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Consequences of self-preservation on the axis of a turbulent round jet

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On the basis of a two-point similarity analysis, the well-known power law variations for the mean kinetic energy dissipation rate $\bar{\epsilon}$ and the longitudinal velocity variance \bar{u}^2 on the axis of a round jet are derived. In particular, the prefactor for $\bar{\epsilon} \propto (x - x_0)^{-4}$, where x_0 is a virtual origin, follows immediately from the variation of the mean velocity, the constancy of the local turbulent intensity and the ratio between the axial and transverse velocity variance.

Second, the limit at small separations of the two-point budget equation, yields an exact relation illustrating the equilibrium between the skewness of longitudinal velocity derivative S and the destruction coefficient G of enstrophy. By comparing the latter relation with that for homogeneous isotropic decaying turbulence, it is shown that the approach towards the asymptotic state at infinite Reynolds number of $S + 2G/R_\lambda$ in the jet differs from that in purely decaying turbulence, although $S + 2G/R_\lambda \propto R_\lambda^{-1}$ in each case. This suggests that, at finite Reynolds numbers, the transport equation for $\bar{\epsilon}$ imposes a fundamental constraint on the balance between S and G which depends on the type of large-scale forcing and may thus differ from flow to flow. This questions the conjecture that S and G follow a universal evolution with R_λ ; instead, S and G must be tested separately in each flow. The implication for the constant $C_{\epsilon 2}$ in the $k - \bar{\epsilon}$ model is also discussed.

1. Introduction

The turbulent round jet is an important canonical decaying shear flow and its study remains valuable in the context of fundamental turbulence research. Also, it features in many practical applications, from pollutant dispersion to the design of efficient chemical reactors (*e.g.* Baldyga *et al.* (1994)). The hypothesis of self-preservation, which assumes that the flow is governed by a single set of length and velocity scales, has been extensively used for describing the spatial evolution of one-point statistics in the far field of a round jet (*e.g.* Panchapakesan & Lumley (1993)). Extending the analysis of self-preservation to two-point statistics provides a deeper insight into the flow details since the evolution of the turbulence structure at a given scale can be assessed. Burattini *et al.* (2005*b,a*) were the first to consider such an analysis on the axis of a round jet using structure functions and spectra. Ewing *et al.* (2007) further extended the similarity analysis to the entire flow using correlation functions. These studies suggested that self-similar solutions for the terms in the two-point budget equation exist, independently of the Reynolds number. Perhaps more importantly, Burattini *et al.* (2005*b*) suggested that since the Reynolds number $R_\lambda = u'\lambda/\nu$ (λ is the Taylor microscale, to be defined later and u' a typical

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velocity fluctuation) is constant during the decay, the round jet flow may assume self-similar solutions, independently of the choice of similarity variables that can be made. This particularity of the round jet flow considerably simplifies the similarity analysis since *inner* scales (*e.g.* the Taylor microscale λ , the integral length scale L_u or the Kolmogorov length-scale η , altogether defined later in the manuscript) can be used interchangeably with *outer* scales (the half-width for instance) because they all behave similarly during the decay. This circumvents the question of which are the most relevant similarity variables to be used for normalization. With the exception of the (very) far field of a plane wake for which the constancy of the Taylor microscale Reynolds number is likely to be attained only asymptotically, and the initial period of decay of grid turbulence at infinitely large Reynolds number, the round jet is the only decaying flow for which $R_\lambda = \text{constant}$.

One important consequence of self-similarity in the far field of a round jet is that single-point turbulent quantities should behave in a power-law fashion. For instance, the *inner* and *outer* length-scales are proportional to the streamwise distance x , the characteristic velocities follow an hyperbolic decay rate proportional to $(x - x_0)^{-1}$ and the mean kinetic energy dissipation rate $\bar{\epsilon}$ varies as $(x - x_0)^{-4}$ (x_0 is a virtual origin). The latter relation is generally derived by assuming that the normalized dissipation rate parameter $C_\epsilon = \bar{\epsilon}L_u/u'^3$ is constant during the decay (Antonia *et al.* 1980). It will be shown in §2 of this paper that this hypothesis is not required to obtain the -4 power-law for $\bar{\epsilon}$.

Since the pioneering work of Kolmogorov (1941), the mean dissipation rate is often considered as one of the most critical quantities in the study of turbulence. However, the transport equation for $\bar{\epsilon}$ has generally been given much less attention by comparison to *e.g.* the one-point kinetic energy budget. This is surprising considering that this equation is involved in the majority of closure models, *e.g.* the $k - \bar{\epsilon}$ model. A reliable parametrization of this type of closure equation requires a basic understanding of the mechanisms underpinning the evolution of $\bar{\epsilon}$ and its relationships to the large-scales. A particular motivation of the present study is to provide, within this perspective, some insight into the transport equation of $\bar{\epsilon}$ along the axis of a round jet.

To this end, we extend the analysis of Burattini *et al.* (2005b,a) in a significant manner by examining in detail some consequences of self-preservation for the two-point energy budget equation on the axis of a round jet. In section §2, it is first shown that the power-law variations for $\bar{\epsilon}$, the longitudinal velocity variance $\overline{u^2}$ and all subsequent quantities can be derived from the constraint imposed by self-preservation of the two-point transport equation. Attention is also paid to the prefactor for $\bar{\epsilon} \propto (x - x_0)^{-4}$. Then, the limit at small separation of the scale-by-scale budget is applied in §3 and yields the transport equation of the enstrophy (or equivalently that of $\bar{\epsilon}$ if local homogeneity holds). More precisely, a relation highlighting the equilibrium between the production S and destruction G of enstrophy (or dissipation rate) is obtained. This relation is compared to that of homogeneous decaying turbulence and implications for the $k - \bar{\epsilon}$ model are discussed.

2. Self-preservation on the axis of a round jet

On the axis of an axisymmetric jet flow, the experimental investigation of Burattini *et al.* (2005b) suggested that the energy budget equation is given by

$$-\frac{3}{r^4} \int_0^r s^4 \left[\overline{U} \frac{\partial \overline{(\Delta u)^2}}{\partial x} + 2 \left(\overline{(\Delta u)^2} - \overline{(\Delta v)^2} \right) \frac{\partial \overline{U}}{\partial x} \right] ds - \overline{(\Delta u)^3} + 6\nu \frac{\partial}{\partial r} \overline{(\Delta u)^2} = \frac{4}{5} \bar{\epsilon} r. \quad (2.1)$$

Eq. (2.1) may be valid over a range of scales over which local isotropy is satisfied. u and v correspond to the fluctuating velocity component in the longitudinal x and transverse direction y respectively. ν is the kinematic viscosity and $\bar{\epsilon} = 15\nu\overline{(\partial u/\partial x)^2}$ stands for the mean energy dissipation rate. The overbar denotes averaging and s is a dummy integration variable. $\Delta u = u(x+r) - u(x)$ is the velocity increment between two points separated by a distance r . Eq. (2.1) is the generalization of the Kolmogorov (1941) equation which accounts for the forcing acting at rather large scales. The first term corresponds to the decay of second-order structure function in the streamwise direction x whilst the second term underlines the effect of production through the longitudinal mean velocity gradient. In the context of infinite Reynolds numbers, and for scales sufficiently small, these terms can be neglected and Eq. (2.1) reduces to the Kolmogorov (1941) equation.

Applying the limit at large separations to Eq. (2.1), yields

$$-\frac{1}{2}\overline{U}\frac{\partial\overline{q^2}}{\partial x} - \left(\overline{u^2} - \overline{v^2}\right)\frac{\partial\overline{U}}{\partial x} = \bar{\epsilon}, \quad (2.2)$$

where $\overline{q^2} = \overline{u^2} + 2\overline{v^2}$ by virtue of axisymmetry. Eq. (2.2) is the one-point kinetic energy budget and indicates that the mean energy dissipation rate compensates both advection and production effects. The pressure-transport term has been neglected in Eq. (2.2) as was done in Burattini *et al.* (2005a); Panchapakesan & Lumley (1993); Taub *et al.* (2013); the DNS data of Taub *et al.* (2013) provide some support for this on the jet axis. For the data of Burattini *et al.* (2005a), the contribution from the production term is about one tenth of that from the advection. Lipari & Stansby (2011) have reviewed the differences between previously published one-point kinetic energy budgets and confirm this.

For the range of scales over which Eq. (2.1) is valid, let us suppose that second- and third-order structure functions can be written as

$$\overline{(\Delta u)^2}(x, r) = u_K^2(x)f^*(r^*) \quad (2.3a)$$

$$\overline{(\Delta v)^2}(x, r) = u_K^2(x)h^*(r^*) \quad (2.3b)$$

$$-\overline{(\Delta u)^3}(x, r) = u_K^3(x)g^*(r^*), \quad (2.3c)$$

where $u_K = (\nu\bar{\epsilon})^{1/4}$ is the Kolmogorov characteristic velocity and $r^* = r/\eta$ with $\eta = (\nu^3/\bar{\epsilon})^{1/4}$ the Kolmogorov length-scale. In Burattini *et al.* (2005b), it was revealed that equilibrium similarity for the scale-by-scale budget yields the Taylor microscale as the relevant normalization length-scale. However, as emphasized by Burattini *et al.* (2005b), the constancy of R_λ suggests that the degree with which structure functions and spectra comply with similarity should be equally satisfactory, regardless of the choice of scaling parameters, and complete self-similarity may be reached. In other words, Kolmogorov scaling can be used interchangeably with George (1992) similarity variables ($\equiv \lambda$ and $\overline{u^2}$) or integral scales ($\equiv L_u$ and $\overline{u^2}$). Use was made here of the Kolmogorov similarity variables, hereafter indicated by an asterisk. The scale-by-scale budget Eq. (2.1) can thus be written as

$$\left[\frac{3\overline{U}}{u_K}\frac{d\eta}{dx}\right]\frac{\Gamma_1^*}{r^{*4}} - \left[\frac{3\overline{U}}{\bar{\epsilon}}\frac{du_K^2}{dx}\right]\frac{\Gamma_2^*}{r^{*4}} - \left[\frac{6u_K^2}{\bar{\epsilon}}\frac{\partial\overline{U}}{\partial x}\right]\frac{\Gamma_2^* - \Gamma_3^*}{r^{*4}} + [1]g^* + [6]f^{*'} = \left[\frac{4}{5}\right]r^*, \quad (2.4)$$

where $\Gamma_1^* = \int_0^{r^*} s^{*5} f^{*'} ds^*$, $\Gamma_2^* = \int_0^{r^*} s^{*4} f^* ds^*$, $\Gamma_3^* = \int_0^{r^*} s^{*4} h^* ds^*$, and the prime denotes differentiation with respect to r^* . Complete self-preservation requires that all terms within brackets behave similarly with x . Since three appear to be constant, this implies

the following conditions for the other three

$$\frac{3\bar{U}}{u_K} \frac{d\eta}{dx} = C_1 \quad (2.5a)$$

$$-\frac{3\bar{U}}{\bar{\epsilon}} \frac{du_K^2}{dx} = C_2 \quad (2.5b)$$

$$-\frac{6u_K^2}{\bar{\epsilon}} \frac{\partial \bar{U}}{\partial x} = C_3, \quad (2.5c)$$

where the constants C_1 , C_2 and C_3 need to be evaluated. As discussed in [Burattini et al. \(2005b\)](#) and [Ewing et al. \(2007\)](#), conservation of mean momentum requires the longitudinal mean velocity \bar{U} to behave as follows

$$\frac{\bar{U}}{U_j} = A_U \left(\frac{x - x_0}{D} \right)^{-1}. \quad (2.6)$$

where U_j is the jet exit velocity, D is the nozzle diameter and A_U is the power-law prefactor for \bar{U} . Combining Eqs. (2.5a) and (2.6) leads to

$$\frac{\bar{\epsilon}D}{U_j^3} = \left[\frac{3A_U}{C_1 Re_D^{1/2}} \right]^2 \left(\frac{x - x_0}{D} \right)^{-4} \quad (2.7)$$

where $Re_D = U_j D / \nu$. Eq. (2.7) emphasizes that the well-known power-law evolution

$$\frac{\bar{\epsilon}D}{U_j^3} = A_\epsilon \left(\frac{x - x_0}{D} \right)^{-4}, \quad (2.8)$$

is recovered. A_ϵ is the power-law prefactor for $\bar{\epsilon}$ and will be evaluated further. Using Eqs. (2.5b) and (2.5c), it can be shown that $C_3 = C_2 = 2C_1$.

In order to assess the evolution of the longitudinal velocity variance, the one-point energy budget of turbulent kinetic energy Eq. (2.2) is used. Injecting Eqs. (2.6) and (2.8) into Eq. (2.2), yields a non-linear first-order differential equation of the form

$$\mathcal{A}\hat{x} \frac{\partial \mathcal{F}}{\partial \hat{x}} + \mathcal{B}\mathcal{F} = \mathcal{C}\hat{x}^{-2}, \quad (2.9)$$

where $\hat{x} = (x - x_0)/D$, $\mathcal{F} = \overline{u^2}/U_j^2$, $\mathcal{A} = -1/2 - \mathcal{R}$, $\mathcal{B} = 1 - \mathcal{R}$, $\mathcal{C} = A_\epsilon/A_U$, $\mathcal{R} = \overline{v^2}/\overline{u^2}$. Seeking a solution for \mathcal{F} of the form $\mathcal{F} = \mathcal{D}\hat{x}^\alpha$, we finally obtain

$$\frac{\overline{u^2}}{U_j^2} = \left[\frac{A_\epsilon}{(2 + \mathcal{R})A_U} \right] \left(\frac{x - x_0}{D} \right)^{-2}. \quad (2.10)$$

Therefore, the power-law decay for the longitudinal velocity variance is also deduced. In addition, the local turbulence intensity

$$A_I = \frac{\sqrt{\overline{u^2}}}{\bar{U}} = \left[\frac{A_\epsilon}{(2 + \mathcal{R})A_U^3} \right]^{1/2} \quad (2.11)$$

is constant with respect to x provided \mathcal{R} does not vary with x (self-preservation for one-point statistics requires \mathcal{R} to be constant). We recall here that a reliable estimation of the mean energy dissipation rate is a challenging task for the experimentalist. Eq. (2.11) provides a simple means for estimating $\bar{\epsilon}$ once \bar{U} , $\overline{u^2}$ and \mathcal{R} are known, *viz.*

$$\frac{\bar{\epsilon}D}{U_j^3} = [(2 + \mathcal{R}) A_U^3 A_I^2] \left(\frac{x - x_0}{D} \right)^{-4}. \quad (2.12)$$

TABLE 1. Experimental and predicted values for the prefactors A_ϵ and A_{R_λ} provided $\mathcal{R} = 2/3$ as in [Burattini et al. \(2005b\)](#) and the respective experimental values for A_U and A_I . Experimental data are taken from [Mi et al. \(2013\)](#) (MXZ), [Friehe et al. \(1972\)](#) (FVAG) and [Burattini et al. \(2005b\)](#) (BAD)

	MXZ								FVAG	BAD
$Re_D(10^3)$	4.05	5.40	6.75	8.05	10.75	13.5	16.35	20.10	120	113
A_ϵ	18.5	28.7	32.0	41.7	47.5	49.2	50.7	49.9	48	42.5
Eq. (2.12)	21.5	27.5	28.9	32.3	34.1	35.5	36.0	35.4	39.7	32.7
Error (%)	16	4	10	29	39	38	41	41	21	30
A_{R_λ}					1.16				1.56	1.33
Eq. (2.15)	1.23	1.25	1.23	1.26	1.28	1.29	1.30	1.29	1.39	1.26
Error (%)	6	8	6	9	10	12	12	11	12	5

Even though assuming $C_\epsilon = \text{constant}$ (which ensues from the hypothesis of self-preservation on the jet axis) also leads to the -4 power-law for $\bar{\epsilon}$, it is worth noting that the magnitude of C_ϵ is not needed here for the prefactor A_ϵ to be derived. It follows that the Kolmogorov characteristic scales should behave as

$$\frac{\eta}{D} = [(2 + \mathcal{R}) Re_D^3 A_U^3 A_I^2]^{-1/4} \left(\frac{x - x_0}{D} \right), \quad (2.13a)$$

$$\frac{u_K}{U_j} = \left[\frac{(2 + \mathcal{R}) A_U^3 A_I^2}{Re_D} \right]^{1/4} \left(\frac{x - x_0}{D} \right)^{-1}, \quad (2.13b)$$

and therefore

$$\overline{u^{*2}} = \frac{\overline{u^2}}{u_K^2} = \left[\frac{A_U A_I^2 Re_D}{2 + \mathcal{R}} \right]^{1/2} \quad (2.14)$$

is constant, independently of x . Since $\overline{u^{*2}} = R_\lambda / \sqrt{15}$, where $R_\lambda = (\overline{u^2})^{1/2} \lambda / \nu$ is the Reynolds number based on the Taylor microscale $\lambda = (15\nu\overline{u^2}/\bar{\epsilon})^{1/2}$, it follows that

$$A_{R_\lambda} = \frac{R_\lambda}{Re_D^{1/2}} = \left[\frac{15 A_U A_I^2}{2 + \mathcal{R}} \right]^{1/2}. \quad (2.15)$$

It is thus not surprising to observe that complete self-preservation requires R_λ to be constant during the decay. The definition of the Taylor microscale leads to

$$\frac{\lambda}{D} = \left[\frac{15}{(2 + \mathcal{R}) Re_D A_U} \right]^{1/2} \left(\frac{x - x_0}{D} \right) \quad (2.16)$$

Since the constant C_1 is related to A_ϵ via Eq. (2.7), the final equation for f^* on the axis of turbulent round jet can be written as

$$\left(\frac{3\sqrt{15}}{2 + \mathcal{R}} \right) R_\lambda^{-1} (\Gamma_1^* + 4\Gamma_2^* - 2\Gamma_3^*) r^{*-4} + g^* + 6f'^* = \frac{4}{5} r^*. \quad (2.17)$$

Note that Eq. (2.17) explicitly indicates that the large-scale terms should vanish when $R_\lambda \rightarrow \infty$.

In Table 1, a sample a different experimental values for A_ϵ and A_{R_λ} are compared with values inferred from Eqs. (2.12) and (2.15) respectively, with $\mathcal{R} = 2/3$ ([Burattini et al.](#)

2005b) and the respective experimental values for A_U and A_I . Predictions are generally in satisfactory agreement with experiments, although lower on average by about 25% for A_ϵ whereas discrepancies for A_{R_λ} are on average of about 10%.

The systematic difference between experimental and predicted values for A_ϵ leads us to question the validity of Eq. (2.1). Indeed, this suggests that the turbulent diffusion and/or pressure transport terms may contribute to the scale-by-scale budget Eq. (2.1) and consequently affect the value of A_ϵ . In addition, apart from the difficulty of measuring $\bar{\epsilon}$ (see for example the detailed review of hot wire measurements of small scale quantities by Antonia *et al.* (1986)) that may also explain the discrepancy for A_ϵ , the inadequate estimation of the virtual origin is likely to have a significant effect. Indeed, since the decay exponent for $\bar{\epsilon}$ is -4 , small variations of x_0 significantly alter the value of the prefactor. A more precise estimation of x_0 could be handled by plotting $\bar{\epsilon}D/U_j^3(x-x_0/D)^4$ for different values of x_0 and then identifying the value which yields the best plateau. Finally, the value of \mathcal{R} which was not measured by Mi *et al.* (2013) and Friehe *et al.* (1972) may differ between different experiments depending on initial/boundary conditions. More generally, it is not out of the question that different initial/boundary conditions may affect the make up of the one-point kinetic energy budget.

It seems clear that the difference between the predicted and measured values of A_ϵ are due to a large extent to an error in x_0 since discrepancies for A_{R_λ} are much less important, the latter being independent of the estimation of the virtual origin. For the data of Mi *et al.* (2013), the range of x ($20D \leq x \leq 30D$) is also likely to be too limited for a reliable fit to be applied. Further, self-similarity is questionable over this range.

In summary, providing that both local isotropy and complete self-similarity hold, and assuming that only the production and advection terms have to be retained for Eq. (2.1) to be valid on the axis of a round jet, the similarity analysis provided here leads to an exact expression for $\bar{\epsilon}$, $\overline{u^2}$ and all subsequent quantities. Further insight is provided into the relationship between the prefactors A_U , A_I , A_ϵ and A_{R_λ} . We show that a knowledge of only A_U , A_I and \mathcal{R} may be sufficient for A_ϵ and A_{R_λ} to be assessed indirectly. Experimental data provide satisfactory support for this. Note also that this analysis indicates that the virtual origin x_0 should be same, *i.e.* independent of the statistical quantity under consideration. This contrasts with the observation of Mi *et al.* (2013) which revealed substantial differences in the value of x_0 depending on the quantity used to assess it.

3. Consequences for the mean energy dissipation rate transport equation

When $r \rightarrow 0$, Eq. (2.1) reduces to the one-point transport equation for $\bar{\epsilon}$ (Antonia *et al.* (2000)). This follows by first writing a Taylor series expansion for $(\Delta u)^2$ and $(\Delta v)^2$

$$\overline{(\Delta\alpha)^2} = \overline{\left(\frac{\partial\alpha}{\partial x}\right)^2} r^2 - \frac{1}{12} \overline{\left(\frac{\partial^2\alpha}{\partial x^2}\right)^2} r^4 + \mathcal{O}(r^6), \quad (3.1)$$

where α stands for either u or v . Local isotropy requires $\overline{(\partial u/\partial x)^2} = \bar{\epsilon}/15\nu$ and $\overline{(\partial v/\partial x)^2} = 2\bar{\epsilon}/15\nu$. After equating terms in r^3 (see Antonia *et al.* (2000)), the final transport equation for $\bar{\epsilon}$ is

$$-\overline{U} \frac{\partial \bar{\epsilon}}{\partial x} + 2 \frac{\partial \overline{U}}{\partial x} \bar{\epsilon} = \frac{7}{3\sqrt{15}} \frac{\bar{\epsilon}^{3/2}}{\nu^{1/2}} \left[S + 2 \frac{G}{R_\lambda} \right] \quad (3.2)$$

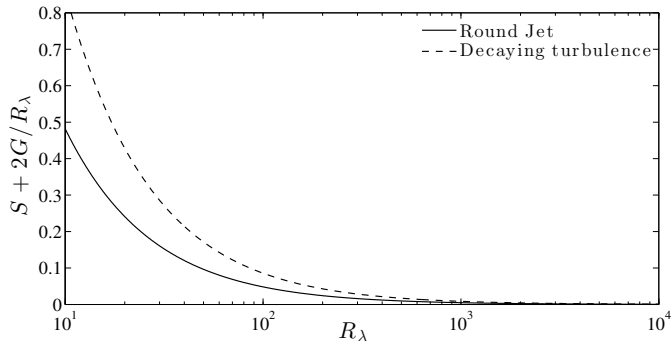


FIGURE 1. $S + 2G/R_\lambda$ as a function of R_λ in decaying (dashed line) and on the axis of a round jet (solid line). For the purely decaying flow, $\mathcal{R} = 1$ and $n_u = 1$ whereas for the jet flow $\mathcal{R} = 2/3$.

with

$$S = \frac{\overline{(\partial u / \partial x)^3}}{\left[\overline{(\partial u / \partial x)^2}\right]^{3/2}} \quad \text{and} \quad G = \frac{\overline{u^2 (\partial^2 u / \partial x^2)^2}}{\left[\overline{(\partial u / \partial x)^2}\right]^2}. \quad (3.3)$$

If similarity applies, \bar{U} behaves as per Eq. (2.6) while S , G and R_λ , cannot vary with x . The magnitudes of these quantities can however depend on the initial conditions. If $\bar{\epsilon} \sim (x - x_0)^{-n}$, then n can only equal 4 for Eq. (3.2) to be satisfied. This is consistent with our earlier deduction from the two-point energy budget.

Injecting Eqs. (2.6) and (2.8) into Eq. (3.2) and further using Eqs. (2.12) and (2.15), one obtains

$$S + \frac{2G}{R_\lambda} = \frac{90}{7(2 + \mathcal{R})} R_\lambda^{-1}, \quad (3.4)$$

which is an exact relation. The main objective in deriving Eq. (3.4) is to assess the way $S + 2G/R_\lambda$ approaches zero along the jet axis as R_λ increases indefinitely and compare with decaying homogeneous isotropic turbulence. In the latter case, $S + 2G/R_\lambda$ is given by (Batchelor & Townsend 1947; Lee *et al.* 2013)

$$S + \frac{2G}{R_\lambda} = \frac{90}{7(1 + 2\mathcal{R})} \left(\frac{1 + n_u}{n_u} \right) R_\lambda^{-1}, \quad (3.5)$$

where $n_u > 0$ is the decay exponent of the longitudinal velocity variance. These two exact equations (Eqs. (3.4) and (3.5)) reveal that $S + 2G/R_\lambda$ will reach an asymptotic state at an infinite Reynolds number corresponding to (locally) stationary turbulence, *i.e.* $S + 2G/R_\lambda = 0$. For finite Reynolds number flows, the equilibrium between S and G , two (very) small scale quantities, is likely to be altered by initial conditions (a large-scale phenomenon), through the large-scale anisotropy ratio \mathcal{R} and the decay exponent n_u for the purely decaying case. Note that $S + 2G/R_\lambda$ in each case approaches zero at the same rate, *i.e.* $S + 2G/R_\lambda \propto R_\lambda^{-1}$. In addition, one important comment is that even though internal intermittency may possibly influence the respective magnitudes of S (Van Atta & Antonia 1980; Sreenivasan & Antonia 1997) and G , the equilibrium between S and G as given by Eq. (3.4) or (3.5) cannot be altered by such effects.

The approach of $S + 2G/R_\lambda$ towards the asymptotic value of zero is represented in Fig. 1. For the purely decaying case, use was made of $\mathcal{R} = 1$ and $n_u = 1$, the latter value leading to complete self-preservation since it implies $R_\lambda = \text{constant}$. The reader is referred to Lee *et al.* (2013) who discussed the effect of varying n_u on $S + 2G/R_\lambda$. For the

jet flow, a value of $\mathcal{R} = 2/3$ is adopted on the basis of data by Burattini *et al.* (2005b). It is clear that the two flows will reach the asymptotic state following two distinct routes, albeit in the same hyperbolic fashion, *viz.* $S + 2G/R_\lambda \propto R_\lambda^{-1}$. On the jet axis $S + 2G/R_\lambda$ is always closer to zero by comparison to the purely decaying case. This difference is only attributed to some large-scale effects, namely (i) the large-scale anisotropy ratio, (ii) the decay exponent for the purely decaying case and (iii) more importantly through the additional effect of the production term in the scale-by-scale budget pertaining to the jet flow. Lindborg (1999); Lundgren (2002); Antonia & Burattini (2006) demonstrated that the maximum value of the non-linear transfer term approaches the asymptote of 4/5 with a scaling of the form $R_\lambda^{-2/3}$. It is thus interesting to note that inertial range scales lose the information about the large-scales at lower rate $R_\lambda^{-2/3}$ by comparison with the smallest scales, for which the scaling is R_λ^{-1} .

In summary, applying the limit at small separation of the scale-by-scale budget, the evolution of $\bar{\epsilon}$ with x on the axis of a round jet is, not surprisingly, recovered in agreement with our previous deduction from the two-point similarity analysis. Further, the transport equation for the mean dissipation rate on the jet axis yields an exact equation for the balance between S and G as a function of the Reynolds number. This equation may be valid independently of R_λ , providing that local isotropy holds, that complete self-preservation is attained and depends on the particular form assumed for the two-point budget equation. If only the production and advection terms are retained, $S + 2G/R_\lambda$ then appears to depend on the large-scales via the anisotropy ratio \mathcal{R} . The trend of $S + 2G/R_\lambda$ towards the asymptote is also compared to the one for a purely decaying flow where production is absent. For these two flows, the asymptotic state is reached along two different routes, although in each case $S + 2G/R_\lambda \propto R_\lambda^{-1}$.

Another important consequence of the present analysis relates to some modeling issues. As discussed by Pope (1978), it is generally found that the constants appearing in closure models such as the $k - \bar{\epsilon}$ model have to be adjusted depending on the flow under consideration. The analysis provided here may explain this *anomaly*. Indeed, let us start by rewriting Eq. (3.2) as

$$-\bar{U} \frac{\partial \bar{\epsilon}}{\partial x} = \mathcal{P} + C_{\epsilon 2} \frac{\bar{\epsilon}^2}{k}, \quad (3.6)$$

which is generally referred to as the $\bar{\epsilon}$ equation in the $k - \bar{\epsilon}$ model where $k = \bar{q}^2/2$ and \mathcal{P} denotes a production term. It thus appears that $C_{\epsilon 2}$ is explicitly related to S and G via

$$C_{\epsilon 2} = \frac{7}{90} (1 + 2\mathcal{R}) [SR_\lambda + 2G] \quad (3.7)$$

Injecting either Eq. (3.4) or (3.5) into Eq. (3.7) leads to

$$\begin{cases} C_{\epsilon 2} = \frac{n_u + 1}{n_u} & \text{in decaying turbulence} \\ C_{\epsilon 2} = \frac{1 + 2\mathcal{R}}{2 + \mathcal{R}} & \text{on the axis of a round jet} \end{cases} \quad (3.8)$$

Therefore, with $\mathcal{R} = 2/3$, one then obtains $C_{\epsilon 2} = 7/8$ in the jet flow which is very different from the value of 2 ($\equiv n_u = 1$) pertaining to a purely decaying turbulence. At this stage, it is clear that the value of $C_{\epsilon 2}$ in the $k - \bar{\epsilon}$ model will differ depending on the form Eq. (3.2) will assume in each flow. In addition, since the balance between the advection, production, diffusion and pressure transport terms varies from location to location in the flow, one should also expect similar variations for $C_{\epsilon 2}$. This confirms that any notion of universality for $C_{\epsilon 2}$ is untenable.

4. Conclusions

All the results presented here rely essentially on three main hypotheses: local isotropy, complete self-preservation, and the particular terms retained in Eq. (2.1). The first hypothesis, which only needs to be valid at the smallest scales (the scale of velocity derivatives) for Eq. (3.4) to be valid, does not appear to be too restrictive. We recall that Antonia & Mi (1993) showed that the departure from isotropy of $\bar{\epsilon}_\theta$, the temperature variance dissipation rate is smaller in the self-preserving region of a round jet than for other shear flows. Recent experiments by Darisse *et al.* (2014) confirm the small departure from isotropy for $\bar{\epsilon}_\theta$ on the jet axis. One expects that $\bar{\epsilon}$ should also exhibit a small departure from isotropy since the scalar field is usually more anisotropic than the velocity field. Even when departures from isotropy cannot be ignored, as for example on the axis of a plane jet, the locally isotropic form of the enstrophy equation appears to be satisfied (Antonia *et al.* 1986). In addition, instead of invoking the isotropic two-point budget equation as given by Eq. (2.1), all the present results may be inferred on the basis of the axisymmetric budget equation as provided by Danaïla *et al.* (2012) with similar deductions. The assumption of complete self-preservation appears to depend only on the constancy of the Reynolds number during the decay. Here again, this assumption is well verified on the axis of a round jet. However, there remains some doubts as to whether the particular forms for the one-point and two-point kinetic energy budget are known with sufficient accuracy. As far as the axis of a round jet is concerned, Burattini *et al.* (2005b); Panchapakesan & Lumley (1993); Taub *et al.* (2013) suggested that only the advection and production terms feature in Eq. (2.2). DNS data obtained at larger x/D and Re_D than in previous simulations could be desirable to confirm this.

One of the main outcomes of the present analysis is to underline the very broad potential offered by the scale-by-scale budget. Since the latter is derived directly from the Navier-Stokes equation, all the results can be considered as exact, provided that the three hypotheses mentioned above are respected. The power-law evolutions for $\bar{\epsilon}$, $\overline{u^2}$ and all subsequent quantities are not based on empirical observations but are derived analytically in an exact manner. The virtual origin is also proven to be the same independently of the quantity under consideration. Further, the analysis yields a very simple way of assessing the mean energy dissipation rate and the ratio between R_λ and Re_D provided A_U , \mathcal{R} and A_I , *i.e.* three readily measurable quantities, are known.

A last important comment ensues from our results. Indeed, Eqs. (3.4) and (3.5) are exact relations and it was shown that the equilibrium between S and G cannot be altered by any intermittency effects. The latter relations thus represent an important constraint that must be considered by any phenomenological theory of small scale turbulence. They also highlight the dependence between large and small scales, in part through the appearance of the large-scale anisotropy ratio. Large and small scales will therefore be statistically uncorrelated only at infinite Reynolds numbers for which $S + 2G/R_\lambda \rightarrow 0$. In addition, it imposes a fundamental dynamical constraint on the way S and G behave with respect to R_λ . The different routes along which $S + 2G/R_\lambda$ approaches 0 reflect the different physical processes associated with the advection, production, diffusion and dissipation of kinetic energy and hence the different forms that Eq. (3.2) will assume in each flow. It should therefore be more meaningful to examine the dependence of S on R_λ in each flow rather than by indiscriminately lumping together data for S from all flows, as was done in the past, *e.g.* Van Atta & Antonia (1980); Sreenivasan & Antonia (1997).

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