 = \tau_1 - \frac{\delta_c}{2L}\hat{\alpha}_j < \hat{\tau}_1 = \tau_1 + \frac{\delta_c}{2L}\hat{\alpha}_j < t_{2,1} < \dots < t_{2,N_2} < \tilde{\tau}_2 \dots \quad (\text{A.6})$$

In addition, for $j > 0$, by definition $R(j) = N_1 + \dots + N_j, r(j) = \max_{i \leq j, N_i > 0} i$, and $\Lambda_j = \tau_j - t_{r(j), N_{r(j)}}$ ($\hat{\Lambda}_j$ and

$\hat{\Lambda}_j$ are defined accordingly by using $\tilde{\tau}_j$ and $\hat{\tau}_j$), see Fig. A.1. Now, by applying the mean-value theorem on the intervals $I_j, j = 1, 2, \dots$, it follows that for all $t \in I_j$ there exists some $\theta \in (0, 1)$ such that $\|\mathbf{x}(t)\| \geq \|\mathbf{x}(\tau_j)\| - \|\dot{\mathbf{x}}(\tau_j + \theta(t - \tau_j))\| |t - \tau_j| \geq \frac{\delta_c}{2}$. From (10), it follows

$$\dot{W}(\sigma, \chi_{R(j)}) \leq -w(\|\chi_{R(j)}(\sigma^+, 0)\|) \leq -w\left(\frac{\delta_c}{2}\right) < 0 \quad (\text{A.7})$$

² Compared with the proof in [18] for retarded differential equations, the sequence elements τ_j are separated $2h$, due the norm used for the functions $\mathcal{PAC}([-h, 0], \mathbb{R}^n)$.

for all $\sigma \in [\tilde{\Lambda}_j, \hat{\Lambda}_j]$. Let i be the smallest value such that $N_i > 0$, then integrating $\dot{W}(\sigma, \chi_{R(i)})$ over the interval $[\tilde{\Lambda}_i, \hat{\Lambda}_i]$, it is obtained $W(\hat{\Lambda}_i, \chi_{R(i)}) - W(\tilde{\Lambda}_i, \chi_{R(i)}) \leq -w(\frac{\delta_c}{2})\frac{\delta_c}{2L}$. From (10), the fact that W is nonincreasing, and condition (7), it is obtained

$$V(\chi_{R(i)}(\Delta_{R(i)}, \cdot)) \leq V(\chi_{R(i)}(0, \cdot)) - w(\frac{\delta_c}{2})\frac{\delta_c}{2L}. \quad (\text{A.8})$$

Since sequence of reset instants is infinite, considering all the intervals I_j with $j \geq i$, it is obtained

$$V(\chi_{R(j)}(\Delta_{R(j)}, \cdot)) \leq V(\chi_{R(i)}(0, \cdot)) - (j-i+1)w(\frac{\delta_c}{2})\frac{\delta_c}{2L}. \quad (\text{A.9})$$

Hence, for a large enough j , it gives $V(\chi_{R(j)}(\Delta_{R(j)}, \cdot)) < 0$, which is a contradiction. Therefore, there exists T such that $\|\mathbf{x}_t\|_{\mathcal{A}} < \delta_b$ for all $t \geq T$, and asymptotic stability follows. Finally, since $\lim_{s \rightarrow \infty} u(s) = \infty$, then δ_a in the above analysis may be chosen arbitrarily large, and ϵ can be set after δ_a to satisfy $v(\delta_a) < u(\epsilon)$. Thus, the trivial solution is globally asymptotically stable. \blacksquare

B Proof of Prop. 3.2

Consider the LK functional proposed in [15], which is reformulated with the proposed notation.

$$\begin{aligned} V(\chi_k(\tau, \cdot)) &= V_1(\chi_k(\tau, \cdot)) + V_2(\chi_k(\tau, \cdot)) + V_3(\chi_k(\tau, \cdot)), \\ V_1(\chi_k(\tau, \cdot)) &= \chi_k^\top(\tau, 0)P\chi_k(\tau, 0), \\ V_2(\chi_k(\tau, \cdot)) &= \int_{-\frac{h}{r}}^0 \chi_k^{e\top}(\tau, \alpha)Q\chi_k^e(\tau, \alpha)d\alpha, \\ V_3(\chi_k(\tau, \cdot)) &= \int_{-\frac{h}{r}}^0 \int_{\beta}^0 \dot{\chi}_k^\top(\tau^+, \alpha)R\dot{\chi}_k(\tau^+, \alpha)d\alpha d\beta, \end{aligned} \quad (\text{B.1})$$

where $P > 0$, $Q > 0$, $R > 0$, $\dot{\chi}_k(\tau^+, \theta)$ is the upper right-hand derivative of $\mathbf{x}(t_k + \tau + \theta)$, and

$$\chi_k^e(\tau, \theta) = \begin{bmatrix} \chi_k(\tau, \theta) \\ \chi_k(\tau, \theta - \frac{1}{r}h) \\ \vdots \\ \chi_k(\tau, \theta - \frac{r-1}{r}h) \end{bmatrix}. \quad (\text{B.2})$$

Considering that $\chi_k(\tau, \cdot)$ is piecewise Lipschitz continuous for every τ (bounded derivative), it can be easily proved that the LK functional (B.1) satisfies condition (5) (see [15]). In addition, since matrices P , Q , and R are positive definite, it is always possible to find $\eta \geq 1$ such that condition (6) is satisfied.

On the other hand, the derivative of (B.1) is as follows:

$$\begin{aligned} \dot{V}(\chi_k(\tau^+, \cdot)) &= \text{He}(\dot{\chi}_k^\top(\tau^+, 0)P\chi_k(\tau^+, 0)) \\ &+ \frac{h}{r}\dot{\chi}_k^\top(\tau^+, 0)R\dot{\chi}_k(\tau^+, 0) - \int_{-\frac{h}{r}}^0 \dot{\chi}_k^\top(\tau^+, \alpha)R\dot{\chi}_k(\tau^+, \alpha)d\alpha \\ &+ \chi_k^{e\top}(\tau^+, 0)Q\chi_k^e(\tau^+, 0) - \chi_k^{e\top}(\tau^+, -\frac{h}{r})Q\chi_k^e(\tau^+, -\frac{h}{r}). \end{aligned} \quad (\text{B.3})$$

Now, the Jensen's inequality and the fundamental theorem of calculus (note that there may be only one reset instant in any interval $[t - \frac{h}{r}, t]$) are applied to get a bound in the derivative³:

$$\begin{aligned} \dot{V}(\chi_k(\tau^+, \cdot)) &\leq \text{He}(\dot{\chi}_k^\top(\tau^+, 0)P\chi_k(\tau^+, 0)) \\ &+ \frac{h}{r}\dot{\chi}_k^\top(\tau^+, 0)R\dot{\chi}_k(\tau^+, 0) - \frac{r}{h}(\chi_k(\tau^+, 0) - \chi_k(\tau^+, -\frac{h}{r})) \\ &- \nu(\tau)(A_R - I)\chi_k(0, 0)^\top R(\chi_k(\tau^+, 0) - \chi_k(\tau^+, -\frac{h}{r})) \\ &- \nu(\tau)(A_R - I)\chi_k(0, 0) + \chi_k^{e\top}(\tau^+, 0)Q\chi_k^e(\tau^+, 0) \\ &- \chi_k^{e\top}(\tau^+, -\frac{h}{r})Q\chi_k^e(\tau^+, -\frac{h}{r}), \end{aligned} \quad (\text{B.4})$$

where $\nu(\tau) = 1$, $0 \leq \tau < \frac{h}{r}$, and $\nu(\tau) = 0$, $\frac{h}{r} \leq \tau \leq \Delta_k$. Now consider for every $k > 0$, the following functional (the functional is adapted from [22, 23] with new terms to deal with impulses)

$$\begin{aligned} \mathcal{V}_k(\tau, \chi_k) &= \tau \frac{1}{\Delta_k} \chi_k^\top(0, 0)A_R^\top P A_R \chi_k(0, 0) \\ &+ (1 - \frac{1}{\Delta_k} \tau) \chi_k^\top(0, 0)P\chi_k(0, 0) \\ &+ (\Delta_k - \tau) [\zeta_k^\top(\tau)S_1\zeta_k(\tau) + \text{He}(\zeta_k^\top(\tau)S_2\chi_k(0, 0))] \\ &+ (\Delta_k - \tau) \int_0^\tau \dot{\chi}_k^\top(s^+, 0)U\dot{\chi}_k(s^+, 0)ds \\ &+ \tau(\Delta_k - \tau)\chi_k^\top(0, 0)X\chi_k(0, 0) + \gamma\|\chi_k(0, 0)\|^2 \end{aligned} \quad (\text{B.5})$$

for some $\gamma > 0$, and where $\zeta_k(\tau) = \chi_k(\tau^+, 0) - \chi_k(0^+, 0)$ and $\tau \in [0, \Delta_k]$. By using simple derivations (omitted due to the space limitation), it can be proved that conditions (7), (8), and (9) are satisfied for $\eta > \frac{1}{\lambda_m(P)} \max(\gamma, \gamma + \lambda_M(P))$. The derivative of \dot{W} is bounded for all $k = 1, 2, \dots$ as follows:

³ Note that similar results can be obtained by the affine Jensen's inequality (see [5] for a comparison of the inequalities), but no substantial improvement has been achieved for the analyzed examples.

$$\begin{aligned}
\dot{W} &\leq \text{He}(\dot{\chi}_k^\top(\tau^+, 0)P\chi_k(\tau^+, 0) + \frac{h}{r}\dot{\chi}_k^\top(\tau^+, 0)R\dot{\chi}_k(\tau^+, 0) \\
&\quad - \frac{r}{h}(\chi_k(\tau^+, 0) - \chi_k(\tau^+, -\frac{h}{r}) - \nu(\tau)(A_R - I)\chi_k(0, 0))^\top R \\
&\quad (\chi_k(\tau^+, 0) - \chi_k(\tau^+, -\frac{h}{r}) - \nu(\tau)(A_R - I)\chi_k(0, 0)) \\
&\quad + \chi_k^{e\top}(\tau^+, 0)Q\chi_k^e(\tau^+, 0) - \chi_k^{e\top}(\tau^+, -\frac{h}{r})Q\chi_k^e(\tau^+, -\frac{h}{r}) \\
&\quad + \frac{1}{\Delta_k}\chi_k^\top(0, 0)(A_R P A_R - P)\chi_k(0, 0) \\
&\quad + (\Delta_k - \tau) (\text{He}(\dot{\chi}_k^\top(\tau^+, 0)S_1\zeta_k(\tau)) \\
&\quad + \text{He}(\dot{\chi}_k^\top(\tau^+, 0)S_2\chi_k(0, 0)) + \dot{\chi}_k^\top(\tau^+, 0)U\dot{\chi}_k(\tau^+, 0)) \\
&\quad - \zeta_k^\top(\tau)S_1\zeta_k(\tau) - \int_0^\tau \dot{\chi}_k^\top(s^+, 0)U\dot{\chi}_k(s^+, 0)ds \\
&\quad - \text{He}(\zeta_k^\top(\tau)S_2\chi_k(0, 0)) + (\Delta_k - 2\tau)\chi_k^\top(0, 0)X_1\chi_k(0, 0).
\end{aligned} \tag{B.6}$$

Let define $\xi_k(\tau) = (\chi_k^e(\tau, 0), \chi_k(\tau, -h), \chi_k(0, 0))$, by applying the affine Jensen's inequality (see [5]) on $-\int_0^\tau \dot{\chi}_k^\top(s^+, 0)U\dot{\chi}_k(s^+, 0)ds$ and using the definition of matrices (18), the following inequality is obtained

$$\begin{aligned}
\dot{W} &\leq \xi_k(\tau^+)^\top (M_0^\top \Pi_0 M_0 + \Pi_1 + \nu(\tau)N_R + (\Delta_k - \tau)\Pi_2 \\
&\quad + (\Delta_k - 2\tau)\Pi_3 + \tau Y U^{-1} Y^\top + \frac{1}{\Delta_k}\Pi_4)\xi_k(\tau^+).
\end{aligned} \tag{B.7}$$

Consider $\Delta_m < \Delta_M$ then $\Pi_4 \leq 0$ from (12). Hence, it is satisfied $\frac{1}{\Delta_k}\Pi_4 \leq \frac{1}{\Delta_M}\Pi_4$. Using this inequality and (B.7), the resulting expression is convex on τ and Δ_k , and thus, it is necessary and sufficient to ensure the negativity at the endpoints of their intervals. Note that there is a discontinuity at $\tau = \frac{h}{r}$ due to $\nu(\tau)$, but it is sufficient to check the negativity in both sides of the discontinuity. Finally, the Schur complement is used to deal with the term U^{-1} and write the conditions in an LMI form. ■