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# On Circularity

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**Abstract**—Circularity is an assumption that was originally introduced for the definition of the probability distribution function of complex normal vectors. However, this concept can be extended in various ways for nonnormal vectors. The first purpose of this paper is to introduce and compare some possible definitions of circularity. From these definitions, it is also possible to introduce the concept of circular signals and to study whether or not the spectral representation of stationary signals introduces circular components. Therefore, the relationships between circularity and stationarity are analyzed in detail. Finally, the theory of linear mean square estimation for complex signals exhibits some connections with circularity, and it is shown that without this assumption, the estimation theory must be reformulated.

## I. INTRODUCTION

COMPLEX random variables, vectors, or signals are widely used in many areas of signal processing. In the time domain, the best example of complex random signal is the analytic signal appearing in the definition of instantaneous amplitude and phase (see p. 229 of [1]). In the frequency domain, it is obvious that the Fourier components of a random signal are usually complex random variables.

The second-order theory of complex random variables or vectors does not introduce any specific difficulty and is extensively presented in many textbooks [1]–[3]. In this context, the variance of a complex random variable  $Z$  is  $E\{ZZ^*}$  instead of  $E\{Z^2\}$ , and the covariance matrix of a zero-mean valued complex random vector  $\mathbf{Z}$  is  $E\{\mathbf{Z}\mathbf{Z}^H\}$  instead of  $E\{\mathbf{Z}\mathbf{Z}^T\}$ , which is valid in the real case. Furthermore, all the concepts of quadratic mean convergence, and then of Hilbert spaces of second-order random variables, can easily be extended from the real to the complex case. Finally, the theory of linear mean square estimation, which is the basis of the methods of statistical linear filtering, can be presented exactly as in the real case.

Some problems, however, appear when leaving the second-order properties. This is especially true when dealing with the concepts of distribution function, of probability density function, or of characteristic function. In fact, the distribution function  $F(x)$  of a real random variable  $X$  is the probability  $P(X \leq x)$ , and the concept of inequality is meaningless for complex numbers. In this case, the standard procedure is to use the real and imaginary parts that are the components of a 2-D real random vector. However, doing so results in considering that complex numbers are nothing else but pairs

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of real numbers, and the complex theory loses most of its interest.

This problem especially appears in the study of the probability distribution of *normal* (or *Gaussian*) complex random vectors. It is always possible to consider that such vectors can be described as a pair of two real normal random vectors without any specific property. On the other hand, if we want to find again the standard structure of probability density, we are lead to introduce the concept of *circular normal* (or *Gaussian*) *random vectors* (see p. 118 of [1] and [4]). This concept can easily be extended to stochastic processes, and, for example, it is well known that the analytic signal of a real Gaussian signal is a complex circular normal signal (see p. 276 of [1]).

Complex circular normal random vectors have several interesting properties that are easily deduced from the normal distribution. However, the question remains open as to whether these properties can be extended to the non-Gaussian case and for higher order moments. This is the purpose of this paper devoted to the study of circularity without introducing the normal assumption. However, as many properties are extensions of some valid in the Gaussian case, Section II presents a systematic review of the principal properties of complex normal circular vectors. Starting from these properties, various definitions of complex circular vectors are presented in Section III, and the relationships between them are analyzed. From vectors, it is possible to pass on to random signals, and this is presented in Sections IV, V, and VI. In particular, relationships between circularity and stationarity are presented. Finally, the last section is devoted to the applications of all these ideas to problems of linear mean square estimation. The main conclusion is that the classical presentation of linear mean square estimation naturally introduces circular vectors. On the other hand, this theory must be transformed to achieve the best performances in the case where the random vectors are not circular.

## II. REVIEW OF PROPERTIES OF COMPLEX NORMAL CIRCULAR VECTORS

Let  $\mathbf{Z}(\omega)$  be a random vector of  $\mathbb{C}^m$ . This vector can be decomposed as  $\mathbf{Z}(\omega) = \mathbf{X}(\omega) + j\mathbf{Y}(\omega)$ , introducing its real and imaginary parts. In order to simplify the presentation, we shall no longer use the letter  $\omega$  to describe randomness and assume that the mean value of  $\mathbf{Z}$  is zero.

The random vector  $\mathbf{Z}$  is said to be *normal* (or *Gaussian*) if  $\mathbf{X}$  and  $\mathbf{Y}$  are a pair of jointly normal random vectors. Let us simply remind that the distribution of this pair is entirely defined by the covariance matrices of  $\mathbf{X}$  and  $\mathbf{Y}$  and by the cross-covariance matrix between  $\mathbf{X}$  and  $\mathbf{Y}$ .

The random vector  $\mathbf{Z}$  is said to be *normal circular* if it is normal, and furthermore, if

$$\mathbf{C} \triangleq E(\mathbf{Z}\mathbf{Z}^T) = \mathbf{0} \quad (2.1)$$

Let us now present the most significant consequences of this definition.

#### A. Rotation Invariance

If  $\mathbf{Z}$  is normal circular, then  $e^{j\alpha}\mathbf{Z}$  and  $\mathbf{Z}$  have the same probability distribution.

*Proof:* Let  $\mathbf{Z}_\alpha$  be the vector  $e^{j\alpha}\mathbf{Z}$ . It is clear that if  $\mathbf{Z}$  is normal,  $\mathbf{Z}_\alpha$  is normal as well. It is obvious that  $\mathbf{Z}$  and  $\mathbf{Z}_\alpha$  have the same covariance matrix, and if (2.1) is valid, the matrix  $\mathbf{C}_\alpha$  corresponding to  $\mathbf{Z}_\alpha$  is also zero, which completes the proof.

Conversely, if  $\mathbf{Z}$  is normal and if there exists at least one  $\alpha$  not equal to a  $k\pi$ ,  $k$  integer such that  $\mathbf{Z}$  and  $\mathbf{Z}_\alpha$  have the same distribution, then  $\mathbf{Z}$  is circular.

*Proof:* It is obvious that  $\mathbf{Z}$  and  $\mathbf{Z}_\alpha$  are normal and have the same covariance matrix. It results from the assumption that the matrices  $\mathbf{C}$  defined by (2.1) corresponding to  $\mathbf{Z}$  and  $\mathbf{Z}_\alpha$  are equal. This yields

$$\mathbf{C}(1 - e^{2j\alpha}) = \mathbf{0} \quad (2.2)$$

and as  $\alpha \neq k\pi$ , then  $\mathbf{C} = \mathbf{0}$  which shows that  $\mathbf{Z}$  is circular.

#### B. Probability Distribution

The probability density and the characteristic functions of a complex normal circular vector are given by (see p. 119 of [1])

$$p(\mathbf{x}, \mathbf{y}) = p(\mathbf{z}) = \pi^{-n} [\det(\mathbb{F}_z)]^{-1} \exp(-z^H \mathbb{F}_z^{-1} \mathbf{z}) \quad (2.3)$$

$$\phi(\mathbf{u}, \mathbf{v}) = \phi(\mathbf{w}) = \exp\{-(1/4)\mathbf{w}^H \mathbb{F}_z \mathbf{w}\} \quad (2.4)$$

where  $\mathbf{z} = \mathbf{x} + j\mathbf{y}$ , and  $\mathbf{w} = \mathbf{u} + j\mathbf{v}$ , and  $\mathbb{F}_z$  is the covariance matrix  $\mathbf{Z}$ . The characteristic function  $\phi(\mathbf{u}, \mathbf{v})$  is defined, as usual, by

$$\phi(\mathbf{u}, \mathbf{v}) = E\{\exp[j(\mathbf{u}^T \mathbf{X} + \mathbf{v}^T \mathbf{Y})]\} \quad (2.5)$$

and  $p(\mathbf{z})$  is a notation for a real function of the complex variable  $\mathbf{z}$  but is meaningless as a PDF of a complex random vector. The main feature to be noted concerning these expressions is that the functions  $p(\mathbf{x}, \mathbf{y})$  or  $\phi(\mathbf{u}, \mathbf{v})$  only depend on quadratic forms constructed either with  $\mathbf{z}$  or with  $\mathbf{w}$ . It is worth pointing out that a normal circular random vector is entirely defined by its covariance matrix  $\mathbb{F}_z = E(\mathbf{Z}\mathbf{Z}^H)$ , as seen in (2.3) or (2.4). However, no constraint is imposed on this matrix, except that it should be nonnegative definite. For example, it can be real. In this case, the two vectors  $\mathbf{X}$  and  $\mathbf{Y}$ , which are the real and imaginary parts of  $\mathbf{Z}$ , are normal, independent, and identically distributed. This is especially the case when  $m = 1$  because  $\Gamma_z$  becomes a variance that is a non-negative number.

#### C. Higher Order Moments

Let us use the notation introduced in p. 122 of [1]. The most generalized moment of order  $k$  can be written as

$$m_k[\{i_j\}; \{\varepsilon_j\}] = E[Z_{i_1}^{\varepsilon_1} Z_{i_2}^{\varepsilon_2} \cdots Z_{i_k}^{\varepsilon_k}]. \quad (2.6)$$

In this expression, the  $i_p$ 's are arbitrary integers satisfying  $1 \leq i_p \leq m$ , where  $m$  is the number of components of the vector  $\mathbf{Z}$ . Furthermore, the  $\varepsilon_p$ 's are equal to  $\pm 1$ , and  $Z_i^{-1}$  is by convention equal to  $Z_i^*$ , whereas  $Z_i^{+1} = Z_i$ .

If  $\mathbf{Z}$  is a complex normal circular vector, then the odd moments  $m_{2k+1}[\ ]$  are zero, and the even moments  $m_{2k}[\ ]$  are nonzero only if

$$\sum_{p=1}^{2k} \varepsilon_p = 0. \quad (2.7)$$

*Proof:* This property is a direct consequence of Property A of rotation invariance. It suffices, in fact, to reason by contradiction. If an odd moment is nonzero, it cannot be invariant when replacing  $\mathbf{Z}$  by  $e^{j\alpha}\mathbf{Z}$ . In fact, the factor  $e^{j\alpha}$  cannot be eliminated from (2.6). The same situation arises when  $k$  is even and when (2.7) is not satisfied.

Conversely, if  $\mathbf{Z}$  is normal and if the moments satisfy these relations,  $\mathbf{Z}$  is circular. In fact (2.7) implies (2.1), which ensures circularity.

### III. DEFINITIONS OF CIRCULAR RANDOM VECTORS

Relaxing the normal assumption, we shall see that the properties analyzed in the previous section can lead to various possible and not equivalent definitions of circularity. Let us first consider the case of a *scalar* complex random variable (RV)  $Z$ . It is said that  $Z$  is circular if for any  $\alpha$ , the RV's  $Z$  and  $\exp(j\alpha)Z$  have the same probability distributions. Let  $A$  and  $\Phi$  be the amplitude (or modulus) and the phase (modulo  $2\pi$ ) of  $Z$ . The circularity of  $Z$  is then characterized by

$$p(a; \phi) = \frac{1}{2\pi} p(a) \quad (3.1)$$

where  $p(a; \phi)$  and  $p(a)$  are the probability density functions (PDF's) of  $A$ . This implies that  $A$  and  $\Phi$  are independent and that  $\Phi$  is uniformly distributed in  $[0, 2\pi]$ . The PDF of  $A$  is of course arbitrary.

#### A. Marginal Circularity

The components  $Z_k$  of a complex random vector  $\mathbf{Z}$  are complex RV's. In the marginal distribution, we consider separately each component independently of each other. This leads to the following definition: A complex random vector  $\mathbf{Z}$  is said to be *marginally circular* if its components  $Z_k$  are complex, scalar, and circular RV's. The marginal PDF of the component  $Z_k$  is then given by (3.1), where  $p(a)$  can now depend on  $k$ .

#### B. Weak Circularity

The random vector  $\mathbf{Z}$  is weakly circular if  $\mathbf{Z}$  and  $\exp(j\alpha)\mathbf{Z}$  have the same probability distribution for any  $\alpha$ . Note that it is this circularity that is introduced in Section II-A for normal random vectors. Let  $p(\mathbf{a}; \phi_1, \phi_2, \dots, \phi_m)$  be the PDF of the

amplitudes  $A_k$  and phases  $\Phi_k$  of  $Z_k$ ,  $1 \leq k \leq m$ . The weak circularity is characterized by

$$\begin{aligned} p(\mathbf{a}; \phi) &= p(\mathbf{a}; \phi_1, \phi_2, \dots, \phi_m) \\ &= p(\mathbf{a}; \phi_1 + \alpha, \phi_2 + \alpha, \dots, \phi_m + \alpha), \forall \alpha. \end{aligned} \quad (3.2)$$

As a consequence,  $p(\mathbf{a}; \phi)$  is only a function of  $(m-1)$  variables  $\phi_k$  or

$$p(\mathbf{a}; \phi) = p(\mathbf{a}; \phi_2 - \phi_1, \phi_3 - \phi_1, \dots, \phi_m - \phi_1). \quad (3.3)$$

Using the fact that phases are defined modulo  $2\pi$ , it is easy to verify from (3.1) and (3.3) that weak circularity implies marginal circularity.

### C. Strong Circularity

The random vector  $\mathbf{Z}$  is strongly circular if

$$p(\mathbf{a}; \phi) = p(\mathbf{a}; \phi_1 + \alpha_1, \phi_2 + \alpha_2, \dots, \phi_m + \alpha_m) \quad (3.4)$$

for arbitrary values of  $\alpha_k$ ,  $1 \leq k \leq m$ . This implies that

$$p(\mathbf{a}; \phi) = \left(\frac{1}{2\pi}\right)^m p_A(\mathbf{a}) \quad (3.5)$$

which means that the phases  $\Phi_k$  are IID RV's with uniform distribution and are independent of the vector  $\mathbf{A}$  of the amplitudes  $A_k$ ,  $1 \leq k \leq m$  as well.

It is obvious that (3.5) implies (3.2), which means that strong circularity implies weak circularity.

### D. Total Circularity

This circularity appears when the RV's  $Z_k$  are independent and circular. This is characterized by the fact that  $p_A(\mathbf{a})$  in (3.5) can be factorized as a product of functions  $p_k(a_k)$ . Note that total circularity implies strong circularity and that marginal circularity and independence give total circularity. This circularity appears especially in the normal case when the matrix  $\mathbb{F}_z$  in (2.3) or (2.4) is diagonal. In this case, the amplitudes  $A_k$  are distributed according to a Rayleigh PDF. In reality, it is easy to show (see p. 138 of [1]) that if a complex *normal* random vector is strongly circular, its covariance matrix is diagonal, and it is then totally circular.

### E. Moment Circularity

Statistical properties of a random vector  $\mathbf{Z}$  can also be described by using the moments of any order. Consider again the moments defined by (2.6). If  $\mathbf{Z}$  is marginally circular, it is only possible to deduce properties of marginal moments such as  $E[Z_k^p Z_k^{*q}]$ . It is obvious that these moments are nonzero only if  $p = q$ . On the other hand, if  $\mathbf{Z}$  is weakly circular, it is obvious that the only nonzero moments (2.6) are those for which  $k$  is even, and (2.7) holds. Finally, the only nonzero moments of a strongly circular vector  $\mathbf{Z}$  are in the form  $E[|Z_{i_1}|^{2p_1} \dots |Z_{i_n}|^{2p_n}]$ , where the  $i_j$ 's are  $n$  distinct integers taken between 1 and  $m$ . The assumption of total circularity implies that the previous moments can be factorized as a product of  $n$  terms  $1 \leq n \leq m$  because of the independence of the  $Z_i$ 's.

Note that moment circularity can be valid only for the moments up to a given order. An example of such a situation will be discussed later.

## IV. CIRCULARITY AND RANDOM SIGNALS

A random signal can be described as a collection of random vectors (see p. 163 of [1]). Therefore, there is no difficulty passing from the definition of circular random vectors to that of circular random signals. More precisely, a random signal  $X(t)$  is said to be circular if its family of finite-dimensional distribution introduces only circular random vectors.

This definition is especially simple in the case of *normal* signals, and it results from (2.1) that a normal signal  $X(t)$  is circular if and only if the second-order moment  $E[X(t)X(t')] = 0$  for any  $t$  and  $t'$ . However, normal circular signals are not the only circular signals that can be introduced, and, for instance, a sequence of IID circular RV's is an example of circular white noise. Such a signal is obviously totally circular by extension of the terminology used for random vectors.

However, for the following discussion, the most interesting point concerning circularity appears in the *frequency domain*, and this justifies other definitions concerning frequency circularity. In all that follows, we only consider harmonizable random signals  $Z(t)$  (see p. 200 of [1]), which means signals with a spectral representation such as

$$Z(t) = \int d\bar{Z}(\nu) \exp(j2\pi\nu t). \quad (4.1)$$

This signal is said to be *marginally circular* in the frequency domain if the increments  $d\bar{Z}(\nu)$  and  $\exp(j\alpha)d\bar{Z}(\nu)$  have the same statistical distributions for any  $\alpha$ . This is valid for any frequency  $\nu$ , but the distribution of  $d\bar{Z}(\nu)$  can obviously depend on the frequency  $\nu$ .

The signal  $Z(t)$  is said to be *weakly circular* if  $Z(t)$  and  $\exp(j\alpha)Z(t)$  have the same probability distribution for any  $\alpha$ . In the frequency domain, this implies that the vector with components  $d\bar{Z}(\nu_1), d\bar{Z}(\nu_2), \dots, d\bar{Z}(\nu_m)$  is weakly circular for any  $m$  and  $\nu_j$ . This of course implies that  $Z(t)$  is marginally circular in the frequency domain. Note at this stage that a real signal cannot be weakly circular. In fact, if  $Y(t) = \exp(j\alpha)X(t)$ , where  $X(t)$  is real, then  $X(t)$  and  $Y(t)$  have the same correlation function. However,  $E[Y(t)Y(t+\tau)] = \exp(j2\alpha)E[X(t)X(t+\tau)]$ , and these moments are not equal.

In order to introduce the *strong circularity*, we must extend to the frequency domain, the definition introduced for vectors. The appropriate tool for this extension is the phase filter. Let us remind that such a filter is characterized by a frequency response  $H(\nu) = \exp[j\phi(\nu)]$ , where  $\phi(\nu)$  is an arbitrary function modulo  $2\pi$ . A signal  $Z(t)$  is then said to be *strongly circular* if its statistical properties are invariant after being filtered in any phase filter. As stated previously, it is obvious that a real signal cannot be strongly circular. Furthermore, as multiplication of the signal by  $\exp(j\alpha)$  is a very simple example of phase filtering, strong circularity implies weak circularity. Finally, results of Section II show that any normal circular signal is strongly circular.

Now  $Z(t)$  is totally circular if it is marginally circular and if  $\bar{Z}(\nu)$  is a random processes with *independent increments*. As for random vectors, total circularity implies strong circularity. Total circularity appears especially in the normal case, and a stationary and circular normal signal is totally circular. In fact, it results from the normality that the function  $\bar{Z}(\nu)$  appearing in (4.1) is also normal. The stationarity implies that the increments  $d\bar{Z}(\nu)$  are uncorrelated. The circularity implies that  $E[d\bar{Z}(\nu_1)d\bar{Z}(\nu_2)] = 0$  for any frequencies  $\nu_1$  and  $\nu_2$ . Consequently, the complex normal increments are independent, which is the total circularity. In this case, the function  $\bar{Z}(\nu)$  is a complex Brownian motion. However, as will be seen later, there are totally circular signals that are not normal.

With all these definitions, we can enter in the core of the discussion concerning relationships between circularity and stationarity of signals.

## V. CIRCULARITY AND STATIONARITY OF CONTINUOUS-TIME SIGNALS

Consider the pure tone signal  $Z(t) = Z \exp(j\omega t)$ , where  $Z$  is a random complex amplitude, and  $\omega$  a deterministic angular frequency. The signal  $Z(t)$  is stationary if for any delay  $\tau$ ,  $Z(t)$  and  $Z(t + \tau)$  are RV's with the same distribution. This obviously implies that the complex scalar RV  $Z$  is circular. The converse is also true. Therefore, in this very specific case of a monofrequency signal, circularity and stationarity are equivalent. It is then interesting to study if this result can be extended to more complex situations.

### A. Signals with Discrete Spectral Components

1) *Marginal Circularity*: Consider the signal  $Z(t)$  written as

$$Z(t) = \sum_{k=1}^N Z_k \exp(j\omega_k t). \quad (5.1)$$

In this expression, which is a particular case of (4.1), the spectral components  $Z_k$  are  $N$  random variables, and the  $N$  frequencies  $\omega_k$  ( $\omega_k = 2\pi\nu_k$ ) are given. The number  $N$  of components is arbitrary. The statistical properties of the signal  $Z(t)$  are entirely defined by the probability distributions of the  $Z_k$ 's, and we assume, for simplicity, that these random variables are continuous. Writing  $Z_k$  as  $A_k \exp(j\Phi_k)$ , it is then possible to introduce, as in (3.2), a PDF

$$p(\mathbf{a}; \phi) = p(a_1, a_2, \dots, a_N; \phi_1, \phi_2, \dots, \phi_N) \quad (5.2)$$

defining the probability distribution of the vector  $\mathbf{Z}$  with components  $Z_i$ ,  $1 < i \leq N$  and then of the signal  $Z(t)$  defined by (5.1). It is clear that  $\mathbf{Z}$  and  $Z(t)$  have the same properties of circularity. This follows directly from the definitions of Sections III and IV.

Suppose that  $Z(t)$  is *stationary* in the strict sense. This implies that any signal deduced from  $Z(t)$  by linear filtering is also stationary. This is especially the case of the signal  $Z_k(t) = Z_k \exp(j\omega_k t)$ . As a result, it appears that  $Z_k$  is a circular scalar RV. Because this is valid for any  $k$ , the result is

that for signals with discrete spectral components, stationarity implies *marginal* circularity in the frequency domain. The converse property is not true, as will be illustrated later by an example. Furthermore, marginal circularity implies that the spectral components of a stationary signal cannot be real since a real RV cannot be circular. In reality, this can be deduced directly from (5.1). In fact, if the  $Z_k$ 's are real, the signal  $Z(t)$  satisfies  $Z(t) = Z^*(-t)$ , and this symmetry with respect to the origin of time is in contradiction with stationarity, which implies invariance of the statistics in any change of this origin.

2) *Weak Circularity*: Weak circularity is characterized by (3.2); it is interesting to investigate whether it has some relationship with stationarity. If  $Z(t)$  is stationary,  $Z(t)$  and  $Z(t + \tau)$  have the same statistical properties. The same is valid for their spectral components. Those of  $Z(t + \tau)$  are deduced from (5.1) and are then  $Z_k \exp(j\omega_k \tau)$ . As a consequence, if all the frequencies  $\omega_k$  are nonzero, the stationarity of  $Z(t)$  is characterized by

$$p(\mathbf{a}, \phi) = p(\mathbf{a}; \phi_1 + \omega_1 \tau, \phi_2 + \omega_2 \tau, \dots, \phi_N + \omega_N \tau), \forall t. \quad (5.3)$$

This relation is similar, but not equivalent, to (3.2). In particular, it implies the same kind of consequence and especially that  $p(\mathbf{a}; \phi)$  is a function of only  $(N - 1)$  independent variables as, for example,  $\phi_2 - (\omega_2/\omega_1)\phi_1, \phi_3 - (\omega_3/\omega_1)\phi_1, \dots, \phi_N - (\omega_N/\omega_1)\phi_1$ . As the phase is defined modulo  $2\pi$ , this obviously implies marginal circularity.

However, if a signal like (5.1) is simultaneously stationary and weakly circular, or if (3.2) and (5.3) hold, then  $p(\mathbf{a}, \phi)$  necessarily takes the form (3.5), which means that the signal is strongly circular. In other words, it is impossible for a signal to be stationary and weakly circular without being strongly circular. This leads us to study this kind of circularity.

3) *Strong Circularity and Signals with Incommensurable Frequencies*: Suppose that  $N = 2$  in (5.1), and let us call  $Z_1(t)$  and  $Z_2(t)$  the two signals of which their sum is equal to  $Z(t)$ . As these signals are deduced from  $Z(t)$  by two linear filters, they are jointly stationary, and then, the moments

$$m_{pqrs}(t) \triangleq E[Z_1^p(t)Z_1^{*q}(t)Z_2^r(t)Z_2^{*s}(t)] \quad (5.4)$$

are time independent. These moments can be written as

$$m_{pqrs}(t) = m_{pqrs} \exp\{j[(p - q)\omega_1 + (r - s)\omega_2]t\} \quad (5.5)$$

with

$$m_{pqrs} \triangleq E[Z_1^p Z_1^{*q} Z_2^r Z_2^{*s}]. \quad (5.6)$$

It results from the stationarity of  $Z(t)$  that  $m_{pqrs}$  is nonzero only if

$$(p - q)\omega_1 + (r - s)\omega_2 = 0. \quad (5.7)$$

Suppose now that the ratio  $\rho = \omega_2/\omega_1$  is not rational. In this case, (5.7) implies  $p = q$  and  $r = s$ . This means that  $Z(t)$  is *strongly circular*. In fact, the only nonzero moments of  $Z_1$  and  $Z_2$  are now in the form  $E[|Z_1|^{2p}|Z_2|^{2r}]$ . Using a phase filter,  $Z_k$  is transformed into  $Z_k \exp[j\phi(\omega_k)]$ , and consequently, all the nonzero moments of the pair of RV's  $Z_1$  and  $Z_2$  are invariant. This is the definition of strong circularity. By using algebraic arguments, it is possible to obtain the same result

directly from (5.3) with  $N = 2$ . Finally, it is easy to verify that the assumption that  $\rho$  is irrational is essential, and without this condition, it is only possible to conclude to marginal circularity. Then, the conclusion is that a stationary signal  $Z(t)$  with two spectral components at incommensurable frequencies is strongly circular. The converse is obviously true. In reality, a strongly circular signal is necessarily stationary. In fact, the assumption of strong circularity is characterized by (3.5), which implies (5.3) and yields stationarity. For this converse property, the assumption of incommensurable frequencies is then not necessary.

The result can be extended to  $N$  frequencies. Suppose that the frequencies appearing in (5.1) are such that the equation

$$n_1\omega_1 + n_2\omega_2 + \cdots + n_N\omega_N = 0 \quad (5.8)$$

implies  $n_i = 0, 1 \leq i \leq N$ , where the  $n_i$ 's are integer. In this case, by considering the  $N$  signals  $Z_k \exp(j\omega_k t)$ , it can be shown that as for  $N = 2$ , stationarity implies strong circularity. In other words, the only nonzero moments of the spectral components are in the form  $E[|Z_1|^{2p_1} \cdots |Z_N|^{2p_N}]$ . However, note that this does not mean that the spectral components are independent RV's.

4) *Total Circularity*: This appears if the components  $Z_k$  in (5.1) are independent. In this case, stationarity and total circularity are equivalent. In fact, stationarity implies marginal circularity, and independence implies that marginal distribution completely defines the PDF (5.2). Conversely, independence and marginal circularity obviously implies (5.3) or stationarity.

### B. Signals with Continuous Spectrum

1) *Marginal Circularity*: Suppose that  $Z(t)$  can be written as in (4.1) and that the function  $\bar{Z}(\nu)$  has no step-wise variations at nonrandom frequencies. This means that there is no term such as (5.1) in  $Z(t)$ . As the differential signal  $d\bar{Z}(\nu) \exp(j2\pi\nu t)$  can be deduced from  $Z(t)$  by linear filtering, it is stationary. As a consequence, the increment  $d\bar{Z}(\nu)$  is a circular RV, which means that the only nonzero moments are in the form  $E[|d\bar{Z}(\nu)|^{2k}]$ . Consequently, the signal  $Z(t)$  is marginally circular in the frequency domain, as defined after (4.1). Without further assumption concerning  $\bar{Z}(\nu)$ , it is the only circularity that is implied by the stationarity of  $Z(t)$ .

2) *Strong Circularity and Normal Manifolds*: At any time instant  $t$ , the most general moment of  $Z(t)$  can be written as

$$m(t; p, q) \triangleq E[Z^p(t)Z^{*q}(t)]. \quad (5.9)$$

This moment must be time independent because of the stationarity of  $Z(t)$ . From (4.1), we deduce that it can be written as

$$m(t; p, q) = \int M(\boldsymbol{\nu}; p, q) \exp[2\pi jt(\nu_1 + \nu_2 + \cdots + \nu_p - \nu_{p+1} - \cdots - \nu_{p+q})] d\boldsymbol{\nu} \quad (5.10)$$

with

$$M(\boldsymbol{\nu}; p, q) d\boldsymbol{\nu} = E[d\bar{Z}(\nu_1) \cdots d\bar{Z}(\nu_p) d\bar{Z}^*(\nu_{p+1}) \cdots d\bar{Z}^*(\nu_{p+q})]. \quad (5.11)$$

As (5.10) must be time independent, it appears that the spectral components  $M(\boldsymbol{\nu}; p, q)$  must be zero outside the manifold of  $\mathbb{R}^{p+q}$  defined by

$$\nu_1 + \nu_2 + \cdots + \nu_p - \nu_{p+1} - \nu_{p+2} - \cdots - \nu_{p+q} = 0. \quad (5.12)$$

This manifold is called the *stationary* manifold of order  $p, q$  (see p. 238 of [1]). This geometrical consequence of stationarity allows us to find the marginal circularity again. In fact, if all the frequencies are equal to  $\nu$ , (5.12) implies that  $p = q$ , and then, the only nonzero moments of the RV  $d\bar{Z}(\nu)$  are those written as  $E[|d\bar{Z}(\nu)|^{2k}]$ , which specifies marginal circularity.

However, it is well known (see p. 279 of [1]) that the spectral moments such as (5.11) of stationary *normal* circular signals are nonzero only on submanifolds of the stationary manifolds; these are known as *normal manifolds*. More precisely, in the normal case, (5.11) is nonzero only if  $p = q$  and if each frequency  $\nu_i$  is associated with a frequency  $\nu_{p+j}$  in such a way that  $\nu_i = \nu_{p+j}$ . There are, of course,  $p$  such equations, and a normal manifold is a  $p$ -dimensional subspace of  $\mathbb{R}^{2p}$ . For example, for  $p = 2$ , there are two normal manifolds defined by

$$\nu_1 = \nu_3; \nu_2 = \nu_4 \text{ and } \nu_1 = \nu_4; \nu_2 = \nu_3. \quad (5.13)$$

For arbitrary  $p$ , there are  $p!$  normal manifolds. It is also known (see p. 280 of [1]) that a signal is normal if the moments  $M(\boldsymbol{\nu}; p, q)$  are not only distributed in the normal manifolds but if the density on these manifolds is normal as well.

This leads to study signals that are not normal but with spectral moments (5.11) distributed only on the normal manifolds. We shall see that such signals are *strongly circular*.

In fact, after filtering of  $Z(t)$  in a filter with frequency response  $H(\nu)$ , the spectral moments  $M(\boldsymbol{\nu}; p, q)$  defined by (5.11) are transformed into

$$M'(\boldsymbol{\nu}; p, q) = H(\nu_1) \cdots H(\nu_p) H^*(\nu_{p+1}) \cdots H^*(\nu_{p+q}) M(\boldsymbol{\nu}; p, q). \quad (5.14)$$

Suppose that  $H(\nu)$  is a phase filter. If  $p = q$  and if to each  $\nu_i$  there is a  $\nu_{p+j}$  equal to  $\nu_i$ , we obtain the term  $|H(\nu_i)|^2$ , which is equal to 1 by definition. Because this is valid for each  $\nu_i$ , we deduce that  $M'(\boldsymbol{\nu}; p, p) = M(\boldsymbol{\nu}; p, p)$ . As a consequence, all the nonzero spectral moments of  $Z(t)$  are invariant through transformation by a phase filter. This is precisely the definition of strong circularity. It is easy to verify that the converse is also true. As a consequence, stationarity and strong circularity are equivalent if and only if the spectral moments  $M(\boldsymbol{\nu}; p, q)$  are zero outside the normal manifolds.

3) *Total Circularity*: Total circularity is defined at the end of Section IV, and it is obvious that if the increments  $d\bar{Z}(\nu)$  are independent, stationarity and circularity are equivalent. The reasoning is the same as in Section V-A-4.

### C. Narrow-Band Signals

We have seen previously that if  $Z(t)$  contains only one spectral component, the assumption of stationarity implies circularity and conversely. It seems obvious that the property of circularity that is valid for a purely monochromatic signal will still partially appear if  $Z(t)$  has a spectral representation

limited to a narrowband in the neighborhood of  $\omega_0$ . This problem is analyzed in p. 241 of [1], and the results can be summarized as follows.

Suppose that the increments of  $d\bar{Z}(\nu)$  appearing in (4.1) are zero outside the frequency domain  $\nu_0 - \Delta\nu, \nu_0 + \Delta\nu$  and that  $Z(t)$  is stationary. By using the same notations as in (2.6), we can introduce the moments defined by

$$m_k[\{t_i\}; \{\varepsilon_j\}] = E[Z^{\varepsilon_1}(t_1)Z^{\varepsilon_2}(t_2) \cdots Z^{\varepsilon_k}(t_k)]. \quad (5.15)$$

It is then shown in the Appendix that if  $k \leq N = \text{Ent}[\nu_0/\Delta\nu]$ , where  $\text{Ent}[x]$  means the entire part of  $x$ , we have  $m_k[\{t_i\}; \{\varepsilon_j\}] = 0$  either if  $k$  is odd or if  $k$  is even, but  $\sum \varepsilon_j \neq 0$ . This means the signal  $Z(t)$  has a moment circularity to the order  $N$  similar to that indicated in Section III-D.

#### D. Discussion and Examples

At the end of this section, let us first discuss the case of *real* stationary signals. The condition ensuring that  $Z(t)$  of (4.1) is real is the Hermitian symmetry  $d\bar{Z}(\nu) = dZ^*(-\nu)$ . If  $Z(t)$  is real and stationary, the increments  $d\bar{Z}(\nu)$  cannot be real. In fact, this reality would imply that  $Z(t) = Z(-t)$ , which is impossible for a stationary signal. This property is also a consequence of the marginal circularity appearing as a consequence of stationarity both for real and complex signals.

It is also of interest to discuss the case of normal real and stationary signals. It is especially interesting to explain why they cannot be strongly circular, as in the complex case. In fact, when filtering such a signal in any phase filter, the correlation function and the normality are preserved. However, the output of such a filter has no reason to remain real, and in order to preserve reality, the phase function  $\phi(\nu)$  must satisfy the constraint  $\phi(\nu) = \phi(-\nu)$ . Without this constraint, the output is complex, and the second-order moment  $E[Z(t)Z(t-\tau)]$  has no reason to remain invariant. Therefore, the statistical properties of a real normal stationary signal are not invariant in any phase filtering, which is in contradiction with the strong circularity. The same reasoning can be applied to real signals with nonzero moments only on the normal manifolds.

Let us now, as indicated above, give an example showing that marginal circularity only does not imply stationarity. Suppose that in (5.1)  $Z_k = Z, 1 \leq k \leq N$ , where  $Z$  is a circular RV. It is obvious that marginal circularity holds. However, an elementary calculation shows that  $E[|Z(t)|^2]$  is not time invariant, which shows that  $Z(t)$  is not stationary.

Let us now give examples of nonnormal signals that are strongly circular. It is especially the case of complex spherically invariant stochastic processes (see p. 299 of [1] and [5], [6]). Let  $u(t)$  be a complex normal circular signal and  $w$  an RV independent of  $u(t)$ . The signal  $z(t) = wu(t)$  is no longer normal and is obviously stationary. Furthermore, as  $u(t)$  is strongly circular, this property also holds for  $z(t)$ . It is easy to show that the spectral higher order moments of  $z(t)$  are nonzero only in the normal manifolds, as for any strongly circular signal.

Finally, let us briefly indicate an example of a signal that is nonnormal and totally circular. Suppose that in (4.1),  $\bar{Z}(\nu)$  is

constructed from a Poisson process (see p. 318 of [1]). This means that  $Z(t)$  can be written as

$$Z(t) = \sum_k Z_k \exp(2\pi j\nu_k t) \quad (5.16)$$

where the frequencies  $\nu_k$  are random and constitute a Poisson process of density  $\lambda(\nu)$ , and the random complex amplitudes  $Z_k$  are independent of each other and of the Poisson process. This expression is clearly a generalization of (5.1). It is obvious that this signal is not normal. However, because of all the independence assumptions, stationarity of  $Z(t)$  implies that it is totally circular. This can easily be verified in the frequency domain. In order to calculate the moment (5.11), it suffices to transpose a procedure well known in the time domain (see p. 367 of [1]). For example, the spectral moment (5.11) calculated for  $p = q = 2$  takes the form

$$\begin{aligned} M(\nu; 2, 2) &= \Gamma(\nu_1)\Gamma(\nu_3)[\delta(\nu_1 - \nu_3)\delta(\nu_2 - \nu_4) \\ &\quad + \delta(\nu_1 - \nu_4)\delta(\nu_2 - \nu_3)] \\ &\quad + \lambda(\nu)E[|Z(\nu)|^4] \\ &\quad \times \delta(\nu_1 - \nu_2)\delta(\nu_2 - \nu_3)\delta(\nu_3 - \nu_4) \end{aligned} \quad (5.17)$$

where  $\Gamma(\nu)$  is the power spectrum and  $Z(\nu)$  is the RV  $Z_k$  when it appears at the frequency  $\nu_k = \nu$ . It is obvious that (5.17) is zero outside the normal manifold. However, its last term is zero in the normal case, which shows that the signal  $Z(t)$  is not normal. This can also be expressed in terms of cumulant *trispectrum*. In fact, it is well known that fourth-order cumulants are obtained from moments by subtracting the normal contribution. This implies that the trispectrum of  $Z(t)$  is given by the last term of (5.17). This term cannot be zero, and then, contrary to the case of normal signals, the trispectrum is also nonzero even if the increments appearing in (4.1) are independent.

At the end of this section, it is worth considering the case of the analytic signal (AS) of a stationary real signal  $x(t)$ . In fact, the AS is one of the most important examples of a complex signal that is used in many areas of signal processing and communications and appears, in particular, in all the problems using narrow-band signals. Without any more specific assumption, the AS is only marginally circular. If the higher order spectra of  $x(t)$  are only dealing with normal manifolds, the AS is strongly circular. If the increments of the spectral representation of  $x(t)$  are independent, the AS is totally circular, and this is the case when  $x(t)$  is normal but can appear with other kinds of signals.

Let us now summarize the main results of this discussion. For any kind of harmonizable signal, stationarity implies *marginal* circularity of the frequency components. This does not mean circularity of the signal itself. If, furthermore, the spectral moments only deal with normal manifolds, which appears in the case of signals with incommensurable discrete frequencies, stationarity implies *strong* circularity of the signal. Now, if the components appearing in the spectral representation of the signal are independent, which particularly appears in the normal case but not in this case alone, stationarity implies total circularity of the signal. Conversely, total and strong circularity of the signal implies stationarity. On the

other hand, marginal circularity in the frequency domain does not imply stationarity, and weak circularity of the signal alone has no specific relationship with stationarity. Finally, stationary narrow-band signals have circular moments up to a certain order.

## VI. CIRCULARITY AND STATIONARITY OF DISCRETE-TIME SIGNALS

The problem immediately appears to be more complicated than for the continuous-time case. In fact, consider the pure monochromatic signal  $Z[k] = Z \exp(2\pi j\nu k)$ ,  $|\nu| < 1/2$  and  $k$  integer. It is stationary if for any integer  $p$ ,  $Z[k]$  and  $Z[k+p]$  are complex RV's with the same distribution. This does not imply that the complex amplitude  $Z$  is a circular RV. For example if  $\nu = 1/4$ , the condition on  $Z$  is that  $\pm Z$  and  $\pm jZ$  have the same distribution, which is insufficient to imply circularity. However, if the frequency  $\nu$  is an *irrational* number, the fact that the set of irrational numbers is dense in  $[0, 1]$  implies that the stationarity of  $Z[k]$  is equivalent to the circularity of  $Z$ . Therefore, the property that is valid for any frequency with continuous-time signals is only valid for irrational frequencies with discrete-time signals. The same kind of difficulty will appear when repeating all the discussion of the previous section, and we shall only outline the principal results.

Let us first consider a signal like (5.1) that is equal to a sum of monochromatic components such as

$$Z_s[k] = Z_s \exp(2\pi j\nu_s k) \quad (6.1)$$

where  $s$  is integer ( $1 \leq s \leq N$ ). Because these components are stationary, we can conclude that if  $\nu_s$  is irrational, the RV  $Z_s$  is circular. Therefore, if all the frequencies  $\nu_s$  are not rational numbers, the stationarity of  $Z(t)$  implies its marginal circularity, as for the continuous-time case.

However, by using moment circularity, a less general result can be obtained even for rational frequencies. For this, consider the component (6.1) and, as in (5.9), its moment

$$m(k; p, q) \triangleq E\{Z_s^p[k] Z_s^{*q}[k]\}. \quad (6.2)$$

From (6.1), we deduce that

$$m(k; p, q) = m_s(p, q) \exp[2\pi j\nu_s k(p - q)] \quad (6.3)$$

where  $m_s(p, q)$  is the moment of order  $p, q$  of the RV  $Z_s$ . Because (6.3) must be independent of  $k$ , we deduce that we must have  $m_s(p, q) = 0$  if  $\nu_s(p - q) \neq l$  with  $l$  integer or zero. If  $\nu_s$  is irrational, this implies that  $p = q$ , and then, the only nonzero moments of  $Z_s$  are  $E[|Z_s|^{2p}]$ . This gives, as seen above, the marginal circularity.

On the other hand, if  $\nu_s$  is a rational number, the relation on  $\nu_s$  does not imply that  $p = q$  and, then, the marginal circularity. However, this can be valid at least for some values of  $p$  and  $q$ . Suppose that we are only interested in moments of an order limited by  $N$ . This means that  $p + q \leq N$ . In many studies concerning higher order statistics,  $N = 4$ , which, for instance, introduces the bi and trispectrum. It is then obvious that for frequencies satisfying  $|\nu_s| < (1/N)$ , we have  $|\nu_s(p - q)| < 1$  and then for those frequencies  $m_s(p, q) = 0$  if  $p \neq q$ . There is

then a moment circularity up to a given order for frequencies smaller than a limit depending on this order.

Let us now consider the case of a two-component signal in order to find under which conditions strong circularity can appear. With the same reasoning as in Section V-A-3, we can start from (5.4), where  $t$  is replaced by an integer  $k$ . The result is then that  $m_{pqrs}$  is nonzero only if  $(p - q)\nu_1 + (r - s)\nu_2 = a$ , where  $a$  is an integer. Therefore, the condition that the ratio  $\nu_2/\nu_1$  is irrational is not sufficient to deduce that  $p = q$ , and  $r = s$ . However, if  $\nu_1$  is a rational number and  $\nu_2$  an irrational number, the previous equation implies  $p = q$  and  $r = s$ , or the strong circularity.

Similarly, if  $Z[k]$  has a continuous spectrum with a spectral representation like (4.1), where the integration is on the range of frequencies  $|\nu| < 1/2$ , then stationarity implies marginal circularity for any frequency  $\nu$  that is an irrational number.

All this discussion shows that the discrete-time case is much more difficult to analyze than the continuous-time case. The main reason for this is that by discrete increments on the unit circle, it is not always possible to reach any point of this circle because of the possible periodic behavior. Except for this specific problem, the main general conclusions are the same as for continuous time signals and are indicated at the end of the previous section.

Another question dealing with spectral representation is whether or not the coefficients appearing in the expansion of a stationary signal in Fourier series are circular.

Two distinct situations must be considered. The first one appears when the signal  $Z(t)$  is stationary and periodic. In this case, it can be expanded as in (5.1), where  $k$  is going from  $-\infty$  to  $+\infty$ , where  $\omega_k = 2\pi/T$ ,  $T$  is the period of the signal. Therefore, the conclusion of the previous discussion remains the same: If  $Z(t)$  is stationary, the  $Z_k$ 's are marginally circular. If, furthermore, the  $Z_k$ 's are independent,  $Z(t)$  is totally circular.

The most common situation appears when the signal is not periodic. In this case, (5.1) is only valid in a specific interval, say  $[0T]$ , and the Fourier coefficients are given by (see p. 209 of [1])

$$Z_k = \int d\bar{Z}(\nu) \text{sinc}\{T[\nu - (k/T)]\}. \quad (6.4)$$

It is clear that when  $Z(t)$  is stationary, this random variable has generally no reason to be circular. However, it has this property when  $\bar{Z}(\nu)$  is a process with independent increments, as can easily be verified. The situation is exactly the same with the coefficients appearing in the discrete Fourier transform of a nonperiodic stationary signal.

Let us finally discuss the relationships between circularity and *sampling*. This operation applied to a stationary continuous-time signal  $Z(t)$  providing a stationary discrete-time signal. It is then of interest to study how marginal circularity is preserved in the aliasing phenomenon due to the sampling procedure. To explain the situation, suppose that  $Z(t)$  is given by (5.1) with only two components at frequencies  $\nu_1$  and  $\nu_2$  with  $\nu_1 < \nu_2$ . Let  $Z_s[k] = Z(kT)$  be the sampled signal. If  $\nu_2 < 1/2T$ , there is no aliasing effect.

Suppose then that  $\nu_1 < 1/2T < \nu_2$ . In this case, we have

$$Z_s[k] = Z_1 \exp(2\pi j\nu_1 k) + Z_2 \exp(2\pi j\nu_2 k) \quad (6.5)$$

with  $\mu_2 = \nu_2 - k/T$ , where  $k$  is such that  $|\mu_2| < 1/2$ . In this case,  $Z_1$  and  $Z_2$  remain circular, and  $Z_s[k]$  is marginally circular in the frequency domain. The only problem appears if  $\mu_2 = \nu_1$ . In this case,  $Z_s[k] = (Z_1 + Z_2) \exp(2\pi j\nu_1 k)$ , and it is left to be explained why  $Z_1 + Z_2$  is circular when  $\omega_1$  is irrational. For this, let us use (5.3), which becomes

$$p(\mathbf{a}; \phi_1, \phi_2) = p(\mathbf{a}; \phi_1 + 2\pi\nu_1\tau, \phi_2 + 2\pi\nu_2\tau), \forall \tau. \quad (6.6)$$

Suppose that  $\nu_2 = \nu_1 + (m/T)$  with  $m$  integer, and let us apply this at the time instants  $\tau = kT$ . As the phase is defined modulo  $2\pi$ , this yields

$$p(\mathbf{a}; \phi_1, \phi_2) = p(\mathbf{a}; \phi_1 + 2\pi\nu_1 kT, \phi_2 + 2\pi\nu_1 kT), \forall k. \quad (6.7)$$

As  $\nu_1$  is irrational, this equation can be transformed into

$$p(\mathbf{a}; \phi_1, \phi_2) = p(\mathbf{a}; \phi_1 + \theta, \phi_2 + \theta), \forall \theta. \quad (6.8)$$

As a result, the RV's  $Z_1$  and  $Z_2$  are the components of a weakly circular vector. Their sum is then circular, and as in (6.1), we again find the marginal circularity of  $Z_s[k]$ . This reasoning can be generalized for an arbitrary number of components and more generally for the increment

$$d\bar{Z}_s(\nu) = (1/T) \sum_n d\bar{Z}(\nu - n/T) \quad (6.9)$$

appearing in the spectral representation of the signal  $Z_s[k]$  obtained by sampling at the period  $T$  of  $Z(t)$ , provided that the frequency  $\nu$  is irrational.

## VII. CIRCULARITY AND LINEAR MEAN SQUARE ESTIMATION

Consider two complex random vectors  $\mathbf{X}$  and  $\mathbf{Y}$  belonging to  $\mathbb{C}^m$  and  $\mathbb{C}^n$ , respectively. Their decomposition in real, and imaginary parts are

$$\mathbf{X} = \mathbf{X}_1 + j\mathbf{X}_2; \mathbf{Y} = \mathbf{Y}_1 + j\mathbf{Y}_2 \quad (7.1)$$

and we assume that all these vectors have a zero mean value. We want to study the problem of linear mean square estimation (LMSE) of  $\mathbf{Y}$  in terms of  $\mathbf{X}$ .

### A. Classical Presentation of Complex LMSE

The problem of complex LMSE is usually presented as an obvious extension of the real situation. In this perspective, the complex estimate  $\hat{\mathbf{Y}}_c$ , which, for each component  $Y_i$ , minimizes the mean square error  $E[|Y_i - \hat{Y}_i|^2]$ ,  $1 \leq i \leq n$ , can be written as  $\hat{\mathbf{Y}}_c = \mathbf{M}_c \mathbf{X}$ , where the estimation matrix is the solution of the equation

$$\mathbf{M}_c \mathbf{F}_c = \mathbf{R}_{y_x, c} \quad (7.2)$$

with

$$\mathbf{F}_c \triangleq E[\mathbf{X}\mathbf{X}^H]; \mathbf{R}_{y_x, c} \triangleq E[\mathbf{Y}\mathbf{X}^H]. \quad (7.3)$$

Writing the real and imaginary parts of  $\mathbf{F}_c$ ,  $\mathbf{R}_{y_x, c}$  and  $\mathbf{M}_c$  as in (7.1), or  $\mathbf{F}_c = \mathbf{F}_1 + j\mathbf{F}_2$ ,  $\mathbf{R}_{y_x, c} = \mathbf{R}_1 + j\mathbf{R}_2$ , and  $\mathbf{M}_c = \mathbf{M}_1 + j\mathbf{M}_2$ , one can express (7.2) in the form

$$\mathbf{M}_1 \mathbf{F}_1 - \mathbf{M}_2 \mathbf{F}_2 = \mathbf{R}_1 \quad (7.4)$$

$$\mathbf{M}_1 \mathbf{F}_2 + \mathbf{M}_2 \mathbf{F}_1 = \mathbf{R}_2$$

and the estimate  $\hat{\mathbf{Y}}_c$  takes the form

$$\hat{\mathbf{Y}}_c = \mathbf{M}_1 \mathbf{X}_1 - \mathbf{M}_2 \mathbf{X}_2 + j(\mathbf{M}_2 \mathbf{X}_1 + \mathbf{M}_1 \mathbf{X}_2). \quad (7.5)$$

### B. Real Presentation of Complex LMSE

As minimizing  $E[|U|^2]$ , where  $U$  is complex, is equivalent to minimizing  $E[U_1^2]$  and  $E[U_2^2]$ , the same problem can be stated in the real field by using vectors of  $\mathbb{R}^{2m}$  and  $\mathbb{R}^{2n}$ . In this perspective, the LMSE of  $\mathbf{Y}$  can be written as

$$\hat{\mathbf{Y}} = \text{LMSE}[\mathbf{Y}_1 | \mathbf{X}_1, \mathbf{X}_2] + j\text{LMSE}[\mathbf{Y}_2 | \mathbf{X}_1, \mathbf{X}_2] \quad (7.6)$$

where  $\text{LMSE}[A | BC]$  means the LMSE of  $A$  in terms of  $B$  and  $C$ , where  $A$ ,  $B$ , and  $C$  are real. Introducing the real and imaginary parts of  $\mathbf{Y}$ , one can write (7.6) in matrix form:

$$\begin{bmatrix} \hat{\mathbf{Y}}_1 \\ \hat{\mathbf{Y}}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix}. \quad (7.7)$$

It is clear that all the  $\mathbf{M}_{ij}$ 's are  $n \times m$  matrices.

Looking at (7.5), we observe that the complex estimation can also be written as  $\hat{\mathbf{Y}}_{c1} + j\hat{\mathbf{Y}}_{c2}$ , where

$$\begin{bmatrix} \hat{\mathbf{Y}}_{c1} \\ \hat{\mathbf{Y}}_{c2} \end{bmatrix} = \begin{bmatrix} \mathbf{M}_1 - \mathbf{M}_2 \\ \mathbf{M}_2 \mathbf{M}_1 \end{bmatrix} \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix} \quad (7.8)$$

and this shows that the classical complex LMSE of  $\mathbf{Y}$  can be written in the form (7.7) with a specific structure of the matrices  $\mathbf{M}_{ij}$ .

Our purpose is to show that for *circular vectors*, the general structure (7.7) takes the form (7.8). On the other hand, if circularity is not introduced, the complex LMSE has no reason to give the best performance that can be obtained with (7.7).

In order to arrive at our main result, the first step consists of calculating the matrices  $\mathbf{M}_{ij}$  appearing in (7.7). For this, let us introduce the vectors  $\mathbf{X}_r$  and  $\mathbf{Y}_r$  of  $\mathbb{R}^{2m}$  and  $\mathbb{R}^{2n}$ , respectively, which are defined by

$$\mathbf{X}_r^T = [\mathbf{X}_1^T, \mathbf{X}_2^T]; \mathbf{Y}_r^T = [\mathbf{Y}_1^T, \mathbf{Y}_2^T]. \quad (7.9)$$

The real LMSE of  $\mathbf{Y}_r$  in terms of  $\mathbf{X}_r$  can be written as

$$\hat{\mathbf{Y}}_r = \mathbf{M}_r \mathbf{X}_r \quad (7.10)$$

where  $\mathbf{M}_r$  is the solution of the equation deduced from the orthogonality principle

$$\mathbf{M}_r \mathbf{F}_{x_r} = \mathbf{R}_{y_x, r}. \quad (7.11)$$

This equation is very similar to (7.2). However, the matrices appearing here are quite different. The matrix  $\mathbf{M}_r$  is a  $2n \times 2m$  matrix, and its block decomposition appears in (7.7). Furthermore

$$\mathbf{F}_{x_r} \triangleq E[\mathbf{X}_r \mathbf{X}_r^T]; \mathbf{R}_{y_x, r} \triangleq E[\mathbf{Y}_r \mathbf{X}_r^T] \quad (7.12)$$

and the block decomposition of these matrices makes use of matrices  $\mathbb{F}_{ij}$  and  $R_{ij}$  in such a way that (7.11) becomes

$$\begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{bmatrix} \begin{bmatrix} \mathbb{F}_1 & \mathbb{F}_2 \\ \mathbb{F}_2 & \mathbb{F}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{R}_{11} & \mathbf{R}_{12} \\ \mathbf{R}_{21} & \mathbf{R}_{22} \end{bmatrix}. \quad (7.13)$$

This gives us four equations that allow the calculation of the matrices  $\mathbf{M}_{ij}$ .

Let us now introduce the assumption of *circularity*. In the present context, second-order circularity is sufficient, and this can be characterized, as in (2.1), by

$$E[\mathbf{X}\mathbf{X}^T] = \mathbf{0}; \quad E[\mathbf{Y}\mathbf{X}^T] = \mathbf{0} \quad (7.14)$$

where  $\mathbf{X}$  and  $\mathbf{Y}$  appear in (7.1). Note that in the normal case, this implies that the pair of vectors  $\mathbf{X}, \mathbf{Y}$  is jointly normal circular, as discussed in Section II. Writing

$$\mathbb{F}_c \triangleq E[\mathbf{X}\mathbf{X}^H] = \mathbb{F}_1 + j\mathbb{F}_2; \quad \mathbf{R}_c \triangleq E[\mathbf{Y}\mathbf{X}^H] = \mathbf{R}_1 + j\mathbf{R}_2 \quad (7.15)$$

it is easy to show (see p. 119 of [1]) that (7.14) yields

$$\mathbb{F}_{11} = \mathbb{F}_{22} = (1/2)\mathbb{F}_1; \quad \mathbb{F}_{12} = -\mathbb{F}_{21} = -(1/2)\mathbb{F}_2 \quad (7.16)$$

and

$$\mathbf{R}_{11} = \mathbf{R}_{12} = (1/2)\mathbf{R}_1; \quad \mathbf{R}_{12} = -\mathbf{R}_{21} = -(1/2)\mathbf{R}_2 \quad (7.17)$$

Introducing these equations in (7.13), we easily obtain

$$\mathbf{M}_{11} = \mathbf{M}_{22} = \mathbf{M}_1; \quad \mathbf{M}_{12} = -\mathbf{M}_{21} = -\mathbf{M}_2 \quad (7.18)$$

where the matrices  $\mathbf{M}_1$  and  $\mathbf{M}_2$  are solutions of (7.4).

This means that when the circularity assumption is introduced, the classical complex LMSE gives the same solution as the real estimation, where the real and imaginary parts of  $\mathbf{Y}$  are separately estimated in terms of  $\mathbf{X}_1$  and  $\mathbf{X}_2$ .

The previous result dealing with LMSE can be extended to problems of matched filtering or of minimum variance estimation. This is mainly due to the fact that LMSE is also a problem of variance minimization. These problems appear especially in spectral estimation or spatial filtering [7] and will not be analyzed here.

### C. Examples of Applications

In order to illustrate the previous results, we shall discuss some examples corresponding to the case where  $Y$  is now a scalar random variable. This especially appears in signal interpolation or prediction.

The classical theory of complex LMSE (see p. 50 of [7]) leads to the following result. The complex estimate  $\hat{Y}_c$  of  $Y$  can be written as

$$\hat{Y}_c = \mathbf{h}^H \mathbf{X} \quad (7.19)$$

where  $\mathbf{h}$  is the solution of

$$\mathbb{F}\mathbf{h} = \mathbf{r} \triangleq E(Y^* \mathbf{X}). \quad (7.20)$$

Introducing the real and imaginary parts of  $\mathbb{F}, \mathbf{h}$  and  $\mathbf{r}$ , this equation becomes

$$\begin{bmatrix} \mathbb{F}_1 & -\mathbb{F}_2 \\ \mathbb{F}_2 & \mathbb{F}_1 \end{bmatrix} \begin{bmatrix} \mathbf{h}_1 \\ \mathbf{h}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \end{bmatrix} \quad (7.21)$$

and  $\hat{Y}_c$  takes the form

$$\hat{Y}_c = \mathbf{h}_1^T \mathbf{X}_1 + \mathbf{h}_2^T \mathbf{X}_2 + j(-\mathbf{h}_2^T \mathbf{X}_1 + \mathbf{h}_1^T \mathbf{X}_2). \quad (7.22)$$

Furthermore, the estimation error is

$$\varepsilon_c^2 = \sigma_y^2 - \mathbf{r}^H \mathbf{h} = \sigma_y^2 - (\mathbf{h}_1^T \mathbf{r}_1 + \mathbf{h}_2^T \mathbf{r}_2) \quad (7.23)$$

where  $\sigma_y^2 = E[|Y|^2]$ .

The "real" solution of the same problem is  $\hat{Y}_r = \hat{Y}_1 + j\hat{Y}_2$ , and these estimates are written as

$$\begin{bmatrix} \hat{Y}_1 \\ \hat{Y}_2 \end{bmatrix} = \mathbf{H}^T \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{h}_{11}^T & \mathbf{h}_{21}^T \\ \mathbf{h}_{12}^T & \mathbf{h}_{22}^T \end{bmatrix} \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix} \quad (7.24)$$

where  $\mathbf{H}$  is a matrix with matrix elements  $\mathbf{h}_{ij}$ . As in (7.8), the real and imaginary parts of  $Y_c$  defined by (7.19) can be written as in (7.24) but with a matrix  $\mathbb{H}_c$  in the form

$$\mathbb{H}_c = \begin{bmatrix} \mathbf{h}_1 & -\mathbf{h}_2 \\ \mathbf{h}_2 & \mathbf{h}_1 \end{bmatrix}. \quad (7.25)$$

Introducing the matrix elements

$$\mathbf{r}_{ij} \triangleq E[\mathbf{X}_i Y_j] \quad (7.26)$$

the orthogonality equations defining  $\hat{Y}_1$  and  $\hat{Y}_2$  yield an equation similar to (7.13), or

$$\begin{bmatrix} \mathbb{F}_{11} & \mathbb{F}_{12} \\ \mathbb{F}_{21} & \mathbb{F}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{h}_{11} & \mathbf{h}_{12} \\ \mathbf{h}_{21} & \mathbf{h}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{r}_{11} & \mathbf{r}_{12} \\ \mathbf{r}_{21} & \mathbf{r}_{22} \end{bmatrix}. \quad (7.27)$$

We again find (7.21) when the circularity assumptions (7.16) and (7.17) are introduced. The estimation error  $\varepsilon_r^2$  is equal to  $\varepsilon_1^2 + \varepsilon_2^2$ , which is sum of the errors in  $\hat{Y}_1$  and  $\hat{Y}_2$ , and the result is

$$\varepsilon_r^2 = \sigma_y^2 - \sum_{i,j} \mathbf{h}_{ij}^T \mathbf{r}_{ij}. \quad (7.28)$$

We deduce from (7.25) and (7.17) that this expression again gives (7.23) when the circularity assumptions are introduced.

Let us now consider some specific examples. Suppose that  $\mathbf{X}$  and  $Y$  are real. This implies that  $\mathbf{X}_2 = \mathbf{0}$  and  $\mathbf{Y}_2 = \mathbf{0}$ , and consequently, the only nonzero matrix elements are  $\mathbb{F}_{11} = \mathbb{F}$  and  $\mathbf{r}_{11} = \mathbf{r}$ . Therefore, (7.27) becomes the classical real relation  $\mathbb{F}\mathbf{h}_{11} = \mathbf{r}$ , and  $\mathbf{h}_{21} = \mathbf{h}_{12} = \mathbf{h}_{22} = \mathbf{0}$ . The same result is deduced from (7.21), and the complex theory applied in the real case leads to the real theory.

Suppose now that  $\mathbf{X}$  is complex, where  $Y$  is still real. This corresponds to the estimation of a real quantity from complex observations. There are several examples of such situations, and a possible one is the estimation of the likelihood ratio, which is real, in terms of spectral components of an observation, which are complex.

In this case, the only consequence of this assumption is

$$\mathbf{r}_{12} = \mathbf{r}_{22} = \mathbf{0}. \quad (7.29)$$

This means that the last column of the matrix of the  $\mathbf{r}_{ij}$ 's appearing in (7.27) is zero. Because the matrix  $\mathbb{F}$  is positive-definite, we deduce that

$$\mathbf{h}_{12} = \mathbf{h}_{22} = \mathbf{0} \quad (7.30)$$

and the estimate  $\hat{Y}_r$  defined by (7.24) is real, which is quite satisfactory for the estimation of a real quantity  $Y$ .

On the contrary, there is no reason for  $\mathbf{h}_2$ , which is defined by (7.21), to be zero, and then, the application of the classical complex theory yields an unsatisfactory result since the estimate of a real quantity is complex. Furthermore, the estimation error is certainly greater than with the real estimate because by construction, it provides the best performance.

Finally, suppose that  $Y$  is still real, but  $\mathbf{X}$  is now circular. Replacing the values given by (7.16) in (7.27), this equation becomes similar to (7.21), where the  $\Gamma_i$ s are divided by 2. Consequently, one obtains

$$\mathbf{h}_{11} = 2\mathbf{h}_1; \quad \mathbf{h}_{21} = 2\mathbf{h}_2 \quad (7.31)$$

and then, the real estimate is

$$\hat{Y}_r = 2(\mathbf{h}_1^T \mathbf{X}_1 + \mathbf{h}_2^T \mathbf{X}_2) \quad (7.32)$$

whereas the complex estimate is still given by (7.22). In these two equations,  $\mathbf{h}_1$  and  $\mathbf{h}_2$  are deduced from (7.21).

The same conclusion appears as in the previous case: The complex estimate of the real quantity is complex, whereas  $\hat{Y}_r$  is real.

Furthermore, it is possible in this case to compare the performances of the two procedures. By using (7.23) and (7.28), we obtain

$$\varepsilon_c^2 = \sigma_y^2 - a; \quad \varepsilon_r^2 = \sigma_y^2 - 2a \quad (7.33)$$

with

$$a = \mathbf{h}_1^T \mathbf{r}_1 + \mathbf{h}_2^T \mathbf{r}_2. \quad (7.34)$$

As  $a > 0$ , because it is, in reality, a quadratic form with a positive definite matrix, it appears clearly that the "real" procedure provides better performances than the "complex" procedure.

The conclusion of this discussion is that these two procedures are equivalent in the case where the observation and the estimation are complex *circular*, at least up to the second order, which justifies the interest and the importance of the assumption, that is often implicit, of circularity.

#### APPENDIX

Let  $M_k\{\{\nu_i\}; \{\varepsilon_j\}\}$  be the Fourier transform of (5.15). As  $Z(t)$  is stationary in the strict sense,  $M_k$  is zero outside the stationary manifold (see p. 238 of [1]) defined by

$$\sum_i \varepsilon_i \nu_i = 0 \quad (A.1)$$

Thus,  $m_k$  is nonzero only if (A.1) holds with frequencies  $\nu_i$  satisfying

$$\nu_0 - \Delta\nu < \nu_0 < \nu_0 + \Delta\nu. \quad (A.2)$$

It is obvious that if  $\Delta\nu = 0$ , this gives  $\sum \varepsilon_i = 0$ , which is impossible for  $k$  odd and gives (2.7) for  $k$  even. It is also obvious that if all the  $\varepsilon_i$ s are equal, it is impossible to have (A.1) and (A.2) simultaneously.

Therefore, let  $q$  be the number of coefficients  $\varepsilon_i$  equal to 1. One can write (A.1) in the form

$$S_1 \triangleq \nu_1 + \nu_2 + \dots + \nu_q = \nu_{q+1} + \nu_{q+2} + \dots + \nu_k \triangleq S_2. \quad (A.3)$$

If  $k = 2q$ , this is not in contradiction with (A.2), even when  $\Delta\nu = 0$ .

Suppose now that  $2q < k$ . This implies that the mean frequency of  $S_1$  is smaller than the one of  $S_2$ . In this case, (A.3) is possible only if the upper bound of  $S_1$  is greater than the lower bound of  $S_2$ , or

$$q(f_0 + \Delta\nu) > (k - q)(f_0 - \Delta\nu). \quad (A.4)$$

This yields

$$\Delta\nu > \frac{k - 2q}{k} f_0. \quad (A.5)$$

Therefore, a sufficient condition to obtain circularity of type  $C$  is that

$$\Delta\nu < L \quad (A.6)$$

where  $L$  is the lower bound of the last term of (A.5).

Taking into account all the values of  $q$  such that  $k - 2q > 0$ , we obtain that  $L = f_0/k$ . The same reasoning can be made if  $2q > k$  by changing the roles of  $S_1$  and  $S_2$ .

In conclusion, if  $\nu_0$  and  $\Delta\nu$  are given, we obtain the  $C$  circularity if  $k < f_0/\Delta\nu$ , and a sufficient condition of  $C$  circularity is then

$$k \leq \text{Ent}[f_0/\Delta\nu]. \quad (A.7)$$

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