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# Kármán-Howarth closure equation on the basis of a universal eddy-viscosity

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The Kármán-Howarth equation [1] can be written in terms of velocity structure functions [2]

$$3\partial_t \overline{(\Delta u)^2} = \frac{1}{r^4} \partial_r \left[ r^4 \left( 6\nu \partial_r \overline{(\Delta u)^2} - \overline{(\Delta u)^3} \right) \right] - 4\bar{\epsilon}. \quad (1)$$

$\Delta u = u(x+r) - u(x)$  is the longitudinal velocity increment between two points separated by a distance  $r$  and  $\partial_{\alpha \bullet} = \partial \bullet / \partial \alpha$ . Further,  $\bar{\epsilon} = 15\nu(\partial_x u)^2$ , is the mean dissipation rate with  $\nu$  the kinematic viscosity and the overbar denotes averaging. Second- and third-order structure functions  $\overline{(\Delta u)^2}$  and  $\overline{(\Delta u)^3}$  appearing in Eq.(1) are usually interpreted as the kinetic energy and the kinetic energy transfer at a given scale respectively, two crucial quantities for modelling turbulent flows.

In spectral space, the equivalent equation known as Lin's equation [3] reads

$$\partial_t E(k) = T(k) - 2\nu k^2 E(k), \quad (2)$$

in which  $E(k)$  is the 3D energy spectrum,  $k$  the wavenumber and  $T(k)$  the spectral energy transfer function. Eq.(2) describes essentially the same physical mechanism as Eq.(1), *i.e.* the decay, the transfer and the dissipation of energy at a given scale or wavenumber.

In the last fifty years, several closures of Eq.(2) have been developed and are still extensively employed. Among others, we can cite the Direct Interaction Approximation model (DIA) proposed by Kraichnan [4] or the Eddy Damped Quasi Normal Markovian (EDQNM) closure [5].

On the contrary, closures of Eq.(1) have not received the same attention. To our knowledge, Millionshchikov [6] (in Russian), Domaradzki & Mellor [7], Effinger & Grossmann [8], Oberlack & Peters [9] and Baev & Chernykh ([10] and references therein) are the only authors who proposed a model (sometimes identical) for  $\overline{(\Delta u)^3}$ . All of them are based on the concept of an *eddy-viscosity*  $\nu_t$ , *i.e.* Eq.(1) is then formally rewritten as

$$3\partial_t \overline{(\Delta u)^2} = \frac{1}{r^4} \partial_r \left[ r^4 6(\nu + \nu_t) \partial_r \overline{(\Delta u)^2} \right] - 4\bar{\epsilon}. \quad (3)$$

The third-order structure function is thus related to  $\nu_t$  and  $\overline{(\Delta u)^2}$  through

$$\overline{(\Delta u)^3} = -6\nu_t \partial_r \overline{(\Delta u)^2}, \quad (4)$$

where  $\nu_t$  is a function of the separation  $r$ . Domaradzki & Mellor [7] proposed an expression for  $\nu_t$  on the basis of inertial range asymptotic relations ( $R_\lambda \rightarrow \infty$ , where  $R_\lambda = \sqrt{u^2} \lambda / \nu$  is the Reynolds number based on the

Taylor microscale  $\lambda \equiv \sqrt{15\nu u^2 / \bar{\epsilon}}$ ). However, as mentioned by the authors, the latter expression was not consistent with the scaling  $\overline{(\Delta u)^3} \propto r^3$  as  $r$  goes to zero. This constraint led Oberlack & Peters [9] to handle another expression for  $\nu_t$ , consistent with both dissipative and inertial range scaling laws. Here again,  $\nu_t$  was parametrized through a constant (called  $\kappa_0$  in their paper) the value of which relies on asymptotic inertial laws. Even though the use of asymptotic relations may be questionable in the context of finite Reynolds numbers flows (for instance, see [11] and references therein), both models were in satisfactory agreement with the third-order correlation functions measured by Stewart & Townsend [12] at (very) low Reynolds numbers ( $R_\lambda < 60$ ).

This intriguing feature indicates that the assumption of infinite Reynolds numbers is not a necessary condition for asymptotic expressions of  $\nu_t$  to be employed. Therefore, there is matter for investigating the approach towards the asymptote and the universal properties, *i.e.* the flow and  $R_\lambda$ -dependence, of the turbulent eddy-viscosity, with the goal of providing an efficient simple closure scheme in physical space.

The results presented in this paper highlight that the Kolmogorov normalized eddy-viscosity reveals a remarkable degree of universality over a wide range of scales. An analytical expression for  $\nu_t$  is provided revealing the existence of two universal parameters, the skewness of velocity derivative  $S$  and a new scale of turbulence called  $r_c$ . In the inertial range and beyond,  $\nu_t$  closely follows the asymptotic scaling even though neither  $\overline{(\Delta u)^2}$  nor  $\overline{(\Delta u)^3}$  indicate any unambiguous scaling. We then take advantage of these properties to model the third-order structure functions in different decaying flows, for Reynolds numbers  $R_\lambda$  lying between 50 and 1100. Finally, the model is numerically time-integrated to predict the decay of second-order structure functions and compared to experiments in grid turbulence ( $R_\lambda \approx 50$ ) for downstream distances up to  $80M$  ( $M$  is the grid mesh size).

In order to derive an analytical expression for  $\nu_t$ , we first recall that at small scales,  $\overline{(\Delta u)^2} = \bar{\epsilon} r^2 / 15\nu$  and  $\overline{(\Delta u)^3} = -S(\bar{\epsilon} r^2 / 15\nu)^{3/2}$ .  $S = -(\partial_x u)^3 / [(\partial_x u)^2]^{3/2}$  is the skewness of the longitudinal velocity derivative with respect to the longitudinal direction  $x$ . It follows that in the dissipative range

$$\frac{\nu_t}{\nu} = \frac{S}{12\sqrt{15}} r^{*2}. \quad (5)$$

Hereafter, the asterisk denotes normalization by the Kolmogorov scales, *i.e.*  $r^* = r/\eta$  with  $\eta = (\nu^3/\bar{\epsilon})^{1/4}$ . Second, in the context of infinite Reynolds numbers and for

90 scales in the range  $\eta \ll r \ll L$  ( $L$  is the integral length-  
 91 scale),  $(\overline{\Delta u})^2 = C_u (\bar{\epsilon} r)^{2/3}$  and  $(\overline{\Delta u})^3 = -A_u \bar{\epsilon} r$  ( $C_u = 2$ ,  
 92  $A_u = 4/5$  [13]). Hence, in the inertial range

$$\frac{\nu_t}{\nu} = \frac{1}{5C_u} r^{*4/3}. \quad (6)$$

93 Equation (6) was already proposed by Domaradzki &  
 94 Mellor [7], even though we became aware of this after we  
 95 derived it. Following *e.g.* [14], we match Eqs.(5) and (6)  
 96 into a single expression

$$\frac{\nu_t}{\nu} = \frac{S r^{*2}}{12\sqrt{15} [1 + \gamma r^{*2}]^{1/3}}. \quad (7)$$

97 Equation (7) generalizes the expression of [7] by covering  
 98 both dissipative and inertial ranges. In Eq.(7), the cross-  
 99 over length-scale between dissipative and inertial range  
 100  $r_c^2 = 1/\gamma$  is determined by equating Eqs.(5) and (6),  
 101 yielding  $r_c^* = (12\sqrt{15}/5C_u S)^{3/2}$ . As for the EDQNM  
 102 spectral closure, dissipative and inertial range intermit-  
 103 tency effects are not taken into account in the present  
 104 analysis. According to the Kolmogorov theory [13],  $S$ ,  
 105  $C_u$  and consequently  $r_c^*$  are universal. However, in the  
 106 context of finite Reynolds number flows,  $S$  and  $r_c^*$  are (*a*  
 107 *priori*) two free parameters. In the following, we turn our  
 108 attention to their evolution with respect to the Reynolds  
 109 number.

110 The analytical expression for  $\nu_t$  (Eq.(7)) is thus com-  
 111 pared to the one inferred from experiments in grid, wake,  
 112 round and plane jet turbulence. The Reynolds number  
 113 is in the range  $50 \leq R_\lambda \leq 1100$ . The grid turbulence  
 114 experiments are described in [15]. The wake flow facility  
 115 is described in [15] while experiments in the round and  
 116 plane jets are outlined in [15]. For the wake, round and  
 117 plane jet experiments, the measurements were made at  
 118 the centerline, thus avoiding to account for any additive  
 119 production terms in Eq.(1) due to the mean shear.

120 The dependence on Reynolds number of the measured  
 121 the eddy-viscosity is presented in Fig.1(a). At small  
 122 scales ( $r^* \lesssim 10$ ), all experimental points converge onto  
 123 a single curve which is well represented by Eq.(5) with  
 124  $S = 0.424$  (Fig.1(a)). The value of  $S$  used here is the  
 125 mean value between the five experiments.  $S$  varies by  
 126 only 5% from one experiment to another. This indicates  
 127 that the skewness of the velocity derivative  $S$  remains  
 128 constant in agreement with the Kolmogorov theory [13].  
 129 For the range of Reynolds numbers investigated, the ef-  
 130 fect of internal intermittency on  $S$  [16] is not discernible.  
 131 Both the constancy and the value itself of  $S$  are quite  
 132 consistent with all experimental values compiled by [17],  
 133 at least for the same range of Reynolds numbers. Fur-  
 134 ther, it is in perfect agreement with EDQNM [16].

135 As we progress through to the larger scales ( $10 \lesssim r^* \lesssim$   
 136  $10^2$ ), even though second-(not shown) and third-order  
 137 structure functions (Fig.1(b)) become  $R_\lambda$ -dependent, the  
 138 eddy-viscosity  $\nu_t$  follows the same evolution indepen-  
 139 dently of the Reynolds number. In other words, the  
 140 Kolmogorov normalized eddy-viscosity collapse over a

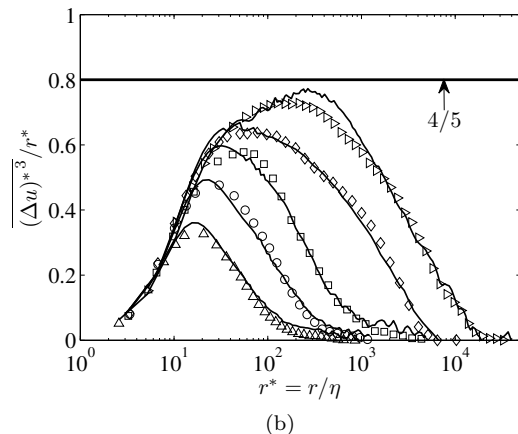
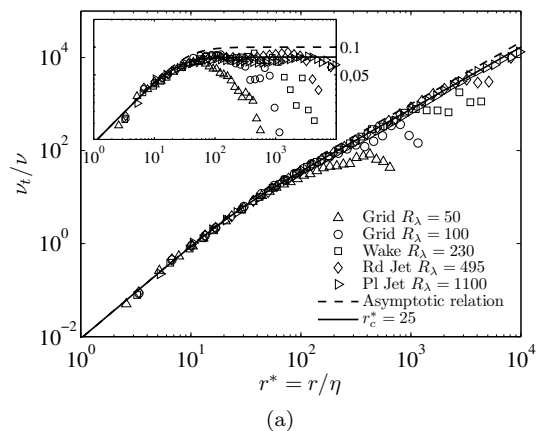


FIG. 1. (a)  $\nu_t/\nu$  as a function of  $r^*$  measured in different flows ( $50 \leq R_\lambda \leq 1100$ ). Eq.(7) (dashed line), Eq.(7) with  $r_c^* = 25$  (solid line). The inset depicts the compensated eddy-viscosity  $(\nu_t/\nu)/r^{*4/3}$ . (b) Kolmogorov-normalized third-order structure functions. Symbols are the same as in Fig.1(a), solid lines represent the present model using  $r_c^* = 25$

141 wider range of separations by comparison to  $(\overline{\Delta u^*})^2$  and  
 142  $(\overline{\Delta u^*})^3$ .

143 Then, for separations  $r^* \gtrsim 10^2$ , the effect of Reynolds  
 144 number becomes discernible and the  $r^{*4/3}$  scaling range  
 145 extends as the Reynolds number increases. Note that  
 146 the separation beyond which the measured eddy-viscosity  
 147 differs from the prediction of Eq.(7) in Fig.1(a) corre-  
 148 sponds to the scale beyond which  $(\overline{\Delta u})^3/r^*$  is almost  
 149 zero in Fig.1(b). Therefore,  $\nu_t$  remains universal in the  
 150 range of separations over which the third-order structure  
 151 function has to be modelled. We further observe that,  
 152 though very close to the asymptotic relation Eq.(7), a  
 153 constant value of  $r_c^* = 25.0$  (instead of 36.3 providing  
 154  $C_u = 2$ ) is more suitable to parametrize  $\nu_t$  over the whole  
 155 range of Reynolds numbers. This supports a universal  
 156 value for  $r_c^*$ , although weaker than the expected (Kol-  
 157 mogorov) value. This is in agreement with the observa-  
 158 tions of [7] revealing that the prefactor in Eq.(6) varies  
 159 by only a few percent in the range  $50 \leq R_\lambda \leq 10^4$  and re-

160 mains always smaller than the expected asymptotic value  
161 even at a very high Reynolds number.

162 Finally, the last observation that one can make is that  
163 at the highest Reynolds number ( $R_\lambda = 1100$ ), the scaling  
164  $\nu_t \propto r^{*4/3}$  is accurately satisfied over almost two decades  
165 of separations ( $10^2 \lesssim r^* \lesssim 10^4$ ) whilst there is no un-  
166 ambiguous scaling range for neither  $\overline{(\Delta u)^2}$  (not shown)  
167 nor  $\overline{(\Delta u)^3}$  (Fig.1(b)). The scaling range of  $\nu_t$  does not  
168 appear to be sensitive to any intermittency effect and is  
169 also much more extended than that of second- and third-  
170 order structure functions.

171 At this stage, we can draw the overall conclusion that,  
172 at least over the range of Reynolds numbers investigated  
173 here,  $S$  and  $r_c^*$  can be reasonably considered as univer-  
174 sal. The constancy of  $S$  relies on the validity of the Kol-  
175 mogorov normalization in the dissipative range, which  
176 holds even at low Reynolds numbers [18]. In contrast,  
177 the constancy of  $r_c^*$  is quite intriguing since it is now  
178 well known that the Kolmogorov 'constant'  $C_u$  and the  
179 scaling exponent of  $\overline{(\Delta u)^2}$  are sensitive to the Reynolds  
180 number variations (at least for  $R_\lambda < 10^4$  [11]). To a large  
181 extent, the observed universality of  $r_c^*$  is thus most likely  
182 due to some compensating effects that occur between  $C_u$ ,  
183  $A_u$ , and the scaling exponent of both  $\overline{(\Delta u)^2}$  and  $\overline{(\Delta u)^3}$   
184 involved in Eq.(6). The consequence is that  $r_c^*$  remains  
185 constant with respect to the Reynolds number.

186 The universality of  $\nu_t$  can be further justified recalling  
187 that  $\nu_t(r) \propto r^2 / \tau(r)$  (see Eq.(19) in [19]), in which the  
188 characteristic time-scale  $\tau(r)$  is representative of the cas-  
189 cade mechanism. In spectral space, one possible expres-  
190 sion for  $\tau(k)$  is that of Batchelor and Kraichnan [20] that  
191 was recently invoked by [21] as a closure for the passive  
192 scalar spectral equation. In [20],  $\tau(k)$  was interpreted as  
193 the time-scale of the strain at a given wavenumber due  
194 to all larger scales. Using Kolmogorov scaling,  $\tau(k)$  can  
195 be expressed as

$$\tau^*(k^*) \propto \left[ \int_0^{k^*} p^{*2} E^*(p^*) dp^* \right]^{-1/2}, \quad (8)$$

196 where  $p$  is a dummy integration variable. In Eq.(8), the  
197 normalized spectrum is multiplied by  $p^{*2}$  so that the  
198 contribution to the integral of the largest scales (low  
199 wavenumbers) is weak. On the contrary, contributions  
200 from the smallest scales are magnified and the range of  
201 scales over which the Kolmogorov scaling is observed is  
202 extended [18]. In other words, the integrand  $p^{*2} E^*(p^*)$  in  
203 Eq.(8) always satisfies Kolmogorov scaling over a larger  
204 range of scales compared to  $E^*(p^*)$  [18]. Therefore, since  
205  $\nu_t$  is intimately related to  $\tau$  via  $\nu_t(r) \propto r^2 / \tau(r)$ , the  
206 same conclusions can be drawn for the eddy-viscosity.

207 The idea of invoking a set of scales which yields a col-  
208 lapse of velocity statistics over a wider range of scales  
209 was already used in [22] for which the relevant scales are  
210  $\lambda$  and  $\overline{q^2} = \overline{u_i u_i}$  (twice the total kinetic energy). Fur-  
211 ther, in the energy-containing and inertial ranges, [23]  
212 demonstrated that the use of  $\overline{u^2}$  and the von Kármán  
213 length-scale ( $\equiv \overline{u^2}^{3/2} / \bar{\epsilon}$ ) leads to a satisfactory collapse of

214 energy spectra. As far as the eddy-viscosity is concerned,  
215 it appears that the relevant normalization is given by the  
216 Kolmogorov scales.

217 We now take advantage of this extended universality to  
218 develop a simple closure equation for Eq.(1). Third-order  
219 structure functions are thus calculated from measured  
220 second-order structure functions using Eqs.(4) and (7).  
221 The comparison between modelled and measured third-  
222 order structure functions is shown in Fig.1(b).

223 Since Eq.(7) accurately represents the measured eddy-  
224 viscosity, it is not surprising to observe that modelled  
225 and measured third-order structure functions are in ex-  
226 cellent agreement (Fig.1(b)). The shape and evolution  
227 of  $\overline{(\Delta u)^3} / r^*$  with respect to the Reynolds number are  
228 very well reproduced. The minor differences that may be  
229 observed are rather due to some slight errors in evalu-  
230 ating the derivative of measured second-order structure  
231 functions.

232 A much more stringent test of the validity of the  
233 present closure is the following. Starting with an ini-  
234 tial condition at a particular position in the flow, can  
235 we reliably predict the decay of second-order structure  
236 functions downstream? To this end, Eq.(3) has to be  
237 time-integrated.

238 Since theory is compared to a spatially decaying tur-  
239 bulence (in this case grid turbulence [15]), we relate the  
240 final time of integration to the downstream distance by  
241 means of Taylor's hypothesis, *i.e.*  $x \equiv Ut$  ( $U$  is the mean  
242 flow velocity). The time-integration of Eq.(3) is handled  
243 using a fourth-order Runge-Kutta algorithm. Derivatives  
244  $\partial_r \bullet$  are approximated by a central second-order finite dif-  
245 ference scheme. Boundary conditions are set as follows,  
246  $\overline{(\Delta u)^2}(r=0) = 0$  and  $\partial_r \overline{(\Delta u)^2}(r \rightarrow \infty) = 0$ .

247 Results are given in Fig.2(a). The initial conditions are  
248 set at  $x = 20M$  behind the grid ( $M = 24.76\text{mm}$  is the  
249 grid mesh size) and predictions are compared with mea-  
250 surements at  $x = 40, 60$  and  $80M$ . The initial Reynolds  
251 number  $R_\lambda$  is about 50 and decreases slightly with  $x$ .

252 Measured and predicted second-order structure func-  
253 tions are in good agreement (Fig.2(a)). Minor differ-  
254 ences can be observed at large separations where the  
255 model very slightly overestimates  $\overline{(\Delta u)^2}$ . From the de-  
256 cay of second-order structure functions, one can obtain  
257 the evolution of one-point statistics, *i.e.* the longitudinal  
258 velocity variance  $2u^2 = \overline{(\Delta u)^2}(r \rightarrow \infty)$ , the mean dis-  
259 sipation rate  $\bar{\epsilon} = 15\nu \lim_{r \rightarrow 0} \overline{(\Delta u)^2} / r^2$ , the Taylor and  
260 Kolmogorov length scales ( $\lambda$  and  $\eta$ ) and the Reynolds  
261 number  $R_\lambda$ . The mean dissipation rate can also be eval-  
262 uated though the one point energy budget

$$\bar{\epsilon} = -\frac{1}{2} \partial_t \overline{q^2}, \quad (9)$$

263 where  $\overline{q^2} = \overline{u^2} + \overline{v^2} + \overline{w^2}$  is twice the total kinetic en-  
264 ergy. The evolution of one-point statistics is depicted in  
265 Fig.2(b). The variation with respect to the downstream  
266 distance of all these quantities is globally very well repro-  
267 duced by the present model. One can further note that

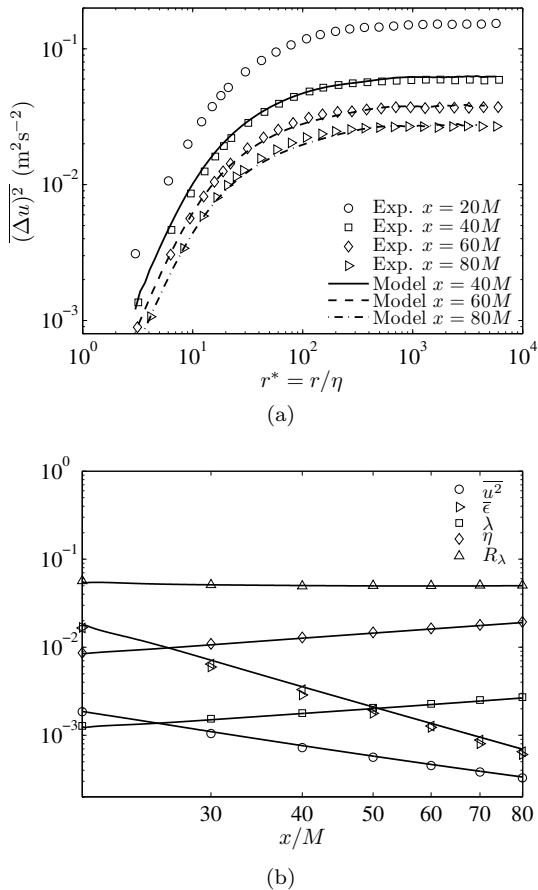


FIG. 2. (a) Comparison between measured and predicted second-order structure functions in grid turbulence ( $R_\lambda \approx 50$ ). The time-integration is started at  $x = 20M$ . (b) Evolution of  $\overline{u^2}/U^2$ ,  $\bar{\epsilon}M/U^3(10^2)$ ,  $\lambda/M(10^{-2})$ ,  $\eta/M$  and  $R_\lambda(10^{-3})$  with  $x/M$ .  $U = 6.4\text{m.s}^{-1}$  is the mean flow velocity. Symbols represent measured values whilst solid lines are the predicted values. The mean dissipation rate  $\bar{\epsilon}$  is estimated from the relation  $\bar{\epsilon} = 15\nu(\partial_x u)^2$  ( $\triangleright$ ) and from Eq.(9) ( $\triangleleft$ ). The measured Taylor and Kolmogorov length-scales were inferred from  $\bar{\epsilon}$  computed from Eq.(9).

the magnitude of the measured mean dissipation rate inferred from  $\bar{\epsilon} = 15\nu(\partial_x u)^2$  is smaller ( $\approx 15\%$ ) than that predicted by the model. This discrepancy may be due to the smallest scales not being sufficiently resolved by the hot wire measurements. Indeed, values of  $\bar{\epsilon}$  using Eq.(9) are only  $\approx 10\%$  smaller than those predicted.

The idea of predicting the decay of one-point statistics from a two-point closure equation was also tackled by Lohse [8], with a closure scheme based on the variable range mean field theory. In the latter study, the prediction of basic quantities, such as the normalized energy dissipation and enstrophy decay rates, compared favourably with experimental results in a particular type of decaying flow where the integral length scale does not vary with time. Obviously, this type of analytical treatment cannot be applied to decaying grid turbulence where the integral length scale grows continuously with time (or distance from the grid).

In summary, the universal facets of the eddy-viscosity for the closure of the Kármán-Howarth equation are examined in detail. It is highlighted that  $\nu_t$  remains impressively universal over a remarkable range of scales. An analytical expression for  $\nu_t$  is further proposed, based on the observed constancy of the skewness of velocity derivatives and highlights the existence of a new scale of turbulence called  $r_c$ . The model is in good agreement with measurements in different types of decaying flows, over a wide range of Reynolds numbers. The closure scheme is finally time-integrated and reproduces measured second-order structure functions in grid turbulence quite favourably.

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